TFY 4345 Computational Assignment 1 : Projectile Motion

Introduction

This is the first compulsory computational exercise for the subject TFY4345 - Classical Mechanics, which has to be returned by October 11th at 13:00 on Blackboard. You may work either alone or in pairs (latter recommended). Refer to your partner when handing in the assignment. Everyone must return a report due to bookkeeping purposes (the same report for pairs, this is subject to changes if we find a more elegant solution in BB). **For crucial background information, please read the sections preceding the tasks.** Prepare your report using either LaTeX or by writing it directly into Jupyter Notebook using Markdown cells. Pay special attention on the quality of figures as these are the most important content (are the labels visible, is the line thickness appropriate, etc.). Attached source code should be zipped. You may program in any language as long as you prepare your own code from scratch. The code should be easy to read with comments and consistent naming schemes. Assessment: Accepted (1) or requires a revision (0).

Numerical calculation of a projectile trajectory

Let us consider two spatial dimensions, ignoring air resistance the equations of motion (EOM) read

$$\begin{aligned} \frac{d^2x}{dt^2} &= 0, \\ \frac{d^2y}{dt^2} &= -g, \end{aligned} \tag{1}$$

where g is the gravitational acceleration. These are second-order differential equations, to solve them numerically we will express them as a set of ordinary differential equations (ODE's), i.e.

$$\begin{aligned}\frac{dx}{dt} &= v_x,\\ \frac{dv_x}{dt} &= 0,\\ \frac{dy}{dt} &= v_y,\\ \frac{dv_y}{dt} &= -g.\end{aligned} \tag{2}$$

We have twice as many equations now, however, we can now solve each of them using a standard ODE solving scheme such as the Euler method or Runge-Kutta 4th-order scheme.

These schemes are based on the finite difference form of the derivative. If we have an ODE involving a function $f(q)$, we can approximate the derivative by discretizing the argument q, resulting in the finite difference approximation of the derivative

$$\frac{df}{dq} \approx \frac{f(q_{i+1}) - f(q_i)}{\Delta q}, \tag{3}$$

where $\Delta q = q_{i+1} - q_i$ is the sampling interval of our discretization. The function evaluated at q_{i+1} is then approximately given by

$$f_{i+1} \approx f_i + \frac{df}{dq}\Delta q, \tag{4}$$

where $f_i \approx f(q_i)$. The accuracy of the approximation is improved by decreasing the sampling interval Δq. Solving Eq. (4) iteratively is the basis of the Euler scheme.

Using the Euler method the EOM (2) can be written in the following way

$$\begin{aligned}x_{i+1} &= x_i + v_{x,i}\Delta t,\\ v_{x,i+1} &= v_{x,i},\\ y_{i+1} &= y_i + v_{y,i}\Delta t,\\ v_{y,i+1} &= v_{y,i} - g\Delta t,\end{aligned} \tag{5}$$

where Δt is the time step of each iteration. For a given set of initial conditions $x_0, v_{x,0}, y_0, v_{y,0}$ the ODE's can be solved iteratively. You should now figure out how to do the same thing for the Runge-Kutta 4th-order scheme.

Since this model does not include any resistive forces when the projectile hits the ground, the motion will be calculated indefinitely as if there is no barrier. Therefore, we need to stop the code when the projectile position is below the ground, i.e. $y < 0$. Depending on the discretization interval Δt used, the last position is very unlikely to be exactly at $y = 0$. To determine the landing point of the projectile we will use linear interpolation. The landing point x_l is then approximately given by

$$x_l = \frac{x_n + r x_{n+1}}{r + 1}, \tag{6}$$

where

$$r = -y_n/y_{n+1}. \tag{7}$$

Air drag

Let us now expand our simple model by adding air resistance. In general, the force from air drag can be written as

$$F_{drag} \approx -B_1 v - B_2 v^2. \tag{8}$$

Here B_1 can be computed for simple objects using Stoke's law, while B_2 is more complicated and has to be estimated. For low velocities the B_1 term dominates, while for high velocities it contributes little compared to the other term and can be neglected. The B_2 term is based on the fact that a moving object must push air in front of it out of the way, after a few steps we obtain (you can check this yourself by considering the mass of displaced air and considering that the work done by the drag force equals to the gained kinetic energy of displaced air)

$$-B_2 v^2 = -\frac{1}{2} C \rho A v^2, \tag{9}$$

where ρ is the air density, A is the frontal-profile area of the object and C is the drag coefficient which depends on the aerodynamics of the object.

Assuming high velocities, we will from now on use only the second term to simulate the air drag force

$$F_{drag} = -B_2 v^2. \tag{10}$$

This force is always directed opposite to the velocity, and both vector components v_x and v_y must be taken into account. Adding this force to the equations of motion yields (check)

$$\begin{aligned}
x_{i+1} &= x_i + v_{x,i}\Delta t, \\
v_{x,i+1} &= v_{x,i} - \frac{B_2 v v_{x,i}}{m}\Delta t, \\
y_{i+1} &= y_i + v_{y,i}\Delta t, \\
v_{y,i+1} &= v_{y,i} - g\Delta t - \frac{B_2 v v_{y,i}}{m}\Delta t.
\end{aligned} \tag{11}$$

More realistic air drag models can be incorporated in the same manner.

The trajectory of a projectile was calculated with and without air drag using the methods described above and can be seen in Fig. 1. Here we can see the dramatic effect air drag can have on a projectile trajectory. However, the model used here is flawed as it does not take the air density at high altitudes into account. Since the air density is lower at high altitudes compared to sea level the air drag should also be reduced accordingly. In the next sections we will look at two different air density models and how to incorporate them into the drag force.

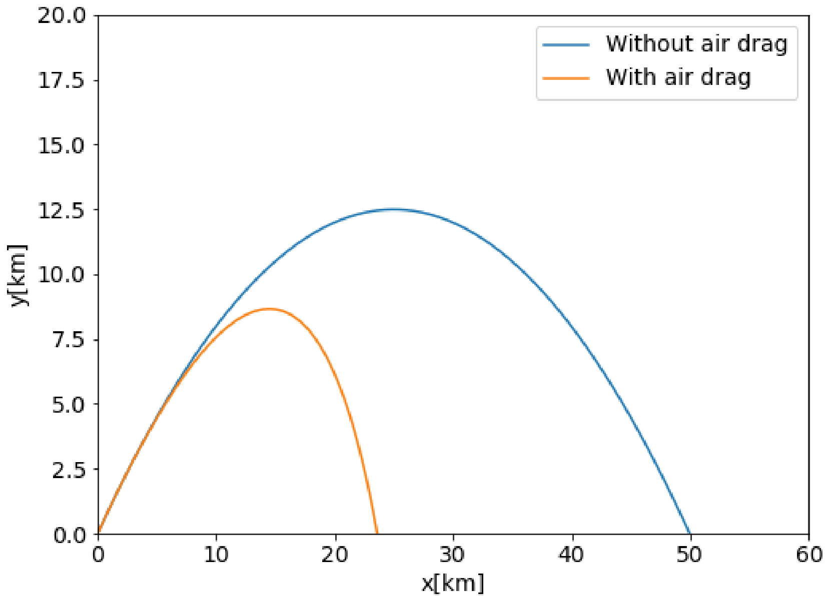

Figure 1: The trajectory of a projectile with and without taking air drag into account. The initial velocity was 700m/s and the firing angle used was 45°.

Air density models

From Eq. (9) we known that the drag force is proportional to the density of the air. If we know how the air density depends on the altitude, the corrected drag force $F^{\star}_{drag}(y)$ can be obtained from

$$F^{\star}_{drag}(y) = \frac{\rho(y)}{\rho_0} F_{drag}(y = 0), \tag{12}$$

where ρ_0 is the air density at sea level ($y = 0$) and $\rho(y)$ is the air density as a function of altitude. Incorporating this into the Eq. (11) in the previous section is as simple as multiplying the drag coefficient B_2 by $\rho(y_i)/\rho_0$. We will now look at two air density models which depend on the altitude.

Isothermal ideal gas

By treating the atmosphere as an *isothermal* ideal gas, one finds that the air density depends on the altitude in the following way

$$\rho = \rho_0 \exp\left(-\frac{y}{y_0}\right). \tag{13}$$

Here $y_0 = k_B T/\bar{m}g$, where k_B is Boltzmann's constant, T is the absolute temperature and $\bar{m}$ is the average mass of an air molecule. This model is not realistic for altitude changes over several kilometers, since we know that air temperature can vary drastically over such distances.

Adiabatic approximation

Another approach is to assume that air is a poor conductor of heat and that convection is slow. This leads to the so called *adiabatic* approximation, which is much better for the troposphere ($y \leq 10$km). The adiabatic approximation leads to the following dependence of air density on altitude

$$\rho = \rho_0 \left(1 - \frac{ay}{T_0}\right)^{\alpha}, \tag{14}$$

where $a \approx 6.5 \times 10^{-3}$K/m fits measured data for our troposphere fairly well. Here T_0 is the sea level temperature in Kelvin and the exponent $\alpha \approx 2.5$ for air.

Task 1: Projectile motion with air resistance

a) First make a program that can calculate **projectile motion in 2D without any drag**. Use the Runge-Kutta algorithm (4th order) when solving the ODE's. Note that solving Eq. (4) iteratively is the basis of the Euler method, not Runge-Kutta. Adapt you algorithm correspondingly. Compare your numerical solution with the analytical solution (derive it) and remember to consider the effect of the time step Δt used. Do a test to validate that the numerical approach converges to the analytical one when Δt is systematically decreased. Use a firing angle of $\theta = 45°$ and a firing velocity of $v_0 = 700$m/s for the initial conditions as a benchmark. Select an appropriate time step for later use.

B_2/m	4×10^{-5} m^{-1}
$k_B T/\bar{m}g$	10^4 m
a	6.5×10^{-3}K/m
α	2.5

Table 1: Constants and parameters.

b) Next, calculate the trajectory of a cannon shell **including both air drag and the reduced air density at high altitudes**. Perform your calculation for different firing angles (θ) and determine the optimal value of the angle that gives the maximum range with the following air drag models:

- Without any air density corrections, see Eq. (10).
- With the isothermic density model, see Eqs. (12), (13).
- With the adiabatic density model, see Eqs. (12), (14).

Use the parameters and constants presented in table 1.

Task 2: The Paris gun

The Germans used a long-range cannon in World War I to bombard Paris with 106-kilogram shells. Its muzzle velocity was 1640 m/s. Find its **predicted range, maximum projectile height and projectile time of flight**. Assume the same air drag model and B_2/m parameter as above. Use the adiabatic air density model.