Math 2414: Mixed Series Solutions

1. $\displaystyle\sum_{n=1}^{\infty} ne^{-2n^2}$

   - Let $f(x) = xe^{-2x^2}$.
   - f is continuous and positive on $[1, \infty)$.

   $$\begin{aligned} f'(x) &= x(-4xe^{-2x^2}) + e^{-2x^2} \\ &= e^{-2x^2}(1 - 4x^2) \end{aligned}$$

   - $f'(x) < 0$ on $(1, \infty)$.

   $$F(b) = \int_1^b xe^{-2x^2}\ dx$$

   $$u = -2x^2 \quad du = -4x\ dx$$

   $$\begin{aligned} &= -\frac{1}{4}\int_{-2}^{-2b^2} e^u\ du \\ &= -\frac{1}{4}e^u\Big]_{-2}^{-2b^2} \\ &= -\frac{1}{4}(e^{-2b^2} - e^{-2}) \\ &= -\frac{1}{4}\left(\frac{1}{e^{2b^2}} - \frac{1}{e^2}\right) \end{aligned}$$

   $$\begin{aligned} \int_1^{\infty} xe^{-2x^2}\ dx &= \lim_{b\to\infty} F(b) \\ &= \frac{1}{4e^2} \end{aligned}$$

   - $\sum a_n$ converges by the IT.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}$

   $\displaystyle\lim_{n\to\infty} \frac{1}{3^n} = 0$

   $a_{n+1} = \dfrac{1}{3^{n+1}} < \dfrac{1}{3^n} = a_n$

   The series converges by the Alternating Series Test.

3. $\displaystyle\sum_{n=0}^{\infty} e^{-2n}$

$$\begin{aligned}\sum_{n=0}^{\infty} e^{-2n} &= \sum_{n=0}^{\infty}\left(\frac{1}{e^2}\right)^n \\ &= \frac{1}{1-\frac{1}{e^2}} \\ &= \frac{e^2}{e^2-1}\end{aligned}$$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$

   Since $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$, the series is a divergent p-series.

5. $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(n+1)!(n+1)!}{(3n+3)!}\cdot\frac{(3n)!}{n!n!} \\ &= \lim_{n\to\infty}\frac{(n+1)n!(n+1)n!}{(3n+3)(3n+2)(3n+1)(3n)!}\cdot\frac{(3n)!}{n!n!} \\ &= \lim_{n\to\infty}\frac{n+1}{(3n+3)(3n+2)(3n+1)} \\ &= 0\end{aligned}$$

   The series converges by the Ratio Test.

6. $\displaystyle\sum_{n=0}^{\infty} \frac{e^n}{n!}$

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{e^{n+1}}{(n+1)!}\cdot\frac{n!}{e^n} \\ &= \lim_{n\to\infty}\frac{e}{n+1} \\ &= \infty\end{aligned}$$

   The series converges by the Ratio Test.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}2n^2}{(3n+1)^2}$

$\displaystyle\lim_{n\to\infty} \frac{2n^2}{(3n+1)^2} = \frac{2}{9}$

$\displaystyle\lim_{n\to\infty} \frac{(-1)^{n+1}2n^2}{(3n+1)^2} = \begin{cases} -\frac{2}{9}, & \text{if } n \text{ is even} \\ \frac{2}{9}, & \text{if } n \text{ is odd} \end{cases}$

$\displaystyle\lim_{n\to\infty} \frac{(-1)^{n+1}2n^2}{(3n+1)^2}$ diverges.

The series converges by the Alternating Series Test.

8. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$

$\displaystyle\frac{\sin^2 n}{2^n} \le \frac{1}{2^n}$ for all $n \ge 0$.

$\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ is a convergent geometric series.

$\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$ converges by the Direct Comparison Test.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n}$

Let $a_n = \dfrac{n+2^n}{n^2 2^n}$ and $b_n = \dfrac{1}{n^2}$.

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n+2^n}{2^n} \\ &= \lim_{n\to\infty} \left(\frac{n}{2^n} + 1\right) \\ &= 1\end{aligned}$$

Since $\sum b_n$ is a convergent p-series, $\sum a_n$ converges by the Limit Comparison Test.

10. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n3^n}$

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{(n+1)!}{(n+1)3^{n+1}} \cdot \frac{n3^n}{n!} \\ &= \lim_{n\to\infty} \frac{(n+1)n!}{(n+1)3^{n+1}} \cdot \frac{n3^n}{n!} \\ &= \lim_{n\to\infty} \frac{n}{3} \\ &= \infty\end{aligned}$$

The series diverges by the Ratio Test.

11. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

$$\begin{aligned} y &= \lim_{x\to\infty} \frac{3^x}{x^3} \quad \infty/\infty \\ &= \lim_{x\to\infty} \frac{(\ln 3)3^x}{3x^2} \quad \infty/\infty \\ &= \lim_{x\to\infty} \frac{(\ln 3)^2 3^x}{6x} \quad \infty/\infty \\ &= \lim_{x\to\infty} \frac{(\ln 3)^3 3^x}{6} \\ &= \infty \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} \frac{3^n}{n^3} &= \infty \\ \sum_{n=1}^{\infty} \frac{3^n}{n^3} & \quad \textit{diverges} \end{aligned}$$

12. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n 5^{2-n}$

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n 5^{2-n} &= \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \frac{5^2}{5^n} \\ &= \sum_{n=0}^{\infty} 25 \left(\frac{1}{20}\right)^n \\ &= \frac{25}{1-\frac{1}{20}} \\ &= \frac{500}{19} \end{aligned}$$

13. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n+1}$

Let $a_n = \dfrac{\ln n}{n+1}$ and $b_n = \dfrac{1}{n}$.

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n\ln n}{n+1} \\ &= \lim_{n\to\infty} \frac{n \cdot \frac{1}{n} + \ln n}{1} \\ &= \lim_{n\to\infty} (1+\ln n) \\ &= \infty\end{aligned}$$

Since $\sum b_n$ is a divergent p-series, $\sum a_n$ diverges by the Limit Comparison Test.

14. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

- Let $f(x) = \dfrac{\ln x}{x^2}$.
- f is continuous and positive on $[2, \infty)$.

$$\begin{aligned} f'(x) &= \frac{x^2 \cdot \frac{1}{x} - 2x \ln x}{x^4} \\ &= \frac{x - 2x \ln x}{x^3} \end{aligned}$$

- $f'(x) < 0$ on $(2, \infty)$.

$$F(b) = \int_2^b \frac{\ln x}{x^2}\, dx$$

$$\begin{array}{rclcrcl} u &=& \ln x & \quad & dv &=& \frac{1}{x^2}\, dx \\ du &=& \frac{1}{x}\, dx & & v &=& -\frac{1}{x} \end{array}$$

$$\begin{aligned} &= -\frac{\ln x}{x} + \int_2^b \frac{1}{x^2}\, dx \\ &= -\frac{\ln x}{x} - \frac{1}{x}\Big]_2^b \\ &= -\frac{\ln x + 1}{x}\Big]_2^b \\ &= -\left(\frac{\ln b + 1}{b} - \frac{\ln 2 + 1}{2}\right) \end{aligned}$$

$$\begin{aligned} \int_2^{\infty} \frac{\ln x}{x^2}\, dx &= \lim_{b\to\infty} F(b) \\ &= \frac{\ln 2 + 1}{2} \end{aligned}$$

- $\sum a_n$ converges by the IT.

15. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+2}$

$\displaystyle\lim_{n\to\infty} \frac{1}{3n+2} = 0$

$a_{n+1} = \dfrac{1}{3n+5} < \dfrac{1}{3n+2} = a_n$

The series converges by the Alternating Series Test.

16. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}2}{e^n - e^{-n}}$

$$\begin{aligned}\lim_{n\to\infty} \frac{2}{e^n - e^{-n}} &= \lim_{n\to\infty} \frac{\frac{2}{e^n}}{1 - \frac{1}{e^{2n}}} \\ &= 0\end{aligned}$$

Let $f(x) = \dfrac{2}{e^x - e^{-x}}$.

$f'(x) = -\dfrac{2(e^x + e^{-x})}{(e^x - e^{-x})^2}$.

Since $f'(x) < 0$ for all $x \geq 1$, $a_{n+1} < a_n$ for all $n \geq 1$.

The series converges by the Alternating Series Test.

17. $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n-1}}{4^{n+1}}$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{3^{n-1}}{4^{n+1}} &= \sum_{n=1}^{\infty} \frac{1}{12}\left(\frac{3}{4}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{12}\left(\frac{3}{4}\right)^n - \frac{1}{12} \\ &= \frac{\frac{1}{12}}{1 - \frac{3}{4}} - \frac{1}{12} \\ &= \frac{1}{3} - \frac{1}{12} \\ &= \frac{1}{4}\end{aligned}$$

18. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{9n^2+15n+4}$

$$\begin{aligned}
\frac{1}{(3n+1)(3n+4)} &= \frac{A}{3n+1} + \frac{B}{3n+4} \\
1 &= A(3n+4) + B(3n+1)
\end{aligned}$$

$$\begin{aligned}
n &= -\tfrac{1}{3} & n &= -\tfrac{4}{3} \\
3A &= 1 & -3B &= 1 \\
A &= \tfrac{1}{3} & B &= -\tfrac{1}{3}
\end{aligned}$$

$$\begin{aligned}
S_n &= \frac{1}{3}\left[\left(1-\frac{1}{4}\right) + \left(\frac{1}{4}-\frac{1}{7}\right) + \cdots + \left(\frac{1}{3n+1} - \frac{1}{3n+4}\right)\right] \\
&= \frac{1}{3}\left(1 - \frac{1}{3n+4}\right) \\
\sum_{n=0}^{\infty} \frac{1}{9n^2+15n+4} &= \lim_{n\to\infty} S_n \\
&= \frac{1}{3}
\end{aligned}$$

19. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{2n^2}$

$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{|a_n|} &= \lim_{n\to\infty}\sqrt[n]{\left[\left(\frac{n}{n+1}\right)^{2n}\right]^n}\\ &= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{2n}\end{aligned}$$

$$\begin{aligned}y &= \lim_{x\to\infty}\left(\frac{x}{x+1}\right)^{2x}\\ \ln y &= \lim_{x\to\infty} 2x\ln\left(\frac{x}{x+1}\right)\\ &= \lim_{x\to\infty}\frac{2\ln\left(\frac{x}{x+1}\right)}{\frac{1}{x}} \quad 0/0\\ &= \lim_{x\to\infty}\frac{2(\ln x-\ln(x+1))}{\frac{1}{x}}\\ &= \lim_{x\to\infty}\frac{2\left(\frac{1}{x}-\frac{1}{x+1}\right)}{-\frac{1}{x^2}}\\ &= \lim_{x\to\infty}\frac{-2x}{x+1}\\ &= -2\\ y &= e^{-2}\end{aligned}$$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = e^{-2}$$

The series converges by the nth-Root Test.

20. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^{n}$

$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{|a_n|} &= \lim_{n\to\infty}\frac{n}{2n+1}\\ &= \frac{1}{2}\end{aligned}$$

The series converges by the nth-Root Test.

21. $\displaystyle\sum_{n=0}^{\infty} e^{-n^2}$

$\dfrac{1}{e^{n^2}} \le \dfrac{1}{e^n}$ for all $n \ge 0$.

Since $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{e}\right)^n$ is a convergent geometric series, $\displaystyle\sum_{n=0}^{\infty} e^{-n^2}$ converges by the Direct Comparison Test.

22. $\displaystyle\sum_{n=1}^{\infty}\frac{2^n n!}{n^n}$

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{2^n n!}\\ &= \lim_{n\to\infty}\frac{2^{n+1}(n+1)n!}{(n+1)(n+1)^n}\cdot\frac{n^n}{2^n n!}\\ &= \lim_{n\to\infty}\frac{2n^n}{(n+1)^n}\\ &= \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^n\end{aligned}$$

$$\begin{aligned}y &= \lim_{x\to\infty}\left(\frac{x}{x+1}\right)^{x}\\ \ln y &= \lim_{x\to\infty} x\ln\left(\frac{x}{x+1}\right)\\ &= \lim_{x\to\infty}\frac{\ln\left(\frac{x}{x+1}\right)}{\frac{1}{x}} \quad 0/0\\ &= \lim_{x\to\infty}\frac{\ln x-\ln(x+1)}{\frac{1}{x}}\\ &= \lim_{x\to\infty}\frac{\frac{1}{x}-\frac{1}{x+1}}{-\frac{1}{x^2}}\\ &= \lim_{x\to\infty}\frac{\frac{1}{x(x+1)}}{-\frac{1}{x^2}}\\ &= \lim_{x\to\infty}\frac{-x}{x+1}\\ &= -1\\ y &= e^{-1}\end{aligned}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{e}$$

The series converges by the Ratio Test.

23. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^2+1}}{\sqrt{n^3+2}}$

$$\begin{aligned} \frac{\sqrt[3]{n^2+1}}{\sqrt{n^3+2}} &> \frac{\sqrt[3]{n^2}}{\sqrt{n^3+2}} \\ &> \frac{\sqrt[3]{n^2}}{\sqrt{n^3+3n^3}} \\ &= \frac{n^{2/3}}{2n^{3/2}} \\ &= \frac{1}{2n^{5/6}} \end{aligned}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{5/6}}$ is a convergent p-series, so $\displaystyle\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^{5/6}}$ converges.

So $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^2+1}}{\sqrt{n^3+2}}$ converges by the Direct Comparison test.

24. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+3}$

- Let $f(x) = \dfrac{1}{x+3}$.
- f is continuous and positive on $[1, \infty)$.
- $f'(x) = -\dfrac{1}{(x+3)^2} < 0$ on $(1, \infty)$.

$$\begin{aligned} F(b) &= \int_1^b \frac{1}{x+3}\,dx \\ &= \ln(x+3)\Big]_1^b \\ &= \ln(b+3) - \ln 4 \end{aligned}$$

$$\begin{aligned} \int_1^{\infty} \frac{1}{x+3}\,dx &= \lim_{b\to\infty} F(b) \\ &= \infty \end{aligned}$$

- $\sum a_n$ diverges by the IT.

25. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$

- The series is a p-series with $p = \dfrac{1}{5} < 1$, so it diverges.

26. $1 + \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} + \dfrac{1}{4\sqrt{4}} + \cdots$

- The series is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is a p-series with $p = \dfrac{3}{2} > 1$, so it converges.

27. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$

$a_n = \dfrac{n}{(n^2+1)^2}$ behaves like $b_n = \dfrac{1}{n^3}$ for large n.

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^4}{n^4+2n^2+1} \\ &= 1 \end{aligned}$$

Since $\sum b_n$ is a convergent p-series, $\sum a_n$ converges by the Limit Comparison Test.

28. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{(2n-1)\pi}{2}$

$$\sin\frac{(2n-1)\pi}{2} = \begin{cases} 1, & \text{if } n \text{ is odd} \\ -1, & \text{if } n \text{ is even} \end{cases}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{(2n-1)\pi}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

$$a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$$

The series converges by the Alternating Series Test.

29. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{3^n}$

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{n+1}{3} \\ &= \infty \end{aligned}$$

The series diverges by the Ratio Test.

30. $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{(n+1)!(n+1)!}{(3n+3)!}\cdot\frac{(3n)!}{n!n!} \\
&= \lim_{n\to\infty} \frac{(n+1)n!(n+1)n!}{(3n+3)(3n+2)(3n+1)(3n)!}\cdot\frac{(3n)!}{n!n!} \\
&= \lim_{n\to\infty} \frac{n+1}{(3n+3)(3n+2)(3n+1)} \\
&= 0
\end{aligned}$$

The series converges by the Ratio Test.