Series Worksheet Solutions : Study these solutions only AFTER you have attempted the problems yourself.

1. Find the EXACT sum:

a) $$\sum_{n=1}^{300}\left[6\cdot\left(\frac{3}{5}\right)^n\right] = 6\cdot\left(\frac{3}{5}\right)\cdot\left[1+\frac{3}{5}+\left(\frac{3}{5}\right)^2+\left(\frac{3}{5}\right)^3+\ldots+\left(\frac{3}{5}\right)^{299}\right] = \frac{18}{5}\cdot\left[\frac{1-\left(\frac{3}{5}\right)^{300}}{1-\frac{3}{5}}\right] = 9\cdot\left[1-\left(\frac{3}{5}\right)^{300}\right]$$

b) $$\sum_{n=25}^{\infty}\left[7\cdot\left(\frac{1}{4}\right)^n\right] = 7\cdot\left(\frac{1}{4}\right)^{25}\cdot\left[1+\frac{1}{4}+\left(\frac{1}{4}\right)^2+\ldots\right] = 7\cdot\left(\frac{1}{4}\right)^{25}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n = 7\cdot\left(\frac{1}{4}\right)^{25}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n$$

$$= 7\cdot\left(\frac{1}{4}\right)^{25}\cdot\left(\frac{1}{1-\frac{1}{4}}\right) = \frac{28}{3}\cdot\left(\frac{1}{4}\right)^{25}$$

c) $$1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\frac{1}{81}\ldots = \sum_{n=0}^{\infty}\left(\frac{-1}{3}\right)^n = \frac{1}{1-\left(\frac{-1}{3}\right)} = \frac{3}{4}$$

d) $$\sum_{n=0}^{\infty}\left(11\cdot\frac{2^{n+3}}{9^{n+2}}\right) = 11\cdot\frac{2^3}{9^2}\cdot\sum_{n=0}^{\infty}\left(\frac{2}{9}\right)^n = \frac{88}{81}\cdot\frac{1}{1-\left(\frac{2}{9}\right)} = \frac{88}{81}\cdot\frac{9}{7} = \frac{88}{63}$$

e) $$\sum_{n=0}^{\infty} e^{-\pi n} = \sum_{n=0}^{\infty}\left(\frac{1}{e^{\pi}}\right)^n = \frac{e^{\pi}}{e^{\pi}-1}$$

2. Use a convergence test to determine if the series converge or diverge. Do not conflate the various tests and their conclusions.

a) $$\sum_{n=1}^{\infty}\left(\frac{4n^2+7n^3}{8n+10n^5}\right) < \sum_{n=1}^{\infty}\left(\frac{4n^3+7n^3}{10n^5}\right) = \frac{11}{10}\cdot\sum_{n=1}^{\infty}\frac{1}{n^2}$$, which converges by the p−series rule ($p = 2$).

Therefore, $\displaystyle\sum_{n=1}^{\infty}\left(\frac{4n^2+7n^3}{8n+10n^5}\right)$ converges by comparison, since the sum is smaller. The first inequality is true since a smaller denominators and larger numerators make the whole fraction larger, and $8n > 0$ for $n > 1$.

b) Applying the root test, $\displaystyle\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\left(2+\frac{3}{n}\right) = 2$. Thus the series $\displaystyle\sum_{n=1}^{\infty}\left(2+\frac{3}{n}\right)^n$ diverges by the root test, since $L > 1$.

c) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\cdot\ln(n)}$; We will apply the integral test: $\displaystyle\int_2^{\infty}\frac{1}{x\cdot\ln(x)}\,dx = \lim_{k\to\infty}\int_2^{k}\frac{1}{x\cdot\ln(x)}\,dx =$

$\displaystyle\lim_{k\to\infty}\int_{\ln(2)}^{\ln(k)}\frac{1}{u}\,du = \lim_{k\to\infty}\left(\ln(\ln(k))-\ln(\ln(2))\right) = \infty$, thus the improper integral diverges. Therefore,

$\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\cdot\ln(n)}$ diverges by the integral test.

d) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{6^n}$; Using the ratio test, we have $\displaystyle\lim_{n\to\infty} \left| \frac{(n+1)!}{6^{n+1}} \cdot \frac{6^n}{n!} \right| = \lim_{n\to\infty} \left| \frac{n+1}{6} \right| = \infty$. Therefore, the series $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{6^n}$ diverges by the ratio test, since $L > 1$.

e) For the series $\displaystyle\sum_{n=1}^{\infty} \left[(-1)^{n+1} \cdot \frac{n}{n^2+3} \right]$, Take the derivative of the mother function, $\dfrac{x}{x^2+3}$ and find the x–values for which $f'(x) < 0$. This will show that the terms a_n will decrease. And Since $\displaystyle\lim_{n\to\infty} \frac{n}{n^2+3} = \lim_{n\to\infty} \frac{1}{2n} = 0$, using L'Hopital's Rule , the AST concludes that $\displaystyle\sum_{n=1}^{\infty} \left[(-1)^{n+1} \cdot \frac{n}{n^2+3} \right]$ converges.

f) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2+\sqrt{n}}{n+10} \right) > \sum_{n=1}^{\infty} \left(\frac{\sqrt{n}}{n+10n} \right) > \frac{1}{11} \cdot \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges by the p–series, with $p = \dfrac{1}{2}$. Thus, $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2+\sqrt{n}}{n+10} \right)$ diverges by comparison. The first inequality is true since smaller numerators and larger denominators yield smaller fractions.

g) $\displaystyle\sum_{n=2}^{\infty} \left[(-1)^{n+1} \cdot \frac{3}{\ln(n)} \right]$; Use the AST, and you can show that the terms decrease by taking the derivative of the mother function $f(x) = \dfrac{3}{\ln(x)}$ and showing that it is negative. Note that $\displaystyle\lim_{n\to\infty} \frac{3}{\ln(n)} = 0$. Thus the series $\displaystyle\sum_{n=2}^{\infty} \left[(-1)^{n+1} \cdot \frac{3}{\ln(n)} \right]$ converges by the AST.

h) $\displaystyle\sum_{n=1}^{10000000} \left(n^2 + 3n \right)$ is a finite series so the sum exists ! In fact, this one you can compute rather easily !!!

i) Applying the nth term test, $\displaystyle\lim_{n\to\infty} \frac{7+2n}{12+5n} = \frac{2}{5}$, thus the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{7+2n}{12+5n} \right)$ diverges by the nth–term test since $\displaystyle\lim_{n\to\infty} a_n \neq 0$.

j) Applying the ratio test gives $\displaystyle\lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right| = \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^n = e$.

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges by the ratio test, since $L > 1$.

k) Since $\displaystyle\lim_{n\to\infty} \sin(n)$ DNE (no HA for $y = \sin(x)$) , the nth term–test says that $\displaystyle\sum_{n=1}^{\infty} \sin(n)$ diverges.

l) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^2} + \frac{1}{n} + \frac{1}{n^4} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} \frac{1}{n^4}$. Note that the first and the last series converges by the p–series rule (p = 2 and p = 4) but the 2nd series diverges by the p–series rule (p = 1). Therefore the original series the p–series rule (p = 2 and p = 4) but the 2nd series diverges by the p–series rule (p = 1). Therefore the original series diverges.

m) Applying the root test, $\lim_{n \to \infty} \sqrt[n]{a_n} = \lim_{n \to \infty} \frac{8+4n^2}{n+n^2} = 4$. Thus the series $\sum_{n=1}^{\infty} \left(\frac{8+4n^2}{n+n^2}\right)^n$ diverges by the root test, since $L > 1$.

n) Simplifying the algebra, we have $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \sum_{n=1}^{\infty} \left[\frac{(n+1)-n}{n(n+1)}\right] = \sum_{n=1}^{\infty} \frac{1}{n^2+n} < \sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges by the p–series $(p = 2)$. The last inequality is true since smaller denominators yield larger fractions. Therefore, the series $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ converges by comparison. In fact, this is an example of a "telescoping" series, and the sum of the series can be computed. $S = 1$!

o) Comparing the graphs, we see that $\ln(x) < x$ for all $x > 2$. This means that $\frac{1}{\ln(x)} > \frac{1}{x}$ for all $x > 2$. Then it follows $\frac{1}{\ln(n)} > \frac{1}{n}$ for all $n > 2$. Thus, $\sum_{n=2}^{\infty} \frac{1}{\ln(n)} > \sum_{n=2}^{\infty} \frac{1}{n}$, which diverges by the p–series $(p = 1)$. Thus the series $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ diverges by comparison.

p) Applying the root test gives $\lim_{n \to \infty} \sqrt[n]{a_n} = \lim_{n \to \infty} \frac{1}{2n+3} = 0$. Therefore the series $\sum_{n=1}^{\infty} \left(\frac{1}{2n+3}\right)^n$ converges by the root test, since $L < 1$.

q) $\sum_{n=1}^{\infty} \frac{(n+3)!}{n!} = \sum_{n=1}^{\infty} [(n+3)\cdot(n+2)\cdot(n+1)] = \infty$ of course, since $\lim_{n \to \infty} [(n+3)\cdot(n+2)\cdot(n+1)] = \infty$

r) Applying the nth term test, $\lim_{n \to \infty} \left(1 - \frac{4}{n+5}\right) = 1$. Therefore the series $\sum_{n=1}^{\infty} \left(1 - \frac{4}{n+5}\right)$ diverges by the nth term test since $\lim_{n \to \infty} a_n \neq 0$.

s) $\sum_{n=0}^{\infty} \frac{1}{n^2+8^n} < \sum_{n=0}^{\infty} \frac{1}{8^n} = \sum_{n=0}^{\infty} \left(\frac{1}{8}\right)^n = \frac{1}{1-\frac{1}{8}} = \frac{8}{7}$. Therefore the series $\sum_{n=0}^{\infty} \left(\frac{1}{n^2+8^n}\right)$ converges by comparison, since the sum is less than $\frac{8}{7}$. Could this series be compared to $\sum_{n=0}^{\infty} \frac{1}{n^2}$?

t) Applying the integral test, we have $\int_1^{\infty} x^3 \cdot e^{-x^4}\, dx = \lim_{k \to \infty} \int_1^{k} x^3 \cdot e^{-x^4}\, dx =$

$\lim_{k \to \infty} \int_{-1}^{-k^4} e^u \cdot \left(\frac{-1}{4}\right) du = \frac{-1}{4} \cdot \lim_{k \to \infty} \left(e^{-k^4} - e^{-1}\right) = \frac{1}{4 \cdot e}$. Therefore, the series $\sum_{n=1}^{\infty} \left(n^3 \cdot e^{-n^4}\right)$ converges by the integral test since the improper integral converges.

u) Applying the ratio test, $\lim\limits_{n \to \infty} \left| \frac{3^{n+1}}{(n+3)!} \cdot \frac{(n+2)!}{3^n} \right| = \lim\limits_{n \to \infty} \left| \frac{3}{n+3} \right| = 0$. Therefore, the series $\sum\limits_{n=1}^{\infty} \frac{3^n}{(n+2)!}$ converges by the ratio test since $L < 1$.

v) Since $\lim\limits_{n \to \infty} n = \infty$, we conclude that the series $\sum\limits_{n=1}^{\infty} n$ diverges by the nth–term test. A little common sense goes a long way. Remember that a series has no chance of being convergent if the nth term does not tend to 0.

w) $\sum\limits_{n=1}^{\infty} \left(\frac{n + 5n^7}{n^{10}} \right) = \sum\limits_{n=1}^{\infty} \frac{1}{n^9} + 5 \cdot \sum\limits_{n=1}^{\infty} \frac{1}{n^3}$, which converges since it is the sum of 2 convergent series, by the p–series rule ($p = 9$ and $p = 3$)

x) Applying the nth term test, $\lim\limits_{n \to \infty} \ln\left(2 + \frac{3}{n}\right) = \ln(2)$. Therefore the series $\sum\limits_{n=1}^{\infty} \ln\left(2 + \frac{3}{n}\right)$ diverges by the nth term–test. since $L \neq 0$.