Calculus 1

Limit

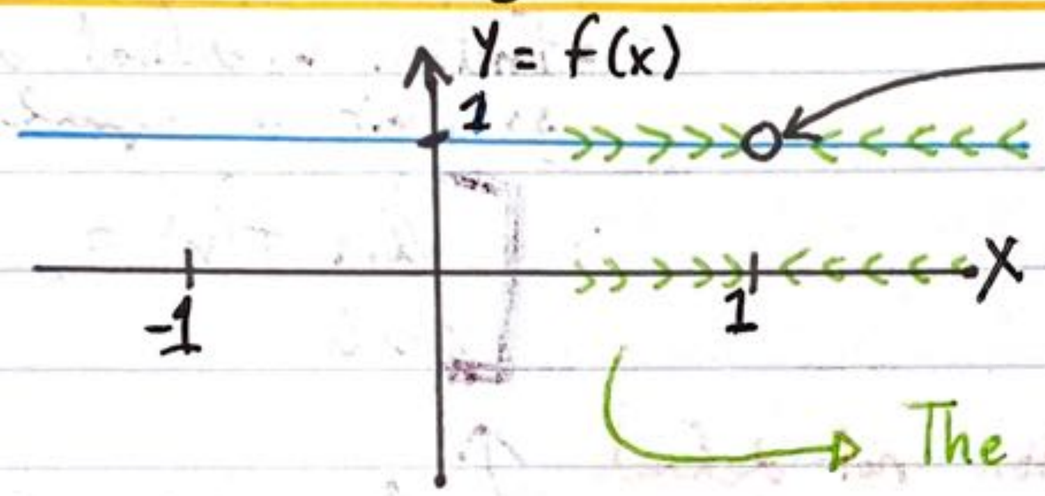
eg1 **Straight** $\longleftrightarrow$

$$f(x) = \frac{x-1}{x-1} \rightarrow f(x) = 1, \quad x \neq 1$$

* Technically $f(x) =$ cancels into $\underline{1}$ but that does not mean its $\underline{1}$

$$f(1) = \frac{0}{0}$$ → * Therefore it is undefined

eg2



o = undefined

lim = limit

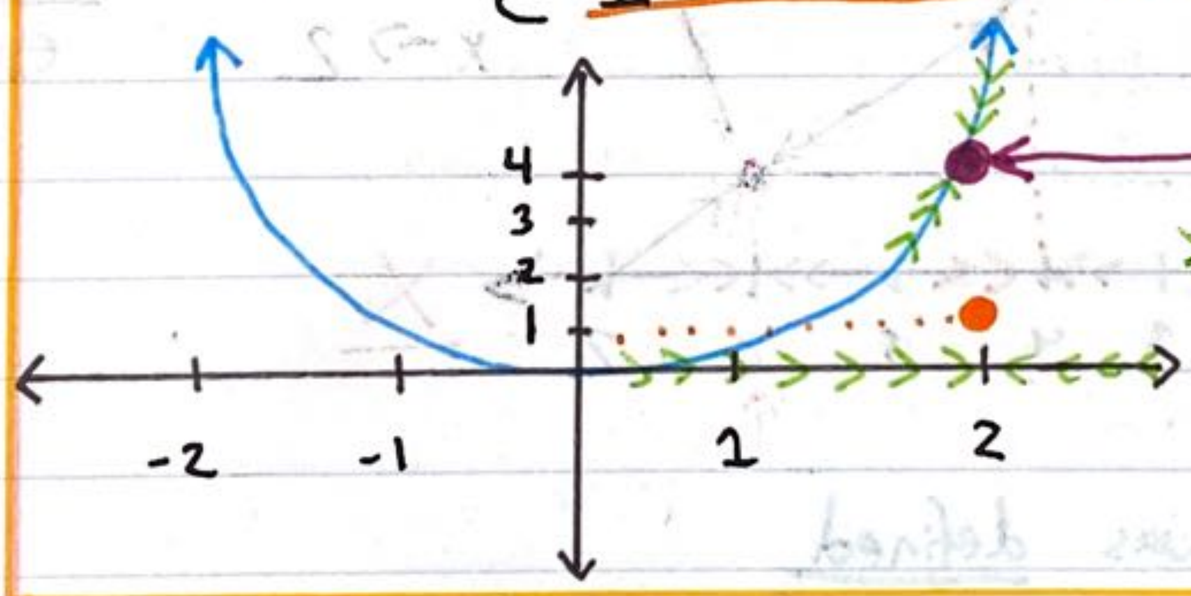
→ The function is approaching 1 $f(x) = 1$

Parabola ↔

$$g(x) = \begin{cases} x^2 = x \neq 2 \\ 1 = x = 2 \end{cases}$$

← This shows a discontinity

← * Y axis and X-axis *



O = undefined

>> → means approaching

$$\lim_{x \to 2} g(x) = 4$$

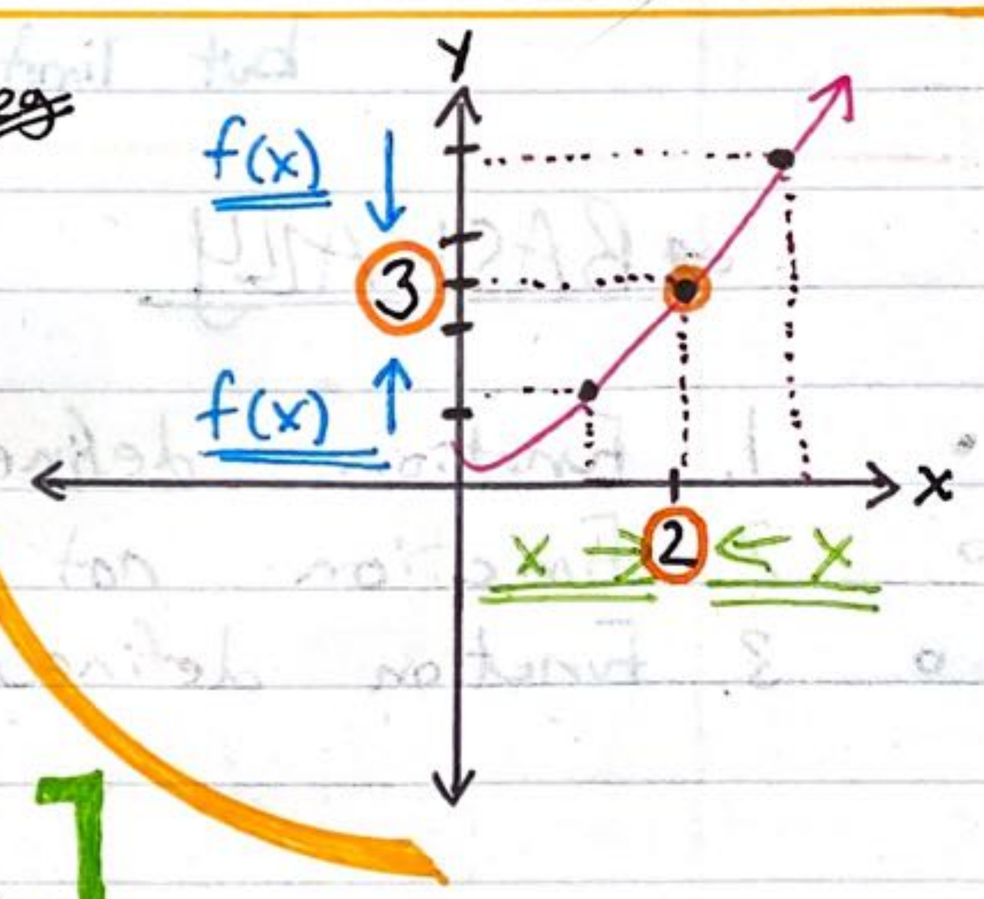
$$g(x) = \begin{cases} x^2 ; x \neq 2 \\ 1 ; x = 2 \end{cases}$$

• First point is at $\underline{y=1}$, $\underline{x=2}$

* This shows discontinity

[$\underline{x=2}$ but in x^2 mean $2^2 = \underline{4}$ meaning the limit for g(x) is $\underline{4}$]

eg

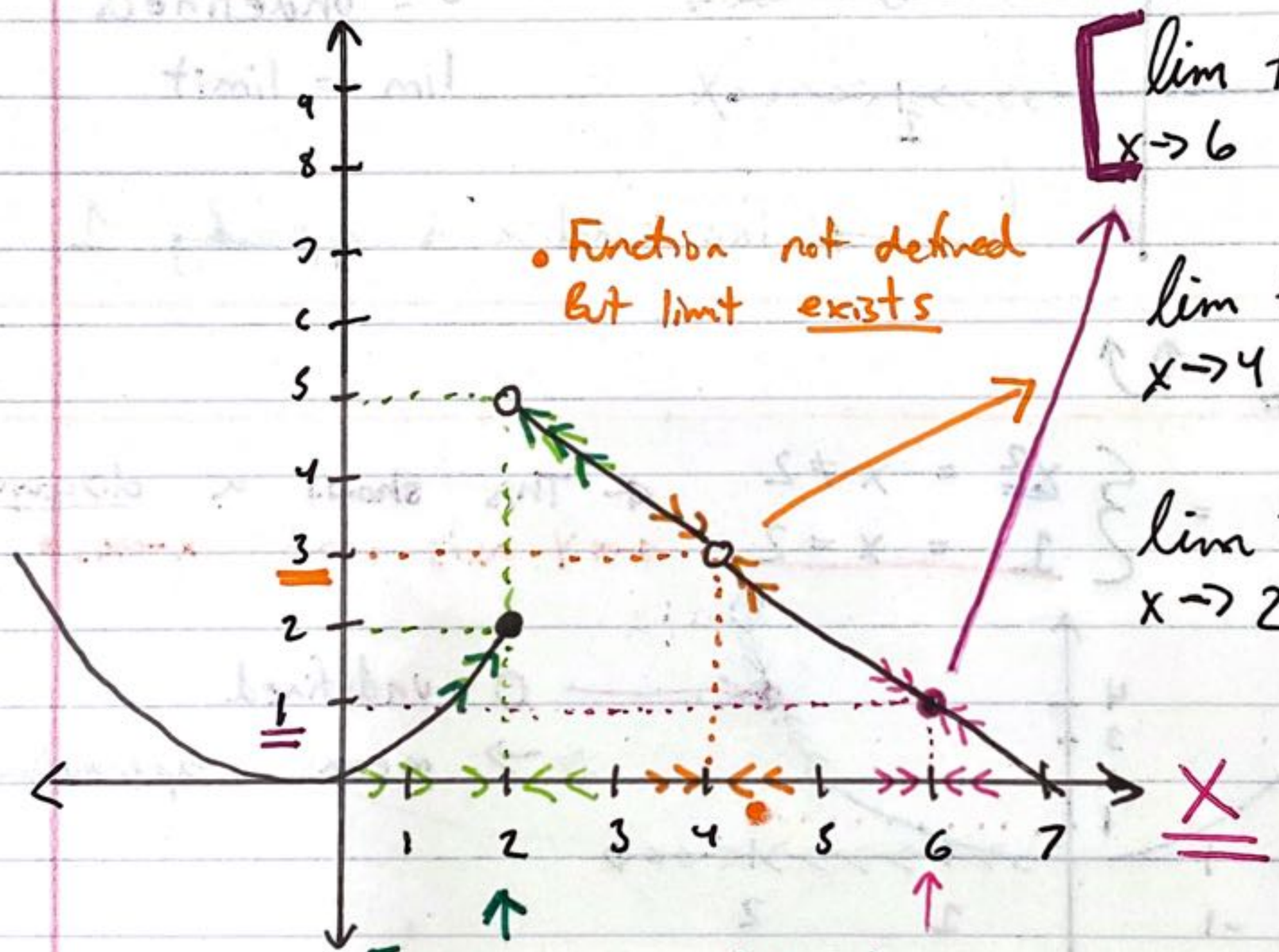


"The limit of ..."

$$\lim_{x \to 3} f(x)$$

"... the function f..."

"... as x approaches 3"



• Function is defined at 6 ✓ so limit is equal to value

$\lim_{x \to 6} f(x) =$ 1

• Function not defined but limit exists

$\lim_{x \to 4} f(x) =$ 3

$\lim_{x \to 2} f(x) =$ does not exist

• function was defined but limit doesnt exist

BASICALLY

1. Function defined, Limit equal to value
2. Function not defined, but limit exists
3. Function defined but LIMIT DOESNT EXIST

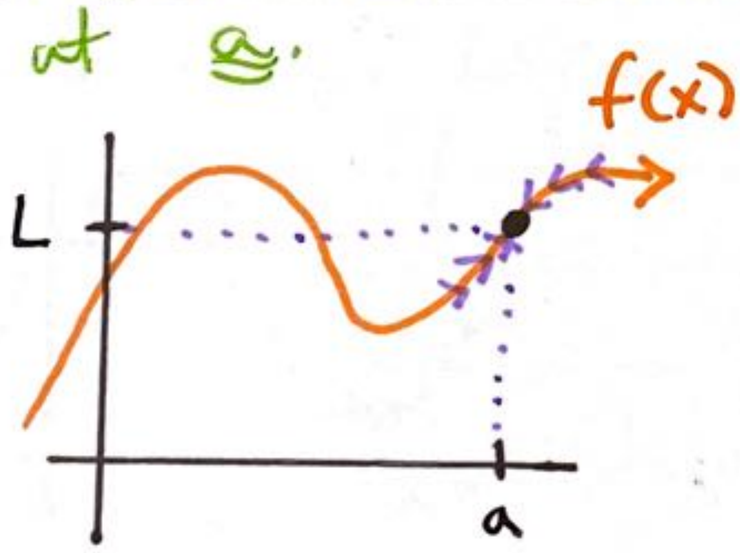
1.1 Limits An Intuitive Approach

Def: Let $\underline{a}$ be a real number and suppose $f(x)$ is defined as being near $\underline{a}$ (but maybe not at a).

* Basically = $\underline{a}$ is a real number and $\underline{f(x)}$ is a number close to $\underline{a}$ but not $\underline{a}$.

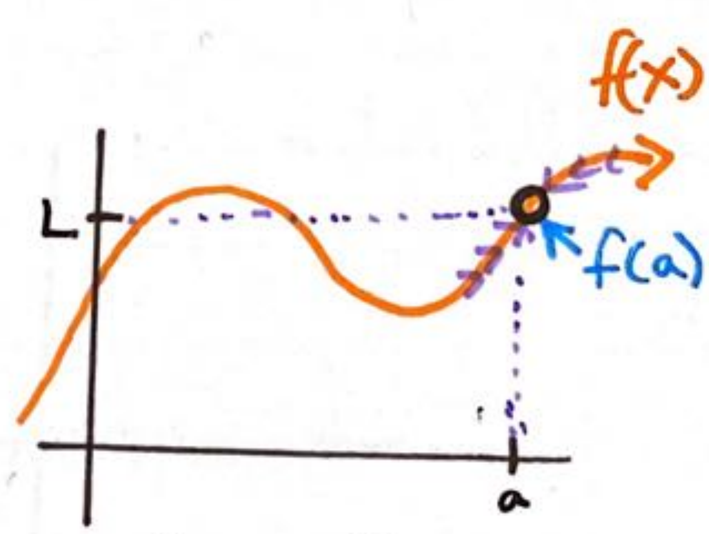
Then $\lim_{x \to a} f(x) = L$ if values of $f(x)$ get arbitrarily close to $\underline{L}$ as $\underline{x}$ gets close to $\underline{a}$.

* This is saying that a <u>Limit</u> is a balance of a <u>function</u> near a particular value but might not actually be the function evaluated at $\underline{a}$.



$\lim_{x \to a} f(x) = L$

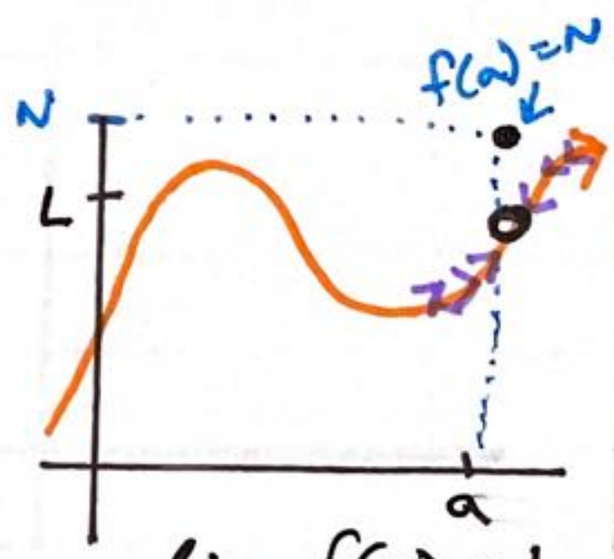
• = defined



$\lim_{x \to a} f(x) = L$

f(a) is <u>undefined</u>

O = undefined



$\lim_{x \to a} f(x) = L$

f(a) is <u>N</u>

Ex Examine the behavior of $f(x) = x^2 - x + 2$ for values near 2 to gives $\lim_{x \to 2} f(x)$ is_

X	f(x)
1.99	3.970100
1.999	3.999001
2	- ?
2.001	4.003001
2.01	4.030100

Guess:

$\lim_{x \to 2} f(x) = \underline{4}$

Ex $f(x) = \frac{x-1}{x^2-1}$ near 1 to give

$\lim_{x \to 1} f(x)$ is ___ ?

X	f(x)
0.99	0.502
0.999	0.500250
1	?
1.001	0.499750
1.01	0.497512

Guess:

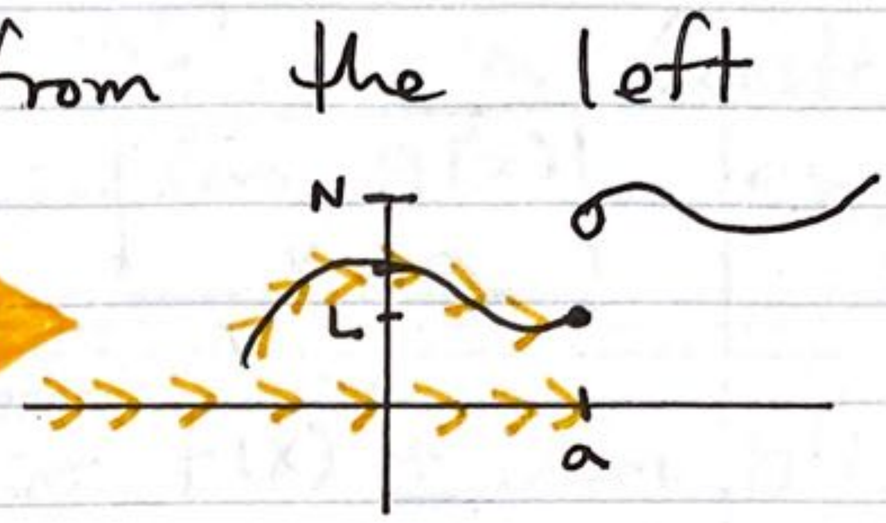
$\lim_{x \to 1} f(x) = \underline{0.5}$

→→→→ BOTH SIDES ←←←←

One Side Limits.

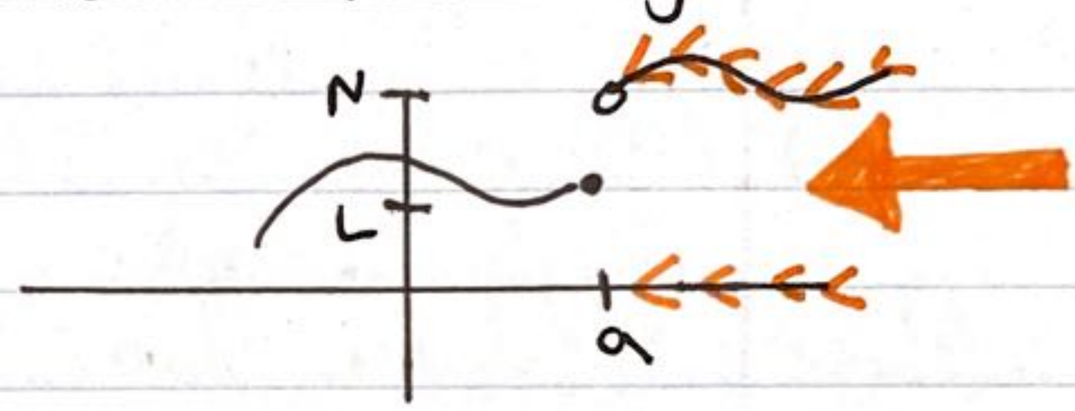
When approaching from the left we write:

$$\boxed{\lim_{x \to a^-} f(x) = L}$$



When approaching from the right we write:

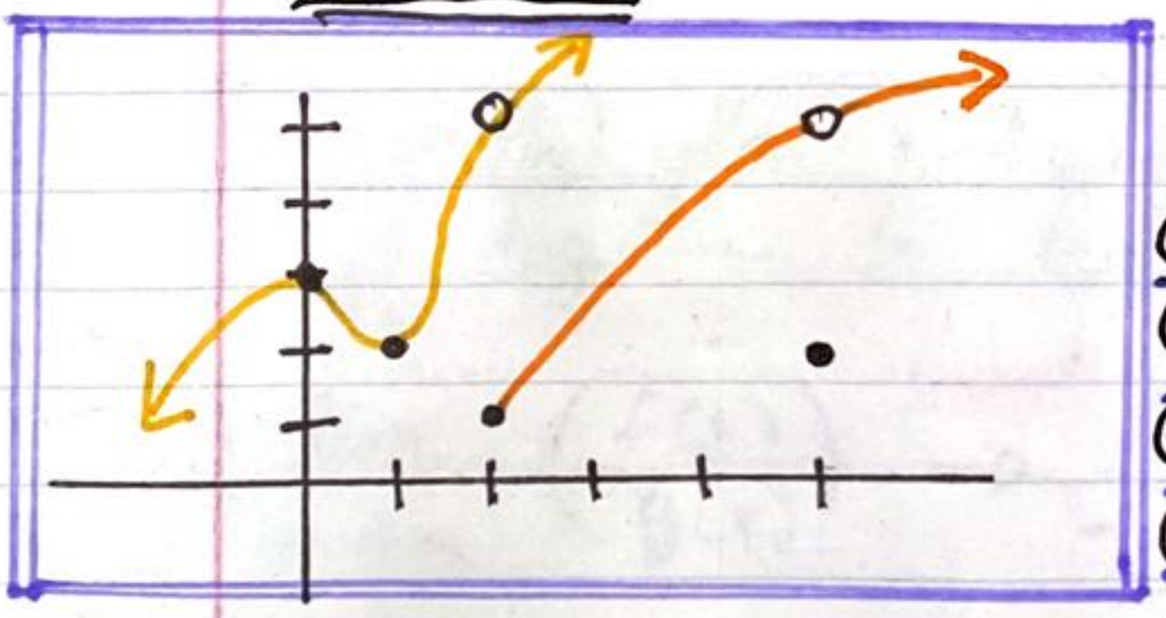
$$\boxed{\lim_{x \to a^+} f(x) = N}$$



Thm: $\boxed{\lim_{x \to a} f(x) = L}$ if and only

$\boxed{\lim_{x \to a^-} f(x) = L}$ and $\boxed{\lim_{x \to a^+} f(x) = L}$

AGREE → Both must agree or it does not EXIST



ⓐ $\lim_{x \to 2^-} g(x) = 5$

ⓑ $\lim_{x \to 2^+} g(x) = 1$

ⓒ $\lim_{x \to 2} g(x) = NA$

ⓓ $\lim g(2) = 1$

ⓔ

ⓕ

ⓖ

ⓗ

1.2 Calculating Limits

Limit LAWS: Suppose c is a Constant and $\boxed{\lim_{x \to a} f(x)}$ and $\boxed{\lim_{x \to a} g(x)}$ exist.

1.) $\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

* The limit of the sums is the sum of the limits OR the limit of the difference is the difference of the limits.
BASICALLY find limit of each and add together

2.) $\lim_{x \to a} (c\, f(x)) = c \lim_{x \to a} f(x)$

* Multiply the function by the constant same as the constant multiplied by the function.

3.) $\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

* Find the limit of each function and multiply together to find the result.

Quotient

4.) $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) \xrightarrow{\text{Quotient}} \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \overset{\text{NOT}}{\neq} 0$

* Makes sure you dont divide by $\underline{0}$ to find answer

Powers 5.) $\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x)\right)^n$, $\boxed{n > 0}$, $\boxed{n \in \mathbb{Z}}$

* MUST USE POSITIVE N
* which mean intergere of positive whole numbers

Root 6.) $\lim_{x \to} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$, $\boxed{n > 0}$, $\boxed{n \in \mathbb{Z}}$

* Must be Postive and whol number *

Ex.

Direct Substitution

1.) $\lim_{x \to 5} (2x^2 - 3x + 4)$ * Always try direct substitution

$2(5)^2 - 3(5) + 4$

$\to 2(25) - 15 + 4$

$\to 50 - 15 + 4$

$= \boxed{39}$

means the "Y" values are closer to 39

2) $\lim_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

$\to \dfrac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)}$

$\to \dfrac{-8 + 8 - 1}{5 + 6} = \boxed{\dfrac{-1}{11}}$ → Y values equal or close to $-\frac{1}{11}$

Trig 3.) $\lim_{x \to \pi} x\cos(x)$

$\to \pi \cos(\pi)$

$\to \pi(-1)$

$= \boxed{-\pi}$

* NOTE
Memorize Unit Circle + Functions

* Direct Substitution only works if your a is continuous.

4) $\lim_{x \to 1} \frac{x^2-1}{x-1}$

$\to \frac{1^2-1}{1-1} = \boxed{\frac{0}{0}}$ * SIGN TO USE ALGEBRA

FACTOR $\to \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x+1$ now direct substitution

$\lim_{x \to 1} (x+1) = 1+1 = \boxed{2}$

* You would have a hole in your graph if $x=1$ ("0") but now this proves "y" is getting closer to 2

★) <u>Thm</u>: If $f(x) = g(x)$ when $x \neq a$ then

$$\boxed{\lim_{x \to a} f(x) = \lim_{x \to a} g(x)}$$ if the limit <u>exists</u>.

<u>Ex</u>. 5) Find $\lim_{x \to 1} g(x)$ when $= \begin{cases} x+1, & \text{if } x \neq 1 \\ \pi, & \text{if } x = 1 \end{cases}$

↳ This would be a random point

PIECEWISE FUNCTION

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} (x+1) = 1+1 = \boxed{2}$$

★ Remember the value is not the same as the actual value but <u>close</u>. You don't want/need to know what G of 1 is equal to.
★ We want to know the <u>BEHAVIOR</u>.

6) $$\lim_{h \to 0} \frac{(3+h)^2 - 9}{h} = \frac{(3+0)^2 - 9}{0} = \frac{3^2 - 9}{0} = \boxed{\frac{0}{0}}$$

★ Distributive Sub. NOT work
★ FACTORING NOT WORK
<u>NEXT</u>
<u>DISTRIBUTE</u>

Distribute → $$\frac{9 + 6h + h^2 - 9}{h}$$

FACTOR → $$\frac{6h + h^2}{h} = \frac{\cancel{h}(6+h)}{\cancel{h}}$$

Direct Subsitution $= 6 + 0 = \boxed{6}$

7) $\lim_{t \to 0} \dfrac{\sqrt{t^2+9} - 3}{t^2} = \dfrac{\sqrt{0^2+9} - 3}{0^2} = \boxed{\dfrac{0}{0}}$

Multiply by the conjugate

$$\rightarrow \frac{(\sqrt{t^2+9} - 3)}{t^2} \cdot \frac{(\sqrt{t^2+9} + 3)}{(\sqrt{t^2+9} + 3)}$$

Cancel

$$\rightarrow \frac{t^2 \cancel{+9} \cancel{+3\sqrt{t^2+9}} \cancel{-3\sqrt{t^2+9}} \cancel{-9}}{t^2(\sqrt{t^2+9} + 3)}$$

Simplify

$$\rightarrow \frac{\cancel{t^2}}{\cancel{t^2}(\sqrt{t^2+9} + 3)} = \frac{1}{\sqrt{t^2+9} + 3}$$

Direct substitution

$$\rightarrow \frac{1}{\sqrt{0^2+9} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{3+3}$$

$$= \boxed{\frac{1}{6}}$$

SQUEEZE Theorem

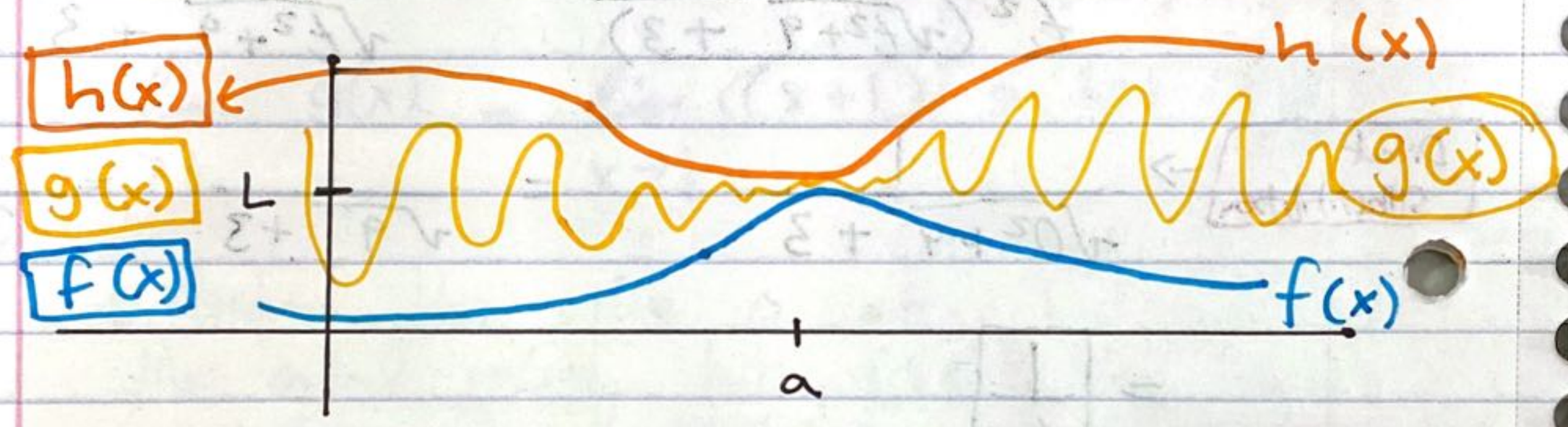
- If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a)

AND

- If $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = \underline{\underline{L}}$

THEN

$$\lim_{x \to a} g(x) = \underline{\underline{L}}$$



NOW TO EVUALATE a LIMIT:

Ex

1.) Show that $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

KNOW: $-1 \le \sin\left(\frac{1}{x}\right) \le 1$ so

$$\underline{\underline{-x^2}} \le x^2 \sin\left(\frac{1}{x}\right) \underline{\underline{\le x^2}}$$

* If you can show the negative equal to positive then that will prove the middle is equal.

We have: $\lim_{x \to 0} (-x^2) = -0^2 = \underline{\underline{0}}$

$\lim_{x \to 0} x^2 = 0^2 = \underline{\underline{0}}$

Therefore it must mean the middle goal $\underline{0}$

→ Proof by Squeeze Theorem

$$\boxed{\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = \underline{\underline{0}}}$$

FACT

- $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = \underline{\underline{1}}$
- $\lim_{\theta \to 0} \frac{\theta}{\sin(\theta)} = \underline{\underline{1}}$

Ex

2) $\lim_{x \to 0} \frac{\sin(7x)}{4x}$

multiply by one → $\frac{\sin(7x)}{4x} \cdot \frac{7}{7}$

Rearrange → $\frac{\sin(7x)}{7x} \cdot \frac{7}{4}$ → 1

$= \boxed{\frac{7}{4}}$

Ex 3.) $\lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta}$ ☆ USE ALGEBRA

Congugate $\rightarrow \frac{(\cos(\theta) - 1)}{\theta} \cdot \frac{(\cos(\theta) + 1)}{(\cos(\theta) + 1)}$

USE TRIG IDENTITY $\rightarrow \frac{\cos^2(\theta) - 1}{\theta(\cos(\theta) + 1)}$ $= \cos^2(\theta) = 1 - \sin^2(\theta)$

Cancel $= \frac{\cancel{1} - \sin^2(\theta) \cancel{- 1}}{\theta(\cos(\theta) + 1)}$

$= \frac{-\sin^2(\theta)}{\theta(\cos(\theta) + 1)}$

Rewrite as a PRODUCT $= \frac{\sin(\theta)}{\theta} \cdot \frac{-\sin(\theta)}{\cos(\theta + 1)}$

Apply Direct Substitution $= 1 \cdot \frac{-\sin(0)}{\cos(0) + 1}$

$= 1 \cdot \frac{0}{1+1} = \boxed{0}$

1.3 Limits Involving Infinity

Ex $\lim_{x \to 0} \frac{1}{x^2}$

x	$f(x)$
± 1	1
± 0.1	100
± 0.01	$10{,}000$

As $x \to 0$

$\frac{1}{x^2}$ gets increasingly <u>LARGE</u>

DEF: $\lim_{x \to a} f(x) = \infty$ means

the values of $f(x)$ can be made arbitrarily large by taking $\underline{x}$ sufficiently close to $\underline{a}$

* In the last <u>ex</u> $\boxed{\lim_{x \to 0} \frac{1}{x^2} = \infty}$

VERTICAL ASYMPTOTE

DEF: The line $x = a$ is a vertical asymptote if at least one of the following is true for lim:

$$\boxed{\lim_{x \to a^+} f(x) = \pm\infty}, \boxed{\lim_{x \to a^-} f(x) = \pm\infty \quad \underline{\text{OR}}}$$

$$\boxed{\lim_{x \to a} f(x) = \pm\infty}$$

EX

1.) Find $\boxed{\lim_{x \to 3^+} \frac{2x}{x-3}}$ and $\boxed{\lim_{x \to 3^-} \frac{2x}{x-3}}$

Positive tiny positive $\lim_{x \to 3^+} \frac{2x}{x-3} = \underline{\underline{\infty}}$ 3^+ slightly <u>larger</u> than $\underline{\underline{3}}$

Positive tiny negative $\lim_{x \to 3^-} \frac{2x}{x-3} = \underline{\underline{-\infty}}$ = 3^- slightly <u>smaller</u> the $\underline{\underline{3}}$

HORIZONTAL ASYMPTOTE

DEF: Let f be defined on some interval (a, ∞).

Then $\lim_{x \to \infty} f(x) = \underline{\underline{L}}$ means that

the values of $f(x)$ become closer to $\underline{\underline{L}}$ as $\underline{\underline{x}}$ gets sufficiently large.

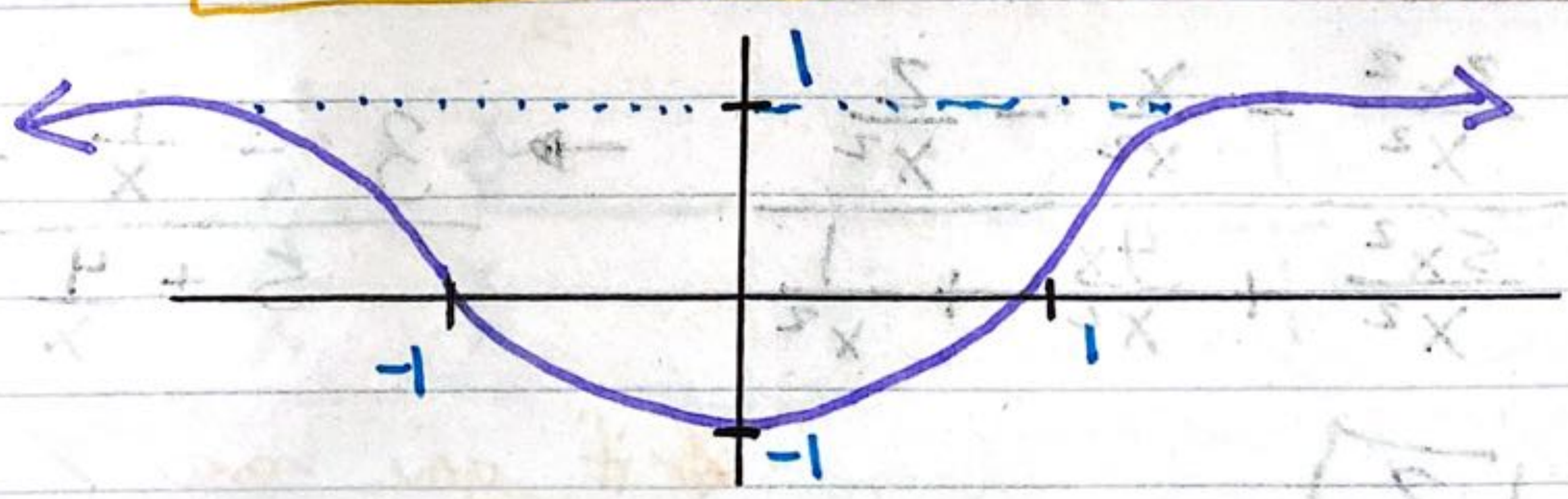
DEF: The line $y = L$ is a <u>horizontal asymptote</u> if either

$$\boxed{\lim_{x \to \infty} f(x) = L} \quad \text{OR} \quad \boxed{\lim_{x \to -\infty} f(x) = L}$$

Ex

1.) Use the graph to find $\boxed{\lim_{x \to \infty} \frac{x^2-1}{x^2+1}}$

and $\boxed{\lim_{x \to -\infty} \frac{x^2-1}{x^2+1}}$

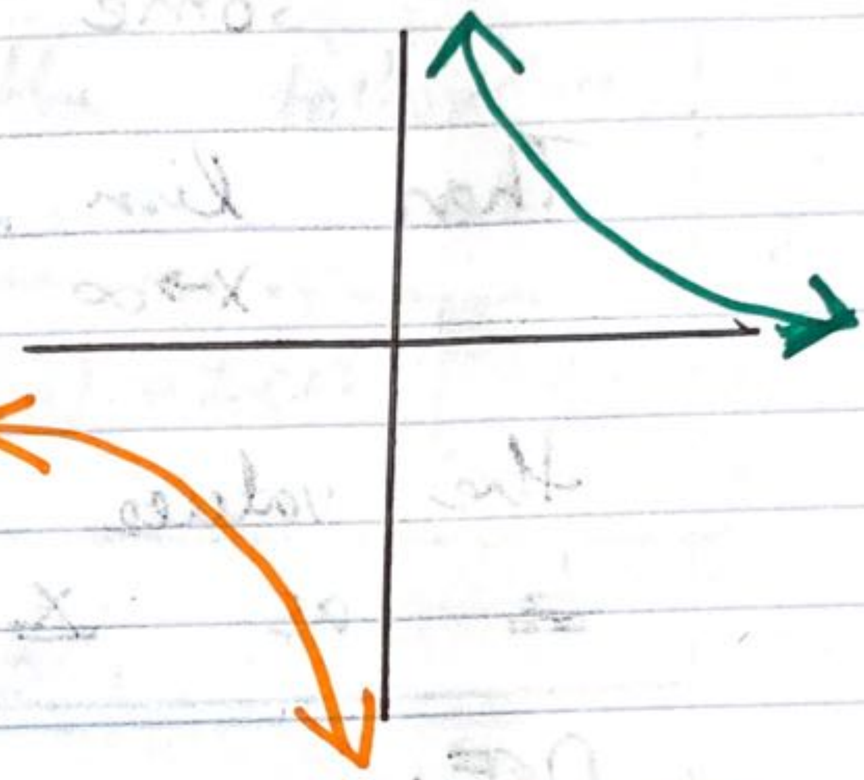


Both limits are 1

<u>Horizontal Asymptote</u> at $\boxed{y = 1}$

EX 2) Find $\boxed{\lim_{x\to\infty} \frac{1}{x}}$ and $\boxed{\lim_{x\to-\infty} \frac{1}{x}}$

$\to \lim_{x\to\infty} \frac{1}{x} = \boxed{0}$



$\to \lim_{x\to-\infty} \frac{1}{x} = \boxed{0}$

FACT: If $n > 0$ then $\boxed{\lim_{x\to\pm\infty} \frac{1}{x^n} = \boxed{0}}$

RATIONAL FUNCTION

POLYNOMIAL / Polynomial

EX 1) $\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$

* Divide every term by x^n where n is the highest exponent in the denominator.

$\to \frac{\frac{3x^2}{x^2} - \frac{x}{x^2} - \frac{2}{x^2}}{\frac{5x^2}{x^2} + \frac{4x}{x^2} + \frac{1}{x^2}} \to \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} - \frac{1}{x^2}}$ ($\frac{1}{x} \to 0$, $\frac{2}{x^2} \to 0$, $\frac{4}{x} \to 0$, $\frac{1}{x^2} \to 0$)

$= \boxed{\frac{3}{5}}$

* if you have x in denominator and just a number on top you will get 0

* ONLY WORKS WITH RATIONALS

2) $\lim_{x \to \infty} (\sqrt{x^2+1} - x)$

$\rightarrow \frac{(\sqrt{x^2+1} - x)}{1} \cdot \frac{(\sqrt{x^2+1} + x)}{(\sqrt{x^2+1} + x)}$

$\rightarrow \frac{\cancel{x^2} + 1 - \cancel{x^2}}{\sqrt{x^2+1} + x}$

"x" in denomator

$\rightarrow \frac{1}{\sqrt{x^2+1} + x} = \boxed{0}$

3.) $\lim_{x \to \infty} (x^2 - x)$

FACTOR $\rightarrow (x(x-1)) = \infty$

$\cdot \lim_{x \to \infty} = (x(x-1)) = \underline{\underline{\infty}}$

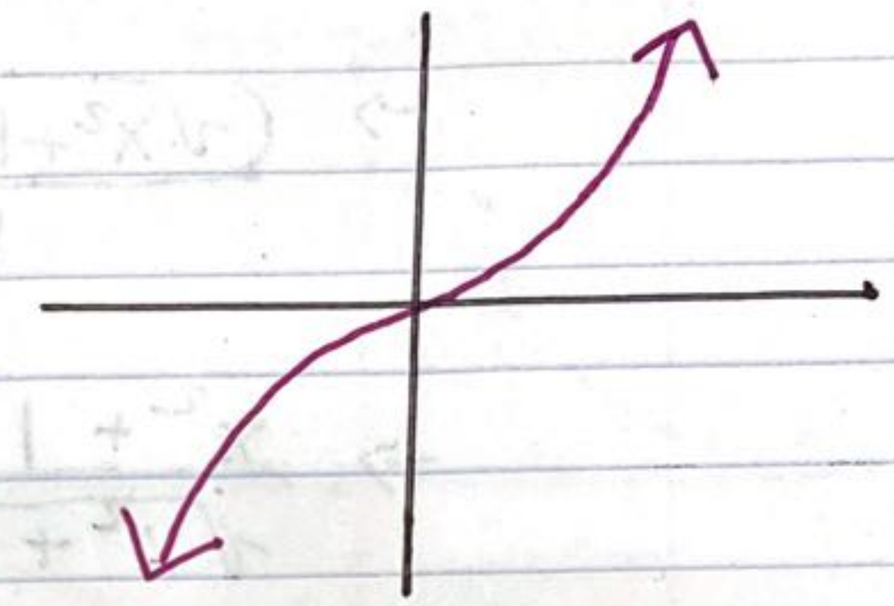
4.) $\lim_{x \to \infty} \frac{x^2 + x}{3 - x}$ Divide by "x"

$\rightarrow \frac{\frac{x^2}{x} + \frac{x}{x}}{\frac{3}{x} - \frac{x}{x}} = \frac{x + 1}{\frac{3}{x} - 1} = \underline{\underline{-\infty}}$

$\rightarrow 0$

5.) Find $\boxed{\lim_{x \to \infty} x^3}$ and $\boxed{\lim_{x \to -\infty} x^3}$

$$\lim_{x \to \infty} x^3 = \underline{\underline{\infty}}$$



$$\lim_{x \to -\infty} x^3 = \underline{\underline{-\infty}}$$

1.5 & 1.6 Limits and Continuity

DEF: A function $f(x)$ is <u>continous</u> at $x = a$ if

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

Ex

1.) Where is $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ discontinous?

$x = 2$ (not defined)

$$\lim_{x \to a} \frac{x^2 - x - 2}{x - 2} = \frac{a^2 - a - 2}{a - 2} = \underline{f(a) \text{ for } a \neq 2}$$

↳ ONLY if <u>a</u> didn't equal <u>2</u>

2) Is $g(x) = \begin{cases} \frac{1}{x^2}, & \text{if } x \neq 0 \\ \underline{1}, & \text{if } x = 0 \end{cases}$ continous at $x = 0$?

- $g(0) = \underline{1}$ is defined ✓
- $\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{1}{x^2} = \underline{\underline{\infty}}$

NOT CONTINOUS at $\underline{x = 0}$

3) Is $h(t) = \begin{cases} \dfrac{t^2 - t - 2}{t-2} & \text{if } t \neq 2 \\ 1 & \text{if } t = 2 \end{cases}$

Continous at $\underline{t=2}$?

- $h(2) = \underline{1}$ is defined ✓
- $\lim\limits_{t \to 2} \dfrac{t^2 - t - 2}{t-2}$ = FACTOR = $\lim\limits_{t \to 2} \dfrac{\cancel{(t-2)}(t+1)}{\cancel{(t-2)}}$

$\rightarrow \lim\limits_{t \to 2} (t+1) = 2+1 = \boxed{3}$

NOT CONTINOUS Because Both do
NOT AGREE

Continous Function

DEF: A function is continous from the right at $x = a$ if $\boxed{\lim\limits_{x \to a^+} f(x) = f(a)}$

It is continous from the left at $x = a$. If $\boxed{\lim\limits_{x \to a^-} f(x) = f(a)}$.

* Must Be Continous from Both sides...

Def: A function is continous on an interval if it is continous at every point in the interval.

Thm: If f and g are continous at a and c is a constant, then $f+g$, $f-g$, cf, fg are contious at a.
If $g(a) \neq 0$ then f/g is continous at a.

Thm: Polynomials

a) any polynomial is cont. for all real numbers.

b) rational functions are contious wherever they are defined.

Thm: Trig functions are continous at every point in their domain.

Intermediate Value Theorem

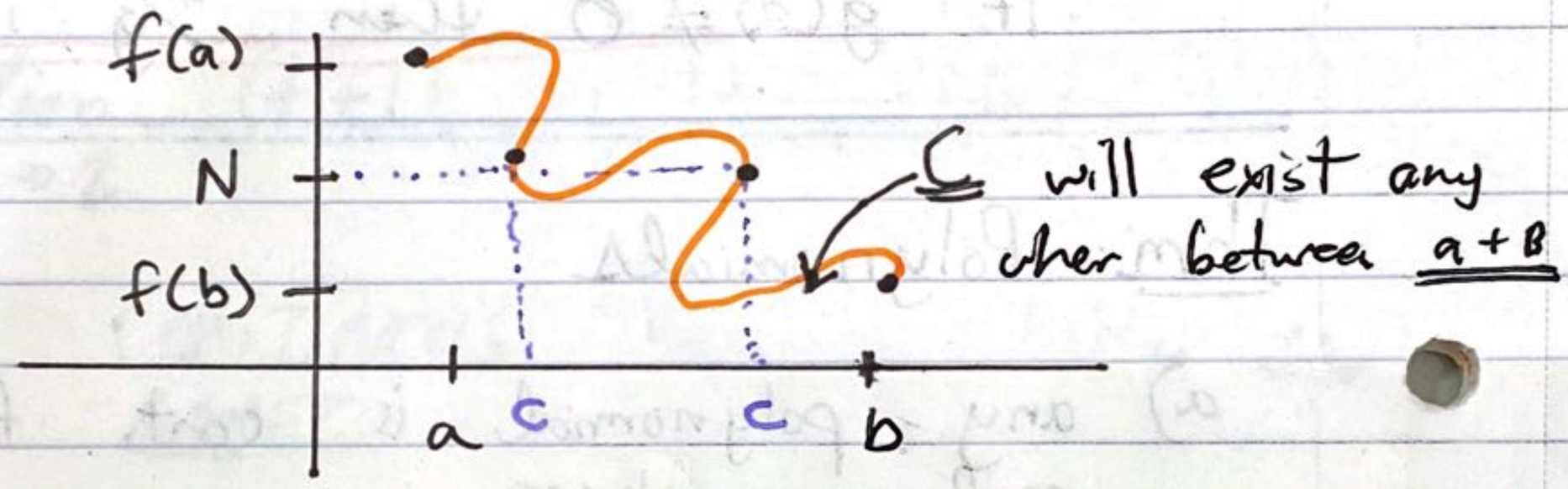
→ Suppose f is continous on $[a, b]$ and let N be any number between $f(a)$ and $f(b)$

- Where $(f(a) \neq f(b))$.

THEN

- There exists c in (a, b) such that $f(c) = N$.

EX



EX

1.) Show that there is a ROOT of $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2

- $f(1) = 4(1)^2 - 6(1)^2 + 3(1) - 2 = \underline{-1}$
- $f(2) = 4(2)^2 - 6(2)^2 + 3(2) - 2 = \underline{12}$

I.V.T = for any N with $-1 < N < 12$ there exists c with $1 < c < 2$ such that $f(c) = N$.

So with $N = 0$ → there is c with $1 < c < 2$ such that $f(c) = 0$

This c is the ROOT

1.7 – 1.8 Review of Inverse Trigonometric, Exponential and Logarithmic Functions.

DEF: The Inverse Sine Function

$\sin^{-1}(x)$ or $\arcsin(x)$ is the inverse of the restricted sine function

$\boxed{\sin(x)}$ $\boxed{-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}}$

* Inverse functions can only exist if you have a <u>one-to-one</u> function, which passes the <u>Horizontal Line Test</u>

* Regular Sine function keeps fluctuating
So by doing Inverse you Restrict the <u>Domain</u> between $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ just to find a small piece.

DEF: The Inverse Cosine Function

$\cos^{-1}(x)$ or $\arccos(x)$ is the inverse of the restricted cosine function ($\cos(x)$)

$\boxed{\cos(x)}$ $\boxed{0 \leq x \leq \pi}$

DEF: The Inverse Tangent Function

$\tan^{-1}(x)$ or $\arctan(x)$ is the inverse of $\boxed{\tan(x)}$ $\boxed{-\frac{\pi}{2} < x < \frac{\pi}{2}}$

Def: The Inverse Secant Function

$\sec^{-1}(x)$ or $\operatorname{arcsec}(x)$ is the inverse $\sec(x)$ $\boxed{0 \le x \le \pi \text{ and } x \neq \frac{\pi}{2}}$

---

* **NOTE**: ALL INVERSE TRIG FUNCTIONS are continuous on their Domain.

FUNCTION	DOMAIN	RANGE
$\sin^{-1}(x)$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos^{-1}(x)$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}(x)$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \pi/2)$
$\sec^{-1}(x)$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi/2) \cup (\frac{\pi}{2}, \pi]$

Ex

1) Where is $f(x) = \dfrac{\sec^{-1}(x)}{\pi + 4\tan^{-1}(x)}$ continous?

***NOTE:**

Num: cont. on $(-\infty, -1] \cup [1, \infty)$

Den: cont. everywhere

NEED: $\pi + 4\tan^{-1}(x) \neq 0$ (need it to not be 0)

$\rightarrow \pi + 4\tan^{-1}(x) = 0$

$\rightarrow \tan^{-1}(x) = -\frac{\pi}{4}$

$\rightarrow x = \tan(-\frac{\pi}{4})$

$= \boxed{x = -1}$ ← not in Domain

FUNCTION is CONTINUOUS on $(-\infty, -1) \cup [1, \infty)$

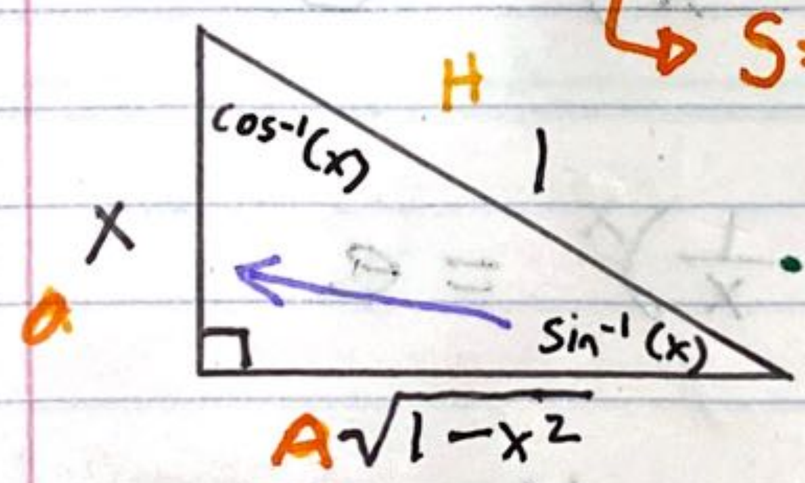
NOT INCLUDING -1

Ex

2) Use the Right Triangle to find

$\boxed{\sin(\cos^{-1}(x))}$

↳ $S = \frac{O}{H} \rightarrow \frac{\sqrt{1-x^2}}{1} = \boxed{\sqrt{1-x^2}}$



- $\boxed{\cos(\cos^{-1}(x))}$

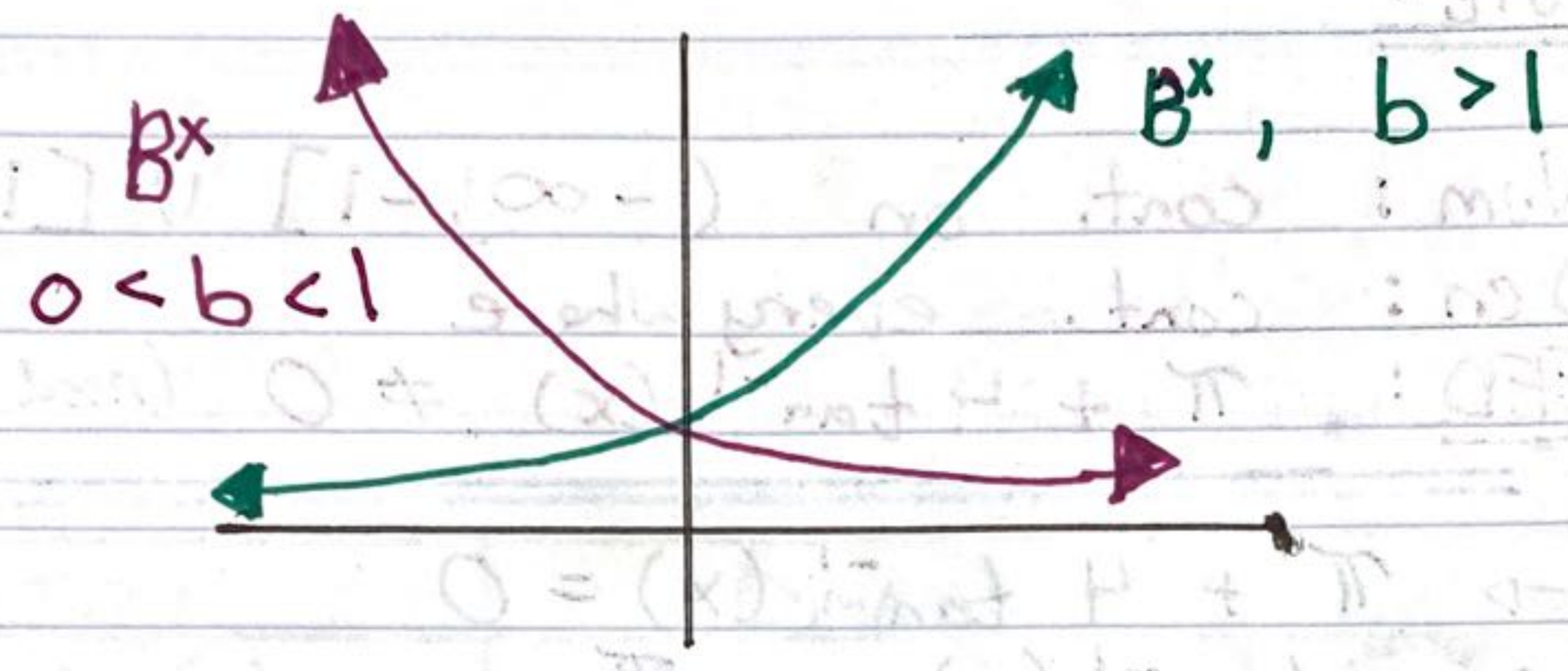
↳ $C = \frac{A}{H} = \frac{x}{1} = \boxed{x}$

- $\boxed{\tan(\sin^{-1}(x))}$

↳ $\frac{x}{\sqrt{1-x^2}}$

Def: <u>Exponential functions</u>

The exponental function with base b is $f(x) = b^x$, $b > 0$, $b \neq 1$



Domain : $(-\infty, \infty)$
Range : $(0, \infty)$

The <u>natural exponential function</u> is $f(x) = e^x$ where

* Euler's number $= e \approx 2.718$

<u>FACTS:</u>

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$$

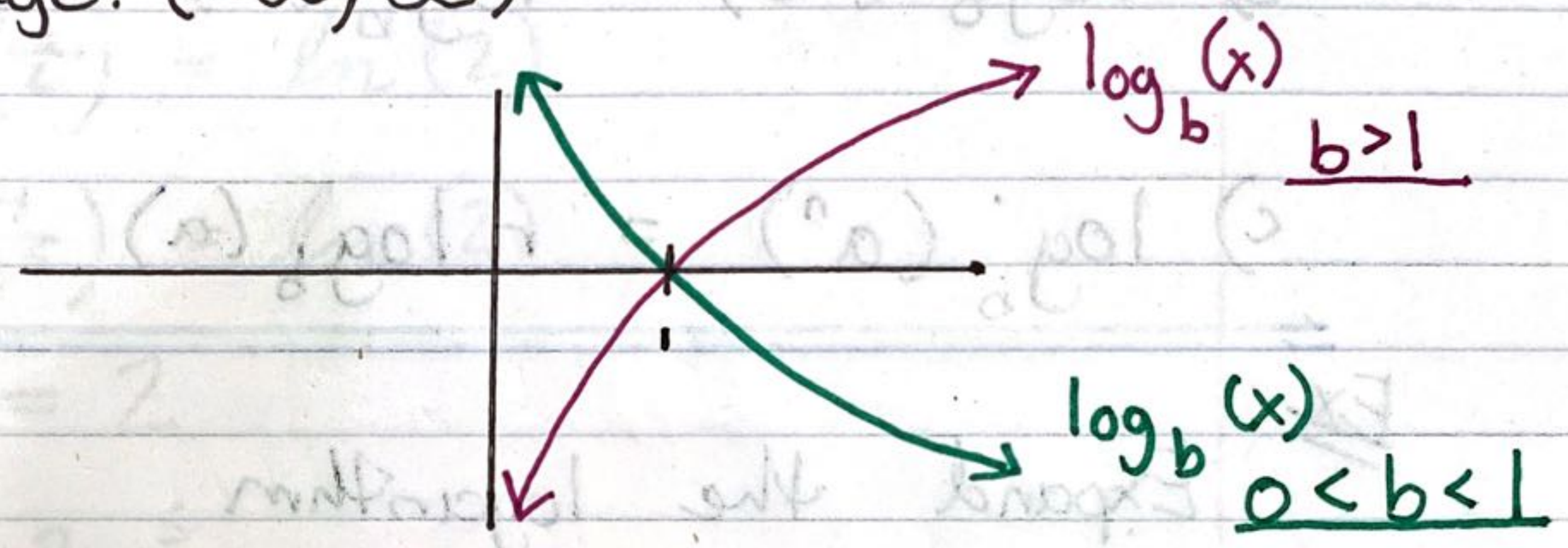
$$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\lim_{x \to 0} (1 + x)^{1/x} = e$$

Def. Logarithmic Function

With base b is $f(x) = \log_b(x)$ which is the inverse of b^x

Domain: $(0, \infty)$
Range: $(-\infty, \infty)$



The natural logarithmic function has base e, and is denoted by $\ln(x)$

In general $\log_b(x) = y$ if and only if $b^y = x$

$\ln(x) = y$ if and only if $e^y = x$

Properties of Logarithms

a) $\log_b(ac) = \log_b(a) + \log_b(c)$

b) $\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c)$

c) $\log_b(a^n) = n\log_b(a)$

Ex

Expand the logarithm

$\ln\left(\frac{x^2 \sin^3(x)}{\sqrt{x^2+1}}\right)$

$\Rightarrow \ln(x^2\sin^3(x)) - \ln(\sqrt{x^2+1})$

$\Rightarrow \ln(x^2) + \ln(\sin^3(x)) - \ln((x^2+1)^{1/2})$

$\Rightarrow 2\ln(x) + 3\ln(\sin(x)) - \frac{1}{2}\ln(x^2+1)$

Ex

Rewrite as a single logarithm

$\frac{1}{2}\log(x) - 3\log(\sin(2x))$

$\Rightarrow \log(x^{\frac{1}{2}}) - \log(\sin^3(2x))$

$\Rightarrow \boxed{\log\left(\frac{\sqrt{x}}{\sin^3(2x)}\right)}$

Ex Solve for x

a) $\ln(4x) - 3\ln(x^2) = \ln(2)$

$\rightarrow \ln(4x) - \ln(x^6) = \ln(2)$

$\rightarrow \ln\left(\frac{4x}{x^6}\right) = \ln(2)$

$\rightarrow \ln\left(\frac{4}{x^5}\right) = \ln(2)$

$\rightarrow \frac{4}{x^5} = 2$

$\rightarrow 4 = 2x^5$

$\rightarrow 2 = x^5$

$= \boxed{\sqrt[5]{2} = x}$

b) $3e^{-2x} = 5$

$e^{-2x} = \frac{5}{3}$

$-2x = \ln(5/3)$

$x = -\frac{1}{2}\ln(5/3)$

c) $e^x - 2xe^x = 0$

$e^x(1-2x) = 0$

$e^x = 0$ OR $1 - 2x = 0$

<u>no solution</u>

$1 = 2x$

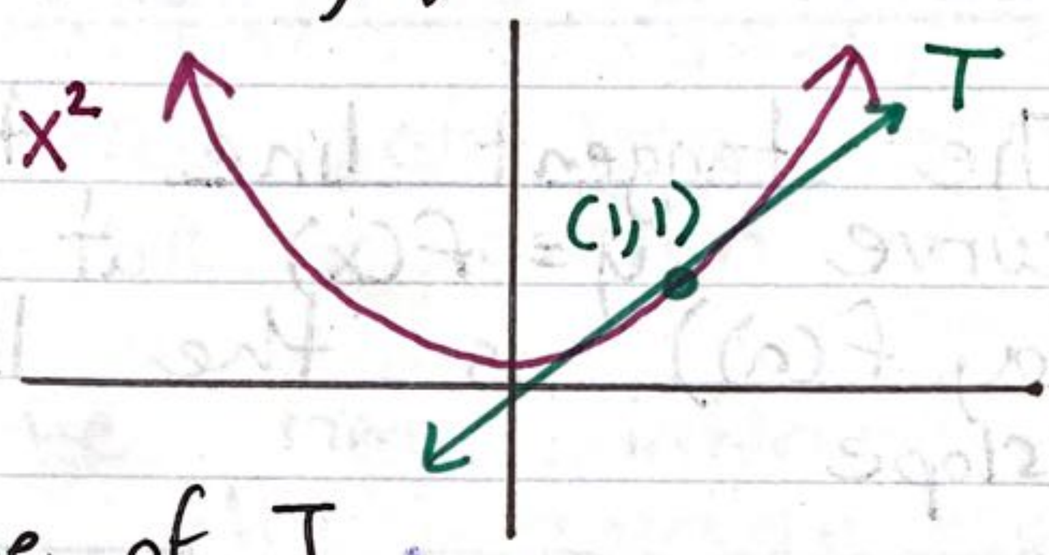
$\boxed{\frac{1}{2} = x}$

2.1/2.2 The Definition of a Derivative

The tangent problem:

Ex

Find the equation of the tangent line to the parabola $y = x^2$ at the point $(1,1)$.



NEED: Slope of T

Estimate using slope of the secant line between $(1,1)$ and the point (x, x^2)

$$\boxed{m \approx \frac{x^2-1}{x-1} \rightarrow m = \lim_{x \to 1} \frac{x^2-1}{x-1}}$$

$$m = \lim_{x \to 1} \frac{x^2-1}{x-1}$$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}$$

$$= \lim_{x \to 1} (x+1)$$

$$= 1 + 1 = \underline{\underline{2}} \leftarrow \text{Slope}$$

$m = 2$, $(1,1)$

Eg: of tan line:

$$\boxed{\begin{gathered} y - 1 = 2(x-1) \\ \text{OR} \\ y = 2x - 1 \end{gathered}}$$

DEF: The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$\boxed{m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}} \quad \text{OR} \quad \boxed{\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

Ex Write the equation of the tangent line to $\underset{f(x)}{y} = 3/x$ at $(\underset{a}{3}, 1)$

$$m = \lim_{x \to 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{3}{3+h} - 1}{h}$$

$$= \lim_{h \to 0} \frac{\frac{3 - (3+h)}{3+h}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3} - \cancel{3} - \cancel{h}}{3+h} \cdot \frac{1}{\cancel{h}} = \underline{\underline{-\frac{1}{3}}}$$

$(3, 1)$ $\underline{\underline{m}}$

$$\boxed{y - y_1 = m(x - x_1)}$$

$y - 1 = -\frac{1}{3}(x-3)$ OR $y = -\frac{1}{3}x + 2$

The Velocity Problem

The function f that describes the motion of an object is called the position function.

From the time interval $t=a$ to $t=a+h$, the average velocity is $\frac{f(a+h)-f(a)}{h}$.

If $h \to 0$ we get the instantaneous velocity at $t=a$, $V(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$.

Ex

Suppose a ball is dropped from 450 m above the ground.

Using the equation of motion $f(t) = 4.9t^2$ What is the velocity after 5 seconds?

$$V(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{4.9(a+h)^2 - 4.9a^2}{h}$$

$$= \frac{\cancel{4.9a^2} + 9.8ah + 4.9h^2 - \cancel{4.9a^2}}{h}$$

$$= \frac{9.8ah + 4.9h^2}{h}$$

$$= \frac{h(9.8a + 4.9h)}{h}$$

$$= 9.8a + 4.9h$$

$$= 9.8a \to V(5)$$

$$= 49 \text{ m/sec}$$

DEF: The <u>derivative</u> of a function f at a number a is

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

$\boxed{\text{OR}}$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Ex Find the derivates of $f(x)$

$f(x) = x^2 - 8x + 9$ at the number a

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{((a+h)^2 - 8(a+h) + 9) - (a^2 - 8a + 9)}{h}$$

$$= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h}$$

$$= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2a + h - 8)}{\cancel{h}} = (2a + h - 8)$$

$$\boxed{= 2a - 8 = f'(a)}$$

Ex Find the equation of the tangent line to $f(x) = x^2 - 8x + 9$ at $(3, -6)$.

$m = f'(3)$

We know $f'(a) = 2a - 8$

$$\begin{aligned} f'(3) &= 2(3) - 8 \\ &= 6 - 8 \\ &= -2 \longleftarrow m \quad (3, -6) \end{aligned}$$

$$\boxed{y + 6 = -2(x - 3) \quad \text{OR} \quad y = -2x}$$

The derivative $f'(a)$ is * slope of the tangent line to $f(x)$ at $x = a$

* Instantaneous rate of change of $f(x)$ with respect to x at $x = a$

* Velocity of a particle at time $t = a$ if $f(t)$ is a position function.

Time
41:10

The Derivative as a Function

Def: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Ex Find $f'(x)$ if $f(x) = x^3 - x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{((x+h)^3 - (x+h)) - (x^3 - x)}{h}$$

$$\lim_{h \to 0} = \frac{\cancel{x^3} + 3x^2h + 3xh^2 - h^3 \cancel{-x} - h \cancel{-x^3} \cancel{+x}}{h}$$

$$= \lim_{h \to 0} = \frac{h(3x^2h + 3xh^2 + h^3 - h)}{h}$$

Near add

$$= \lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 1)}{\cancel{h}}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2 - 1)$$

$$= 3x^2 - 1$$

$$\boxed{f'(x) = 3x^2 - 1}$$

Ex

Find $f'(x)$ if $f(x) = \sqrt{x}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})} = \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x+0} + \sqrt{x}}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

OTHER NOTATIONS : If $y = f(x)$ then

Most Common

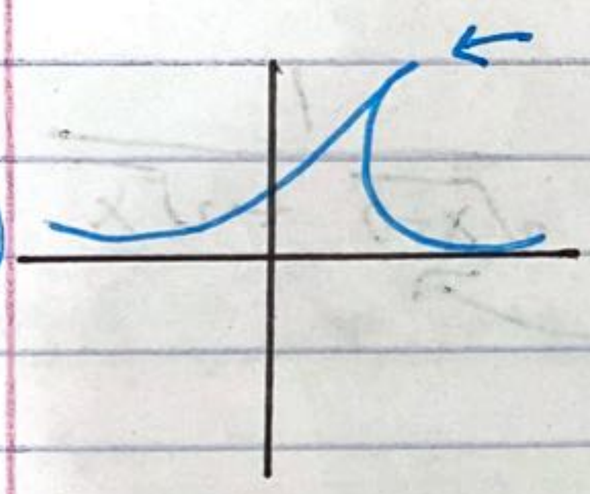
$$\boxed{f'(x) = y' = \frac{dy}{dx}} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x)$$

$$= D_x f(x)$$

Def: f is <u>differentiable</u> at $x=a$ if $f'(a)$ exists. It is differentiable an open interval if it is differtiable at every point in the interval.

Thm:
If f is differentiable at a then it is continuous at $x=a$.

FAILING to be Differentiable.

1


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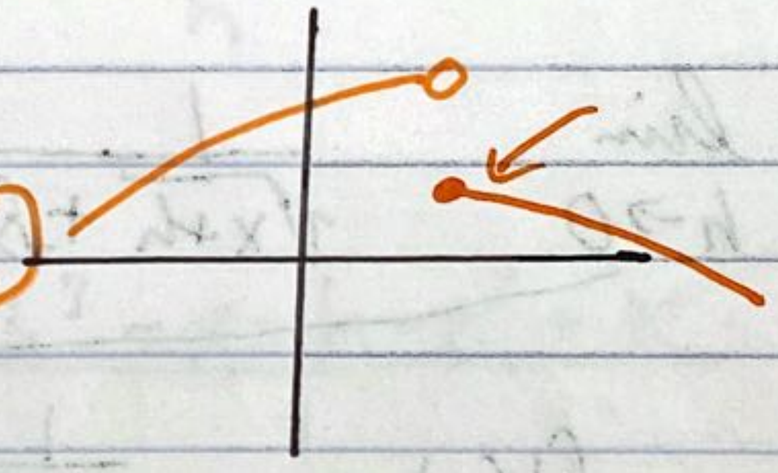
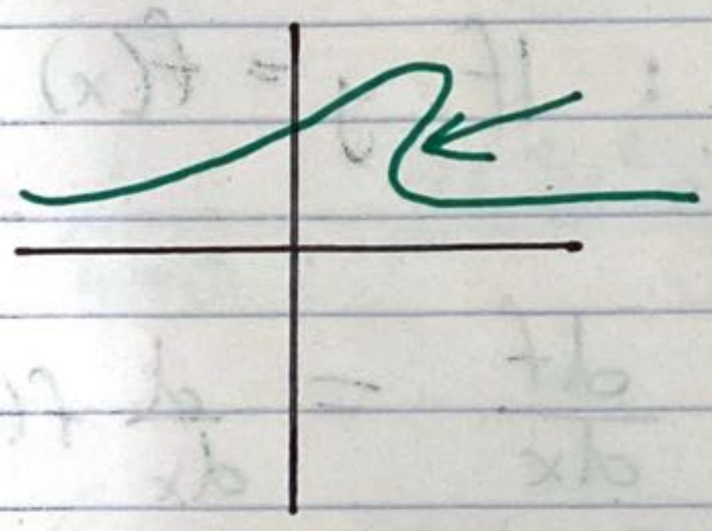


- Corner / cusp
- discontinous

3


- Vertical tangent

Higher Order Derivates

- Second derivates, $f''(x)$ or $\frac{d^2y}{dx^2}$ is the derivate of $f'(x)$.

- Third derivates $f'''(x)$ or $\frac{d^3y}{dx^3}$ is the derivative of $f''(x)$.

- n^{th} derivative $f^{(n)}(x)$ or $\frac{d^ny}{dx^n}$ is the result of differentiating $f(x)$ n times.

Ex Find the 2nd, 3rd and 4th derivates of

$$f(x) = x^3 - x$$

Know: $f'(x) = 3x^2 - 1$

Second Derivative

- $f''(x) = \lim_{h \to 0} \frac{(3(x+h)^2 - 1) - (3x^2 - 1)}{h}$

$$f''(x) = \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 1 - 3x^2 + 1}{h}$$

$$f''(x) = \lim_{h \to 0} \frac{6xh + 3h^2}{h} = \frac{h(6x + 3h)}{h}$$

$$f''(x) = \lim_{h \to 0} (6x + 3h) = \boxed{6x}$$

- <u>Third Derivative</u>

$$f'''(x) = \lim_{h \to 0} \frac{6(x+h) - 6x}{h}$$

$$f'''(x) = \lim_{h \to 0} \frac{6x + 6h - 6x}{h}$$

$$f'''(x) = \lim_{h \to 0} \frac{6\cancel{h}}{\cancel{h}} = \boxed{6}$$

$$\boxed{f'''(x) = 6}$$

- <u>Fourth Derivative</u>

$$f^{(4)}(x) = \lim_{h \to 0} = \frac{6-6}{h} = \frac{0}{h}$$

$$f^{4}(x) = \lim_{h \to 0} (0) = \boxed{0}$$

$$\boxed{f^{4}(x) = 0}$$

$f'(x) = 3x^2 - 1$
$f''(x) = 6x$
$f'''(x) = 6$
$f^{4}(x) = 0$

Derivative of position is Velocity

$v'(t) = a(t)$ acceleration

↑

second derivate of position function

Tme 9:30

2.3 & 2.5 : Basic Differentiation Function

Constant Rule: $\frac{d}{dx} c = 0$

Power Rule: If n is a real number then $f(x) = x^n$, we have $f'(x) = nx^{n-1}$

Ex Find the derivates:

ⓐ $f(x) = 5 \rightarrow f'(x) = 0$

How 4?

ⓑ $f(x) = x^5 \rightarrow f'(x) = 5x^4$

ⓒ $f(x) = \pi \rightarrow f'(x) = 0$ continous

ⓓ $f(x) = x^1 \rightarrow f'(x) = 1x^0 \rightarrow f'(x) = 1$

ⓔ $f(x) = \sqrt{x}$

$f(x) = x^{1/2} \rightarrow f'(x) = \frac{1}{2}x^{-1/2}$

$f'(x) = \frac{1}{2x^{1/2}}$

$f'(x) = \frac{1}{2\sqrt{x}}$

ⓕ $f(x) = \frac{1}{x^2}$

$f(x) = x^{-2} \rightarrow f'(x) = -2x^{-3}$ or $\frac{2}{x^3}$

How 3?

ⓖ $f(x) = \pi^3 \rightarrow f'(x) = 0$

Constant Multiple Rule

$$\frac{d}{dx}(c\,f(x)) = c\,f'(x)$$

Sum and Difference Rule:

$$\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$$

Ex Differentiate

$$f(x) = x^8 + 12x^5 - 4x^4 + 10x^3 - 6x - 5$$ constant

$$f'(x) = 8x^7 + 12 \cdot 5x^4 - 4 \cdot 4x^3 + 10 \cdot 3x^2 - 6 \cdot 1x^0$$

$$= 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$$

Ex Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

Need Slope of tangent line $= 0$

$\rightarrow f'(x) = 0$

$y' = 4x^3 - 6 \cdot 2x^1 + 0$

$= 4x^3 - 12 = 0$

$= 4x(x^2 - 3) = 0$

$4x = 0$ or $x^2 - 3 = 0$

$\underline{x = 0}$ $\underline{x = \pm\sqrt{3}}$

* Plug into y

Find y coordinates : $y = x^4 - 6x^2 + 4$

$x = 0$: $y = 0^4 - 6(0)^2 + 4 = \underline{\underline{4}}$

$x = \sqrt{3}$ $y = (\sqrt{3})^4 - 6(\sqrt{3})^2 + 4 = \underline{\underline{-5}}$

$x = -\sqrt{3}$ $y = (-\sqrt{3})^4 - 6(-\sqrt{3})^2 + 4 = \underline{\underline{-5}}$

$\boxed{\text{Therefore: } (0,4), (\sqrt{3}, -5), (-\sqrt{3}, -5)}$

Derivatives of sine and cosine

- $\frac{d}{dx} \sin(x) = \cos(x)$
- $\frac{d}{dx} \cos(x) = -\sin(x)$

example
$\theta = x$

Ex Differentiate

$y = 3\sin(\theta) + 4\cos(\theta)$

$y' = 3 \cdot \cos(\theta) + 4 \cdot (-\sin(\theta))$

Ans $\boxed{y' = 3\cos(\theta) - 4\sin(\theta)}$

Ex Find the 27th derivate of $f(x) = \cos(x)$

$f(x) = \cos(x)$

$f'(x) = -\sin(x)$

$f''(x) = -\cos(x)$

$f'''(x) = -(-\sin(x)) = +\sin(x)$

$f''''(x) = \cos(x)$

* Notice Pattern Repeating
* Every 4th derivative $= \cos(x)$

Find 27th derivative

$f^{(24)}(x) = \cos(x)$
$f^{(25)}(x) = -\sin(x)$
$f^{(26)}(x) = -\cos(x)$
$\boxed{f^{(27)}(x) = \sin(x)}$

Repeated pattern can use shortcut. repeats every 4

Ex

The position of a partical (in meters) is given by $f(t) = t^3 - 6t^2 + 9t$ where t is time in seconds

ⓐ What is the velocity after 2 sec? After 4 sec?

$v(t) = 3t^2 - 6 \cdot 2t^1 + 9 \cdot t^0 = (1)$

$= 3t^2 - 12t + 9$

$V(2) = \underset{3\times4}{3(2)^2} - \underset{12\times2}{12(2)} + 9 = \boxed{-3\ m/s}$

$= 12 - 24 + 9$

$V(4) = 3(4)^2 - 12(4) + 9 = \boxed{9\ m/s}$

$48 - 48 + 9$

ⓑ When is the particle at rest?

$V(t) = 0$

$3t^2 - 12t + 9 = 0$

$3(t^2 - 4t + 3) = 0$

$3(t-1)(t-3) = 0$

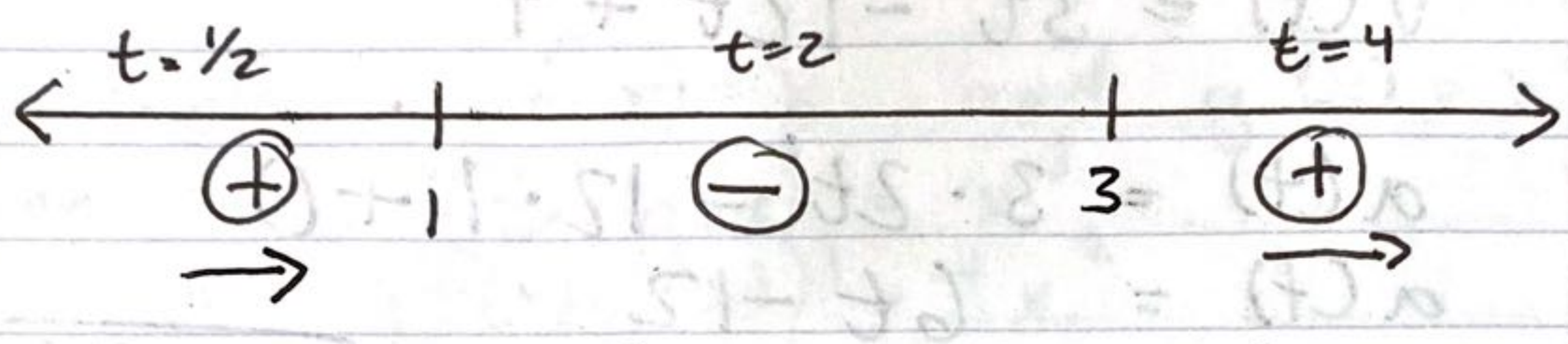
$t - 1 = 0$ or $t - 3 = 0$

$\underline{t = 1 sec}$ or $\underline{t = 3 sec}$

ⓒ When is the particle moving forward?

$v(t) > 0$ * needs to be positive to move forward

$3(t-1)(t-3) > 0$



It is moving forward when $\underline{\underline{t<1}}$ or $\underline{\underline{t>3}}$.

ⓓ Find the total distance traveled in the first 5 seconds?

$\underline{t=0 \text{ to } t=1}$: forward

$|f(1) - f(0)| = |(1^3 - 6(1)^2 + 9(1)) - (0^3 - 6(0)^2 + 9(0))|$

$= \underline{\underline{4m}}$

$t=1$ to $t=3$: backwards

$|f(3) - f(1)| = 4m$

$t=3$ to $t=5$ = forward

$|f(5) - f(3)| = \underline{\underline{20m}}$

Total $= 4m + 4m + 20m$

$\boxed{= \underline{28m}}$

(e) Find the acceleration of the particle after 4 seconds.

$$a(t) = v'(t)$$

$$v(t) = 3t^2 - 12t + 9$$

$$a(t) = 3 \cdot 2t^1 - 12 \cdot 1 + 0$$

$$a(t) = 6t - 12$$

$$a(4) = 6(4) - 12 = \boxed{12 \ m/s^2}$$

2.4 & 2.5 The Product and Quotient Rules

Is $(f(x)g(x))' = f'(x)\,g'(x)$?

NO.

for example = $f(x) = x$ and $g(x) = x^2$

Then $\rightarrow f(x)\,g(x) = x^3$

$\rightarrow (f(x)g(x))' = 3x^2$

but $f'(x) = 1$ and $g'(x) = 2x$

$\rightarrow f'(x)\,g'(x) = 2x$

and

$\rightarrow 3x^2 \neq 2x$

Product Rule

$$(fg)' = fg' + gf'$$

Ex Differiante $y = \underbrace{x^2}_{f}\,\underbrace{\sin(x)}_{g}$

$$y' = x^2(\cos(x)) + \sin(x)(2x)$$

$$\boxed{y' = x^2\cos(x) + 2x\sin(x)}$$

Ex If $h(x) = x\,g(x)$ and $g(3) = 5$ and $g'(3) = 2$

find $h'(3)$:

Find $h'(3)$

$$h(x) = \underbrace{x}_{f}\,\underbrace{g(x)}_{g}$$

$$h'(x) = x(g'(x)) + g(x)(1)$$
$$= x\,g'(x) + g(x)$$

$$h'(3) = 3g'(3) + g(3)$$
$$= 3(2) + 5$$

$$\boxed{h'(3) = 11}$$

Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

$$\left(\frac{\text{high}}{\text{low}}\right)' = \frac{\text{low} \cdot \text{D-high} - \text{high} \cdot \text{D-low}}{\text{square the bottom}}$$

Ex

Differentiate

$$y = \frac{x^2 + x - 2}{x^3 + 6} \quad \begin{matrix} \leftarrow \text{high} \\ \leftarrow \text{low} \end{matrix}$$

$$y' = \frac{(x^3+6)(2x+1) - (x^2+x-2)(3x^2)}{(x^3+6)^2}$$

simplified $= \dfrac{2x^4 + x^3 + 12x + 6 - 3x^4 - 3x^3 + 6x^2}{(x^3+6)^2}$

Simplified

$$\rightarrow \quad y' = \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3+6)^2}$$

Ex

Find the equatition of the tangent line to $y = \frac{\sqrt{x}}{(1+x^2)}$ at $(1, \frac{1}{2})$.

$$y = \frac{x^{1/2}}{1+x^2} \quad \begin{matrix} \leftarrow \text{high} \\ \leftarrow \text{low} \end{matrix}$$

$$y' = \frac{(1+x^2)(\frac{1}{2}x^{-1/2}) - x^{1/2}(2x)}{(1+x^2)^2}$$

$$y'(1) = \frac{(1+1^2)(\frac{1}{2}(1)^{-1/2}) - 1^{1/2}(2(1))}{(1+1^2)^2}$$

$$y'(1) = \frac{2(\frac{1}{2}) - 2}{2^2}$$

$$y'(1) = \boxed{-\frac{1}{4}} \leftarrow \text{slope}$$

$m = -\frac{1}{4}$, $(1, \frac{1}{2})$ use slope

$$y - y_1 = m(x - x_1)$$

$$\boxed{y - \frac{1}{2} = -\frac{1}{4}(x-1)}$$

OR

$$\boxed{y = -\frac{1}{4}x + \frac{3}{4}}$$

Tm: 27:41

Ex: Find derivative of $\tan(x)$

$$f(x) = \tan(x)$$

$$f(x) = \frac{\sin(x)}{\cos(x)}$$

$$f'(x) = \frac{\cos(x)(\cos(x)) - \sin(x)(-\sin(x))}{\cos^2(x)}$$

$$f'(x) = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

$$f'(x) = \frac{1}{\cos^2(x)} = \underline{\underline{\sec^2(x)}}$$

$$\boxed{\frac{d}{dx}\tan(x) = \sec^2(x)}$$

Derivates of <u>Trig functions</u>

Derivates of Trig functions

$\frac{d}{dx} \sin(x) = \cos(x)$	$\frac{d}{dx} \cos(x) = -\sin(x)$
$\frac{d}{dx} \tan(x) = \sec^2(x)$	$\frac{d}{dx} \cot(x) = -\csc^2(x)$
$\frac{d}{dx} \sec(x) = \sec(x)\tan(x)$	$\frac{d}{dx} \csc(x) = -\csc(x)\cot(x)$

Ex Differiantiate

$$f(x) = \frac{\sec(x)}{1+\tan(x)}$$

$$f'(x) = \frac{(1+\tan(x))(\sec(x)\tan(x)) - \sec(x)(\sec^2(x))}{(1+\tan(x))^2}$$

$$f'(x) = \frac{\sec(x)(\tan^2(x) + \tan(x) - \sec^2(x))}{(1+\tan(x))^2}$$

NOTE = $\tan^2(x) + 1 = \sec^2(x)$

$\tan^2(x) - \sec^2(x) = -1$

$$\boxed{f'(x) = \frac{\sec(x)(\tan(x)-1)}{(1+\tan(x))^2}}$$

2.6 The Chain Rule

If $f(x) = (f \circ g)(x)$
$= f(g(x))$ then

$$F'(x) = f'(g(x)) \cdot g'(x)$$

If $y = f(u)$ and $u = g(x)$ then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

- Outside function x inside function

Ex Differentiate $f(x) = \sqrt{x^2+1}$

$$f(x) = (x^2+1)^{1/2}$$

outside $= (\quad)^{1/2}$
inside $= x^2+1$

$$F'(x) = \tfrac{1}{2}(x^2+1)^{1/2} \cdot 2x$$

OR

$$F'(x) = \underline{x(x^2+1)^{-1/2}} \quad \text{or} \quad \underline{\frac{x}{\sqrt{x^2+1}}}$$

Ex Differentiate $y = \sin(x^2)$ and $y = \sin^2(x)$

$y = \sin(x^2)$ → outside $\sin(\quad)$
→ inside x^2

$$y' = \cos(x^2) \cdot 2x$$
$$y' = 2x\cos(x^2)$$

$y = \sin^2(x) \rightarrow y\,(\sin(x))^2$

$y' = 2(\sin(x))^1 \cdot \cos(x)$

$y' = 2\sin(x)\cos(x)$

Ex

Differentiate $y = (x^3-1)^{100}$

$$y' = \underbrace{100\,(x^3-1)^{99}}_{\text{outside}} \cdot \underbrace{3x^2}_{\text{inside}}$$

$$y' = 300x^2\,(x^3-1)^{99}$$

Ex Differentiate $f(x) = \dfrac{1}{\sqrt[3]{x^3+x+1}}$

$= f(x) = (x^2+x+1)^{-1/3}$

$= f'(x) = -\frac{1}{3}(x^2+x+1)^{-4/3} \cdot (2x+1)$

Ex Differentiate $g(t) = \left(\dfrac{t-2}{2t-1}\right)^9$

$$g'(t) = \underbrace{9\left(\frac{t-2}{2t+1}\right)^8}_{\text{out}} \cdot \underbrace{\frac{(2t+1)(1) - (t-2)(2)}{(2t\cdot 1)}}_{\text{in}}$$

$=$

Simplified

$$= 9\left(\frac{t-2}{2t+1}\right)^1 \cdot \frac{2t+1-2t+4}{(2t+1)^2}$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \frac{5}{(2t+1)^2}$$

$$= \frac{45(t-2)^8}{(2t+1)^8} \cdot \frac{1}{(2t+1)^2} = \frac{45(t-2)^8}{(2t-1)^{10}}$$

Ex Differentiate (Product Rule) with chain Rule

$$y = (2x+1)^5 (x^3 - x + 1)^4$$

$$y' = (2x+1)^5 \cdot \left(4(x^3-x+1)^3 \cdot (3x^2-1)\right) + (x^3-x+1)^4 \cdot \left(5(2x+1)^4 \cdot 2\right)$$

Ex Differentiate (Nested chain Rule) find $F'(x)$

$$F(x) = \sin(\cos(\tan(x)))$$

$$F'(x) = \cos(\cos(\tan(x))) \cdot \underbrace{\overbrace{-\sin(\tan(x))}^{\text{outside 2}} \cdot \overbrace{\sec^2(x^2)}^{\text{inside 2}}}_{\text{inside}}$$

Find y' if $y = \sqrt{\sec(x^3)}$

$$y = (\sec(x^3))^{1/2}$$

$$y' = \frac{1}{2}(\sec(x^3))^{-1/2} \cdot \sec(x^3)\tan(x^3) \cdot 3x^2$$

$$= \frac{3}{2}x^2 (\sec(x^3))^{1/2} \tan(x^3)$$

3.1 Implicit Differentiation

1) Differentiate both sides of your equation

2) When taking a derivative of a term involving "y", multiply by $\frac{dy}{dx}$ or y'

3) Solve for $\frac{dy}{dx}$ or y'

Ex Find $\frac{dy}{dx}$ when $x^2+y^2=25$

$x^2+y^2 = 25$ ← differentiate

$2x + 2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = -2x$

$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$

Ex Find the equation of the tangent line to $x^2+y^2=25$ at $(3,4)$.

slope: $\frac{dy}{dx} = \frac{-x}{y}$ at $(3,4)$

$m = -\frac{3}{4}$

$\boxed{y-4 = -\frac{3}{4}(x-3)}$ or $\boxed{y = -\frac{3}{4}x + \frac{25}{4}}$

Ex Find the equation of the tangent line to $x^3 + y^3 = 6xy$ at $(3, 3)$.

$$x^3 + y^3 = 6xy \quad \leftarrow \text{differentiate}$$

$$3x^2 + 3y^2 y' = 6x(y') + y(6)$$

$$3x^2 + 3y^2 y' = 6xy' + 6y$$

$$3y^2 y' - 6xy' = 6y - 3x^2$$

$$y'(3y^2 - 6x) = 6y - 3x^2$$

$$y' = \frac{6y - 3x^2}{3y^2 - 6x}$$

$$y' = \frac{2y - x^2}{y^2 - 2x}$$

at $(3,3)$: $y' = \frac{2(3) - 3^2}{3^2 - 2(3)} = -1 \leftarrow m$

$$\boxed{\begin{gathered} y - 3 = -1(x - 3) \\ \text{or} \\ y = -x + 6 \end{gathered}}$$

Ex Find $\frac{dy}{dx}$ if $\sin(x+y)\, y^2 \cos(x)$

$\sin(x+y) = y^2 \cos(x)$ ← differ.

$\cos(x+y) \cdot \left(1 + \frac{dy}{dx}\right) = y^2(-\sin(x)) + \cos(x)(2y\frac{dy}{dx})$

$\cos(x+y)\left(1+\frac{dy}{dx}\right) = -y^2\sin(x) + 2y\cos(x)\frac{dy}{dx}$

$\cos(x+y) + \cos(x+y)\frac{dy}{dx} = -y^2\sin(x) + 2y\cos(x)\frac{dy}{dx}$

$\cos(x+y)\frac{dy}{dx} - 2y\cos(x)\frac{dy}{dx} = -y^2\sin(x) - \cos(x+y)$

$\frac{dy}{dx}\left(\cos(x+y) - 2y\cos(x)\right) = -y^2\sin(x) - \cos(x+y)$

$$\boxed{\frac{dy}{dx} = \frac{-y^2\sin(x) - \cos(x+y)}{\cos(x+y) - 2y\cos(x)}}$$

Second Derivatives

- find $\frac{dy}{dx}$ first
- Do implicit differentiation again
- Substitute $\frac{dy}{dx}$ as needed

Ex A particle moves along the curve $8x^2+y^2=8$. Find an expression for the acceleration.

$$8x^2+y^2=8 \quad \leftarrow \underline{\text{diff}}$$

$$16x+2y\frac{dy}{dx}=0$$

$$2y\frac{dy}{dx}=-16x$$

$$\frac{dy}{dx}=-\frac{8x}{y}$$

$$\boxed{\frac{dy}{dx}=\frac{-8x}{y} \qquad 8x^2+y^2=8}$$

diff

$$\frac{d^2y}{dx^2}=\frac{y(-8)-(-8x)\left(\frac{dy}{dx}\right)}{y^2}$$

$$\frac{d^2y}{dx^2}=\frac{-8y+8x\frac{dy}{dx}}{y^2}$$

$$\frac{d^2y}{dx^2}=\frac{-8y+8x\left(-\frac{8x}{y}\right)}{y^2}\cdot\frac{y}{y}$$

$$\frac{d^2y}{dx^2}=\frac{-8y^2-64x^2}{y^3}$$

$$\frac{d^2y}{dx^2}=\frac{-8(y^2+8x^2)}{y^3}$$

$$\frac{d^2y}{dx^2}=\frac{-8(8)}{y^3}=\boxed{\frac{-64}{y^3}}$$

3.2 & 3.3

Derivatives of Logarithmic, Exponential and Inverse Trigonometric functions.

$$\frac{d}{dx} \ln(x) = \frac{1}{x}$$

For $\log_b(x)$ recall $\log_b(x) = \frac{\ln(x)}{\ln(b)}$

So $\frac{d}{dx} \log b(x) = \frac{d}{dx}\left(\frac{\ln(x)}{\ln(b)}\right) = \frac{1}{\ln(b)} \frac{d}{dx} \ln(x)$

$$\boxed{= \frac{1}{x \ln(b)}}$$ ← memorize

Ex $\frac{d}{dx} \ln(x^2+1)$

$$\frac{d}{dx} \ln(x^2+1) = \frac{1}{x^2+1} \cdot 2x = \boxed{\frac{2x}{x^2+1}}$$

Ex $\frac{d}{dx} \ln(3x^2-x)$

$$\frac{d}{dx} \ln(3x^2-x) = \frac{1}{3x^2-x} \cdot (6x-1) = \boxed{\frac{6x-1}{3x^2-x}}$$

Ex Find $\frac{d}{dx} \ln\left(\frac{x^2 \sin(x)}{\sqrt{1+x}}\right)$

Use properties of logs.

$$\frac{d}{dx}\left(\ln(x^2\sin(x)) - \ln\left((1+x)^{1/2}\right)\right)$$

$$\frac{d}{dx}\left(\ln x^2 + \ln \sin x - \frac{1}{2}\ln(1+x)\right)$$

$$\frac{d}{dx}\left(2\ln(x) + \ln \sin x - \frac{1}{2}\ln(1+x)\right)$$

$$2 \cdot \frac{1}{x} + \frac{1}{\sin(x)} \cdot \cos(x) - \frac{1}{2} \cdot \frac{1}{1+x} \cdot 1$$

$$\boxed{= \frac{2}{x} + \cot(x) - \frac{1}{2(1+x)}}$$

Logarithmic Differentiation

* take natural log of both sides of the equation
* use implicit differentiation
* solve for $\frac{dy}{dx}$

Ex Differentiate

$$y = \frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}$$

$$\ln(y) = \ln\left(\frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}\right)$$

$$\ln(y) = \boxed{2\ln x} + \boxed{\tfrac{1}{3}\ln(7x-14)} - \boxed{4\ln(1+x^2)}$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \boxed{2\cdot\frac{1}{x}} + \boxed{\frac{1}{3}\cdot\frac{1}{7x-14}\cdot 7} - \boxed{4\cdot\frac{1}{1+x^2}\cdot 2x}$$

$\frac{1}{7x-14} \times \frac{7}{3(x-2)}$ $\quad$ $4\times 2 = 8$ $\quad = \frac{8x}{1+x^2}$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x}\cdot\frac{1}{3(x-2)} - \frac{8x}{1+x^2}$$

$$\frac{dy}{dx} = y\left(\frac{2}{x}\cdot\frac{1}{3(x-2)} - \frac{8x}{1+x^2}\right)$$

$$\boxed{\frac{dy}{dx} = \frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}\left(\frac{2}{x} + \frac{1}{3(x-2)} - \frac{8x}{1-x^2}\right)}$$

Exponential Functions

$$\frac{d}{dx} e^x = e^x \qquad \frac{d}{dx} b^x = b^x \ln(b)$$

Ex Differentiate

ⓐ $y = 2^x$

$\boxed{y' = 2^x \ln(2)}$ differentiate

ⓑ $y = e^{-2x}$

$\boxed{y' = e^{-2x} \cdot (-2) \rightarrow -2e^{-2x}}$ chain rule

ⓒ $y = e^{x^3}$

$y' = e^{x^3} \cdot 3x^2$

$\boxed{y' = 3x^2 e^{x^3}}$ chain Rule

ⓓ $y = e^{\cos(x)}$

$y' = e^{\cos(x)} \cdot (-\sin(x))$

$\boxed{y' = -\sin(x) e^{\cos(x)}}$ chain rule

Derivatives of Inverse Trig Functions

① $\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$

② $\frac{d}{dx} \cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$

③ $\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}$

④ $\frac{d}{dx} \cot^{-1}(x) = -\frac{1}{1+x^2}$

⑤ $\frac{d}{dx} \sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$

⑥ $\frac{d}{dx} \csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$

Ex Differentiate

ⓐ $y = \sin^{-1}(x^3)$

$$y' = \frac{1}{\sqrt{1-(x^3)^2}} \cdot 3x^2$$

$$\boxed{y' = \frac{3x^2}{\sqrt{1-x^6}}}$$

ⓑ $y = \sec^{-1}(e^x)$

$$y' = \frac{1}{|e^x|\sqrt{(e^x)^2-1}} \cdot e^x$$

$$y' = \frac{e^x}{e^x\sqrt{e^{2x}-1}} = \boxed{\frac{1}{\sqrt{e^{2x}-1}}}$$

3.4 Related Rates

* will use alot of implicit differtiation

1.) Find an equation that relates the quantities

2) Use implicit differentiation to differentiate both sides with respect to time.

Ex Air is being pumped into sperical balloon so that its volume increases at a rate of 100 cm³/sec. How fast is the radius increasing when the diameter is 50cm?

$$\boxed{V = \frac{4}{3}\pi R^3}$$

Know $\frac{dV}{dt} = 100$ Want $= \frac{dR}{dt}$ when = $r = 25$ ~~seconds~~

Know $V = \frac{4}{3}\pi r^3$ ← Volume of Sphere

$$\frac{dV}{dt} = 100$$

WANT $\frac{dr}{dt}$ = when $r = 25$

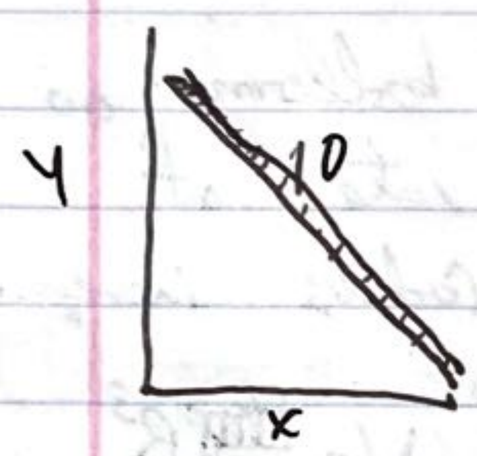
$$V = \frac{4}{3}\pi r^3 \qquad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$100 = 4\pi (25)^2 \frac{dr}{dt}$$

$$\frac{100}{4\pi(25)^2} = \frac{dr}{dt} = \boxed{\frac{1}{25\pi} \text{ cm/sec} = \frac{dr}{dt}}$$

Ex

A ladder 10ft long rests against a verbal wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/sec, how fast is the top of the ladder sliding down when the bottom is 6 ft away?



$x^2+y^2=100$

$\frac{dx}{dt} = 1$ ft/sec

Know $= x^2+y^2=100$

$\frac{dx}{dt} = 1$

Want $= \frac{dy}{dt}$ when $x=6$

(when $x=6$, $6^2+y^2=100 \rightarrow y=8$)

$$x^2+y^2 = 100$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2(6)(1) + 2(8)\frac{dy}{dt} = 0$$

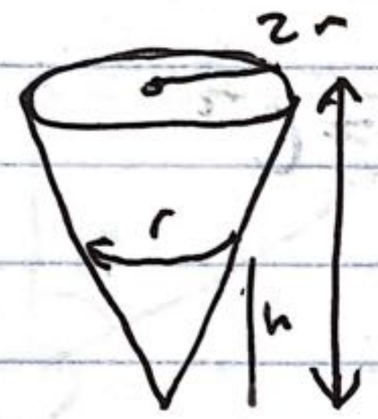
$$12 + 16\frac{dy}{dt} = 0$$

$$16\frac{dy}{dt} = -12$$

$$\boxed{\frac{dy}{dt} = \frac{-3}{4} \text{ ft/sec}}$$

Ex

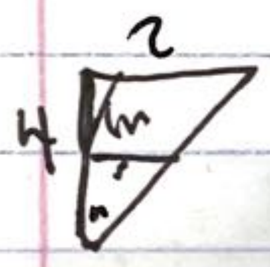
A water tank has a shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of $2 \frac{m^3}{s}$, find the rate at which the water level is rising when it is 3 ft deep.



$$V = \frac{1}{3}\pi r^2 h$$

$V_{know} = 2 \frac{m^3}{s}$ is going to be $\frac{dv}{dt}$

$V_{want} = \text{Find} = dh/dt$



← look at it as a triangle

$$\frac{2}{h} = \frac{r}{h} \qquad \frac{1}{2}h = r \qquad \text{So } V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$$

Rewrite

$$V = \frac{1}{12}\pi h^3$$

$$V = \frac{1}{12}\pi h^3$$

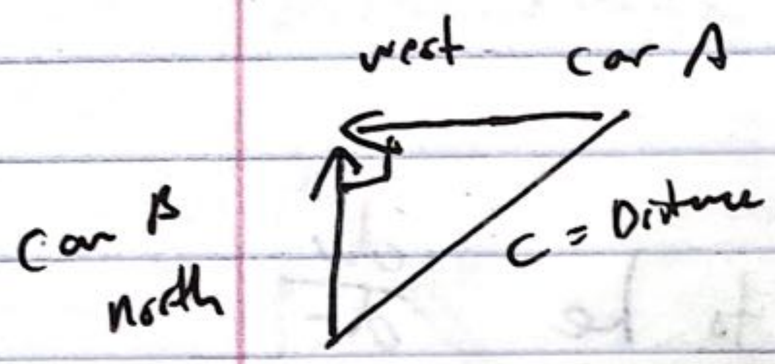
$$\frac{dv}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

$$2 = \frac{1}{4}\pi (3)^2 \frac{dh}{dt}$$

$$2 = \frac{9\pi}{4}\frac{dh}{dt}$$

$$\boxed{\frac{8}{9\pi} \ m/min = \frac{dh}{dt}}$$

Ex Car A is traveling west at $\boxed{50 \text{ m/hr}}$ and Car B is traveling north at $\boxed{60 \text{ m/hr}}$. Both cars are headed toward the intersection of crossroads. At what rate are the cars approaching each other, when Car A is 0.3 miles from intersect and car B is 0.4 miles away?



Know $= A^2 + B^2 = C^2$

$\frac{dA}{dt} = 50$ (A)

$\frac{dB}{dt} = 60$ (B)

Want $= \frac{dc}{dt}$ when $A = 0.3$, $B = 0.4$

Find C : $0.3^2 + 0.4^2 = C^2$

$0.25 = c^2 \rightarrow \underline{\underline{0.5}}$

$$A^2 + B^2 = C^2$$

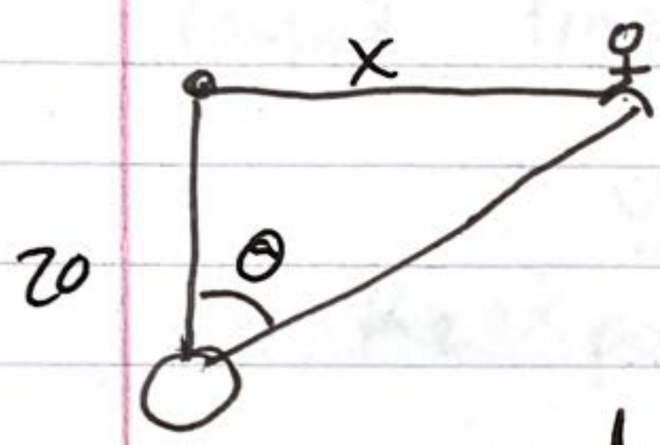
$$2A\frac{da}{dt} + 2B\frac{db}{dt} = 2c\frac{dc}{dt}$$

$$2(0.3)(50) + 2(0.4)(60) = 2.(0.5)\frac{dc}{dt}$$

$$30 + 48 = \frac{dc}{dt}$$

$$\boxed{= 78 \text{ m/hr } \frac{dc}{dt}}$$

Ex A man walks along a straight path at a speed of 4 ft/sec. A searchlight is on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the light?



$$\text{know} = \tan\theta = \frac{x}{20}$$

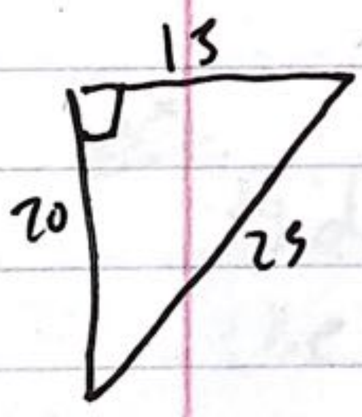
$$\frac{dx}{dt} = 4$$

want: $\frac{d\theta}{dt}$ when $x = 15$

$$\tan(\theta) = \frac{x}{20}$$

$$\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{20}\frac{dx}{dt}$$

• find $\sec^2(\theta)$ to finish •



$$15^2 + 20^2 = ?^2 \qquad \sqrt{\ } \text{ sqrt}$$

$$25 = ?$$

$$\sec(\theta) = \frac{1}{\cos\theta} = \frac{\text{hypotenuse}}{\text{adj.}}$$

$$\sec(\theta) = \frac{25}{20} = \frac{5}{4}$$

$$\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{20}\cdot\frac{dx}{dt}$$

$$\left(\frac{5}{4}\right)^2\frac{d\theta}{dt} = \frac{1}{20}(4)$$

$$\frac{25}{16}\frac{d\theta}{dt} = \frac{1}{5} \rightarrow \frac{d\theta}{dt} = \frac{1}{5}\cdot\frac{16}{25}$$

$$\boxed{\frac{d\theta}{dt} = \frac{16}{125} \text{ radians per second}}$$

3.5 Linear Approximation

We can use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a.

Tangent line : $y - y_1 = m(x - x_1)$ * Point slope form

$$\underline{y = y_1 + m(x - x_1)}$$

at the point $\underline{(a, f(a))}$ with Slope $\underline{m = f'(a)}$

$$y = f(a) + f'(a)(x - a)$$

The <u>Linear Approximation</u> or <u>linearization</u> of $f(x)$ near $x = a$ is

$$f(x) \approx \underbrace{f(a) + f'(a)(x - a)}_{\text{Tangent line}}$$

Ex

Find the linearization of $f(x) = \sqrt{x+3}$ at $a = 1$. Use this to approximate $\sqrt{3.98}$ and $\sqrt{4.05}$.

Need : $f(a)$, $f'(a)$

$$f(1) = \sqrt{1+3} = \sqrt{4} = \underline{2}$$

chain Rule

$$f(x) = (x+3)^{\frac{1}{2}}$$
$$f'(x) = \frac{1}{2}(x+3)^{-1/2} \cdot 1$$
$$f'(x) = \frac{1}{2\sqrt{x+3}}$$
$$f'(1) = \frac{1}{2\sqrt{1+3}} = \frac{1}{4} \longrightarrow$$

$f(x) \approx f(a) + f'(a)(x-a)$

$f(x) = 2 + \frac{1}{4}(x-1)$

$f(x) \approx 2 + \frac{1}{4}x - \frac{1}{4}$

$f(x) \approx \frac{1}{4}x + \frac{7}{4}$

$a=1$

$\sqrt{x+3} \approx \frac{1}{4}x + \frac{7}{4}$

$\sqrt{3.98} = \sqrt{\underbrace{0.98}_{x \text{ close to } 1} + 3}$ so $\sqrt{3.98} \approx \frac{1}{4}(0.98 + \frac{7}{4})$

$= \frac{7.98}{4} = \underline{\underline{1.995}}$

$\sqrt{4.05} = \sqrt{\underbrace{1.05}_{x \text{ close to } 1} + 3}$ so $\sqrt{1.05+3} = \frac{1}{4}(1.05 + \frac{7}{4})$

$= \frac{8.05}{4} = \underline{\underline{2.0125}}$

<u>Ex</u> Find linear Approx for $f(x) = \sin(x)$ near $x=0$

$f(0) = \sin(0) = 0$

deriv $f'(x) = \cos(x)$

$f'(0) = \cos(0) = 1$

* $f(x) \approx f(a) + f'(a)(x-a)$

$\sin(x) \approx 0 + 1(x-0)$

$\sin(x) \approx x$ for values near 0

* this is why we use $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

3.~~6~~ L'Hôpital Rule

- Suppose that $f(x)$ and $g(x)$ are differentiable and $g'(x) \neq 0$ near a, except possibly at a.
- Suppose $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ OR
- Suppose $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$

$$\boxed{\text{then } \lim_{x \to} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

* Can only be used in correct form limit $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\frac{-\infty}{-\infty}$, $\frac{\pm\infty}{\pm\infty}$ and functions differentiable and derivative of the denominator not zero near value

Ex $\lim_{x \to 1} \frac{\ln(x)}{x-1} = \frac{0}{0}$ ✓ appropriate ✓ ∴ can take derivative of both $\ln(x)$ and $x-1$

$$\lim_{x \to 1} \frac{\ln(x)}{x-1} \overset{L'H}{=} \lim_{x \to 1} \frac{\frac{1}{x}}{1}$$

$$= \lim_{x \to 1} \frac{1}{x} = \boxed{1}$$

Ex $\lim_{x \to \infty} \frac{e^x}{x^2}$ $\frac{\infty}{\infty}$ $\overset{L'H}{\Rightarrow}$ $\frac{e^x}{2x}$ do again

$$\to \lim_{x \to \infty} \frac{e^x}{2} = \underline{\underline{\infty}}$$

Ex $\lim_{x\to\infty} \frac{\ln(x)}{\sqrt[3]{x}} \overset{L'H}{=} \frac{\ln(x)}{x^{1/3}}$ ← non zero ✓

$\lim_{x\to\infty} \frac{\ln(x)}{x^{1/3}} \overset{L'H}{=} \frac{\frac{1}{x}}{\frac{1}{3}x^{-2/3}}$

Indeterminate form

$= \lim_{x\to\infty} \frac{3}{x^{1/3}} = \boxed{0}$

If we have $0 \cdot \pm\infty$ we need to rewrite so it is $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

Ex $\lim_{x\to 0^+} x\ln(x) = 0 \cdot -\infty$ ← rewrite as quotient

$\lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} = \frac{-\infty}{\infty} \Rightarrow \frac{\ln(x)}{x^{-1}} = \frac{\frac{1}{x}}{-x^{-2}}$

$\lim_{x\to 0^+} (-x) = \boxed{0}$

If we have $\infty - \infty$, we need to rewrite as quotient

Ex $\lim_{x\to\frac{\pi}{2}^-} (\sec(x) - \tan(x))$ convert cos + sin

$\lim \left(\frac{1}{\cos x} - \frac{\sin(x)}{\cos(x)}\right)$

$$\lim_{x \to \frac{\pi}{2}^+} \left(\frac{1 - \sin(x)}{\cos(x)} \right) \quad \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x \to \frac{\pi}{2}^-} \frac{-\cos(x)}{-\sin(x)} \quad \frac{0}{1} = \boxed{0}$$

Ex Difficult

If you have 0^0, ∞^0 or 1^∞ need natural log

Ex $\lim_{x \to 0^+} (1 + \sin(4x))^{\cot(x)}$

$\to 1^\infty$. Use log

let $y = (1 + \sin(4x))^{\cot x}$ then

$\ln(y) = \ln((1 + \sin(4x))^{\cot x})$

$\ln(y) = \cot(x) \ln(1 + \sin(4x))$

$\ln(y) = \lim \cot(x) \ln(1 + \sin(4x))$ $\infty \cdot 0$

$$\lim_{x \to 0^+} \frac{\ln(1 + \sin(4x))}{\tan(x)}$$

$$= \frac{\left(\frac{1}{1 + \sin(4x)}\right) \cdot \left(\cos(4x) \cdot 4\right)}{\sec^2(x) \to 1} = \underline{\underline{4}}$$

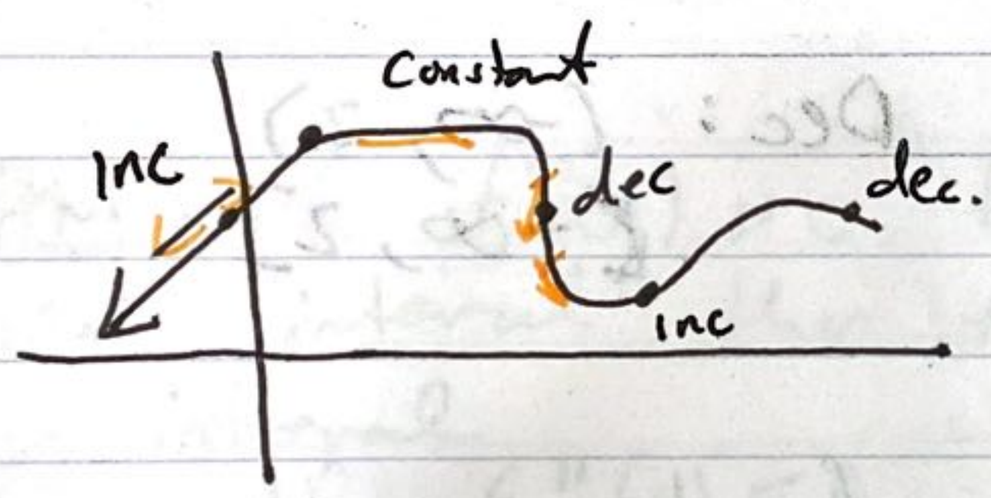
We want $\lim_{x \to 0^+} y$

$$\lim_{x \to 0^+} y = \boxed{e^4}$$

4.1/4.2 Analysis Functions - Relative Extrema & Inflection Points

Def: Let f be defined in an interval and let x_1, x_2 be in interval. Then f is <u>increasing</u> on the interval if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in the interval.

f is decreasing on the interval if $f(x_1) > f(x_2)$ for all $x_1 < x_2$ in the interval.
f is constant on interval if $f(x_1) = f(x_2)$ for all x_1, x_2 in the interval.



→ tangent lines
↗ positive
↘ negative
→ none

<u>THM:</u> Let f be continuous on $[a, b]$ and differentiable on (a, b)

ⓐ If $f'(x) > 0$ (positive) for every x in (a, b) then f is increasing on (a, b) (Horizontal $([a, b])$)

ⓑ If $f'(x) < 0$ (negative) for every x in (a, b) then f is decreasing on (a, b) ($[a, b]$)

ⓒ If $f'(x) = 0$ for every x in (a, b) then f is constant on (a, b) ($[a, b]$)

Ex Find the intervals where $f(x)$ is increasing/decreasing for $f(x) = x^2 - 4x + 3$

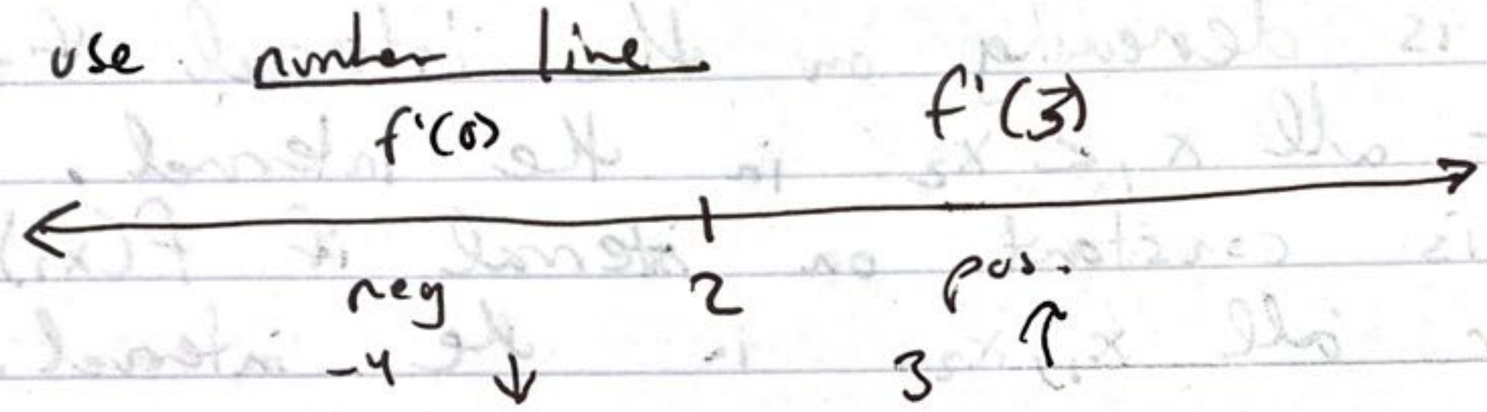
$\rightarrow f'(x) = 2x - 4$

$0 = 2x - 4$

$4 = 2x$

$2 = x$

$f'(x)$ use number line



Inc: $(2, \infty)$ Dec: $(-\infty, 2)$

$[2, \infty)$ $(-\infty, 2]$

Ex Find where the function is increasing or decreasing. $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$

$f'(x) = 12x^3 + 12x^2 - 24x$

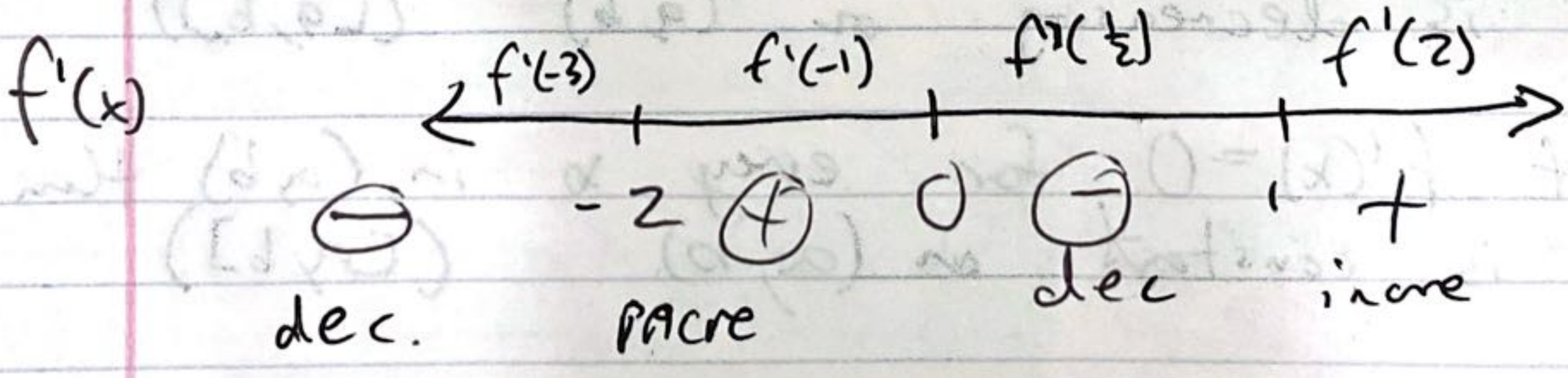
$0 = 12x^3 - 12x^2 - 24x$

$0 = 12x(x^2 + x - 2)$

$0 = 12x(x+2)(x-1)$

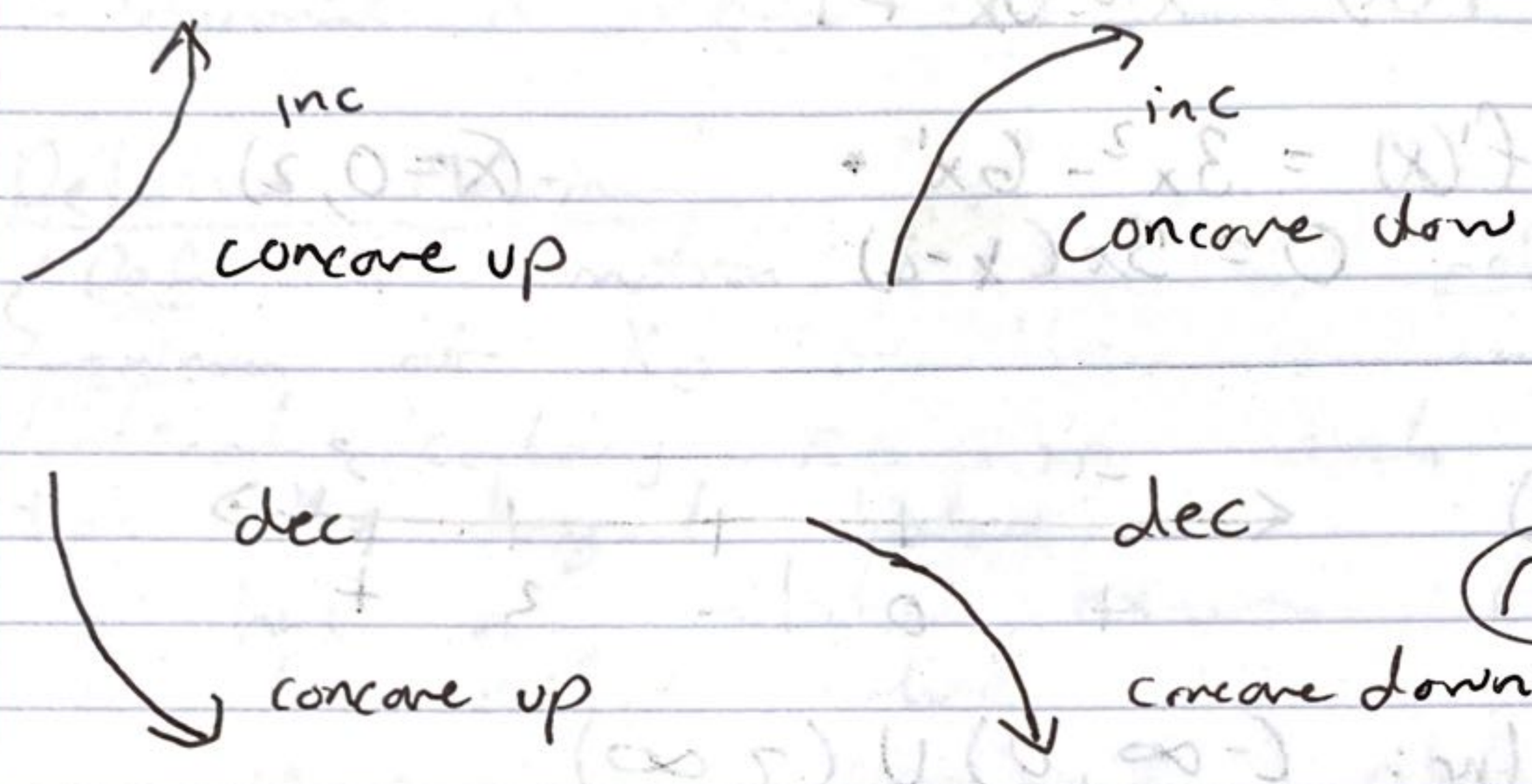
$x = 0, 1, -2$

$f'(x)$



Ex

Inc: $(2,0) \cup (1,\infty)$
Dec: $(-\infty,-2) \cup (0,1)$

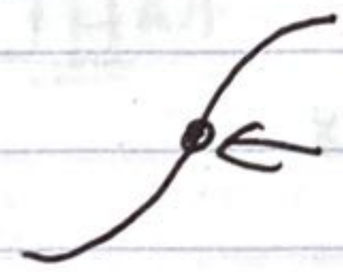


THM: If $f''(x) > 0$ for every x in an open interval then f is concave up (CCU) on interval

If $f''(x) < 0$ for every x in an open interval then f is concave down (CCD) on interval

Inflection Point

Def: A point where f changes concavity is called an inflection point.



- Where it changes direction

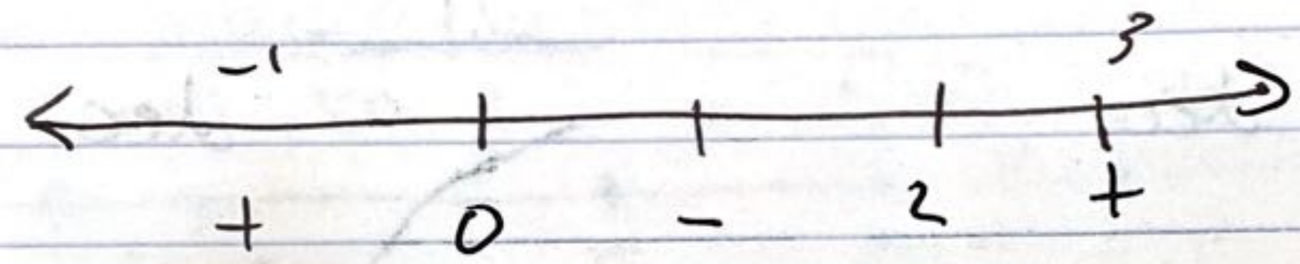
Ex Find intervals where f is inc, dec, ccu ccd. and any inflection points

$f(x) = x^3 - 3x^2 + 1$

$f'(x) = 3x^2 - 6x$ $\quad (x = 0, 2)$

$0 = 3x(x-2)$

$f'(x)$



Inc: $(-\infty, 0) \cup (2, \infty)$

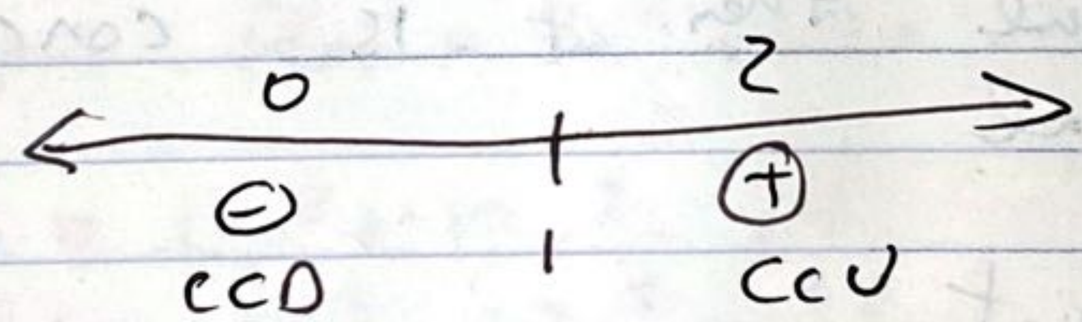
Dec: $(0, 2)$

$f'(x) = 3x^2 - 6x$ $\quad x = 1$

$f''(x) = 6x - 6$

$0 = 6(x-1)$

$f''(x)$



CCU : $(1, \infty)$

CCD : $(-\infty, 1)$

Inflection point : $\underline{\underline{x = 1}}$

original problem $f(1) = 1^3 - 3(1)^2 + 1$ $\quad \boxed{(1, -1)}$

$f(1) = -1$

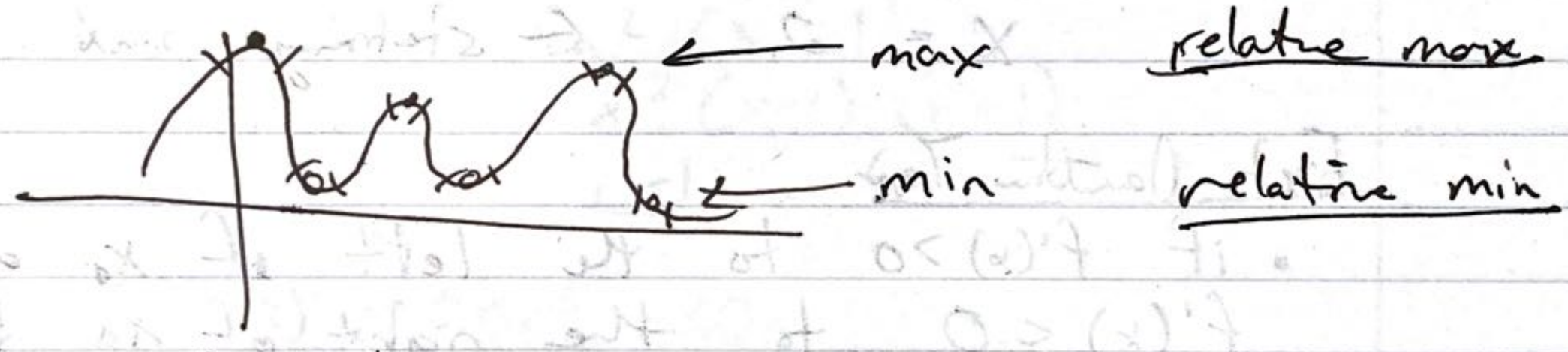
If there is an inflection point, it occurs where $f''(x_0)=0$ or is undefined.

* Polynomial is defined everywhere

Relative Maximum

tallest { **Def** A function f has a <u>relative maximum</u> at X_0 if there is an open interval containing x_0 on which $f(x_0)$ has the largest value.

smallest { f has a relative minimum if $f(x_0)$ is the smallest value.



Critical Point

Def: is a point in the domain of f such that $f'(x)=0$ or is undefined.
If $f'(x_0)=0$ we sometimes call that critical point a <u>stationary point</u>

THM: If f has a relative extremum at x_0 then $X=X_0$ is a critical point

Ex →

Ex Find the x-coordinates of all critical points of $f(x) = 3x^{5/3} - 15x^{2/3}$

$$f'(x) = 5x^{2/3} - 10x^{-1/3}$$
$$f'(x) = 5x^{-1/3}(x-2)$$

$$f'(x) = \frac{5(x-2)}{x^{1/3}}$$

$f'(x)$ = indefinite at $x=0$

$$0 = \frac{5(x-2)}{x^{1/3}}$$

$$0 = 5(x-2)$$
$$x = 2 \quad \leftarrow \text{stationary point}$$

First Derivative Test

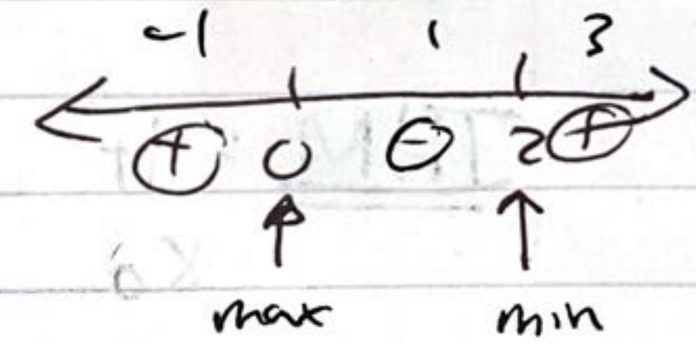
- if $f'(x) > 0$ to the left of x_0 and $f'(x) < 0$ to the right of x_0 then x_0 is a relative ~~minimum~~ maximum

- if $f'(x) < 0$ to the left "
  "$f'(x) > 0$ to right then ~~minimum~~

Second Der

Example

$$f'(x) = \frac{5(x-2)}{x^{1/3}}$$



Second Derivative Test

If $f'(x_0) = 0$ and $f''(x_0) > 0$ then x_0 is minimum

If $f'(x_0) = 0$ and $f''(x_0) < 0$ then x_0 is MAX

If $f'(x_0) = 0$ and $f''(x_0) = 0$ the x_0 is inconclusive

Ex Find the x coordinate of the relat extrema of

$$f(x) = 3x^5 - 5x^3$$

- $f'(x) = 15x^4 - 15x^2$

$$0 = 15x^2(x^2 - 1)$$

$$0 = 15x^2(x-1)(x+1)$$

$$x = 0, 1, -1$$

- $f''(x) = 60x^3 - 30x$

$$= 30x(2x^2 - 1)$$

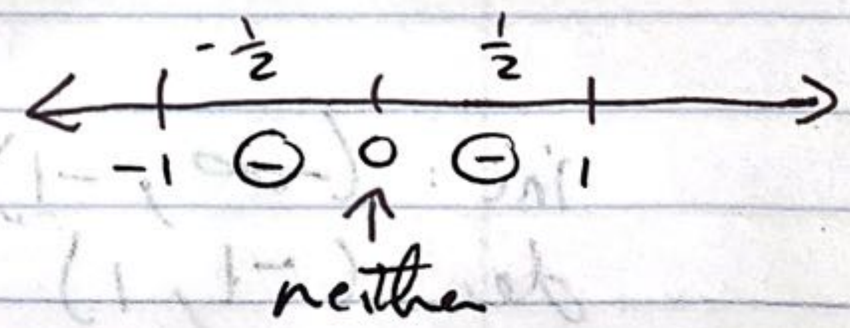
$x=0$: $f''(0) = 30(0)(2(0)^2 - 1) = 0$ $\boxed{\text{inclusive}}$

$x=1$: $f''(1) = 30(1)(2(1)^2 - 1) > 0$ MIN

$x=-1$: $f''(-1) = 30(-1)(2(-1)^2 - 1) < 0$ MAX

$x=1 \rightarrow$ rel min, $x=-1 \rightarrow$ rel max

$f'(x) = 15x^2(x+1)(x-1)$



Ex Sketch $y = x^3 - 3x + 2$ including all intercepts, relative extrema, inflection points

y intercept = $y = 0$

$$y = (0)^3 - 3(0) + 2 = (0, 2)$$

X int = $x = 2$

$$0 = x^3 - 3x + 2$$

Factor $= (x+2)(x-1)^2$

$x = -2, 1$ $\quad (-2, 0), (1, 0)$

Points

$\boxed{(0,2), (-2,0), (1,0)}$

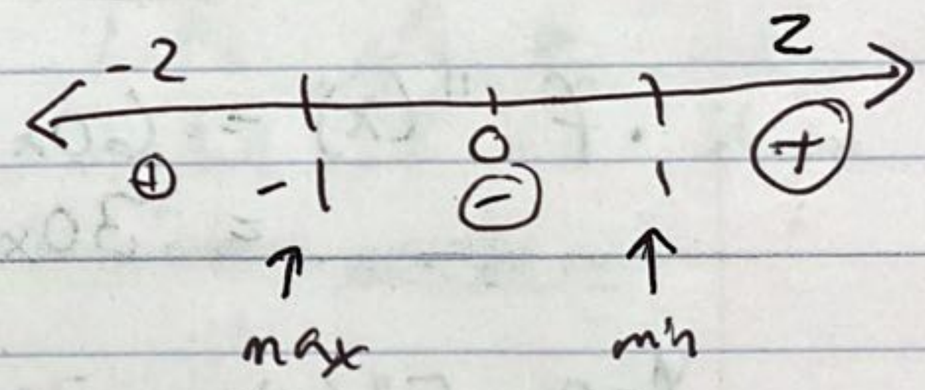
$$y = x^3 - 3x + 2$$

derivative $\rightarrow y' = 3x^2 - 3$

$$0 = 3(x^2 - 1)$$

$$0 = 3(x-1)(x+1)$$

$$x = 1, -1$$



Original function

$$y = (-1)^3 - 3(-1) + 2$$

$$\underline{y = 4}$$

$(1, 0)$ min

$(-1, 4)$ max

inc: $(-\infty, -1) \cup (1, \infty)$

dec $(-1, 1)$

$y' = 3x^2 - 3$

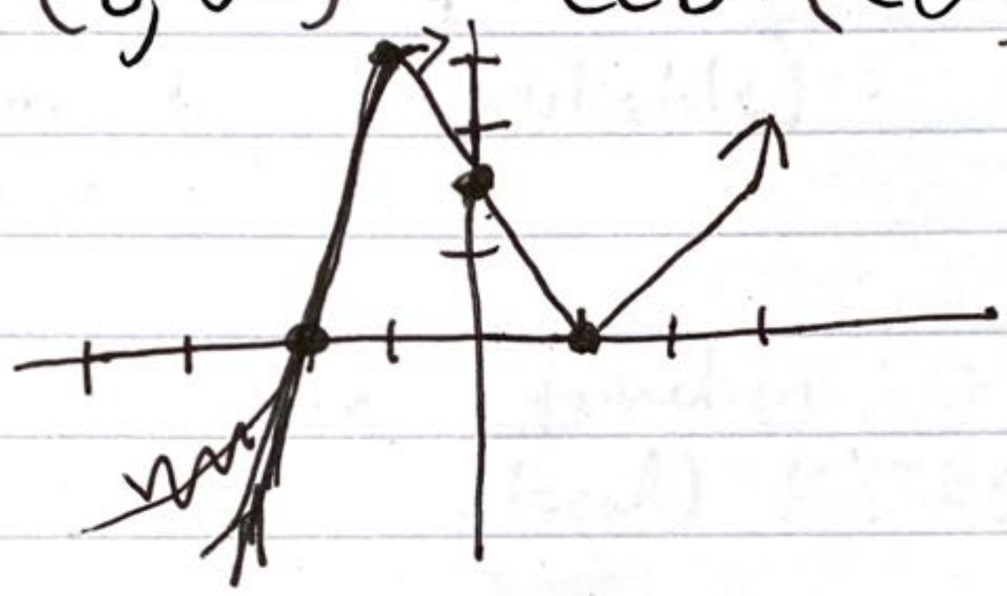
$y'' = 6x$

$0 = 6x$

$0 = x$

y'': $\ominus$ -1 0 1 $\oplus$

CCU: $(0, \infty)$ CCD: $(-\infty, 0)$



4.3 Graphing With Calculus

1.) Find the domain
2) Find all intercepts
   - y int → $x=0$
   - x int → $y=0$
3) Find Symmetry (if applicable)
4) Asymptotes

5.) Find where the function is Inc ↑ or Dec ↓
"Find relative (local) EXTREMA"

6) Find where the function is concave up $\boxed{CCU}$ or Concave down $\boxed{CCD}$ and inflection points.

7) Sketch Graph.

Ex Sketch $y = \frac{2x^2}{x^2-1}$ ← rational function

- Domain: $x^2-1=0$ • factor
  $(x-)(x+1)=0$
  $\boxed{x=1}$ $\boxed{x=-1}$
  → all real numbers except $\boxed{x=\pm 1}$

- Intercepts: y int: $y = \frac{2(0)^2}{0^2-1} = 0$
  $\boxed{=(0,0)}$
  x int: $0 \quad \frac{2x^2}{x^2-1}$
  $=0=2x^2$
  $\boxed{=x=0}$

$$y = \frac{2x^2}{x^2-1}$$

- Asymptote

vertical → VA: $x^2-1=0 \rightarrow x = \pm 1$ $\boxed{VA = \pm 1}$

HA: $\lim_{x\to\infty} \frac{2x^2}{x^2-1} = \frac{\frac{2x^2}{x^2}}{\frac{x^2}{x^2}-\frac{1}{x^2}} = \frac{2}{1-\frac{1}{x^2}} = \underline{\underline{2}}$

$HA = \boxed{y=2}$

Inc / Dec Find derivative

$$y' = \frac{(x^2-1)(4x) - 2x^2(2x)}{(x^2-1)^2}$$

Distribute

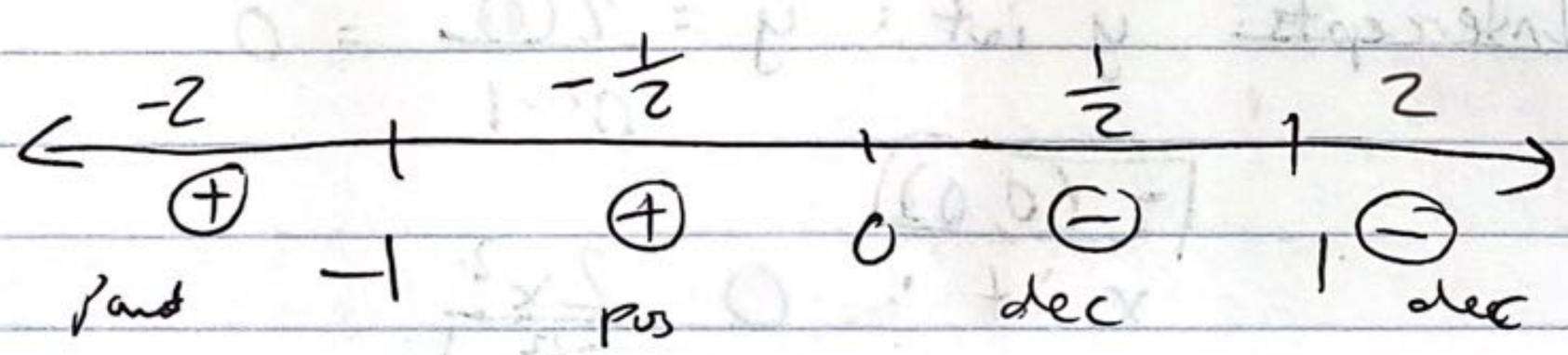
$$y' = \frac{4x^3 - 4x - 4x^3}{(x^2-1)^2}$$

stationary point
Derivative = 0

$$y' = \frac{-4x}{(x^2-1)^2}$$

Critical Values: y' is undefined at $\underline{x = \pm 1}$

$y' = 0$ when $-4x = 0 \rightarrow \underline{x=0}$ ⇐ stationary points



$$-\frac{\cancel{2}(-\cancel{2})^2}{(-\cancel{2})^2-1} =$$

$$= \frac{-4(-2)}{(-2^2-1)^2}$$

Inc: $(-\infty, -1), (-1, 0)$
Dec: $(0, 1), (1, 0)$

$$y' = \frac{-4x}{(x^2-1)^2}$$

$$y'' = \frac{(x^2-1)^2(-4) - (-4x)\,2\cancel{(x^2-1)}(2x)}{(x^2-1)^{\cancel{4}3}}$$

$$y'' = \frac{(x^2-1)(-4) + (+4x)(2)(2x)}{(x^2-1)^3} = \frac{-4x^3 - 1 + 16x^2}{(x^2-1)^3}$$

$$y'' = \frac{12x^2+14}{(x^2-1)^3}$$

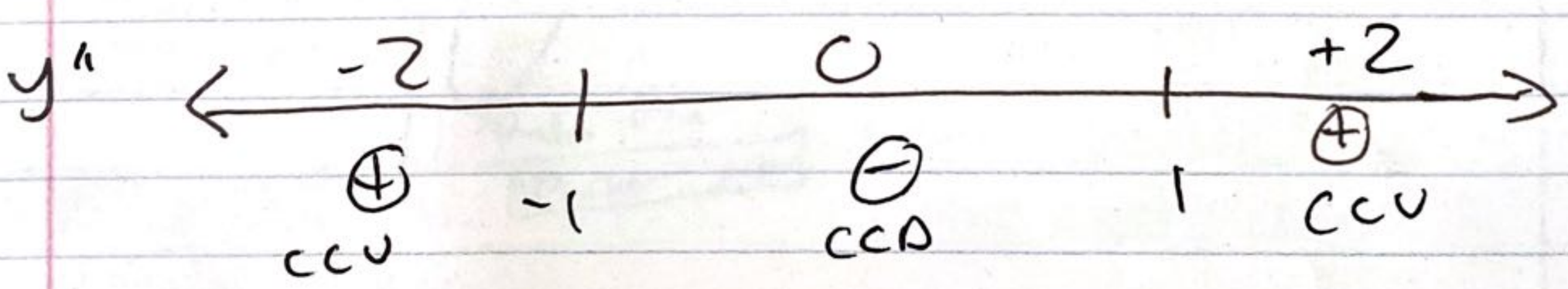
Undefined : $x = \pm 1$

$y'' = 0$ when $12x^2 + 4 = 2$

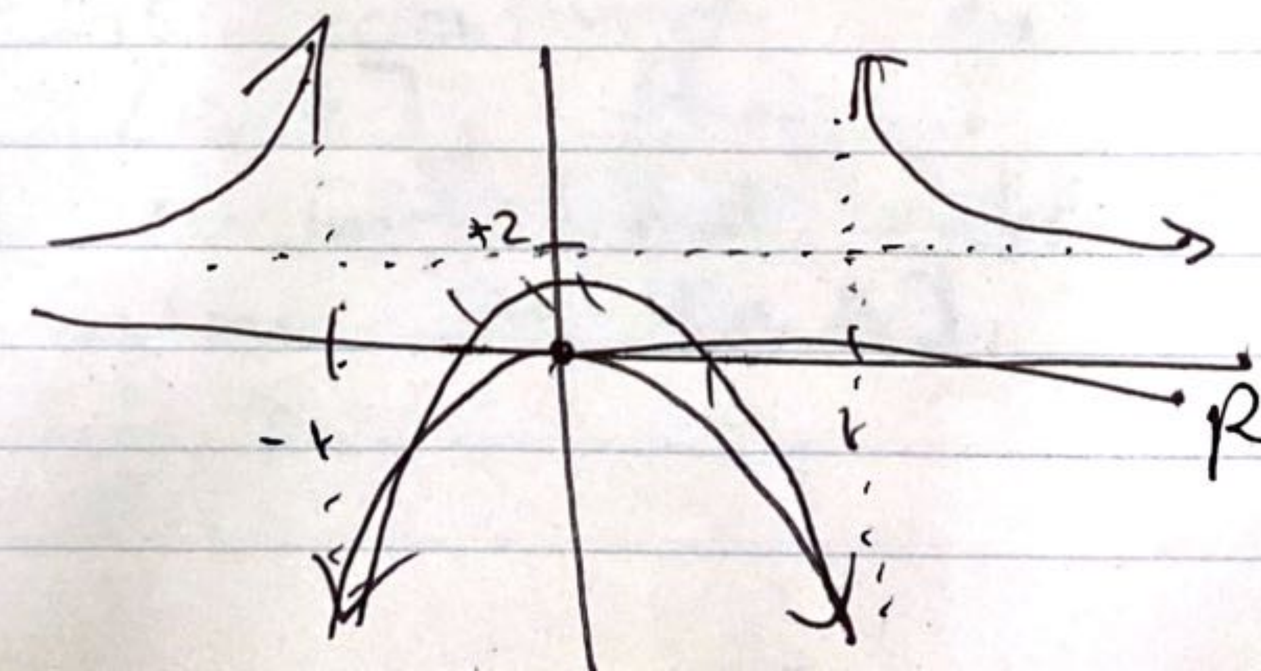
$12x^2 = -4$

$x^2 = -1/3$

No real solution



CCU : $(-\infty, -1), (1, \infty)$ CCD = $(-1, 1)$



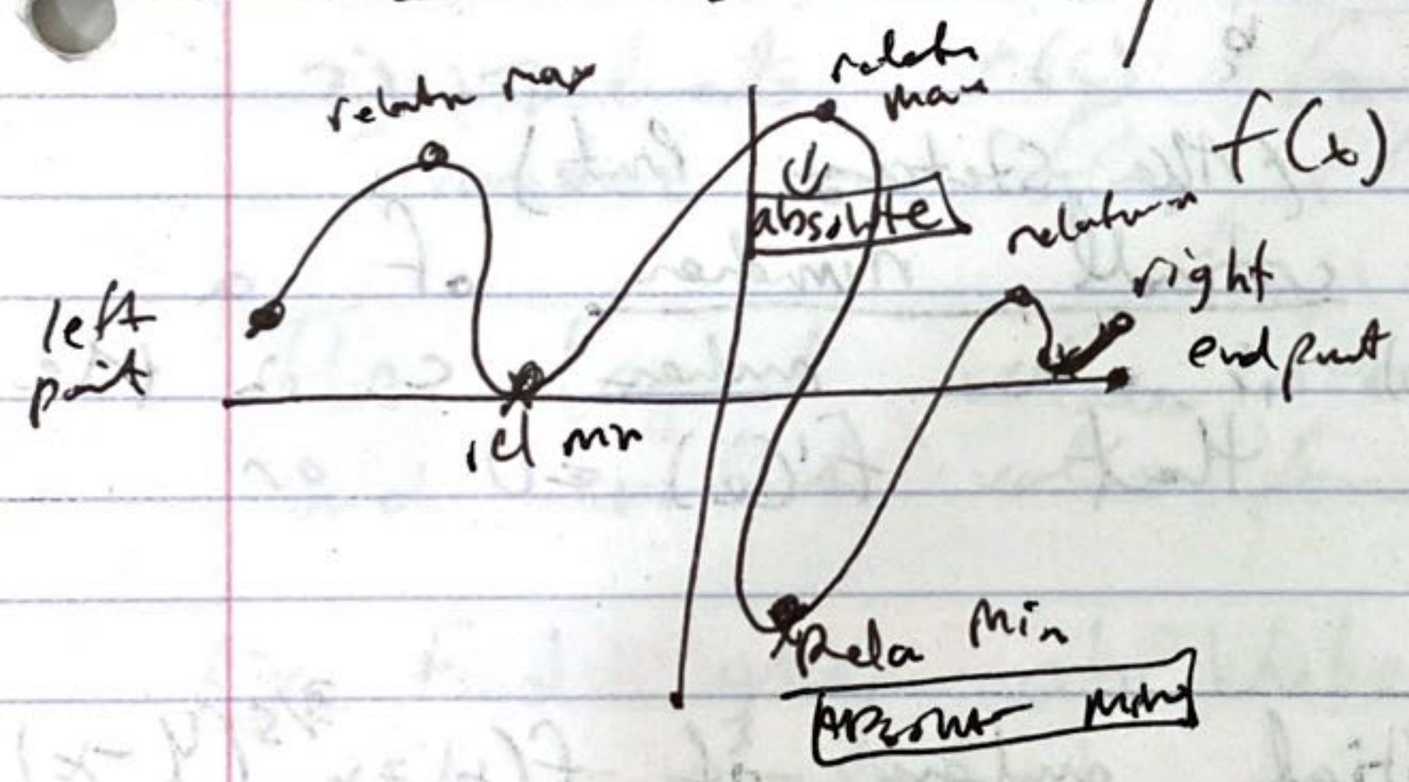
Relative maximum $(0,0)$

4.4 Absolute Maximum & Minimum Values

Let c be a number in the domain D of $f(x)$

- $f(c)$ is the absolute minimum value of $f(x)$ on D if $f(c) = f(x)$ for all x in D.
- $f(c)$ is the absolute maximum value of $f(x)$ on D if $f(c) \geq f(x)$ for all x in D.

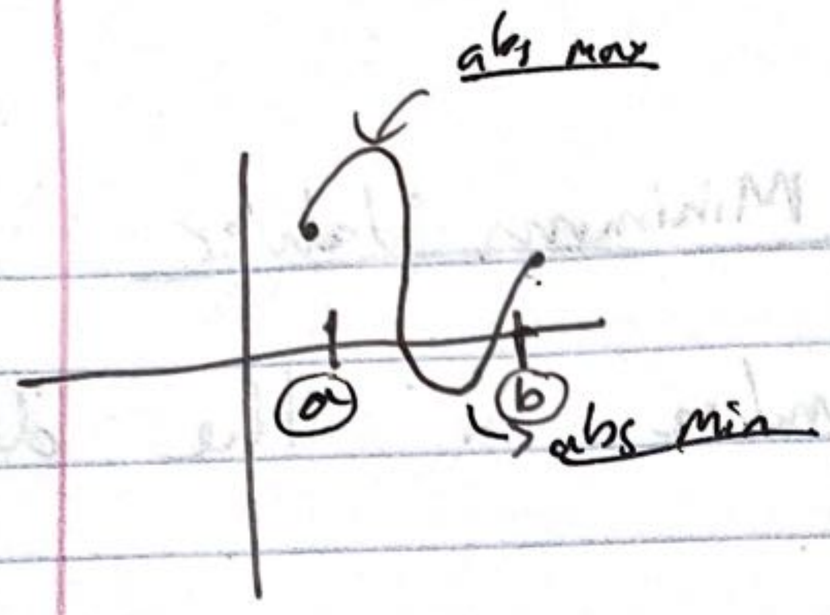
* ABSOLUTE MAX/MIN $\longleftrightarrow$ (Global Max/min)

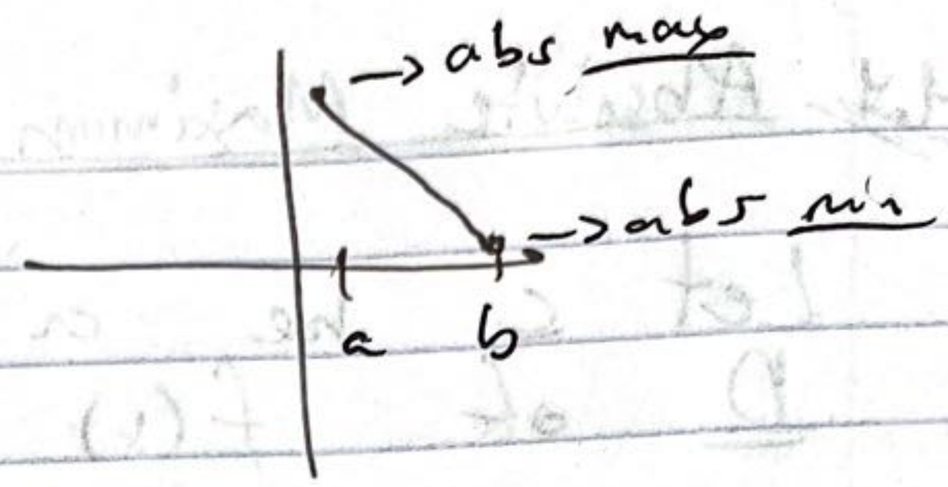


THM: EXTREME VALUE Theorem

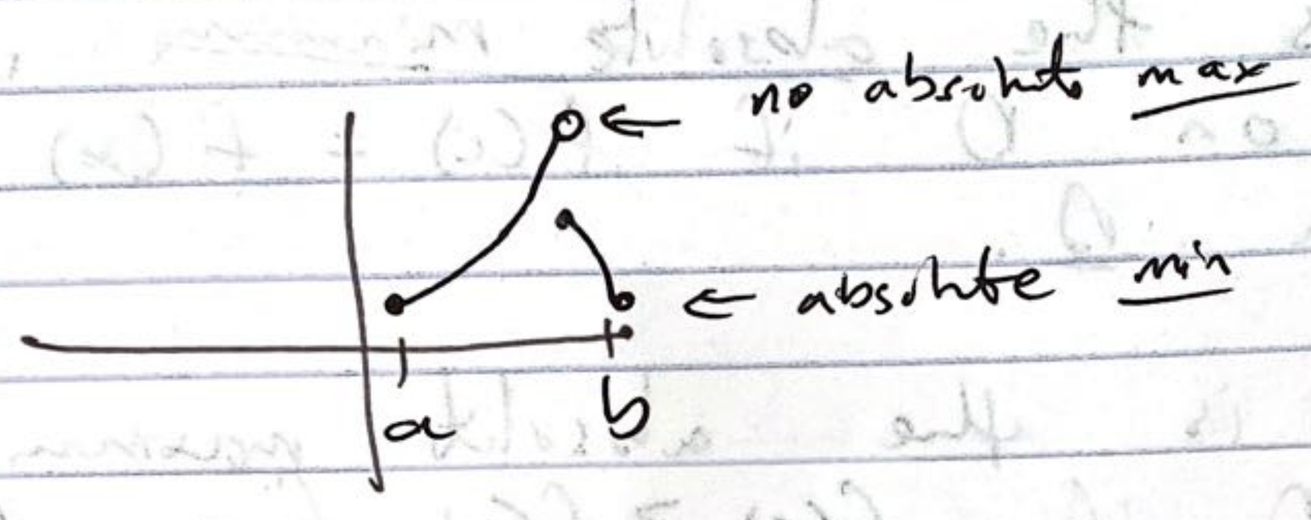
If f is continuous on a closed interval $[a, b]$, then it attains an absolute maximum and absolute min for some number in $[a, b]$.

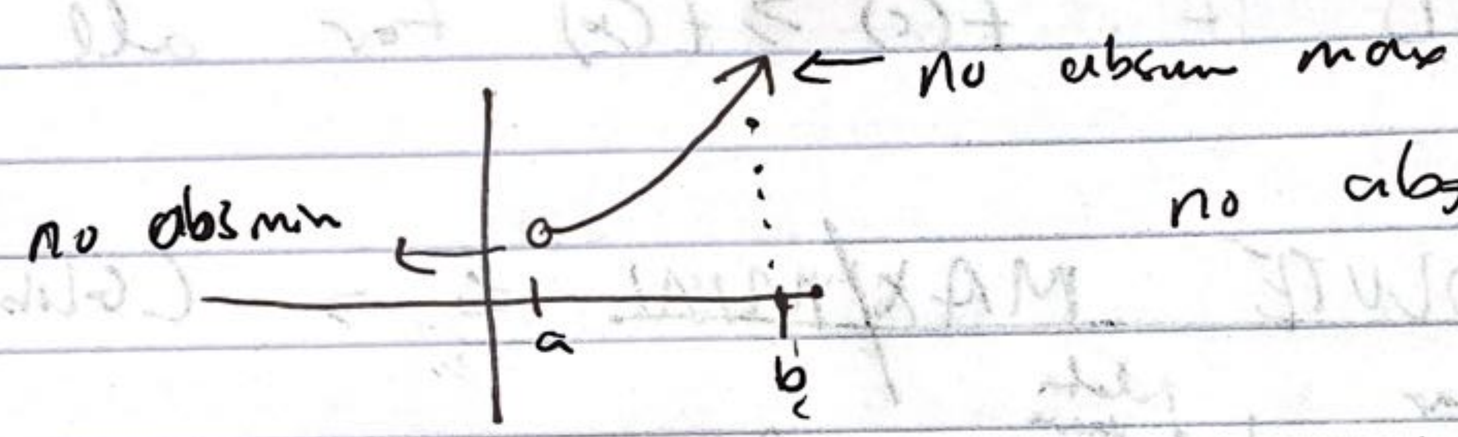
Closed





continuous
closed





no absolute extrema.

(Also Stationary Points)

Reminder: A critical number of a function $f(x)$ is a number c in the domain such that $f'(c) = 0$ or undefined.

referring to stationary point

Ex: Find the critical numbers of $f(x) = x^{3/5}(4-x)$

Rewrite by distributing

$f(x) \to 4x^{3/5} - x^{8/5}$

$f'(x) = \frac{12}{5}x^{-2/5} - \frac{8}{5}x^{3/5}$

$f'(x) = \frac{12}{5x^{2/5}} - \frac{8}{5}x^{3/5} \longrightarrow \frac{12}{5x^{2/5}} - \frac{8x}{5x^{2/5}}$

$\boxed{\frac{12-8x}{5x^{2/5}}}$

$$f'(x) = \frac{12-8x}{5x^{2/5}}$$

undefined

$5x^{2/5} = 0$

$\boxed{x = 0}$

$f'(x) = 0$

$12 - 8x = 0$

$12 = 8x$

$\boxed{\frac{3}{2} = x}$ ← stationary point

* Closed Interval Method

To find absolute max/min values of a certain funct on $[a, b]$

1) Find all the critical numbers (stationary points) of f on $[a, b]$
2) Evaluate $f(c)$ for each critical number on $f(a)$, $f(b)$

The largest value from Step 2 is the abs max and smallest value from Step 2 is abs min.

Ex Find the absolute extrema of $f(x) = x^3 - 3x^2 + 1$ on $[-1/2, 4]$

$f'(x) = 3x^2 - 6x$

$0 = 3x^2 - 6x$

$0 = 3x(x-2)$

$0 = 3x$ or $0 = x - 2$

$0 = x$ or $2 = x$

within $[1/2, 4]$

$f(x) = x^3 - 3x^2 + 1$

$f(0) = 0^3 - 3(0)^2 + 1 = 1$

min → $f(2) = 2^3 - 3(2)^3 + 1 = -3$

min → $f(1/2) = (1/2)^3 - 3(1/2)^3 + 1 = 1/8$

max → $f(4) = 4^3 - 3(4)^2 + 1 = 17$

abs max $= 17$ at $x = 4$

abs min $= -3$ at $x = 2$

Second Derivative Test

For a critical number c, if $f''(c) > 0$, then c corresponds to a minimum.

If $f''(c) < 0$, then c corresponds to a maximum.

Ex Find any absolute extrema of ~~f(x)=c~~ $f(x) = e^{x^3 - 3x^2}$ on $(0, \infty)$

$$f'(x) = e^{x^3-3x^2} \cdot (3x^2 - 6x)$$
$$= (3x^2 - 6x)\, e^{x^3 - 3x^2}$$

$$0 = (3x^2 - 6x)\, e^{x^3 - 3x^2} \rightarrow \text{always positive}$$
$$0 = 3x(x-2)$$
$$x \geq 0, \quad \boxed{x = 2}$$

$$\rightarrow f'(x) = (3x^2 - 6x)\, e^{x^3 - 3x^2}$$

$$f''(x) = (3x^2 - 6x)\, e^{x^3 - 3x^2}(3x^2 - 6x) + e^{x^3 - 3x^2}(6x - 6)$$

$$f''(x) = e^{x^3 - 3x^2}\left((3x^2 - 6x)^2 + 6x - 6\right)$$

$$\rightarrow f''(2) = e^{2^3 - 3(2)^3}\left((3(2)^2 - 6(2))\right)^2 + 6(2-6)$$

$$f''(2) = \underset{\frac{1}{e^4}}{e^{-4}}(12 - 6) > 0$$

$\rightarrow$ 2nd deriv test, <u>absolute min at $x=2$</u>

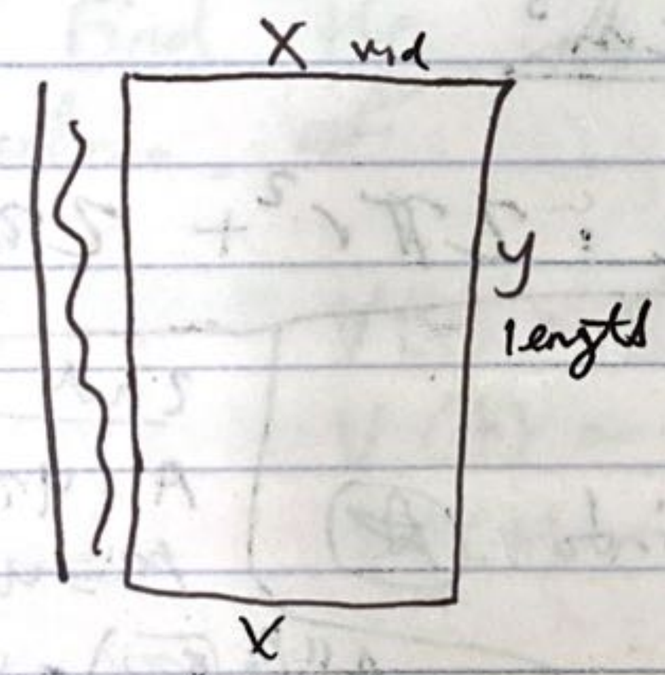
$$f(2) = e^{(2)^3 - 3(2)^2} = e^{-4} \text{ at } \underline{\underline{x = 2}}$$

4.5 Optimization Problems

1.) Understand the problem
2) Draw a picture
3) Introduce Notation
4) Express what we want MAX / MINIMIZED in terms of the other variables.

5) Find the absolute Max & min

Ex: A farmer has 2400 ft of fencing wants to fence off a rectangular field that borders a straight river. No fence is needed along the river. What are the dimension of the field that has the largest area?



$A = xy$ ⊛

$= 2x + y = 2400$

$y = 2400 - 2x$

So we have $A = xy$

$A = x(2400 - 2x)$

Maximizing $= A = x(2400 - 2x)$

distribute $A = 2400x - 2x^2$

$A' = 2400 - 4x$

$0 = 2400 - 4x$

$4x = 2400$

$x = 600$ ← verify that it gives max →

2nd Deriv test

$A'' = -4$

$A''(600) = -4 < 0$ max ✓

$x = 600$ gives max area

$y = 2400 - 2(600) = 1200$

$\boxed{600 \text{ ft} \times 1200 \text{ ft}}$

Ex. A cylindrical can is to be made to hold 1L of oil.

Find the dimensions that will min the cost of the metal to manufacture the can.

$\pi r^2 h = \underline{1000 \text{ cm}^3}$

Surface area : $2\pi r^2 + 2\pi r h$ ⊛

$\pi r^2 h = 1000$

$h = \dfrac{1000}{\pi r^2}$ ← Plug into ⊛

$A = 2\pi r^2 + 2\pi r\left(\dfrac{1000}{\pi r^2}\right)$

$A = 2\pi r^2 + \dfrac{2000}{r}$

Manage: $A = 2\pi r^2 + 2000 r^{-1}$

~~mang~~ = $A' = 4\pi r - 2000 r^{-2} = 4\pi r - \dfrac{2000}{r^2}$

$0 = 4\pi r^3 - 2000$

$2000 = 4\pi r^3$

$\dfrac{500}{\pi} = r^3 \longrightarrow \boxed{\sqrt[3]{500/\pi} = r}$ ← verify it gives min →

2nd derivative

$A' = 4\pi r - 2000 r^{-2}$

$A'' = 4\pi + 4000 r^{-3}$

$A''\left(\sqrt[3]{\frac{500}{\pi}}\right) = 4\pi + \dfrac{4000}{\frac{500}{\pi}}$

$= \underline{\text{min}}$ ✓

SA is minimized to radius $= \sqrt[3]{\frac{500}{\pi}}$ and $h = \dfrac{1000}{\pi\sqrt{\frac{500}{\pi}}^2} = \dfrac{1000}{\pi\left(\frac{500}{\pi}\right)^{2/3}}$

$h = 2\sqrt[3]{\frac{500}{\pi}}$

4.6 Rectilinear Motion

Recall if $S(t)$ is the position of an object at the time t then the velocity is $v(t) = s'(t)$ and the acceleration is $a(t) = s''(t)$.

- $V(t) > 0 \rightarrow$ object moving positive
  $V(t) < 0 \rightarrow$ negative direction ~~also~~
  ~~that you~~ !

Define the speed of the object to be $|V(t)|$

Ex Let $S(t) = t^3 - 6t^2$ be the position function of a particle moving along an s-axis where s is in the meters and t is in seconds.

Find the velocity, acceleration and speed functions. ~~of~~

Since $V(t) = S'(t)$ we have
$V(t) = 3t^2 - 12t$
Speed $|v(t)| = |3t^2 - 12t|$

Since $a(t) = s''(t) = V'(t)$ we have $a(t) = 6t - 12$

We say the object is speeding up when the speed is increasing. This happens when Velocity and acceleration have the same sign.

The object is slowing down when the speed is decreasing when velocity and acceleration are opposite sign.

$V(+)$ $A(-)$ → slowing down

$V(-)$ $A(+)$ → still slowing down

Ex For $s(t) = t^3 - 6t^2$ when is the object speeding up or slowing down.

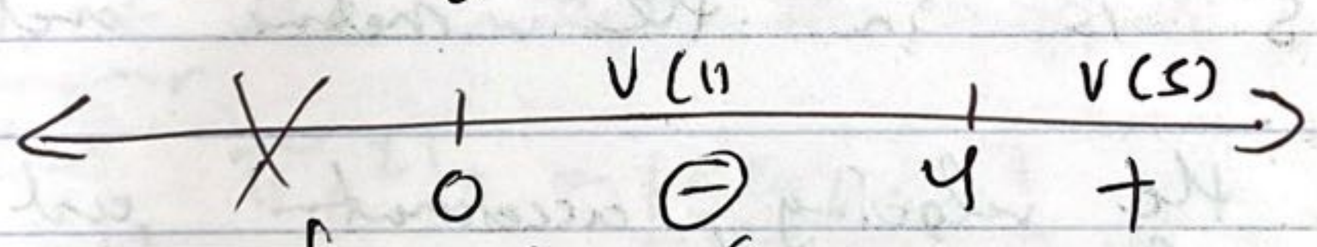
$V(t) = 3t^2 - 12t$ $\quad$ 0, 4

$0 = 3t^2 - 12t$

$0 = 3t(t-4)$ → factored

$0 = t$, $4 = t$

$V(t)$



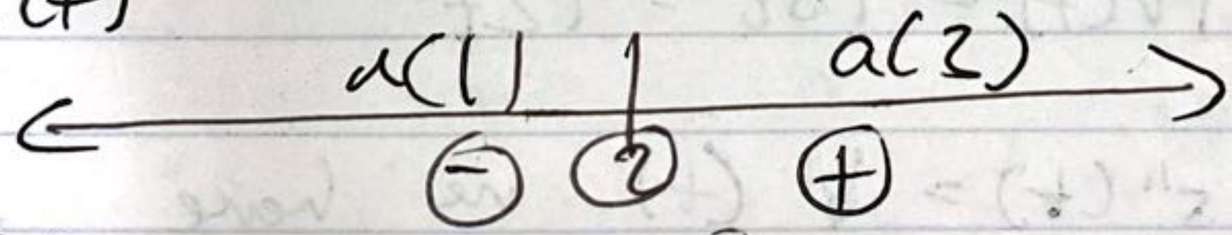
decreasing

Need to make for $a(t) = 6t - 12$

$0 = 6t - 12$

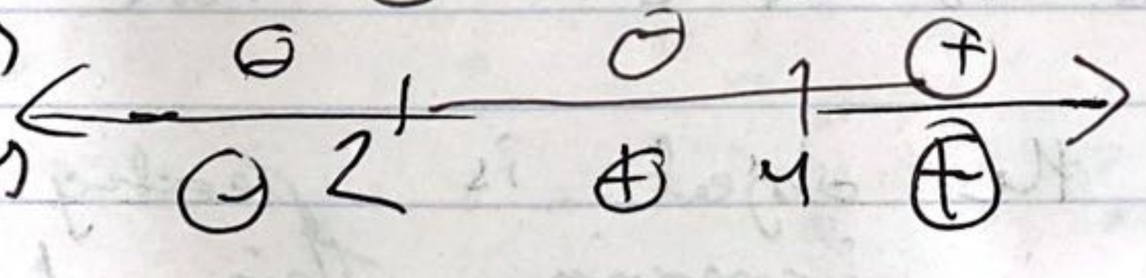
$t = 2$

$a(t)$



Velocity $V(t)$

Acceleration $a(t)$



Speeding up : $(0, 2) \cup (4, \infty)$

Slowing down : $(2, 4)$

Ex find the maximum speed of an object with the position function
$s(t) = 2.5t^2 - 17t$ where s is in feet and t is in second one the interval $\underline{\underline{1 \le t \le 5.}}$

$$s(t) = 2.5t^2 - 17t$$
$$v(t) = 5t - 17$$

Speed $|v(t)| = |5t-17|$

If $5t-17 > 0$, speed $5t-17$
$(\text{Speed})' = 5$
no critical values

If $5t-17 < 0$, speed $-(5t-17)$
Speed $= -5t + 17$
$(\text{Speed})' = -5$

So when $5t-17=0$, the derivative of $|v(t)| = |5t-17|$ does not exist

$$5t - 17 = 0$$
$$t = \frac{17}{5} = 3.4$$ critical value

$$|v(t)| = |5t-17|$$

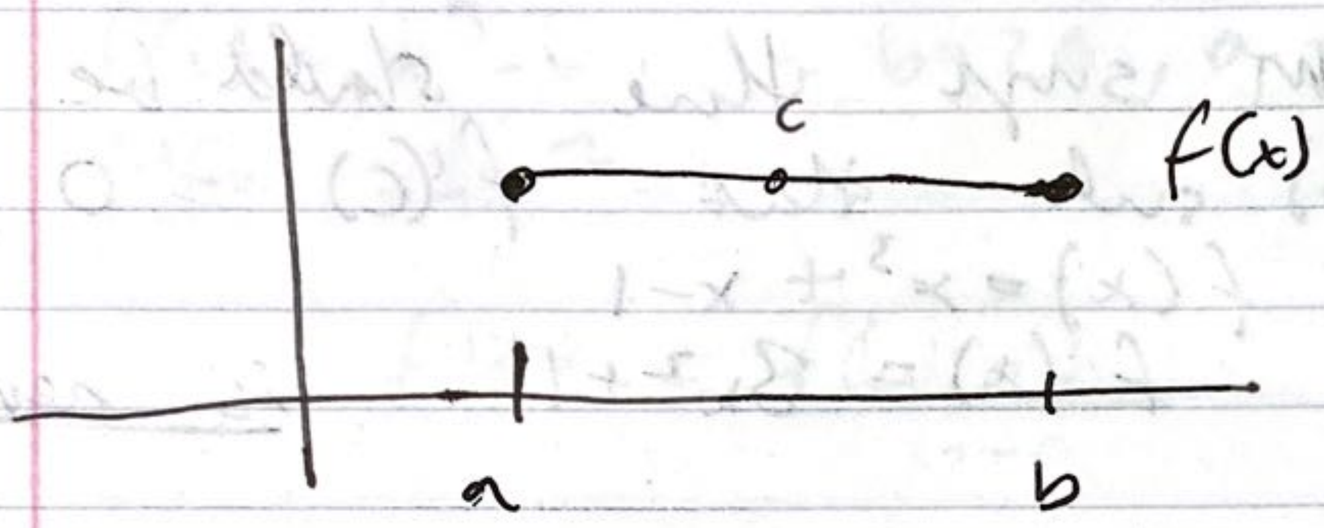
speed
at $|v(3.4)|$ → critical value $= |5(3.4)-17| = 0$
left $|v(1)|$ $= |5-17| = \underline{12}$ (-12)
right $|v(5)|$ $= \underline{8}$

Max speed at 12 ft/s at $t=1$ s

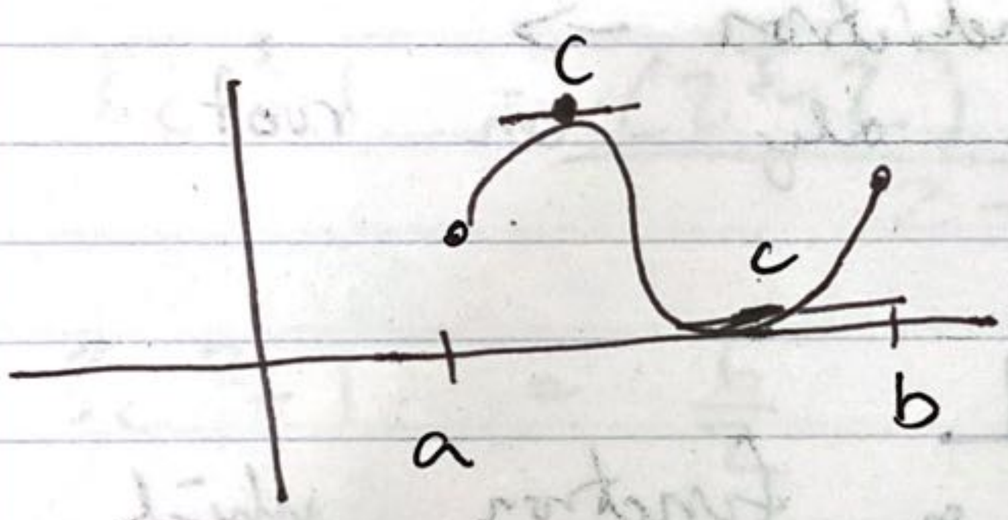
Min speed is 0 ft/s at time $t=3.4$ s

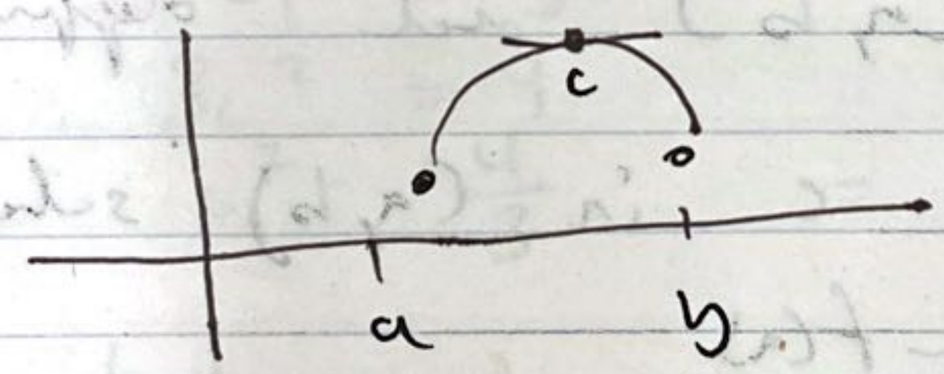
Rolle's Theorem and the Mean Value Theorem

If f is a function which is continuous on $[a,b]$ and differentiable on (a,b) and $f(a)=f(b)$, then there exists c in (a,b) such that $f'(c)=0$

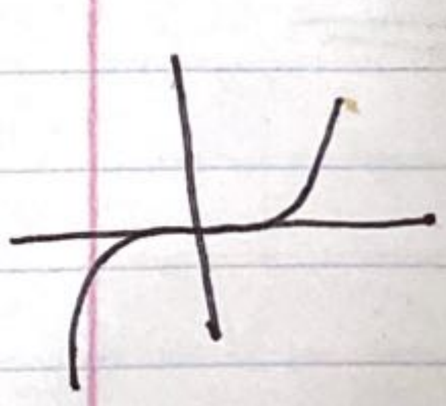


H=





Ex Prove then $f(x)=x^3+x-1$ has exactly one real root



We have at least one real root since $f(x)$ is continuous and $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x)=\infty$

Assume we have some then one root
- $f(a) = 0$ and $f(b) = 0$

$f(x)$ is a polynomial so it is a polynomial
So its continuous on $[a, b]$
and differentiable on (a, b).

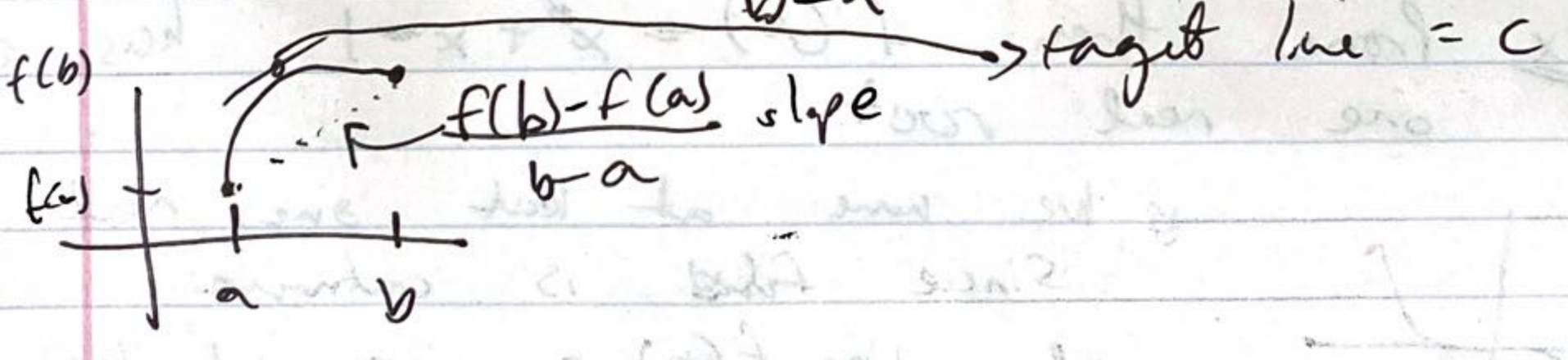
→ Rolle's THM says there should be a
c in (a, b) such that $f'(c) = 0$
But for $f(x) = x^3 + x - 1$
$f'(x) = 3x^2 + 1$ <u>is never 0</u>

We have a contradiction →
There is is only on root

<u>Mean Value THM</u>

Let f be a function which is continuous on $[a, b]$ and differentiable on (a, b)
There exist c in (a, b) s.t. that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Ex Let $f(x) = x^3 - x$, $a = 0$, $b = 2$

The MVT says there exists c in $(0, 2)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

What is c?

$f(x) = x^3 - x$, $b = 2$, $a = 0$

$f'(x) = 3x^2 - 1$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$3c^2 - 1 = \frac{(2^3 - 2) - (0^3 - 0)}{2 - 0}$$

$$3c^2 - 1 = \frac{6}{2}$$

$$3c^2 - 1 = 3$$

$$3c^2 = 4$$

$$c^2 = \frac{4}{3} = \pm\sqrt{\frac{4}{3}}$$

but c is in $(0, 2)$ so

$$c = \sqrt{\frac{4}{3}} \quad \text{or} \quad \frac{2}{\sqrt{3}}$$

MVT THM:

If $f'(x) = 0$ for all x in (a,b) then $f(x)$ is constant on (a,b)

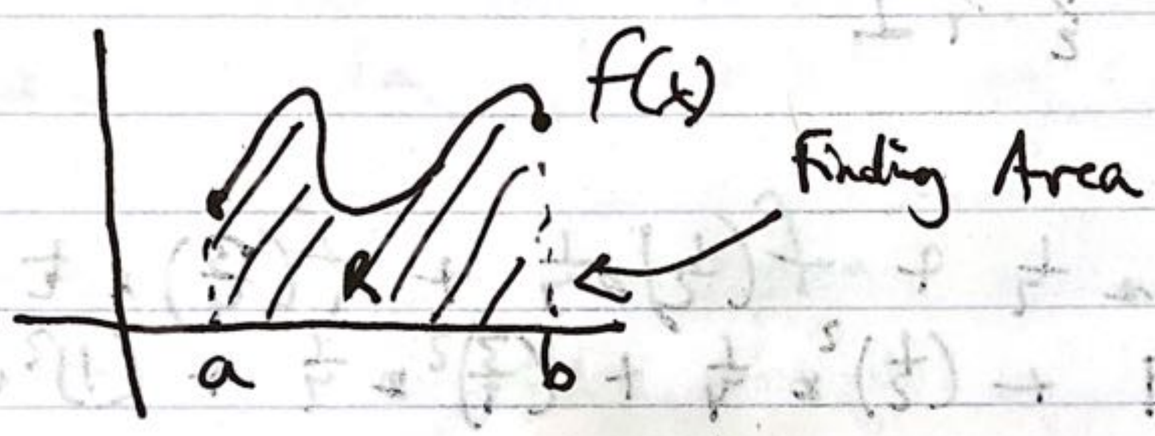
Corollary: If $f'(x) = g'(x)$ for all x in (a,b) then $f(x) - g(x)$ is constant on (a,b). So $f(x) = g(x) + c$

An Overview of the Area Problem

Area Problem:

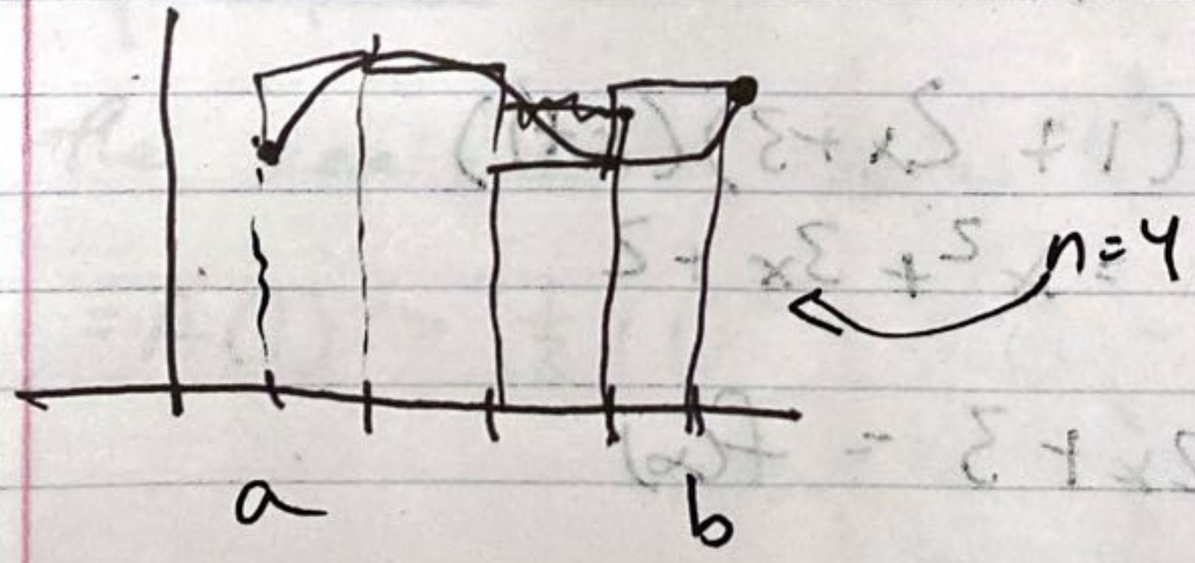
- If $f(x)$ is continous and non negative on $[a, b]$ then what is the area of the region R under the curve $f(x)$ and above the x-axis on $[a, b]$?

ex



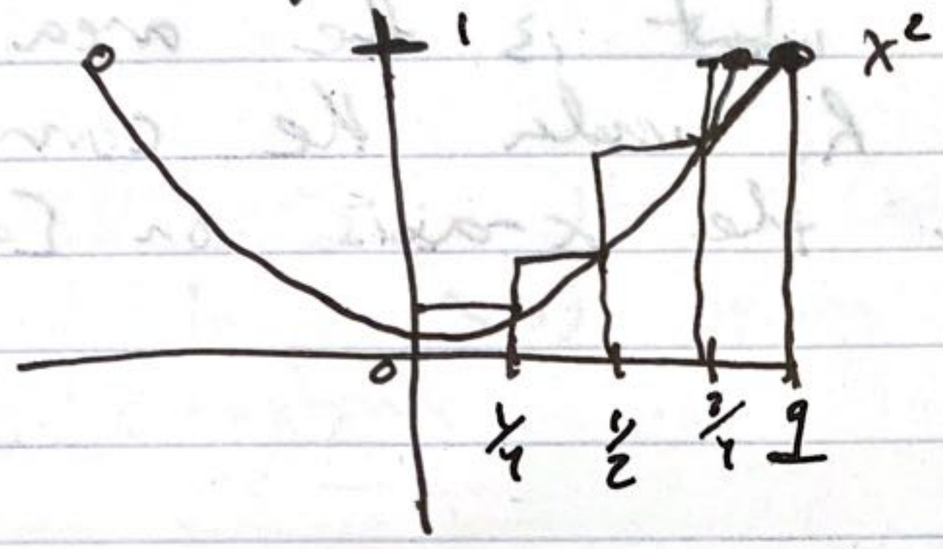
Rectangle Area Method

- take $[a, b]$ and divide into n equal subintervals and construct a rectangle of height $f(x_i)$ for each subinterval.
- The ~~interval~~ total area of the n rectangle is an approximation of the area R.



Ex Use the rectangle method with $n=4$ rectangles to estimate the area beneath $f(x)=x^2$ on $[0,1]$.
Rectangles have width $1/4$.

Sol

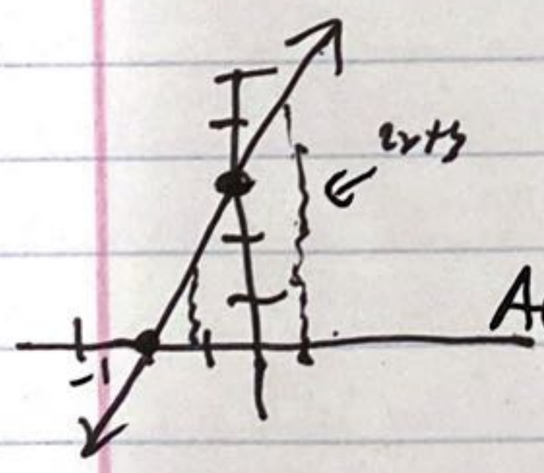


$$A(x) = f(\tfrac{1}{4})\cdot\tfrac{1}{4} + f(\tfrac{1}{2})\cdot\tfrac{1}{4} + f(\tfrac{3}{4})\cdot\tfrac{1}{4} + f(1)\cdot\tfrac{1}{4}$$
$$= (\tfrac{1}{4})^2\cdot\tfrac{1}{4} + (\tfrac{1}{2})^2\cdot\tfrac{1}{4} + (\tfrac{3}{4})^2\cdot\tfrac{1}{4} + (1)^2\cdot\tfrac{1}{4}$$

estimate $= \frac{15}{32} = 0.46875$

* exact area is $\frac{1}{3}$

Ex Find the area between $f(x)=2x+3$ and the x axis on $[-1,x]$. Then find the derivative $A'(x)$ of the area function.



Area Trapizoid $= \frac{1}{2}(b_1+b_2)h$

$$A(x) = \frac{1}{2}(1 + 2x+3)(x+1)$$
$$A(x) = x^2+3x+2$$

find derivative $= A'(x) = 2x+3 = f(x)$

- The derivative of $A(x)$ has a $f(x)$
- The process of finding a function from its derivative form its derivative is called a <u>anti-differentiation</u>
- $A(x)$ is the antiderivative of $f(x)$

<u>Ex</u> Use the antidiferentiation to find the area under $f(x) = x^2$ on $[0, 1]$.

<u>Know</u>: $A(x) =$ antiderv of $f(x)$ on the interval $[0, x]$ where x is in $[0, 1]$

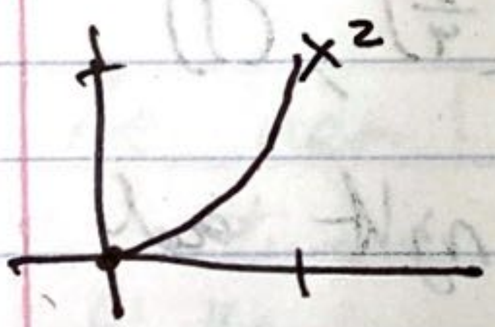
$f(x) = x^2$

antiderivative increase power by 1

$A(x) = \frac{1}{3}x^3 + C$ ⇐ constant

if $x = 0$, $A(0) = \frac{1}{3}(0)^3 + C$

$A(0) = C$



$A(0) = C \Rightarrow 0 = C$

So $= A(x) \frac{1}{3}x^3$

the area under $f(x) = x^2$ on $[0, 1]$ is $A(1)$

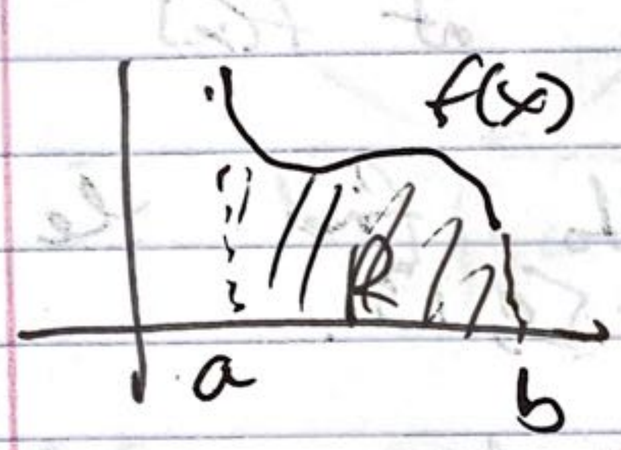
$= A(1) = \frac{1}{3}(1)^3 = \boxed{\frac{1}{3}}$

5.2 Area and Riemann Sums

In general, if you have a region under $y=f(x)$ from a to b

Find the area of the Region R that lies under a curve $f(x)$ from a to b?

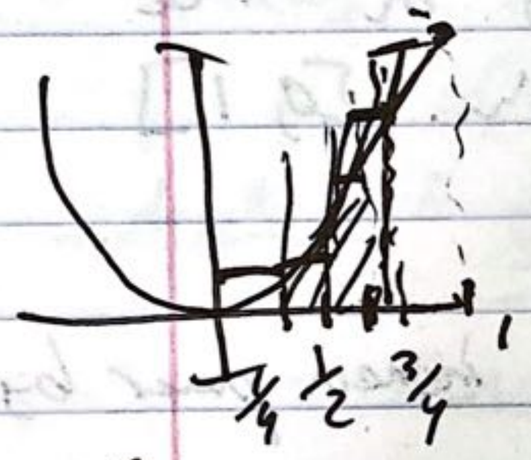
Ex



$\cancel{y=x^2}$

Use 4 rectangles

$y=x^2$



= Use rectangles to estimate the area under $y=x^2$ from 0 to 1

If we add the area of 4 rectangles we have an approximation for the area of R.

width of rectangles $= \frac{1}{4}$

Height of " " $= (1/4)^2, (\frac{1}{2})^2, (\frac{3}{4})^2, (1)^2$

and using 4 rectangles and right end

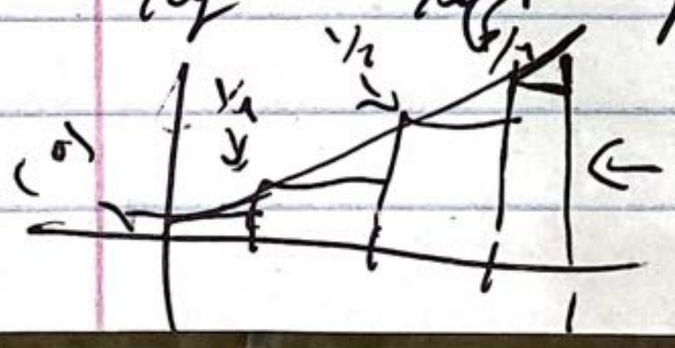
$A \approx \frac{1}{4}(\frac{1}{4})^2 + \frac{1}{4}(\frac{1}{2})^2 + \frac{1}{4}(\frac{3}{4})^2 + \frac{1}{4}(1)^2$

$= \frac{15}{32}$ 0.46875 over

$\boxed{0.21875 < \text{Area } R < 0.46875}$

Area $= R \leq 0.46875$

Try left points

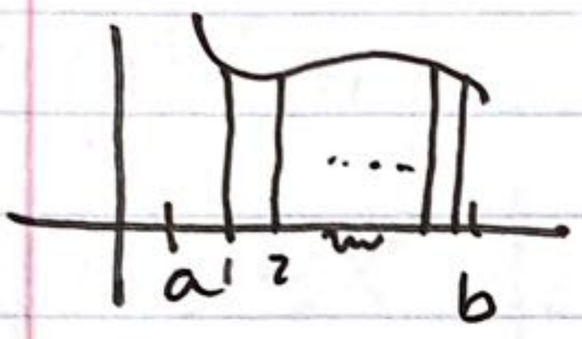


$= 0^2 + (\frac{1}{4})^2, (\frac{1}{2})^2, (\frac{3}{4})^2$

$= 0^2 \cdot \frac{1}{4} \cdots\cdots$

$= \frac{7}{32} = 0.21875$ under

In general, if you have a region S under $y = f(x)$ from a to b and you want the area, begin by dividing S into n strips of equal width.



width of strip $= \Delta x = \frac{b-a}{n}$

The strips divide $[a, b]$ into n subintervals.
$[x_0, x_1], [x_1, x_2] \ldots [x_{n-1}, x_n]$
where $x_0 = a$, $x_n = b$

- The right end point of the strips are
  $x_1 = a + \Delta x$
  $x_2 = a + 2\Delta x$
  $x_{n-1} = a + (n-1)\Delta x$

Estimate the area of Strip S_i by rectangle of width Δx and height $f(x_i)$
Then $A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n) + \Delta x$

The area of the region S under the graph of a continuous funct. $f(x)$ is the limit of the sum of the area of the rectangles.

- $A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$

Sigma notation is shorter version

$= \sum_{i=1}^{n} f(x_i)\Delta x$ $\boxed{= \text{so } A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x}$

Formulas

1) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

2) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

3) $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

4) $\sum_{i=1}^{n} c = nc$

5) $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

6) $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

$\lim_{n\to\infty} \left(\frac{81}{n^4}\left(\frac{n(n+1)}{n}\right)^2 - \frac{54}{n^2}\left(\frac{n(n+1)}{2}\right)\right)$

$= \frac{81(n^2+n)^2}{4n^4} - \frac{54(n^2+n)}{2n^2}$

$= \frac{81}{4} - \frac{54}{2} = \frac{81}{4} - 27 = -\frac{27}{4}$

Ex Find the area under: $f(x) = x^3 - 6x$ on $[0,3]$.

With n subintervals, the width is Δx

$= \frac{3-0}{n} = \frac{3}{n}$

then $= x_0 = 0$, $x_1 = \frac{3}{n}$, $x_2 = \frac{6}{n}$ ..., $x_i = \frac{3i}{n}$

$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right) \cdot \underbrace{\frac{3}{n}}_{\text{constant}}$

$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right) \to \frac{3}{n}\left(\frac{27}{n^3}\sum i^3 - \frac{18}{n}\sum i\right)$

$\left(\frac{27}{n^3} i^3 - \frac{18}{n} i\right)$ $\quad \frac{81}{n^4}\sum_{i=1}^{n} i^3 - \frac{54}{n^2}\sum i$

Antiderivative and Indefinite Integrals

<u>Def</u>: A function F is an antiderivative of f on an interval I if $F'(x) = f(x)$ for all $x \in I$

Ex Let $f(x) = x^2$ find an antiderivative of $f(x)$.

not x^2 ✗ $F(x) = x^3 \mid f'(x) = 3x^2$

✓ $F(x) = \frac{1}{3}x^3 \mid f'(x) = x^2$

number constant always an $\underline{0}$

If F is an antiderivative of f on I then the most general antiderivative of $f(x)$ on I is $F(x) + c$, where c is constant.

Ex find most general antiderivative of the following

(a) $f(x) = \sin(x)$

$\boxed{F(x) = -\cos(x) + c}$

(b) $f(x) = x^n, \quad n \geq 0$

$\boxed{F(x) = \frac{x^{n+1}}{x+1} + c}$

(c) $f(x) = x^{-3}$

$F(x) = \frac{x^{-3+1}}{-3+1} + c$

$\boxed{F(x) = \frac{x^{-2}}{-2} + c}$

(d)

$\boxed{g(x) = -4\cos(x) + \frac{2x^5}{5} - 2x^{1/2} + c}$

Ex find all functions g such that

$g'(x) = 4\sin(x) + \frac{2x^5 - \sqrt{x}}{x}$

$= 4\sin(x) + \frac{2x^5}{x} - \frac{x^{\frac{1}{2}}}{x}$

$= 4\sin(x) + 2x^4 - x^{-\frac{1}{2}}$

$= -4\cos(x) + \frac{2x^5}{5} - \frac{x^{1/2}}{1/2} + c$

Indefinite Integral

Def) The indefinite integral of $f(x)$ is the antiderivative of $f(x)$.

tell's its an integral $\rightarrow \int f(x)\,dx = F(x) + C$ where $F'(x) = f(x)$

Basically say find antiderivative

1) $\int x^n dx$ where $n \neq -1$

$= \frac{x^{n+1}}{n+1} + C$

natural log 2) $\int \frac{1}{x} dx = \ln|x| + C$

3) $\int (10x^4 - 2\sec^2(x))\,dx$

$= \frac{10x^5}{5} - 2\tan(x) + C$

An equation that involves the derivative of a function is called <u>differential equation</u>.

Ex Find if $f'(x) = x\sqrt{x}$ and $f(1) = 2$.

$F'(x) = xx^{1/2} = x^{3/2}$ so $f(x) = \frac{2}{5}x^{5/2} + C$

Plug in $f(1) = 2$:

$2 = \frac{2}{5}(1)^{5/2} + C = \frac{2}{5} + C \rightarrow \frac{8}{5} + C$

$\boxed{f(x) = \frac{2}{5}x^{5/2} + \frac{8}{5}}$

Ex Find $f(x)$ if $f''(x) = 12x^2 + 6x - 4$ and $f(0) = 4$ and $f(1) = 1$

$$f''(x) = 12x^2 + 6x - 4$$
$$f'(x) = \frac{12x^3}{3} + \frac{6x^2}{2} - \frac{4x}{1} + c$$
$$f'(x) = 4x^3 + 3x^2 - 4x + c$$
$$f(x) = \frac{4x^4}{4} + \frac{3x^3}{3} - \frac{4x^2}{2} + cx + d$$
$$f(x) = x^4 + x^3 - 2x^2 + \underline{cx} + \underline{d} \quad ?$$

$f(0) = 4 \rightarrow 0^4 + 0^3 - 2(0)^2 + c(0) + d = 4$ $\boxed{d = 4}$

$f(x) = x^4 + x^3 - 2x^2 + cx + \underline{4}$

$f(1) = 1 \rightarrow 1^4 + 1^3 - 2(1)^2 + c(1) + 4 = c + 4$ $\boxed{-3 = c}$

$$\boxed{f(x) = x^4 + x^3 - 2x^2 - 3x + 4}$$

Integration with Substitution

If $u = g(x)$ is a differentiable function whose range is I and f is continuous on I then

Undoing chain rules →

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

Ex $\int 2x\sqrt{1+x^2}\, dx$

$\to u = 1 + x^2$

$\to \frac{du}{dx} = 2x = du = 2x\,dx$

$= \int \sqrt{u}\, du = \int u^{1/2}\, du = \frac{2}{3} u^{3/2} + c$

$= \frac{2}{3}(1+x^2)^{3/2} + c$

check derivat

$\frac{3}{2} \cdot \frac{2}{3}(1+x^2)^{1/2} \cdot 2x = 2x\sqrt{1+x^2}$

Ex $\int x^3 \cos(x^4+2)\, dx$ $\quad u = x^4 + 2 \quad du = 4x^3\, dx$

$\frac{du}{dx} = 4x^3 \to 4x^3 dx = \frac{du}{dx} = dx$

$\int \cancel{x^3} \cos(u) \frac{du}{4\cancel{x^3}} = \int \cos(u) \frac{du}{4} = \int \frac{1}{4} \cos(u)\, du =$

$\frac{1}{4}\sin(u) + c \to \boxed{\frac{1}{4}\sin(x^4+2) + c}$

Ex $\int \sqrt{2x+1}\, dx \to u = 2x+1$

$du = 2\,dx$

$\frac{du}{2} = dx$

$= \int \sqrt{u}\, \frac{du}{2} =$

$\int \frac{1}{2} u^{1/2}\, du$

$= \left(\frac{2}{3}\right) \frac{1}{2} u^{3/2} + c = \boxed{\frac{1}{3}(2x+1)^{3/2} + c}$