TTK 4130 - Modeling and Simulation

Assignement 7

Ingebrigt Stamnes Reinsborg

1a) Dynamic model:

$$\dot{x} = f(x, u, t)$$

Assume:

$$u = u_k \quad \text{for} \quad t \in [t_k, t_{k+1}]$$

ERK2 based on two evaluations of f, the first on $[x(t_k), u_k]$ can be written as:

$$K_1 = f(x(t_k), u(t_k), t_k)$$

$$K_2 = f(x(t_k) + a \cdot \Delta t \cdot K_1, \; u(t_k + c\Delta t), \; t_k + c\Delta t)$$

$$x_{k+1} = x(t_k) + \Delta t \sum_{i=1}^{2} b_i \cdot k_i$$

Butcher tableau

$$\begin{array}{c|cc} 0 & & \\ c & a & \\ \hline & b_1 & b_2 \end{array}$$

ERZ numerical error:

$$e_K = X_{K+1} - X(t_{K+1}|K)$$

$X(t|K)$ actual traj. of ODE with init. cond. $X(t_K|K) = X_K$

I am to provide cond. on a, b_1, b_2 and c such that e_K of the method is of order 3.

$$X(t_{K+1}) = X(t_K) + \Delta t \cdot f(X(t_K), u_K) + \frac{\Delta t^2}{2} \cdot \dot{f}(X(t_K), u_K) + O(\Delta t^3)$$

(2c) $X_{k+1} = X(t_k) + \Delta t\,(b_1 K_1 + b_2 K_2)$

(3) $\Downarrow$

$$X_{k+1} - X(t_{k+1}) = \Delta t\, b_1 K_1 + \Delta t\, b_2 K_2 - \Delta t\, K_1 - \frac{\Delta t^2}{2} \underbrace{\dot{f}(X(t_k), u_k)}_{(\approx \dot{f} \text{ from now})} + O(\Delta t^3)$$

$$= e_k$$

We do a Taylor exp. on K_2

$$K_2 = K_1 + a\Delta t\, \dot{f}(X(t_k), u_k) + O(\Delta t^2)$$

$\Downarrow$

$$e_k = \Delta t\, b_1 K_1 + \Delta t\, b_2 K_1 + \Delta t^2 b_2 a \dot{f} + \Delta t\, b_2\, O(\Delta t^2) - \Delta t K_1 - \frac{\Delta t^2}{2} \dot{f} - O(\Delta t^3)$$

$$= \Delta t\,(b_1 + b_2 - 1) K_1 + \Delta t^2 \left(b_2 a - \tfrac{1}{2}\right) + O(\Delta t^3)$$

$$\Rightarrow \underline{\underline{b_1 + b_2 = 1, \quad b_2 a = \tfrac{1}{2}, \quad c \in [0,1]}}$$

b) $e_K = O(\Delta t^3)$

$$\Downarrow$$

$$\|X_N - X(T)\| = O(\Delta t^2)$$

(if $\|X_{K+1} - X(t_{K+1})\|$, for each ~~step~~ step has an error $O(\Delta t^3)$ and integration to T, we get $\frac{T}{\Delta t} = N$ steps in total.

After these, the error is:

$$\frac{T}{\Delta t}\Delta t^3 = T\Delta t^2 = O(\Delta t^2)$$

c) Modus operandi for surviving third year spring-semester for MTTK, dictates that:

(optional) := "Do not"

Some day in the future, I'll have the time to enjoy what I learn.

2a) Butcher Tableau:

RK1:

$$\begin{array}{c|c} 0 & \\ \hline & 1 \end{array}$$

RK2:

$$\begin{array}{c|cc} 0 & & \\ \frac{1}{2} & \frac{1}{2} & \\ \hline & 0 & 1 \end{array}$$

RK3:

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & & \frac{1}{2} & & \\ 1 & & & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

Test system:

$$\dot{X} = \lambda X$$

we'll use:

$$X_{K+1} = X_K + \Delta t \sum_{i}^{s} b_i K_i$$

$$K_s = f(X_K + \Delta t \sum_{i}^{s} a_{si} K_i \,,\ u(t_K + C_s \Delta t)$$

The code was tested for

$\lambda = -2$, $t \in [0,2]$, $\Delta t = 0.4$, $X(0) = 1$

See three first figures for

RK1, RK2 and RK4

b) I got better results for lower Δt's

Very low Δt's (like 0.1) gave very good results for all methods (see the 4. 5. and 6. fig)

Actual order of methods as a function of Δt:

RK	Order	$\frac{1}{2}\Delta t$-error
1	1	$\frac{1}{2}$
2	2	$\frac{1}{4}$
4	4	$\frac{1}{16}$

c) At $\lambda < -5$ RK1 and 2 is unstable

At $\lambda < -6.97$ RK4 is unstable

(I just twiddled with the code for a bit until they became unstable.)

3a) $\dot{x} = y$

$\dot{y} = u(1 - x^2)y - x$

$u = 5$

$x(0) = 2$

$y(0) = 0$

with ODE45 it seems both x and y are marginally stable

b) Increasing beyond $\Delta t = 16$ doesn't work.

Seems lower values work the best

$\Delta t < 0.12$ gives that RK4 and ODE45 gives somewhat equal plots.