TTK4130 - Modeling and Simulation
Spring 2020 (Trial exam)

This exam contains 8 pages (including this cover page) and 4 problems.

All material is allowed, but...

SHOW US THAT YOU UNDERSTOOD THE MATERIAL

Answers directly copied from your book or lecture notes will not be accepted. Answers will be checked for plagiarism from the internet. The copies will be cross-checked for plagiarism. Suspicions of cheating will be reported to the faculty for further investigations.

- You can answer in Inspera by typing the answer or uploading a file. This file should be a .pdf or .jpg.
- You should deliver photographs or scans that are readable, otherwise we cannot count them as correct.
- Mysterious or unsupported answers will not receive credit.
- If you have questions during the exam, you can contact us on (0047)41187277 or (0047)45917969. Please do not use this resource for no reason, that would deprive other students from asking their own questions.
- None of the proposed questions require extremely long computations. If you get caught in endless algebra, you have probably missed the simple way of doing it.
- The passing grade will a priori be given at 41 % of the maximum amount of points. This limit may be lowered depending on the outcome of the exam.

Problem	Points	Score
1	8	
2	8	
3	9	
4	6	
Total:	31	

Best of luck to all !!

1. **Lagrange modelling** Consider two masses of mass m (the black balls in Fig. 1) linked by a massless rigid rod of length L. The masses are gliding without friction on a surface of equation:

$$z = \frac{1}{2}x^2 + \frac{1}{2}y^2 \tag{1}$$

   (a) (2 points) Write the Lagrange function describing the system

   (b) (2 points) Derive the equations of motion

   (c) (2 points) What are the consistency conditions associated to the equations?

   (d) (2 points) What condition must be respected for the balls to never fall of the surface?

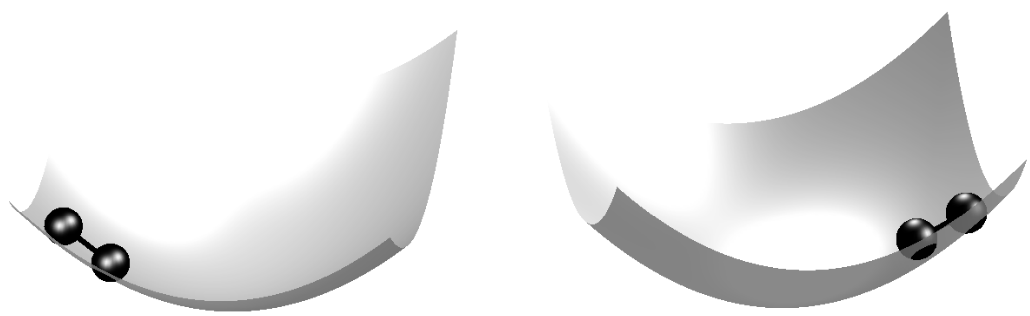

Figure 1: Illustration of the rollercoaster

Solution:

(a) The Lagrange function comprises the kinetic (T) and potential energy (V) of the balls, and the constraint (1). Let us describe the position of the balls via

$$\mathbf{q}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \qquad \mathbf{q}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \tag{2}$$

We then have:

$$T = \frac{1}{2}m\dot{\mathbf{q}}_1^\top \dot{\mathbf{q}}_1 + \frac{1}{2}m\dot{\mathbf{q}}_2^\top \dot{\mathbf{q}}_2, \qquad V = mg\left(z_1 + z_2\right) \tag{3a}$$

$$\mathcal{L} = T - V + \lambda^\top \begin{bmatrix} \frac{1}{2}x_1^2 + \frac{1}{2}y_1^2 - z_1 \\ \frac{1}{2}x_2^2 + \frac{1}{2}y_2^2 - z_2 \\ \frac{1}{2}\left(\left\|\mathbf{q}_1 - \mathbf{q}_2\right\|^2 - L^2\right) \end{bmatrix} \tag{3b}$$

(b) The equations of motion are given by a simple application of the Lagrange equations, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = 0 \tag{4}$$

Where the generalized coordinates $\mathbf{q}$ are $\mathbf{q} = \left[\mathbf{q}_1^\top, \mathbf{q}_2^\top\right]^\top = [x_1, y_1, z_1, x_2, y_2, z_2]^\top$. We can therefore compute:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}^\top = m \begin{bmatrix} \ddot{\mathbf{q}}_1 \\ \ddot{\mathbf{q}}_2 \end{bmatrix}, \qquad \frac{\partial \mathcal{L}}{\partial \mathbf{q}}^\top = \begin{bmatrix} \boldsymbol{\lambda}_1 x_1 \\ \boldsymbol{\lambda}_1 y_1 \\ -\boldsymbol{\lambda}_1 - gm \\ \boldsymbol{\lambda}_2 x_2 \\ \boldsymbol{\lambda}_2 y_2 \\ -\boldsymbol{\lambda}_2 - gm \end{bmatrix} + \boldsymbol{\lambda}_3 \begin{bmatrix} \mathbf{q}_1 - \mathbf{q}_2 \\ \mathbf{q}_2 - \mathbf{q}_1 \end{bmatrix} \tag{5a}$$

such that the equations of motion read as:

$$m \begin{bmatrix} \ddot{\mathbf{q}}_1 \\ \ddot{\mathbf{q}}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\lambda}_1 x_1 \\ \boldsymbol{\lambda}_1 y_1 \\ -\boldsymbol{\lambda}_1 - gm \\ \boldsymbol{\lambda}_2 x_2 \\ \boldsymbol{\lambda}_2 y_2 \\ -\boldsymbol{\lambda}_2 - gm \end{bmatrix} + \boldsymbol{\lambda}_3 \begin{bmatrix} \mathbf{q}_1 - \mathbf{q}_2 \\ \mathbf{q}_2 - \mathbf{q}_1 \end{bmatrix} \tag{6}$$

(c) Let us define:

$$\mathbf{c} = \begin{bmatrix} \frac{1}{2}x_1^2 + \frac{1}{2}y_1^2 - z_1 \\ \frac{1}{2}x_2^2 + \frac{1}{2}y_2^2 - z_2 \\ \frac{1}{2}\left(\|\mathbf{q}_1 - \mathbf{q}_2\|^2 - L^2\right) \end{bmatrix} \tag{7}$$

The consistency conditions require that the initial conditions of the system must satisfy $\mathbf{c} = 0$ and $\dot{\mathbf{c}} = 0$. The latter equation requires that:

$$\begin{bmatrix} x_1\dot{x}_1 + y_1\dot{y}_1 - \dot{z}_1 \\ x_2\dot{x}_2 + y_2\dot{y}_2 - \dot{z}_2 \\ (\mathbf{q}_1 - \mathbf{q}_2)^\top (\dot{\mathbf{q}}_1 - \dot{\mathbf{q}}_2) \end{bmatrix} = 0 \tag{8}$$

holds on the initial conditions.

(d) The force exerted by the surface on the balls is proportional to the multipliers $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$, and their orientation is given by their signs. We observe from the terms $-\boldsymbol{\lambda}_i - gm$ that $\boldsymbol{\lambda}_i > 0$ corresponds to the surface "pulling" the masses downwards (a force acting in the same direction as gravity). It follows that the condition for the masses staying in contact with the surface is $\boldsymbol{\lambda}_{1,2} \leq 0$ at all time along the trajectories of the system.

2. **Differential-Algebraic and Implicit Differential Equations**

(a) (5 points) Consider the DAE:

$$\dot{\mathbf{x}}_1 = \mathbf{x}_1 + \mathbf{x}_2 + z \tag{9a}$$

$$\dot{\mathbf{x}}_2 = z + u \tag{9b}$$

$$0 = \frac{1}{2}\left(\mathbf{x}_1^2 + \mathbf{x}_2^2 - 1\right) \tag{9c}$$

1. Why is it a DAE?
2. What is the differential index of (9)?
3. Perform an index-reduction of (9).

(b) (3 points) Consider the DAE:

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}z \tag{10a}$$

$$0 = \mathbf{g}(\mathbf{x}) \tag{10b}$$

where $z \in \mathbb{R}$ and function $\mathbf{g} : \mathbb{R}^{n_\mathbf{x}} \times \mathbb{R} \mapsto \mathbb{R}^m$. Assume that $\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{b}$ is full rank.

1. Specify m.
2. Perform an index reduction of the DAE
3. What are the consistency conditions?
4. What condition is needed for the DAE to be of index larger than 2?

Solution:

(a) 1. We can answer in a simple or formal way:
- The simple way relies on observing that variable z does not enter as time-differentiated in (9), it is an algebraic variable and therefore
- For the complex and formal way, we observe that (9) is given by the fully implicit differential equation:

$$F(\dot{s}, s, u) = \begin{bmatrix} \dot{\mathbf{x}}_1 - \mathbf{x}_1 - \mathbf{x}_2 - z \\ \dot{\mathbf{x}}_2 - z - u \\ \frac{1}{2}\left(\mathbf{x}_1^2 + \mathbf{x}_2^2 - 1\right) \end{bmatrix} \tag{11}$$

where $\mathbf{s} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & z \end{bmatrix}^\top$. And observe that matrix

$$\frac{\partial F}{\partial \dot{\mathbf{s}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{12}$$

is rank-deficient

2. We observe that (9) is a semi-explicit DAE with

$$g(\mathbf{x}, z, u) = \frac{1}{2}\left(\mathbf{x}_1^2 + \mathbf{x}_2^2 - 1\right) \tag{13}$$

and $\frac{\partial g}{\partial z} = 0$, hence it is of index larger than 1. In order to compute precisely the differential index, we need to perform time-differentiations on (9) until it is transformed in an ODE. Because (9a)-(9b) are already ODEs (functions of z), we can leave them alone, and focus on (9c). We then observe that:

$$\frac{\mathrm{d}}{\mathrm{d}t} g(\mathbf{x}, z, u) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left(\mathbf{x}_1^2 + \mathbf{x}_2^2 - 1\right)\right) = \mathbf{x}_1\dot{\mathbf{x}}_1 + \mathbf{x}_2\dot{\mathbf{x}}_2 \tag{14}$$

Replacing $\dot{\mathbf{x}}_1$, $\dot{\mathbf{x}}_2$ by there expressions from (9a)-(9b), we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t} g\left(\mathbf{x}, z, u\right) = \mathbf{x}_1 \left(\mathbf{x}_1 + \mathbf{x}_2 + z\right) + \mathbf{x}_2 \left(z + u\right) \tag{15}$$

which is not yet a differential equations. A second time-derivative delivers:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} g\left(\mathbf{x}, z, u\right) = \dot{\mathbf{x}}_1 \left(\mathbf{x}_1 + \mathbf{x}_2 + z\right) + \dot{\mathbf{x}}_2 \left(z + u\right) + \mathbf{x}_1 \left(\dot{\mathbf{x}}_1 + \dot{\mathbf{x}}_2 + \dot{z}\right) + \mathbf{x}_2 \left(\dot{z} + \dot{u}\right) \tag{16}$$

we then can solve $\frac{\mathrm{d}^2}{\mathrm{d}t^2} g\left(\mathbf{x}, z, u\right) = 0$ for $\dot{z}$:

$$\dot{z} = \frac{-\dot{\mathbf{x}}_1 \left(\mathbf{x}_1 + \mathbf{x}_2 + z\right) - \dot{\mathbf{x}}_2 \left(z + u\right) - \mathbf{x}_1 \dot{\mathbf{x}}_1 - \mathbf{x}_1 \dot{\mathbf{x}}_2 - \mathbf{x}_2 \dot{u}}{\mathbf{x}_1 + \mathbf{x}_2} \tag{17}$$

which is an ODE as long as $\mathbf{x}_1 + \mathbf{x}_2 \neq 0$.

3. We have already performed this task in the previous question. The index-reduced DAE is the one occurring "one step before getting an ODE", i.e. we can write:

$$\dot{\mathbf{x}}_1 = \mathbf{x}_1 + \mathbf{x}_2 + z \tag{18a}$$
$$\dot{\mathbf{x}}_2 = z + u \tag{18b}$$
$$0 = \mathbf{x}_1 \left(\mathbf{x}_1 + \mathbf{x}_2 + z\right) + \mathbf{x}_2 \left(z + u\right) \tag{18c}$$

Here as well we need $\mathbf{x}_1 + \mathbf{x}_2 \neq 0$ to be able to solve (18c).

(b) We are dealing with a semi-explicit DAE, hence the index reduction requires time-differentiations of the algebraic equation (10b). The first step of the index reduction reads as:

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}z \tag{19a}$$
$$0 = \frac{\partial \mathbf{g}\left(\mathbf{x}\right)}{\partial \mathbf{x}} \dot{\mathbf{x}} = \frac{\partial \mathbf{g}\left(\mathbf{x}\right)}{\partial \mathbf{x}} \left(A\mathbf{x} + \mathbf{b}z\right) \tag{19b}$$

We note that $\frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}} \mathbf{b}$ is scalar here, and different than zero since it is full rank. It follows that (19) is of index 1. The consistency condition is simply:

$$\mathbf{g}\left(\mathbf{x}(0)\right) = 0 \tag{20}$$

DAE (10) is of index 2 if $\frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}} \mathbf{b} \neq 0$, if $\frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}} \mathbf{b} = 0$, then it is of index at least 3.

3. **Simulation**

(a) (4 points) Write a pseudo-code (algorithm) that would deploy an IRK scheme for an implicit autonomous DAE

$$\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{z}) = 0 \tag{21}$$

Be specific enough that someone could code it without knowing what the algorithm is about.

(b) (2 points) What necessary conditions the DAE (21) should satisfy to ensure that the scheme works?

(c) (3 points) Describe how (21) can be numerically simulated using an explicit RK scheme. Why is this approach not preferred to the IRK one?

Solution:

(a) The pseudo-code will look like

Algorithm: Integration of implicit ODE

Input: $\mathbf{x}_0$, α and Δt

Set $K = 0$

for $k = 0 : N-1$ **do**

  while $\|\mathbf{r}(\mathbf{K}, \mathbf{x}_k, \mathbf{u}(.))\| > \text{tol}$ **do**

    Evaluate:

$$\mathbf{r}(\mathbf{K}, \mathbf{z}, \mathbf{x}_k) = \begin{bmatrix} \mathbf{F}\left(\mathbf{K}_1, \mathbf{x}_k + \Delta t \sum_{i=1}^{s} a_{1i}\mathbf{K}_i,\ \mathbf{z}_1\right) \\ \vdots \\ \mathbf{F}\left(\mathbf{K}_s, \mathbf{x}_k + \Delta t \sum_{i=1}^{s} a_{si}\mathbf{K}_i,\ \mathbf{z}_s\right) \end{bmatrix} = 0$$

    and

$$M = \begin{bmatrix} \frac{\partial \mathbf{r}(\mathbf{K},\mathbf{z},\mathbf{x}_k)}{\partial \mathbf{K}} & \frac{\partial \mathbf{r}(\mathbf{K},\mathbf{z},\mathbf{x}_k)}{\partial \mathbf{z}} \end{bmatrix}$$

    Take the Newton step

$$\begin{bmatrix} \mathbf{K} \\ \mathbf{z} \end{bmatrix} \leftarrow \begin{bmatrix} \mathbf{K} \\ \mathbf{z} \end{bmatrix} - \alpha M^{-1}\mathbf{r} \tag{22}$$

  Take the integrator step:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \sum_{i=1}^{s} b_i \mathbf{K}_i \tag{23}$$

return $\mathbf{x}_{0,\dots N}$

Obs: pseudo-code adequately tailored to implicit DAEs will also be counted as correct.

(b) DAE (21) must be such that M is square and full rank throughout the integration. This is guaranteed if the DAE is of index 1. If the DAE is not of index 1, matrix M may still be full rank, but the integration scheme will, in general, deliver incorrect trajectories.

(c) DAE (21) can be numerically simulated via ERK methods by computing for each instance in the algorithm the $\dot{\mathbf{x}}$ via a Newton iteration. I.e. each time the ERK method requires an $\dot{\mathbf{x}}$, a Newton iteration of dimension $\mathbb{R}^{n_{\mathbf{x}}+n_{\mathbf{z}}}$ (number of unknowns to compute) is run. For, e.g., and RK4 methods, for each RK step this Newton iteration would be called 4 times (one per model evaluation in the RK method). While it is difficult to compare the efficiency (accuracy vs. CPU time) of this approach vs. an IRK method, since Newton iterations are required for both, and since the IRK method provides a high-order integration, the latter is generally preferred.

4. **Newton** The Newton methods aims at solving a set of equation $\mathbf{r}(\mathbf{x}) = 0$ numerically. To that end, iterates the recursion:

$$\frac{\partial \mathbf{r}(\mathbf{x})}{\partial \mathbf{x}} \Delta \mathbf{x} + \mathbf{r}(\mathbf{x}) = 0 \tag{24a}$$

$$\mathbf{x} \leftarrow \mathbf{x} + \alpha \Delta \mathbf{x} \tag{24b}$$

where $\alpha \in]0,\, 1]$ is the step-size.

(a) (2 points) When do full Newton steps converge to a solution of $\mathbf{r}(\mathbf{x}) = 0$.

(b) (2 points) When is a quadratic convergence rate of the Newton iteration possible? When is it linear? What is better?

(c) (2 points) What happens to the Newton iteration if $\mathbf{r}(\mathbf{x}) = 0$ holds for several distinct values of $\mathbf{x}$? How can one deal with that?

Solution:

(a) Full Newton steps are guaranteed to converge in a neighborhood of a solution only. The "size" of that neighborhood depends on how nonlinear $\mathbf{r}(\mathbf{x})$ is, and the Jacobian $\frac{\partial \mathbf{r}(\mathbf{x})}{\partial \mathbf{x}}$ must be full rank throughout this neighborhood.

(b) The quadratic convergence occurs when it is possible to take full Newton steps ($\alpha = 1$), i.e. when the iterate is close enough to a solution. When that is not possible, reduced steps must be used, yielding a linear convergence rate. Quadratic convergence is much faster than linear, and therefore better.

(c) If the Newton iteration converges, it converges to one of the solutions to $\mathbf{r}(\mathbf{x}) = 0$. It often converges to the "closest solution" to the initial guess, though this statement is not necessarily true. In practice, one needs to provide an initial guess to the Newton iteration that is close to the solution one wants to attain.