TTK4130 - Modeling and Simulation
Assignement 6
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1a)
$$\dot{x}_1 = x_1 + x_2 + z$$
$$\dot{x}_2 = z + u$$
$$0 = \frac{1}{2}(x_1^2 + x_2^2 - 1)$$

We can tell that this isn't an ODE by observing that ~~[illegible]~~ $\dot{z}$ (time-diff form of z) isn't given. This is therefore a DAE.

b) We set:
c)
$$g(x, z, u) = \frac{1}{2}(x_1^2 + x_2^2 - 1) = 0$$
$$\dot{g}(x, z, u) = \dot{x}_1 x_1 + \dot{x}_2 x_2 = 0$$
$$= x_1(x_1 + x_2 + z) + x_2(z + u) = 0$$

This delivers $\dot{x}$ and z, so this ~~[illegible]~~ means that the differential index of (1) was 2.

2a)

$$\dot{x} = -\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x - z$$

$$\varepsilon \dot{z} = \frac{1}{10} x - A z$$

$$A = \begin{bmatrix} x_1^2 & x_2 \\ 0 & x_2^2 \end{bmatrix} + \alpha I$$

We start by rewriting $\dot{z}$ as

$$\dot{z} = \frac{1}{\varepsilon}\left(\frac{1}{10} x - \begin{bmatrix} x_1^2+\alpha & x_2 \\ 0 & x_2^2+\alpha \end{bmatrix} z\right)$$

$$\dot{z} = A^* z + Bu$$

$$\| $$

$$\frac{1}{\varepsilon}\begin{bmatrix} -(x_1^2+\alpha) & -x_2 \\ 0 & -(x_2^2+\alpha) \end{bmatrix}$$

This have only negative eig-values, so

$\dot{z}$ is stable $\forall x$

$$\dot{z} = g(x, z)$$

$\frac{\partial g}{\partial z}$ has full rank everywhere

$$F(\dot{x}, x, z, u, t) = \begin{bmatrix} \dot{x} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + z \\ \varepsilon \dot{z} - \frac{1}{10} x + A z \end{bmatrix} = 0$$

$$\det\left(\frac{\partial F}{\partial \dot{x} \partial \dot{z}}\right) \neq 0$$

$\Rightarrow$ ODE

b) See the added plots.
We see that decreasing $\varepsilon \to 0$
makes them more similar.

c) Done

d) This makes sense, since
Tikhonovs theorem states that
the ODE sol. will be equal to
the DAE as $\varepsilon \to 0$

In c, A becomes singular
when $\alpha = 0$, and then the
DAE crash. (Around the 1.2-mark)

3a) $$\dot{x}_1 + u + x_1 + x_2 = 0$$

$$\underbrace{u + x_2 + \dot{x}_2\dot{x}_1 + \dot{x}_2 u + \dot{x}_2 x_1 + \dot{x}_2 x_2 + u^2 = 0}_{F}$$

$$\frac{\partial F}{\partial \dot{x}} = \begin{bmatrix} 1 & 0 \\ \dot{x}_2 & \dot{x}_1 + x_1 + x_2 + u \end{bmatrix}$$

$$\det\left(\frac{\partial F}{\partial \dot{x}}\right) = \dot{x}_1 + x_1 + x_2 + u$$
$$= 0 \quad (6a)$$

Jacobian rank-deficient.
$\Rightarrow$ DAE

6a) $\dot{x}_1 = -x_1 - x_2 - u$
$\downarrow$ insert into 6b)

$$u + x_2 + \dot{x}_2(-x_1 - x_2 - u) + \dot{x}_2(x_1 + x_2 + u) + u^2 = 0$$
$$x_2 = -u - u^2$$

$$\dot{x}_1 = -x_1 - x_2 - u$$

$$x_2 = -u - u^2$$

x_1 : diff. state.

x_2 : Alg. state

b) $u + \dot{x}_1 x_1 + \dot{x}_2 x_2 = 0$

$$u \dot{x}_1 x_1 + \dot{x}_2 u x_2 = 0$$

$$\frac{\partial F}{\partial \dot{x}} = \begin{bmatrix} x_1 & x_2 \\ u x_1 & u x_2 \end{bmatrix}$$

$$\det\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 \quad \Rightarrow \quad \text{DAE}$$

Not possible to determine algebraic or differential states.

4a) $\dot{x} + u + \tanh(\dot{x}) + xz = 0$

$\tanh(\underbrace{2u - z}_{(=0)}) = 0$

Let's say $\dot{x} = v$

$F(x, v, u, z)$

$$g(\cancel{x}, v, u, z) = \begin{matrix} v + u + \tanh(uv) + xz = 0 \\ 2u - z = 0 \end{matrix}$$

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b) $\begin{bmatrix} \frac{\partial g}{\partial \dot{x}} & \frac{\partial g}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 + \frac{u}{\cosh^2(u\dot{x})} & x \\ 0 & -\frac{1}{\cosh^2(2u-z)} \end{bmatrix}$

Well-defined since this stud has full rank (IFT)

c) We pull a fast $\frac{d}{dt}$ on the system from a)

$$\dot{v} + \dot{u} + \frac{\dot{u}v + \dot{v}u}{\cosh^2(uv)} + \dot{x}z + x\dot{z} = 0$$

$$2\dot{u} - \dot{z} = 0$$

$$\dot{v} = - \frac{\dot{u}\left(1 + 2x\dot{u} + \frac{v}{\cosh^2(uv)} + vz\right.}{1 + \frac{u}{\cosh^2(uv)}}$$

$$z = 2u$$

$$\dot{x} = v$$

$\Rightarrow$ <u>Diff index : 2</u>