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For questions during the exam:
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Exam in TTK4130 Modeling and Simulation

Thursday, May 15 2019
09:00 – 13:00

Permitted aids (code A): All written and handwritten examination support materials are permitted.

Answers in English, Norwegian, or a mixture of the two are accepted.

Grades available: As specified by regulations.

Problem 1 (11 %)

(2 %) (a) Explain briefly how the Modelica commands `replaceable` and `redeclare` are connected.

> **Solution:** With `replaceable` exchangeable components are declared; with `redeclare` such a exchangeable component, e.g. in a new derived model, is exchanged with another component.

(6 %) (b) You have available the three models: `RedModel`, `WhiteModel`, and `BlueModel`. You like to implement a flexible model `Austria`, where the `RedModel`, the `WhiteModel` and another `RedModel` are connected and can be exchanged (Fig. 1). The models are connected by just one connector variable.

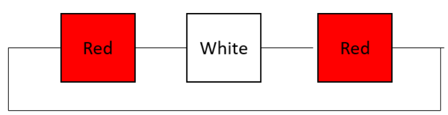

Figure 1: Model `Austria`.

Moreover, you like to set the parameter $p1$ in one of the `RedModel` to $p1 = 5$ and in the other to $p1 = 10$. Write down the Modelica code of the *flexible* model `Austria`.

Solution:

```
model Austria

  replaceable RedModel comp1(p1=5);
  replaceable WhiteModel comp2;
  replaceable RedModel comp3(p1=10);

equation
  connect(comp1.qout, comp2.qin);
  connect(comp2.qout, comp3.qin);
  connect(comp3.qout, comp1.qin);

end Austria;
```

(3 %) (c) A new model `Russia` should be implemented with the structure shown in Fig. 2. Write the Modelica code of the new model by using model `Austria`.

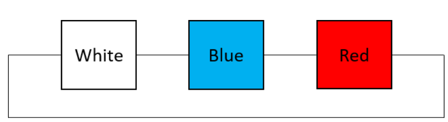

Figure 2: Model `Russia`.

Solution:

```
model Russia = Austria(redeclare WhiteModel comp1, redeclare BlueModel comp2);
```

Problem 2 (10 %)

Consider the following differential-algebraic system with input u

$$\begin{aligned}\dot{x}_1 &= x_3 - x_1 z,\\ \dot{x}_2 &= u - x_2 z,\\ \dot{x}_3 &= -x_1,\\ 0 &= {}^1\!/\!_2(1 - x_1^2 - x_2^2).\end{aligned}$$

(4 %) (a) Derive mathematically the differential index of the system.

Solution: We have to take two differentiations to transform the DAE into an ODE system. Consequently, the differential index is two.
First differentiation results in

$$\begin{aligned}-x_1\dot{x}_1 - x_2\dot{x}_2 &= 0,\\ -x_1(x_3 - x_1 z) - x_2(u - x_2 z) &= 0.\end{aligned}$$

Obviously the second differentiation will result in an equation $\dot{z} = \dots$.

For the rest of the problem you can neglect the algebraic equation and assume $z = {}^1\!/\!_2$

(6 %) (b) Is the ODE system passive for input u and output x_2?

Solution: The system is linear given by

$$\begin{aligned}\dot{x} &= \begin{pmatrix} -{}^1\!/\!_2 & 0 & 1\\ 0 & -{}^1\!/\!_2 & 0\\ -1 & 0 & 0\end{pmatrix} x + \begin{pmatrix}0\\1\\0\end{pmatrix} u,\\ y &= \begin{pmatrix}0 & 1 & 0\end{pmatrix} x.\end{aligned}$$

The transfer function is given by

$$H(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}.$$

It becomes apparent from the structure of $\mathbf{B}$ and $\mathbf{C}$ that we only have to find the value (2,2) of $(s\mathbf{I} - \mathbf{A})^{-1}$. Consequently, the transfer function is $H(s) = \frac{2}{1+2s}$. It is a rational transfer function with pol at $s = -{}^1\!/\!_2$. The real part of the transfer function is

$$\begin{aligned}H(j\omega) &= \frac{2}{1+2j\omega} = \frac{2-4j\omega}{1+4\omega^2},\\ \mathcal{R}(H(j\omega)) &= \frac{2}{1+4\omega^2} \geq 0.\end{aligned}$$

Since there are no pols on the imaginary axis the system with input u and output x_2 is passive.

Problem 3 (26 %)

Consider the following Runge-Kutta method with the Butcher tableau

$$
\begin{array}{c|cc}
\gamma & \gamma & 0 \\
1-\gamma & 1-2\gamma & \gamma \\
\hline
 & 1/2 & 1/2
\end{array} .
$$

(5 %) (a) Write down the equations of the method. What are the limits for γ, and is the method explicit or implicit?

> **Solution:** The method is given by
>
> $$
> \begin{aligned}
> \mathbf{k}_1 &= \mathbf{f}(\mathbf{y}_n + \gamma h \mathbf{k}_1, t_n + \gamma h), \\
> \mathbf{k}_2 &= \mathbf{f}(\mathbf{y}_n + (1-2\gamma) h \mathbf{k}_1 + \gamma h \mathbf{k}_2, t_n + (1-\gamma) h), \\
> \mathbf{y}_{n+1} &= \mathbf{y}_n + \frac{1}{2} h \mathbf{k}_1 + \frac{1}{2} h \mathbf{k}_2.
> \end{aligned}
> $$
>
> The interpolation parameters c_i form an increasing sequence and are in the range $0 \leq c_i \leq 1$, therefore, $0 \leq \gamma \leq 0.5$. The method is implicit if $\gamma \neq 0$, otherwise it reduces to the improved Euler method (an explicit method).

(7 %) (b) Find the stability function of the method.

> **Solution:** Both (14.142) and (14.143) can be used to find the stability function of the method. Both solutions will be presented.
>
> $$
> R(s) = \frac{\det\left(\mathbf{I} - s(\mathbf{A} - \mathbf{1}\mathbf{b}^T)\right)}{\det(\mathbf{I} - s\mathbf{A})}
> $$
>
> The denominator is
>
> $$
> \det(\mathbf{I} - s\mathbf{A}) = \det\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - s \begin{pmatrix} \gamma & 0 \\ 1-2\gamma & \gamma \end{pmatrix} \right) = (1 - s\gamma)^2.
> $$
>
> The nominator is
>
> $$
> \begin{aligned}
> \det\left((\mathbf{I} - s(\mathbf{A} - \mathbf{1}\mathbf{b}^T)\right) &= \det\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - s \left[\begin{pmatrix} \gamma & 0 \\ 1-2\gamma & \gamma \end{pmatrix} - \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \right] \right) \\
> &= \det\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - s \begin{pmatrix} \gamma - 1/2 & -1/2 \\ 1/2 - 2\gamma & \gamma - 1/2 \end{pmatrix} \right) \\
> &= \det\left(\begin{pmatrix} 1 - s(\gamma - 1/2) & -1/2 s \\ -s(1/2 - 2\gamma) & 1 - s(\gamma - 1/2) \end{pmatrix} \right) \\
> &= (1 - s(\gamma - 1/2))^2 + 1/2 s^2 (1/2 - s\gamma) \\
> &= 1 + s(1 - 2\gamma) + s^2(\gamma^2 - 2\gamma + 1/2)
> \end{aligned}
> $$
>
> The stability function
>
> $$
> R(s) = \frac{1 + s(1-2\gamma) + s^2(\gamma^2 - 2\gamma + 1/2)}{(1 - s\gamma)^2}
> $$

Alternatively (14.142) can be used

$$
\begin{aligned}
R(s) &= \left(1 + s\mathbf{b}^T(\mathbf{I} - s\mathbf{A})^{-1}\mathbf{1}\right) \\
&= 1 + s\begin{pmatrix} 1/2 & 1/2 \end{pmatrix}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - s\begin{pmatrix} \gamma & 0 \\ 1-2\gamma & \gamma \end{pmatrix}\right)^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= 1 + s\begin{pmatrix} 1/2 & 1/2 \end{pmatrix}\frac{1}{(1-s\gamma)^2}\begin{pmatrix} 1-s\gamma & 0 \\ s(1-2\gamma) & 1-s\gamma \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= 1 + s\begin{pmatrix} 1/2 & 1/2 \end{pmatrix}\frac{1}{(1-s\gamma)^2}\begin{pmatrix} 1-s\gamma \\ 1+s-3s\gamma \end{pmatrix} \\
&= 1 + \frac{s(1 + 1/2s - 2\gamma s)}{(1-s\gamma)^2} \\
&= \frac{1 + s(1-2\gamma) + s^2(\gamma^2 - 2\gamma + 1/2)}{(1-s\gamma)^2}
\end{aligned}
$$

(6 %) (c) Find the limits on γ so the method is B-stable.
Hint: Make appropriate assumptions.

Solution: Instead of checking B-stability we can use algebraic stability which implies B-stability. A method is algebraic stable if $b_i \geq 0$ for $i = 1, \dots, \sigma$ (which is fulfilled) and $\mathbf{M} = \operatorname{diag}(\mathbf{b})\mathbf{A} + \mathbf{A}^T \operatorname{diag}(\mathbf{b}) - \mathbf{b}\mathbf{b}^T \geq 0$.
The matrix $\mathbf{M}$ is

$$
\begin{aligned}
\mathbf{M} &= \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} \gamma & 0 \\ 1-2\gamma & \gamma \end{pmatrix} + \begin{pmatrix} \gamma & 1-2\gamma \\ 0 & \gamma \end{pmatrix}\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \end{pmatrix} \geq 0 \\
&= \begin{pmatrix} 1/2\gamma & 0 \\ 1/2 - \gamma & 1/2\gamma \end{pmatrix} + \begin{pmatrix} 1/2\gamma & 1/2 - \gamma \\ 0 & 1/2\gamma \end{pmatrix} - \begin{pmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix} \geq 0 \\
&= \begin{pmatrix} \gamma - 1/4 & 1/4 - \gamma \\ 1/4 - \gamma & \gamma - 1/4 \end{pmatrix} \geq 0
\end{aligned}
$$

It can already be seen that the limit will be $\gamma = 1/4$, but mathematically it can be proven with

$$
\begin{aligned}
\det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{pmatrix} \gamma - 1/4 - \lambda & 1/4 - \gamma \\ 1/4 - \gamma & \gamma - 1/4 - \lambda \end{pmatrix} = 0 \\
&= (\gamma - 1/4 - \lambda)^2 - (1/4 - \gamma)^2 = 0 \\
&= \lambda^2 - 1/2\lambda - 2\gamma\lambda = 0 \\
&= \lambda(\lambda + 1/2 - 2\gamma) = 0
\end{aligned}
$$

The eigenvalues of the matrix have to fulfill $\lambda_i \geq 0$, therefore, $\gamma \geq 1/4$.

(4 %) (d) Find the limits on γ for which the method is B- and L-stable.

Solution: B-stability guarantees A-stability so a sensible first limit is given by the previous task. Moreover, for L-stability the denominator of the stability function must have a higher order than the nominator. Therefore, $\gamma^2 - 2\gamma + 1/2 \overset{!}{=} 0$ and $\gamma \neq 0$. Solving the first equality leads to $\gamma_{1/2} = \frac{2 \pm \sqrt{2}}{2}$. So the method is B- and L-stable only for $\gamma_1 = \frac{2-\sqrt{2}}{2}$ since $\gamma_2 = \frac{2+\sqrt{2}}{2} \geq 0.5$.

(4 %) (e) Explain why the method can be seen as numerical efficient for $\gamma \neq 0$.

Solution: The method belongs to the class of DIRK (diagonal IRK) methods for $\gamma \neq 0$ since $\mathbf{A}$ is a lower triangular matrix. For these methods each stage can be solved individually rather than simultaneously in a full IRK method. This leads to solve σ equations of dimensions $d \times d$ rather than solving a $(\sigma d) \times (\sigma d)$ equation system.

Problem 4 (25 %)

The following system consisting of a slender beam with mass m_1 and length l and a disc with mass m_2 and radius R should be modeled. The disc rolls without slipping on the ground (Fig. 3). The system is frictionless.

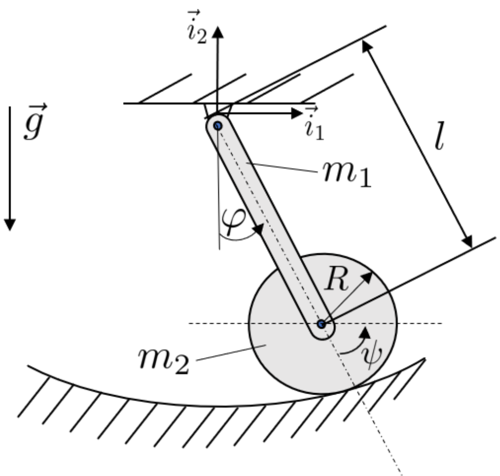

Figure 3: Slender beam with disc.

Hint: The moment of inertia of a disc: $I = {}^1\!/_2 mR^2$.

(3 %) (a) Find a connection between φ and ψ. How many degrees of freedom have the system?

Solution: Since the disc does not slide the connection between φ and ψ can be found by expressing the velocity in the point of connection between beam and disc:

$$l\dot{\varphi} = -R\dot{\psi}$$
$$\dot{\psi} = -{}^l\!/_R\dot{\varphi}.$$

The system has one degree of freedom.

If you were not able to find a connection between φ and ψ use $\varphi = \psi$ [which is not necessarily the correct answer to (a)]

(6 %) (b) Find the position, velocity and acceleration vector of the center of mass of both body parts.

Solution: The position, velocity and acceleration vector of the slender beam are given by $\vec{r}_1 = {}^l\!/_2\vec{b}_1$, $\vec{v}_1 = {}^l\!/_2\dot{\varphi}\vec{b}_2$, and $\vec{a}_1 = {}^l\!/_2\ddot{\varphi}\vec{b}_2 - {}^l\!/_2\dot{\varphi}^2\vec{b}_1$. Similary the position, velocity and acceleration vector of the disc is $\vec{r}_1 = l\vec{b}_1$, $\vec{v}_1 = l\dot{\varphi}\vec{b}_2$, and $\vec{a}_1 = l\ddot{\varphi}\vec{b}_2 - l\dot{\varphi}^2\vec{b}_1$, where in both cases a body coordinate system is used with $\vec{b}_1$ along the longitudinal axis of the slender beam and $\vec{b}_2$ pointing in direction of the movement of the disc (simple rotation of the inertia coordinate system in Fig. 3). *For more information about the solution check the book section 6.12.*

(13 %) (c) Find the equation of motion of the system as a function of the given variables (Fig. 3) with the Newton-Euler approach.

Caution: Solutions using the Lagrange approach will give no points in this task.

Hint 1: Given variables: $\vec{g}, m_1, m_2, l, R, \psi, \varphi$, including their derivatives.

Hint 2: The equation of motion does not necessary contain all given variables.

Solution: The system consist of two rigid body parts. Both have to be cut free and the forces and torque on both determined.

We begin with the slender beam (see Fig. 4)

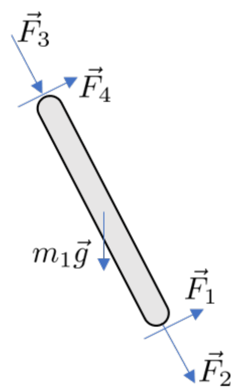

Figure 4: Free-cut of slender beam and attacking forces.

The resulting forces on the center of mass of the slender beam is

$$F_1^r = -m_1 g\vec{i}_2 + (F_2 + F_3)\vec{b}_1 + (F_1 + F_4)\vec{b}_2.$$

We will derive the torque around the attachment of the slender beam at the ceiling (around CoM is also possible):

$$T_{1,o}^r = (-m_1 g\,{}^{l}\!/\!{}_{2} \sin\varphi + F_1 l)\vec{b}_3.$$

Applying the Newton-Euler equations with the momentum balance around point o (7.59) to the slender beam using the acceleration found in b) results in

$$m_1({}^{l}\!/\!{}_{2}\ddot{\varphi}\vec{b}_2 - {}^{l}\!/\!{}_{2}\dot{\varphi}^2\vec{b}_1) = -m_1 g\vec{i}_2 + (F_2 + F_3)\vec{b}_1 + (F_1 + F_4)\vec{b}_2 \tag{1}$$

$$ {}^{1}\!/\!{}_{3} ml^2\ddot{\varphi}\vec{b}_3 = (-m_1 g\,{}^{l}\!/\!{}_{2} \sin\varphi + F_1 l)\vec{b}_3, \tag{2}$$

where we used that the moment of inertia of a slender beam rotating around the tip is $I = {}^{1}\!/\!{}_{3} ml^2$ (book Example 121; Lecture 14) and that the acceleration of the point at the ceiling is $a_o = 0$.

Secondly, we cut free the disc (Fig. 5) and find forces and torques acting on it

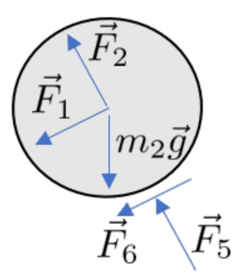

Figure 5: Free-cut of disc and attacking forces.

Resulting force on the disc:

$$F_2^r = -m_2 g \vec{i}_2 - (F_2 + F_5)\vec{b}_1 - (F_1 + F_6)\vec{b}_2.$$

Resulting torque around the center of mass:

$$T_{2,c}^r = -RF_6 \vec{b}_3.$$

Applying Newton-Euler equations (7.41,7.42) using the acceleration found in b)

$$m_2(l\ddot{\varphi}\vec{b}_2 - l\dot{\varphi}^2\vec{b}_1) = -m_2 g\vec{i}_2 - (F_2 + F_5)\vec{b}_1 - (F_1 + F_6)\vec{b}_2 \tag{3}$$
$${}^1\!/\!_2 m_2 R^2 \ddot{\psi}\vec{b}_3 = -RF_6\vec{b}_3, \tag{4}$$
$$-{}^{lR}\!/\!_2 m_2 \ddot{\varphi}\vec{b}_3 = -RF_6\vec{b}_3, \tag{5}$$

where we used the hint about the moment of inertia of a disc (Eq. 4) and the connection between φ and ψ from a) (to get from Eq. 4 to Eq. 5).
Multiplying (2) and (5) with $\vec{b}_3$ gives

$$F_1 = {}^1\!/\!_3 m_1 l\ddot{\varphi} + {}^1\!/\!_2 m_1 g \sin\varphi$$
$$F_6 = {}^1\!/\!_2 m_2 l\ddot{\varphi}.$$

Multiplying (3) with $\vec{b}_2$ gives

$$m_2 l\ddot{\varphi} = -m_2 g \sin\varphi - (F_1 + F_6),$$

which together with the information about F_1 and F_6 results in

$$\ddot{\varphi} + \frac{g}{l}\frac{2m_2 + m_1}{3m_2 + {}^2\!/\!_3 m_1}\sin\varphi = 0.$$

Alternative: If not the torque around the attachment but around the center of mass is used:

$$T_{1,c}^r = {}^l\!/\!_2(F_1 - F_4)\vec{b}_3,$$

which results applying Newton-Euler equation (7.42) and multiplying with $\vec{b}_3$ in

$$F_1 - F_4 = {}^1\!/\!_6 l m_1 \ddot{\varphi}, \tag{6}$$

where we used that the moment of inertia of a slender beam is $I = {}^1\!/\!_{12} ml^2$ (Book Example 121). We have to find expressions for F_4 and F_1.
Multiplying (1) with $\vec{b}_2$ results in

$$m_1 {}^l\!/\!_2 \ddot{\varphi} = -m_1 g \sin\varphi + F_4 + F_1. \tag{7}$$

From (6) we get

$$F_1 = {}^1\!/\!_6 l m_1 \ddot{\varphi} + F_4, . \tag{8}$$

Using (8) in (7) results in

$$F_4 = {}^1\!/\!_6 l m_1 \ddot{\varphi} + {}^1\!/\!_2 m_1 g \sin\varphi, \tag{9}$$

and we get $F_1 = {}^1\!/\!_3 m_1 l\ddot{\varphi} + {}^1\!/\!_2 m_1 g \sin\varphi$ (same as before).

If you were not able to find the EoM use in the following $\ddot{\varphi} l^2(2m_1 + m_2) + {}^1\!/\!{}_3 lg(3m_1 + {}^1\!/\!{}_2 m_2)\sin\varphi = 0$ *[which is not necessarily the correct answer to (c)]*

(3 %) (d) Find the period of oscillation.
Hint: Make appropriate assumptions.

Solution: For a small deflection $\sin\varphi \approx \varphi$, consequently the period of oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}\frac{3m_2 + {}^2\!/\!{}_3 m_1}{2m_2 + m_1}}.$$

Problem 5 (12 %)

Consider a thin plate (Fig. 6) of height h with a constant density ρ and the mass $m = {}^\pi\!/\!{}_4 abh\rho$. The outer contour of the cross-section can be described by $z^2 = b^2 - {}^{b^2y^2}\!/\!{}_{a^2}$, where $a \gg h$ and $b \gg h$.

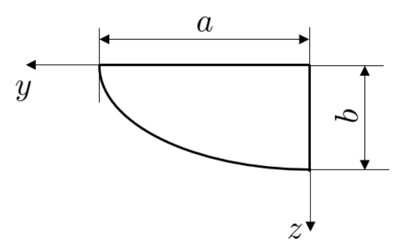

Figure 6: Thin plate

The following may be useful

$$\int \sqrt{X}dx = \frac{1}{2}(x\sqrt{X} + c^2 \arcsin\frac{x}{c})$$
$$\int x\sqrt{X}dx = -\frac{1}{3}\sqrt{X^3}$$
$$\int x^2\sqrt{X}dx = -\frac{x}{4}\sqrt{X^3} + \frac{c^2}{8}(x\sqrt{X} + c^2 \arcsin\frac{x}{c}),$$

with $X = (c^2 - x^2)$.

(7 %) (a) Find the moment of inertia $I_{y,o}$ of the plate for the given coordinate system as a function of mass m.

Solution: The moment of inertia I_y is given by

$$I_y = \int z^2 dm$$

in the Cartesian coordinate system, where the x-component is neglected. The infinitesimal mass dm is given by $dm = \rho dydzdx$. So we get with the correct integration boundaries

$$I_{y,o} = \int_0^b \int_0^{\eta(z)} \int_0^h z^2 \rho dxdydz,$$

where $\eta(z) = \sqrt{(1 - {z^2}/{b^2})a^2}$. Consequently, we get

$$
\begin{aligned}
I_{y,o} &= h\rho \int_0^b \int_0^{\eta(z)} z^2 dydz, \\
&= h\rho \int_0^b z^2 \sqrt{\left(1 - \frac{z^2}{b^2}\right) a^2} dz \\
&= h\rho \frac{a}{b} \int_0^b z^2 \sqrt{b^2 - z^2} dz \\
&= h\rho \frac{a}{b} \left[-\frac{z}{4} \sqrt{(b^2 - z^2)^3} + \frac{b^2}{8} (z\sqrt{b^2 - z^2} + b^2 \arcsin \frac{z}{b}) \right]_0^b \\
&= h\rho \frac{a}{b} \left[\frac{b^4}{8} (\arcsin(1) - \arcsin(0)) \right] \\
&= \frac{\pi}{16} ab^3 \rho h = \frac{1}{4} b^2 m
\end{aligned}
$$

If you were not able to find $I_{y,o}$ use in the following $I_{y,o} = {1}/{6} a^2 b^2 m$ [which is not necessarily the correct answer to (a)]

(5 %) (b) Find the moment of inertia $I_{y,c}$ around the center of mass of the plate as a function of mass m.

Solution: We first have to find the location z_c of the center of mass

$$
\begin{aligned}
z_c &= \frac{1}{m} \int z dm \\
&= \frac{1}{m} \frac{a}{b} \rho l \int_0^b z\sqrt{b^2 - z^2} dz \\
&= \frac{1}{m} \frac{a}{b} \rho l \left[-\frac{1}{3} \sqrt{(b^2 - z^2)^3} \right]_0^b \\
&= \frac{1}{m} \frac{ab^2 \rho l}{3} = \frac{4b}{3\pi}.
\end{aligned}
$$

Note the calculation is similar to a). Note also that it would be possible to check if you are on the right track by calculating the mass m given in the task.

With the parallel axis theorem we get for this problem

$$
\begin{aligned}
I_{y,c} &= I_{y,o} - z_c^2 m \\
&= \frac{1}{4} b^2 m - \frac{16}{9\pi^2} b^2 m \\
&= b^2 m \left(\frac{1}{4} - \frac{16}{9\pi^2} \right) \left[= \rho l a b^3 \left(\frac{\pi}{16} - \frac{4}{9\pi} \right) \right]
\end{aligned}
$$

Problem 6 (16 %)

In an open wind channel of diameter D the air flow around a wind turbine with diameter d is investigated (Fig. 7). The air flow has a constant density ρ. The upstream wind velocity is u_∞, the following velocity field over the cross-section of the channel is measured at a specific downstream position $x = \text{const}$.

$$u_e(r,\phi) = \begin{cases} u_\infty\left((1-c) - c\cos(\pi r/R)\right), & 0 \le r \le R \\ u_\infty, \quad R < r \le D/2 \end{cases}$$

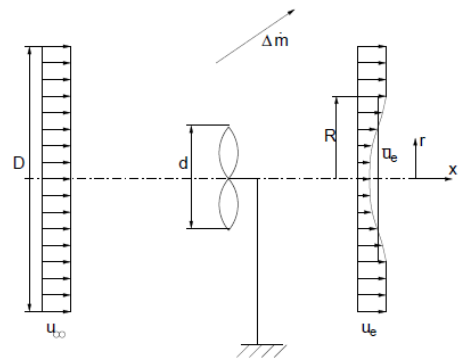

Figure 7: Open wind channel with wind turbine

Assume that the wind channel is in steady-state, $d \ll D$, $d/2 < R < D/2$ and $0 < c < 1$. Moreover, friction forces can be neglected.

The following may be useful

$$\int x\cos(ax)dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a}.$$

(7 %) (a) Determine the displaced mass flow $\Delta\dot{m}$ in the wind channel.
Hint: As usually for these problems you first have to define the control volume.

Solution: The control volume we will use in this task is shown in Fig. 8.

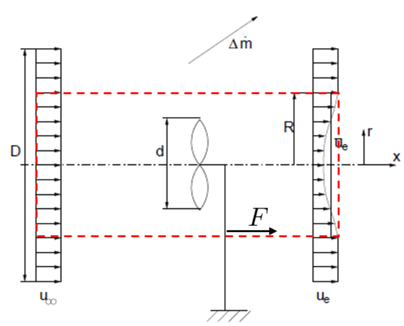

Figure 8: The control volume to determine the mass displacement.

$$\frac{d}{dt}\iiint_{V_c} \rho dV = -\iint_{\partial V_c} \rho \vec{v}\vec{n} dA,$$

$$0 = \rho u_\infty \pi R^2 - \rho \int_0^{2\pi}\int_0^R u_e(r,\phi) r dr d\phi - \Delta\dot{m},$$

where we used that the process is in steady-state. The determination of $\Delta\dot{m}$ is straight forward

$$\begin{aligned}\Delta\dot{m} &= \rho u_\infty \pi R^2 - 2\pi\rho u_\infty \int_0^R \left[(1-c)r - cr\cos(\pi r/R)\right] dr \\ &= \rho u_\infty \pi R^2 - 2\pi\rho u_\infty \left[(1-c)\frac{r^2}{2} - c\frac{\cos(\pi r/R)}{\left(\pi/R\right)^2} - c\frac{r\sin(\pi r/R)}{\pi/R}\right]_0^R \\ &= \rho u_\infty \pi R^2 c\left(1 - \frac{4}{\pi^2}\right)\end{aligned}$$

If you were not able to find the displaced mass flow continue with $\Delta\dot{m} = \rho u_\infty c\left(1 - \frac{2}{\pi}\right)$ [which is not necessarily the correct answer to (a)]

(3 %) (b) The velocity profile $u_e(r,\phi)$ should be replaced with an piecewise constant velocity $u_e(r,\phi) = \bar{u}_e,\ 0 \le r \le R$ and $u_e(r,\phi) = u_\infty,\ R < r \le D/2$. Determine the average velocity $\bar{u}_e$ such that the displaced mass flow $\Delta\dot{m}$ stays the same as in a).

Solution: We can again use the control volume shown in Fig. 8 and apply the mass balance

$$0 = \rho u_\infty \pi R^2 - \rho\bar{u}_e \pi R^2 - \Delta\dot{m}.$$

Consequently, we get

$$\bar{u}_e = u_\infty\left(1 - c\left(1 - \frac{4}{\pi^2}\right)\right)$$

With the hint we get $\bar{u}_e = u_\infty\left(1 - {}^{c}/_{(\pi R^2)}\left(1 - {}^{2}/_{\pi}\right)\right)$

For the following task assume the average velocity $\bar{u}_e$ is known.

(6 %) (c) Determine the force F produced from the wind turbine using the piecewise velocity profile.

Solution: Again we use the same control volume as before (Fig. 8). The mass balance gives

$$\Delta\dot{m} = \rho(u_\infty - \bar{u}_e)\pi R^2.$$

For determining the force F we have to use the momentum balance

$$\frac{d}{dt}\iiint_{V_c} \rho\vec{v} dV = -\iint_{\partial V_c} \rho\vec{v}\vec{v}\vec{n} dA + \vec{F}^{(r)},$$

$$0 = \rho u_\infty^2 \pi R^2 - \rho\bar{u}_e^2 \pi R^2 - \Delta\dot{m} u_\infty + F.$$

It follows that the force F from the turbine is given by

$$F = (\bar{u}_e - u_\infty)\bar{u}_e \rho\pi R^2.$$