Chapter 4

Lyapunov Stability

Stability theory plays a central role in systems theory and engineering. There are different kinds of stability problems that arise in the study of dynamical systems. This chapter is concerned mainly with stability of equilibrium points. In later chapters, we shall see other kinds of stability, such as input–output stability and stability of periodic orbits. Stability of equilibrium points is usually characterized in the sense of Lyapunov, a Russian mathematician and engineer who laid the foundation of the theory, which now carries his name. An equilibrium point is stable if all solutions starting at nearby points stay nearby; otherwise, it is unstable. It is asymptotically stable if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium point as time approaches infinity. These notions are made precise in Section 4.1, where the basic theorems of Lyapunov's method for autonomous systems are given. An extension of the basic theory, due to LaSalle, is given in Section 4.2. For a linear time-invariant system $\dot{x}(t) = Ax(t)$, the stability of the equilibrium point $x = 0$ can be completely characterized by the location of the eigenvalues of A. This is discussed in Section 4.3. In the same section, it is shown when and how the stability of an equilibrium point can be determined by linearization about that point. In Section 4.4, we introduce class $\mathcal{K}$ and class $\mathcal{KL}$ functions, which are used extensively in the rest of the chapter, and indeed the rest of the book. In Sections 4.5 and 4.6, we extend Lyapunov's method to nonautonomous systems. In Section 4.5, we define the concepts of uniform stability, uniform asymptotic stability, and exponential stability for nonautonomous systems, and give Lyapunov's method for testing them. In Section 4.6, we study linear time-varying systems and linearization.

Lyapunov stability theorems give sufficient conditions for stability, asymptotic stability, and so on. They do not say whether the given conditions are also necessary. There are theorems which establish, at least conceptually, that for many of Lyapunov stability theorems, the given conditions are indeed necessary. Such theorems are usually called converse theorems. We present three converse theorems in Section 4.7. Moreover, we use the converse theorem for exponential stability to show that an equilibrium point of a nonlinear system is exponentially stable if and only if the linearization of the system about that point has an exponentially stable equilibrium at the origin.

Lyapunov stability analysis can be used to show boundedness of the solution, even when the system has no equilibrium points. This is shown in Section 4.8 where the notions of uniform boundedness and ultimate boundedness are introduced. Finally, in Section 4.9, we introduce the notion of input-to-state stability, which provides a natural extension of Lyapunov stability to systems with inputs.

4.1 Autonomous Systems

Consider the autonomous system

$$\dot{x} = f(x) \tag{4.1}$$

where $f : D \to R^n$ is a locally Lipschitz map from a domain $D \subset R^n$ into R^n. Suppose $\bar{x} \in D$ is an equilibrium point of (4.1); that is, $f(\bar{x}) = 0$. Our goal is to characterize and study the stability of $\bar{x}$. For convenience, we state all definitions and theorems for the case when the equilibrium point is at the origin of R^n; that is, $\bar{x} = 0$. There is no loss of generality in doing so because any equilibrium point can be shifted to the origin via a change of variables. Suppose $\bar{x} \neq 0$ and consider the change of variables $y = x - \bar{x}$. The derivative of y is given by

$$\dot{y} = \dot{x} = f(x) = f(y + \bar{x}) \stackrel{\text{def}}{=} g(y), \quad \text{where } g(0) = 0$$

In the new variable y, the system has equilibrium at the origin. Therefore, without loss of generality, we will always assume that $f(x)$ satisfies $f(0) = 0$ and study the stability of the origin $x = 0$.

Definition 4.1 *The equilibrium point $x = 0$ of (4.1) is*

- *stable if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that*

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall\, t \geq 0$$

- *unstable if it is not stable.*

- *asymptotically stable if it is stable and δ can be chosen such that*

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

The ε–δ requirement for stability takes a challenge-answer form. To demonstrate that the origin is stable, then, for any value of ε that a challenger may care to designate, we must produce a value of δ, possibly dependent on ε, such that a trajectory starting in a δ neighborhood of the origin will never leave the ε neighborhood. The three types of stability properties can be illustrated by the pendulum example of Section 1.2.1. The pendulum equation

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -a\sin x_1 - bx_2\end{aligned}$$

has two equilibrium points at $(x_1 = 0,\ x_2 = 0)$ and $(x_1 = \pi,\ x_2 = 0)$. Neglecting friction, by setting $b = 0$, we have seen in Chapter 2 (Figure 2.2) that trajectories in the neighborhood of the first equilibrium are closed orbits. Therefore, by starting sufficiently close to the equilibrium point, trajectories can be guaranteed to stay within any specified ball centered at the equilibrium point. Hence, the ε–δ requirement for stability is satisfied. The equilibrium point, however, is not asymptotically stable since trajectories starting off the equilibrium point do not tend to it eventually. Instead, they remain in their closed orbits. When friction is taken into consideration ($b > 0$), the equilibrium point at the origin becomes a stable focus. Inspection of the phase portrait of a stable focus shows that the ε–δ requirement for stability is satisfied. In addition, trajectories starting close to the equilibrium point tend to it as t tends to ∞. The second equilibrium point at $x_1 = \pi$ is a saddle point. Clearly the ε–δ requirement cannot be satisfied since, for any $\varepsilon > 0$, there is always a trajectory that will leave the ball $\{x \in R^n \mid \|x - \bar{x}\| \le \varepsilon\}$ even when $x(0)$ is arbitrarily close to the equilibrium point $\bar{x}$.

Implicit in Definition 4.1 is a requirement that solutions of (4.1) be defined for all $t \ge 0$.[1] Such global existence of the solution is not guaranteed by the local Lipschitz property of f. It will be shown, however, that the additional conditions needed in Lyapunov's theorem will ensure global existence of the solution. This will come as an application of Theorem 3.3.

Having defined stability and asymptotic stability of equilibrium points, our task now is to find ways to determine stability. The approach we used in the pendulum example relied on our knowledge of the phase portrait of the pendulum equation. Trying to generalize that approach amounts to actually finding all solutions of (4.1), which may be difficult or even impossible. However, the conclusions we reached about the stable equilibrium point of the pendulum can also be reached by using energy concepts. Let us define the energy of the pendulum $E(x)$ as the sum of its potential and kinetic energies, with the reference of the potential energy chosen such that $E(0) = 0$; that is,

$$E(x) = \int_0^{x_1} a\sin y\ dy + \tfrac{1}{2}x_2^2 = a(1 - \cos x_1) + \tfrac{1}{2}x_2^2$$

When friction is neglected ($b = 0$), the system is conservative; that is, there is no dissipation of energy. Hence, $E =$ constant during the motion of the system or, in

[1] It is possible to change the definition to alleviate the implication of global existence of the solution. In [154], stability is defined on the maximal interval of existence $[0, t_1)$, without assuming that $t_1 = \infty$.

other words, $dE/dt = 0$ along the trajectories of the system. Since $E(x) = c$ forms a closed contour around $x = 0$ for small c, we can again arrive at the conclusion that $x = 0$ is a stable equilibrium point. When friction is accounted for $(b > 0)$, energy will dissipate during the motion of the system, that is, $dE/dt \leq 0$ along the trajectories of the system. Due to friction, E cannot remain constant indefinitely while the system is in motion. Hence, it keeps decreasing until it eventually reaches zero, showing that the trajectory tends to $x = 0$ as t tends to ∞. Thus, by examining the derivative of E along the trajectories of the system, it is possible to determine the stability of the equilibrium point. In 1892, Lyapunov showed that certain other functions could be used instead of energy to determine stability of an equilibrium point. Let $V : D \to R$ be a continuously differentiable function defined in a domain $D \subset R^n$ that contains the origin. The derivative of V along the trajectories of (4.1), denoted by $\dot{V}(x)$, is given by

$$\begin{aligned} \dot{V}(x) &= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) \\ &= \left[\frac{\partial V}{\partial x_1}, \ \frac{\partial V}{\partial x_2}, \ \ldots, \ \frac{\partial V}{\partial x_n} \right] \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} = \frac{\partial V}{\partial x} f(x) \end{aligned}$$

The derivative of V along the trajectories of a system is dependent on the system's equation. Hence, $\dot{V}(x)$ will be different for different systems. If $\phi(t; x)$ is the solution of (4.1) that starts at initial state x at time $t = 0$, then

$$\dot{V}(x) = \left. \frac{d}{dt} V(\phi(t; x)) \right|_{t=0}$$

Therefore, if $\dot{V}(x)$ is negative, V will decrease along the solution of (4.1). We are now ready to state Lyapunov's stability theorem.

Theorem 4.1 *Let $x = 0$ be an equilibrium point for (4.1) and $D \subset R^n$ be a domain containing $x = 0$. Let $V : D \to R$ be a continuously differentiable function such that*

$$V(0) = 0 \ \text{ and } \ V(x) > 0 \text{ in } D - \{0\} \tag{4.2}$$

$$\dot{V}(x) \leq 0 \text{ in } D \tag{4.3}$$

Then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \text{ in } D - \{0\} \tag{4.4}$$

then $x = 0$ is asymptotically stable. ◇

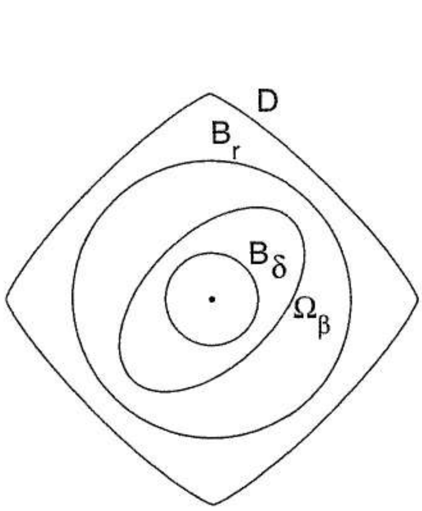


Figure 4.1: Geometric representation of sets in the proof of Theorem 4.1.

Proof: Given $\varepsilon > 0$, choose $r \in (0, \varepsilon]$ such that

$$B_r = \{x \in R^n \mid \|x\| \leq r\} \subset D$$

Let $\alpha = \min_{\|x\|=r} V(x)$. Then, $\alpha > 0$ by (4.2). Take $\beta \in (0, \alpha)$ and let

$$\Omega_\beta = \{x \in B_r \mid V(x) \leq \beta\}$$

Then, Ω_β is in the interior of B_r.[2] (See Figure 4.1.) The set Ω_β has the property that any trajectory starting in Ω_β at $t = 0$ stays in Ω_β for all $t \geq 0$. This follows from (4.3) since

$$\dot{V}(x(t)) \leq 0 \ \Rightarrow \ V(x(t)) \leq V(x(0)) \leq \beta, \ \forall \ t \geq 0$$

Because Ω_β is a compact set,[3] we conclude from Theorem 3.3 that (4.1) has a unique solution defined for all $t \geq 0$ whenever $x(0) \in \Omega_\beta$. As $V(x)$ is continuous and $V(0) = 0$, there is $\delta > 0$ such that

$$\|x\| \leq \delta \ \Rightarrow \ V(x) < \beta$$

Then,

$$B_\delta \subset \Omega_\beta \subset B_r$$

and

$$x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in B_r$$

Therefore,

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < r \leq \varepsilon, \ \ \forall \ t \geq 0$$

[2]This fact can be shown by contradiction. Suppose Ω_β is not in the interior of B_r, then there is a point $p \in \Omega_\beta$ that lies on the boundary of B_r. At this point, $V(p) \geq \alpha > \beta$, but for all $x \in \Omega_\beta$, $V(x) \leq \beta$, which is a contradiction.

[3]Ω_β is closed by definition and bounded, since it is contained in B_r.

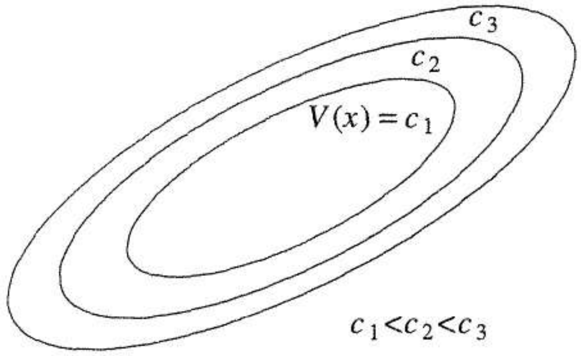


Figure 4.2: Level surfaces of a Lyapunov function.

which shows that the equilibrium point $x = 0$ is stable. Now, assume that (4.4) holds as well. To show asymptotic stability, we need to show that $x(t) \to 0$ as $t \to \infty$; that is, for every $a > 0$, there is $T > 0$ such that $\|x(t)\| < a$, for all $t > T$. By repetition of previous arguments, we know that for every $a > 0$, we can choose $b > 0$ such that $\Omega_b \subset B_a$. Therefore, it is sufficient to show that $V(x(t)) \to 0$ as $t \to \infty$. Since $V(x(t))$ is monotonically decreasing and bounded from below by zero,

$$V(x(t)) \to c \geq 0 \;\text{ as }\; t \to \infty$$

To show that $c = 0$, we use a contradiction argument. Suppose $c > 0$. By continuity of $V(x)$, there is $d > 0$ such that $B_d \subset \Omega_c$. The limit $V(x(t)) \to c > 0$ implies that the trajectory $x(t)$ lies outside the ball B_d for all $t \geq 0$. Let $-\gamma = \max_{d \leq \|x\| \leq r} \dot{V}(x)$, which exists because the continuous function $\dot{V}(x)$ has a maximum over the compact set $\{d \leq \|x\| \leq r\}$.[4] By (4.4), $-\gamma < 0$. It follows that

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))\, d\tau \leq V(x(0)) - \gamma t$$

Since the right-hand side will eventually become negative, the inequality contradicts the assumption that $c > 0$. □

A continuously differentiable function $V(x)$ satisfying (4.2) and (4.3) is called a *Lyapunov function.* The surface $V(x) = c$, for some $c > 0$, is called a *Lyapunov surface* or a *level surface.* Using Lyapunov surfaces, we notice that Figure 4.2 makes the theorem intuitively clear. It shows Lyapunov surfaces for increasing values of c. The condition $\dot{V} \leq 0$ implies that when a trajectory crosses a Lyapunov surface $V(x) = c$, it moves inside the set $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$ and can never come out again. When $\dot{V} < 0$, the trajectory moves from one Lyapunov surface to an inner Lyapunov surface with a smaller c. As c decreases, the Lyapunov surface $V(x) = c$ shrinks to the origin, showing that the trajectory approaches the origin as

[4] See [10, Theorem 4-20].

time progresses. If we only know that $\dot{V} \leq 0$, we cannot be sure that the trajectory will approach the origin,[5] but we can conclude that the origin is stable since the trajectory can be contained inside any ball B_ε by requiring the initial state $x(0)$ to lie inside a Lyapunov surface contained in that ball.

A function $V(x)$ satisfying condition (4.2)—that is, $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$—is said to be *positive definite.* If it satisfies the weaker condition $V(x) \geq 0$ for $x \neq 0$, it is said to be *positive semidefinite.* A function $V(x)$ is said to be *negative definite* or *negative semidefinite* if $-V(x)$ is positive definite or positive semidefinite, respectively. If $V(x)$ does not have a definite sign as per one of these four cases, it is said to be indefinite. With this terminology, we can rephrase Lyapunov's theorem to say that *the origin is stable if there is a continuously differentiable positive definite function $V(x)$ so that $\dot{V}(x)$ is negative semidefinite, and it is asymptotically stable if $\dot{V}(x)$ is negative definite.*

A class of scalar functions for which sign definiteness can be easily checked is the class of functions of the quadratic form

$$V(x) = x^T P x = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j$$

where P is a real symmetric matrix. In this case, $V(x)$ is positive definite (positive semidefinite) if and only if all the eigenvalues of P are positive (nonnegative), which is true if and only if all the leading principal minors of P are positive (all principal minors of P are nonnegative).[6] If $V(x) = x^T P x$ is positive definite (positive semidefinite), we say that the matrix P is positive definite (positive semidefinite) and write $P > 0$ ($P \geq 0$).

Example 4.1 Consider

$$\begin{aligned} V(x) &= a x_1^2 + 2x_1 x_3 + a x_2^2 + 4 x_2 x_3 + a x_3^2 \\ &= [x_1 \ x_2 \ x_3] \begin{bmatrix} a & 0 & 1 \\ 0 & a & 2 \\ 1 & 2 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x^T P x \end{aligned}$$

The leading principal minors of P are a, a^2, and $a(a^2-5)$. Therefore, $V(x)$ is positive definite if $a > \sqrt{5}$. For negative definiteness, the leading principal minors of $-P$ should be positive; that is, the leading principal minors of P should have alternating signs, with the odd-numbered minors being negative and the even-numbered minors being positive. Consequently, $V(x)$ is negative definite if $a < -\sqrt{5}$. By calculating all principal minors, it can be seen that $V(x)$ is positive semidefinite if $a \geq \sqrt{5}$ and negative semidefinite if $a \leq -\sqrt{5}$. For $a \in (-\sqrt{5}, \sqrt{5})$, $V(x)$ is indefinite. $\triangle$

Lyapunov's theorem can be applied without solving the differential equation (4.1). On the other hand, there is no systematic method for finding Lyapunov

[5] See, however, LaSalle's theorem in Section 4.2

[6] This is a well-known fact in matrix theory. Its proof can be found in [21] or [63].

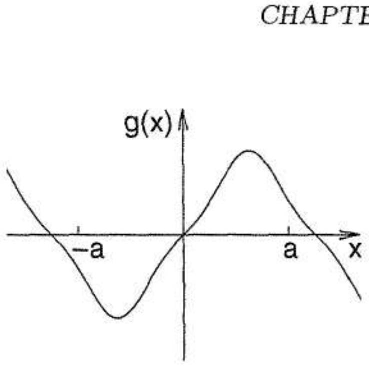


Figure 4.3: A possible nonlinearity in Example 4.2.

functions. In some cases, there are natural Lyapunov function candidates like energy functions in electrical or mechanical systems. In other cases, it is basically a matter of trial and error. The situation, however, is not as bad as it might seem. As we go over various examples and applications throughout the book, some ideas and approaches for searching for Lyapunov functions will be delineated.

Example 4.2 Consider the first-order differential equation

$$\dot{x} = -g(x)$$

where $g(x)$ is locally Lipschitz on $(-a, a)$ and satisfies

$$g(0) = 0; \quad xg(x) > 0, \quad \forall\ x \neq 0 \ \text{ and } \ x \in (-a, a)$$

A sketch of a possible $g(x)$ is shown in Figure 4.3. The system has an isolated equilibrium point at the origin. It is not difficult in this simple example to see that the origin is asymptotically stable, because solutions starting on either side of the origin will have to move toward the origin due to the sign of the derivative $\dot{x}$. To arrive at the same conclusion using Lyapunov's theorem, consider the function

$$V(x) = \int_0^x g(y)\ dy$$

Over the domain $D = (-a, a)$, $V(x)$ is continuously differentiable, $V(0) = 0$, and $V(x) > 0$ for all $x \neq 0$. Thus, $V(x)$ is a valid Lyapunov function candidate. To see whether or not $V(x)$ is indeed a Lyapunov function, we calculate its derivative along the trajectories of the system.

$$\dot{V}(x) = \frac{\partial V}{\partial x}[-g(x)] = -g^2(x) < 0, \ \forall\ x \in D - \{0\}$$

Hence, by Theorem 4.1 we conclude that the origin is asymptotically stable. $\triangle$

Example 4.3 Consider the pendulum equation without friction, namely,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\ a \sin x_1 \end{aligned}$$

and let us study the stability of the equilibrium point at the origin. A natural Lyapunov function candidate is the energy function

$$V(x) = a(1 - \cos x_1) + \tfrac{1}{2}x_2^2$$

Clearly, $V(0) = 0$ and $V(x)$ is positive definite over the domain $-2\pi < x_1 < 2\pi$. The derivative of $V(x)$ along the trajectories of the system is given by

$$\dot{V}(x) = a\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = ax_2 \sin x_1 - ax_2 \sin x_1 = 0$$

Thus, conditions (4.2) and (4.3) of Theorem 4.1 are satisfied, and we conclude that the origin is stable. Since $\dot{V}(x) \equiv 0$, we can also conclude that the origin is not asymptotically stable; for trajectories starting on a Lyapunov surface $V(x) = c$ remain on the same surface for all future time. $\triangle$

Example 4.4 Consider again the pendulum equation, but this time with friction, namely,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\,a \sin x_1 - bx_2 \end{aligned}$$

Again, let us try $V(x) = a(1 - \cos x_1) + (1/2)x_2^2$ as a Lyapunov function candidate.

$$\dot{V}(x) = a\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = -\,bx_2^2$$

The derivative $\dot{V}(x)$ is negative semidefinite. It is not negative definite because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1; that is, $\dot{V}(x) = 0$ along the x_1-axis. Therefore, we can only conclude that the origin is stable. However, using the phase portrait of the pendulum equation, we have seen that when $b > 0$, the origin is asymptotically stable. The energy Lyapunov function fails to show this fact. We will see later in Section 4.2 that LaSalle's theorem will enable us to arrive at a different conclusion. For now, let us look for a Lyapunov function $V(x)$ that would have a negative definite $\dot{V}(x)$. Starting from the energy Lyapunov function, let us replace the term $(1/2)x_2^2$ by the more general quadratic form $(1/2)x^TPx$ for some 2×2 positive definite matrix P:

$$\begin{aligned} V(x) &= \tfrac{1}{2}x^TPx + a(1 - \cos x_1) \\ &= \tfrac{1}{2}[x_1\ x_2]\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + a(1 - \cos x_1) \end{aligned}$$

For the quadratic form $(1/2)x^TPx$ to be positive definite, the elements of the matrix P must satisfy

$$p_{11} > 0, \quad p_{11}p_{22} - p_{12}^2 > 0$$

The derivative $\dot{V}(x)$ is given by

$$\begin{aligned} \dot{V}(x) &= (p_{11}x_1 + p_{12}x_2 + a \sin x_1)\, x_2 + (p_{12}x_1 + p_{22}x_2)\,(-a \sin x_1 - bx_2) \\ &= a(1 - p_{22})x_2 \sin x_1 - ap_{12}x_1 \sin x_1 + (p_{11} - p_{12}b)\, x_1x_2 + (p_{12} - p_{22}b)\, x_2^2 \end{aligned}$$

Now we want to choose p_{11}, p_{12}, and p_{22} such that $\dot{V}(x)$ is negative definite. Since the cross product terms $x_2 \sin x_1$ and $x_1 x_2$ are sign indefinite, we will cancel them by taking $p_{22} = 1$ and $p_{11} = bp_{12}$. With these choices, p_{12} must satisfy $0 < p_{12} < b$ for $V(x)$ to be positive definite. Let us take $p_{12} = b/2$. Then, $\dot{V}(x)$ is given by

$$\dot{V}(x) = -\ \tfrac{1}{2} a b x_1 \sin x_1 - \tfrac{1}{2} b x_2^2$$

The term $x_1 \sin x_1 > 0$ for all $0 < |x_1| < \pi$. Taking $D = \{x \in R^2 \mid |x_1| < \pi\}$, we see that $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite over D. Thus, by Theorem 4.1, we conclude that the origin is asymptotically stable. △

This example emphasizes an important feature of Lyapunov's stability theorem; namely, *the theorem's conditions are only sufficient.* Failure of a Lyapunov function candidate to satisfy the conditions for stability or asymptotic stability does not mean that the equilibrium is not stable or asymptotically stable. It only means that such stability property cannot be established by using this Lyapunov function candidate. Whether the equilibrium point is stable (asymptotically stable) or not can be determined only by further investigation.

In searching for a Lyapunov function in Example 4.4, we approached the problem in a backward manner. We investigated an expression for the derivative $\dot{V}(x)$ and went back to choose the parameters of $V(x)$ so as to make $\dot{V}(x)$ negative definite. This is a useful idea in searching for a Lyapunov function. A procedure that exploits this idea is known as the *variable gradient method.* To describe the procedure, let $V(x)$ be a scalar function of x and $g(x) = \nabla V = (\partial V/\partial x)^T$. The derivative $\dot{V}(x)$ along the trajectories of (4.1) is given by

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = g^T(x) f(x)$$

The idea now is to try to choose $g(x)$ such that it would be the gradient of a positive definite function $V(x)$ and, at the same time, $\dot{V}(x)$ would be negative definite. It is not difficult (Exercise 4.5) to verify that $g(x)$ is the gradient of a scalar function if and only if the Jacobian matrix $[\partial g/\partial x]$ is symmetric; that is,

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \quad \forall\ i, j = 1, \ldots, n$$

Under this constraint, we start by choosing $g(x)$ such that $g^T(x) f(x)$ is negative definite. The function $V(x)$ is then computed from the integral

$$V(x) = \int_0^x g^T(y)\ dy = \int_0^x \sum_{i=1}^n g_i(y)\ dy_i$$

The integration is taken over any path joining the origin to x.[7] Usually, this is done along the axes; that is,

$$V(x) \quad = \quad \int_0^{x_1} g_1(y_1, 0, \ldots, 0)\ dy_1 + \int_0^{x_2} g_2(x_1, y_2, 0, \ldots, 0)\ dy_2$$

[7]The line integral of a gradient vector is independent of the path. (See [10, Theorem 10-37].)

$$+ \cdots + \int_0^{x_n} g_n(x_1, x_2, \ldots, x_{n-1}, y_n) \, dy_n$$

By leaving some parameters of $g(x)$ undetermined, one would try to choose them to ensure that $V(x)$ is positive definite. The variable gradient method can be used to arrive at the Lyapunov function of Example 4.4. Instead of repeating the example, we illustrate the method on a slightly more general system.

Example 4.5 Consider the second-order system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1) - ax_2 \end{aligned}$$

where $a > 0$, $h(\cdot)$ is locally Lipschitz, $h(0) = 0$, and $yh(y) > 0$ for all $y \neq 0$, $y \in (-b, c)$ for some positive constants b and c. The pendulum equation is a special case of this system. To apply the variable gradient method, we want to choose a second-order vector $g(x)$ that satisfies

$$\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1}$$

$$\dot{V}(x) = g_1(x)x_2 - g_2(x)[h(x_1) + ax_2] < 0, \quad \text{for } x \neq 0$$

and

$$V(x) = \int_0^x g^T(y) \, dy > 0, \quad \text{for } x \neq 0$$

Let us try

$$g(x) = \begin{bmatrix} \alpha(x)x_1 + \beta(x)x_2 \\ \gamma(x)x_1 + \delta(x)x_2 \end{bmatrix}$$

where the scalar functions $\alpha(\cdot)$, $\beta(\cdot)$, $\gamma(\cdot)$, and $\delta(\cdot)$ are to be determined. To satisfy the symmetry requirement, we must have

$$\beta(x) + \frac{\partial \alpha}{\partial x_2} x_1 + \frac{\partial \beta}{\partial x_2} x_2 = \gamma(x) + \frac{\partial \gamma}{\partial x_1} x_1 + \frac{\partial \delta}{\partial x_1} x_2$$

The derivative $\dot{V}(x)$ is given by

$$\dot{V}(x) = \alpha(x)x_1x_2 + \beta(x)x_2^2 - a\gamma(x)x_1x_2 - a\delta(x)x_2^2 - \delta(x)x_2h(x_1) - \gamma(x)x_1h(x_1)$$

To cancel the cross-product terms, we choose

$$\alpha(x)x_1 - a\gamma(x)x_1 - \delta(x)h(x_1) = 0$$

so that

$$\dot{V}(x) = -[a\delta(x) - \beta(x)]x_2^2 - \gamma(x)x_1h(x_1)$$

To simplify our choices, let us take $\delta(x) = \delta$ = constant, $\gamma(x) = \gamma$ = constant, and $\beta(x) = \beta$ = constant. Then, $\alpha(x)$ depends only on x_1, and the symmetry requirement is satisfied by choosing $\beta = \gamma$. The expression for $g(x)$ reduces to

$$g(x) = \begin{bmatrix} a\gamma x_1 + \delta h(x_1) + \gamma x_2 \\ \gamma x_1 + \delta x_2 \end{bmatrix}$$

By integration, we obtain

$$\begin{aligned} V(x) &= \int_0^{x_1} [a\gamma y_1 + \delta h(y_1)]\, dy_1 + \int_0^{x_2} (\gamma x_1 + \delta y_2)\, dy_2 \\ &= \tfrac{1}{2} a\gamma x_1^2 + \delta \int_0^{x_1} h(y)\, dy + \gamma x_1 x_2 + \tfrac{1}{2}\delta x_2^2 \;=\; \tfrac{1}{2} x^T P x + \delta \int_0^{x_1} h(y)\, dy \end{aligned}$$

where

$$P = \begin{bmatrix} a\gamma & \gamma \\ \gamma & \delta \end{bmatrix}$$

Choosing $\delta > 0$ and $0 < \gamma < a\delta$ ensures that $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite. For example, taking $\gamma = ak\delta$ for $0 < k < 1$ yields the Lyapunov function

$$V(x) = \frac{\delta}{2} x^T \begin{bmatrix} ka^2 & ka \\ ka & 1 \end{bmatrix} x + \delta \int_0^{x_1} h(y)\, dy$$

which satisfies conditions (4.2) and (4.4) of Theorem 4.1 over the domain $D = \{x \in R^2 \mid -b < x_1 < c\}$. Therefore, the origin is asymptotically stable. △

When the origin $x = 0$ is asymptotically stable, we are often interested in determining how far from the origin the trajectory can be and still converge to the origin as t approaches ∞. This gives rise to the definition of the *region of attraction* (also called *region of asymptotic stability, domain of attraction*, or *basin*). Let $\phi(t; x)$ be the solution of (4.1) that starts at initial state x at time $t = 0$. Then, the region of attraction is defined as the set of all points x such that $\phi(t; x)$ is defined for all $t \geq 0$ and $\lim_{t\to\infty} \phi(t; x) = 0$. Finding the exact region of attraction analytically might be difficult or even impossible. However, Lyapunov functions can be used to estimate the region of attraction, that is, to find sets contained in the region of attraction. From the proof of Theorem 4.1, we see that if there is a Lyapunov function that satisfies the conditions of asymptotic stability over a domain D and, if $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$ is bounded and contained in D, then every trajectory starting in Ω_c remains in Ω_c and approaches the origin as $t \to \infty$. Thus, Ω_c is an estimate of the region of attraction. The estimate, however, may be conservative; that is, it may be much smaller than the actual region of attraction. In Section 8.2, we will solve examples on estimating the region of attraction and see some ideas to enlarge the estimates. Here, we want to pursue another question: Under what conditions will the region of attraction be the whole space R^n? It will be the case if we can show that for any initial state x, the trajectory $\phi(t; x)$ approaches the origin

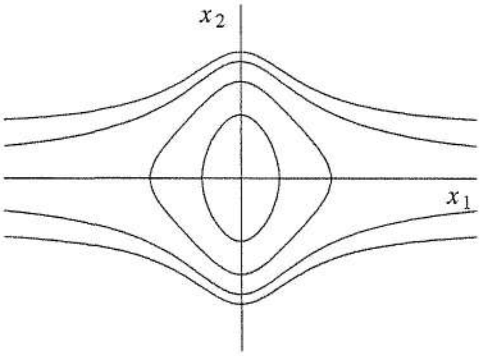


Figure 4.4: Lyapunov surfaces for $V(x) = x_1^2/(1+x_1^2) + x_2^2$.

as $t \to \infty$, no matter how large $\|x\|$ is. If an asymptotically stable equilibrium point at the origin has this property, it is said to be *globally asymptotically stable.* Recalling again the proof of Theorem 4.1, we can see that global asymptotic stability can be established if any point $x \in R^n$ can be included in the interior of a bounded set Ω_c. It is obvious that for this condition to hold, the conditions of the theorem must hold globally, that is, $D = R^n$; but, is that enough? It turns out that we need more conditions to ensure that any point in R^n can be included in a bounded set Ω_c. The problem is that for large c, the set Ω_c need not be bounded. Consider, for example, the function

$$V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$$

Figure 4.4 shows the surfaces $V(x) = c$ for various positive values of c. For small c, the surface $V(x) = c$ is closed; hence, Ω_c is bounded since it is contained in a closed ball B_r for some $r > 0$. This is a consequence of the continuity and positive definiteness of $V(x)$. As c increases, a value is reached after which the surface $V(x) = c$ is open and Ω_c is unbounded. For Ω_c to be in the interior of a ball B_r, c must satisfy $c < \inf_{\|x\| \geq r} V(x)$. If

$$l = \lim_{r\to\infty} \inf_{\|x\| \geq r} V(x) < \infty$$

then Ω_c will be bounded if $c < l$. In the preceding example,

$$l = \lim_{r\to\infty} \min_{\|x\|=r} \left[\frac{x_1^2}{1+x_1^2} + x_2^2\right] = \lim_{|x_1|\to\infty} \frac{x_1^2}{1+x_1^2} = 1$$

Thus, Ω_c is bounded only for $c < 1$. An extra condition that ensures that Ω_c is bounded for all values of $c > 0$ is

$$V(x) \to \infty \ \text{ as } \ \|x\| \to \infty$$

A function satisfying this condition is said to be *radially unbounded.*

Theorem 4.2 *Let $x = 0$ be an equilibrium point for (4.1). Let $V : R^n \to R$ be a continuously differentiable function such that*

$$V(0) = 0 \text{ and } V(x) > 0, \quad \forall\, x \neq 0 \tag{4.5}$$

$$\|x\| \to \infty \;\Rightarrow\; V(x) \to \infty \tag{4.6}$$

$$\dot{V}(x) < 0, \quad \forall\, x \neq 0 \tag{4.7}$$

then $x = 0$ is globally asymptotically stable. ◇

Proof: Given any point $p \in R^n$, let $c = V(p)$. Condition (4.6) implies that for any $c > 0$, there is $r > 0$ such that $V(x) > c$ whenever $\|x\| > r$. Thus, $\Omega_c \subset B_r$, which implies that Ω_c is bounded. The rest of the proof is similar to that of Theorem 4.1. □

Theorem 4.2 is known as Barbashin–Krasovskii theorem. Exercise 4.8 gives a counterexample to show that the radial unboundedness condition of the theorem is indeed needed.

Example 4.6 Consider again the system of Example 4.5, but this time, assume that the condition $yh(y) > 0$ holds for all $y \neq 0$. The Lyapunov function

$$V(x) = \frac{\delta}{2} x^T \begin{bmatrix} ka^2 & ka \\ ka & 1 \end{bmatrix} x + \delta \int_0^{x_1} h(y)\; dy$$

is positive definite for all $x \in R^2$ and radially unbounded. The derivative

$$\dot{V}(x) = -a\delta(1-k)x_2^2 - a\delta k x_1 h(x_1)$$

is negative definite for all $x \in R^2$ since $0 < k < 1$. Therefore, the origin is globally asymptotically stable. △

If the origin $x = 0$ is a globally asymptotically stable equilibrium point of a system, then it must be the unique equilibrium point of the system. For if there were another equilibrium point $\bar{x}$, the trajectory starting at $\bar{x}$ would remain at $\bar{x}$ for all $t \geq 0$; hence, it would not approach the origin, which contradicts the claim that the origin is globally asymptotically stable. Therefore, global asymptotic stability is not studied for multiple equilibria systems like the pendulum equation.

Theorems 4.1 and 4.2 are concerned with establishing the stability or asymptotic stability of an equilibrium point. There are also instability theorems for establishing that an equilibrium point is unstable. The most powerful of these theorems is Chetaev's theorem, which will be stated as Theorem 4.3. Before we state the theorem, let us introduce some terminology that will be used in the theorem's statement. Let $V : D \to R$ be a continuously differentiable function on a domain $D \subset R^n$ that contains the origin $x = 0$. Suppose $V(0) = 0$ and there is a point x_0

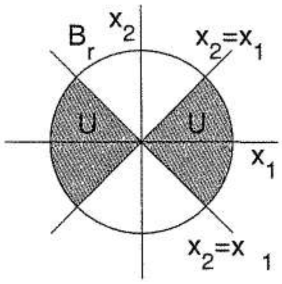


Figure 4.5: The set U for $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$.

arbitrarily close to the origin such that $V(x_0) > 0$. Choose $r > 0$ such that the ball $B_r = \{x \in R^n \mid \|x\| \leq r\}$ is contained in D, and let

$$U = \{x \in B_r \mid V(x) > 0\} \tag{4.8}$$

The set U is a nonempty set contained in B_r. Its boundary is the surface $V(x) = 0$ and the sphere $\|x\| = r$. Since $V(0) = 0$, the origin lies on the boundary of U inside B_r. Notice that U may contain more than one component. For example, Figure 4.5 shows the set U for $V(x) = (x_1^2 - x_2^2)/2$. The set U can be always constructed provided that $V(0) = 0$ and $V(x_0) > 0$ for some x_0 arbitrarily close to the origin.

Theorem 4.3 *Let $x = 0$ be an equilibrium point for (4.1). Let $V : D \to R$ be a continuously differentiable function such that $V(0) = 0$ and $V(x_0) > 0$ for some x_0 with arbitrarily small $\|x_0\|$. Define a set U as in (4.8) and suppose that $\dot{V}(x) > 0$ in U. Then, $x = 0$ is unstable.* ◇

Proof: The point x_0 is in the interior of U and $V(x_0) = a > 0$. The trajectory $x(t)$ starting at $x(0) = x_0$ must leave the set U. To see this point, notice that as long as $x(t)$ is inside U, $V(x(t)) \geq a$, since $\dot{V}(x) > 0$ in U. Let

$$\gamma = \min\{\dot{V}(x) \mid x \in U \text{ and } V(x) \geq a\}$$

which exists since the continuous function $\dot{V}(x)$ has a minimum over the compact set $\{x \in U \text{ and } V(x) \geq a\} = \{x \in B_r \text{ and } V(x) \geq a\}$.[8] Then, $\gamma > 0$ and

$$V(x(t)) = V(x_0) + \int_0^t \dot{V}(x(s))\, ds \geq a + \int_0^t \gamma\, ds = a + \gamma t$$

This inequality shows that $x(t)$ cannot stay forever in U because $V(x)$ is bounded on U. Now, $x(t)$ cannot leave U through the surface $V(x) = 0$ since $V(x(t)) \geq a$. Hence, it must leave U through the sphere $\|x\| = r$. Because this can happen for an arbitrarily small $\|x_0\|$, the origin is unstable. □

There are other instability theorems that were proved before Chetaev's theorem, but they are corollaries of the theorem. (See Exercises 4.11 and 4.12.)

[8] See [10, Theorem 4-20].

Example 4.7 Consider the second-order system

$$\begin{aligned} \dot{x}_1 &= x_1 + g_1(x) \\ \dot{x}_2 &= -x_2 + g_2(x) \end{aligned}$$

where $g_1(\cdot)$ and $g_2(\cdot)$ are locally Lipschitz functions that satisfy the inequalities

$$|g_1(x)| \le k\|x\|_2^2, \qquad |g_2(x)| \le k\|x\|_2^2$$

in a neighborhood D of the origin. These inequalities imply that $g_1(0) = g_2(0) = 0$. Hence, the origin is an equilibrium point. Consider the function

$$V(x) = \tfrac{1}{2}(x_1^2 - x_2^2)$$

On the line $x_2 = 0$, $V(x) > 0$ at points arbitrarily close to the origin. The set U is shown in Figure 4.5. The derivative of $V(x)$ along the trajectories of the system is given by

$$\dot{V}(x) = x_1^2 + x_2^2 + x_1 g_1(x) - x_2 g_2(x)$$

The magnitude of the term $x_1 g_1(x) - x_2 g_2(x)$ satisfies the inequality

$$|x_1 g_1(x) - x_2 g_2(x)| \le \sum_{i=1}^{2} |x_i| \cdot |g_i(x)| \le 2k\|x\|_2^3$$

Hence,

$$\dot{V}(x) \ge \|x\|_2^2 - 2k\|x\|_2^3 = \|x\|_2^2(1 - 2k\|x\|_2)$$

Choosing r such that $B_r \subset D$ and $r < 1/(2k)$, it is seen that all the conditions of Theorem 4.3 are satisfied. Therefore, the origin is unstable. △

4.2 The Invariance Principle

In our study of the pendulum equation with friction (Example 4.4), we saw that the energy Lyapunov function fails to satisfy the asymptotic stability condition of Theorem 4.1 because $\dot{V}(x) = -bx_2^2$ is only negative semidefinite. Notice, however, that $\dot{V}(x)$ is negative everywhere, except on the line $x_2 = 0$, where $\dot{V}(x) = 0$. For the system to maintain the $\dot{V}(x) = 0$ condition, the trajectory of the system must be confined to the line $x_2 = 0$. Unless $x_1 = 0$, this is impossible because from the pendulum equation

$$x_2(t) \equiv 0 \;\Rightarrow\; \dot{x}_2(t) \equiv 0 \;\Rightarrow\; \sin x_1(t) \equiv 0$$

Hence, on the segment $-\pi < x_1 < \pi$ of the $x_2 = 0$ line, the system can maintain the $\dot{V}(x) = 0$ condition only at the origin $x = 0$. Therefore, $V(x(t))$ must decrease toward 0 and, consequently, $x(t) \to 0$ as $t \to \infty$, which is consistent with the fact that, due to friction, energy cannot remain constant while the system is in motion.

The foregoing argument shows, formally, that if in a domain about the origin we can find a Lyapunov function whose derivative along the trajectories of the system is negative semidefinite, and if we can establish that no trajectory can stay identically at points where $\dot{V}(x) = 0$, except at the origin, then the origin is asymptotically stable. This idea follows from LaSalle's *invariance principle*, which is the subject of this section. To state and prove LaSalle's invariance theorem, we need to introduce a few definitions. Let $x(t)$ be a solution of (4.1). A point p is said to be a *positive limit point* of $x(t)$ if there is a sequence $\{t_n\}$, with $t_n \to \infty$ as $n \to \infty$, such that $x(t_n) \to p$ as $n \to \infty$. The set of all positive limit points of $x(t)$ is called the *positive limit set* of $x(t)$. A set M is said to be an *invariant set* with respect to (4.1) if

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall\, t \in R$$

That is, if a solution belongs to M at some time instant, then it belongs to M for all future and past time. A set M is said to be a *positively invariant set* if

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall\, t \geq 0$$

We also say that $x(t)$ approaches a set M as t approaches infinity, if for each $\varepsilon > 0$ there is $T > 0$ such that

$$\text{dist}(x(t), M) < \varepsilon, \quad \forall\, t > T$$

where $\text{dist}(p, M)$ denotes the distance from a point p to a set M, that is, the smallest distance from p to any point in M. More precisely,

$$\text{dist}(p, M) = \inf_{x \in M} \|p - x\|$$

These few concepts can be illustrated by examining an asymptotically stable equilibrium point and a stable limit cycle in the plane. The asymptotically stable equilibrium is the positive limit set of every solution starting sufficiently near the equilibrium point. The stable limit cycle is the positive limit set of every solution starting sufficiently near the limit cycle. The solution approaches the limit cycle as $t \to \infty$. Notice, however, that the solution does not approach any specific point on the limit cycle. In other words, the statement $x(t)$ approaches M as $t \to \infty$ does not imply that $\lim_{t\to\infty} x(t)$ exists. The equilibrium point and the limit cycle are invariant sets, since any solution starting in either set remains in the set for all $t \in R$. The set $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$ with $\dot{V}(x) \leq 0$ for all $x \in \Omega_c$ is a positively invariant set since, as we saw in the proof of Theorem 4.1, a solution starting in Ω_c remains in Ω_c for all $t \geq 0$.

A fundamental property of limit sets is stated in the next lemma, whose proof is given in Appendix C.3.

Lemma 4.1 *If a solution $x(t)$ of (4.1) is bounded and belongs to D for $t \geq 0$, then its positive limit set L^+ is a nonempty, compact, invariant set. Moreover, $x(t)$ approaches L^+ as $t \to \infty$.* ◇

We are now ready to state LaSalle's theorem.

Theorem 4.4 *Let $\Omega \subset D$ be a compact set that is positively invariant with respect to* (4.1). *Let $V : D \to R$ be a continuously differentiable function such that $\dot{V}(x) \le 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.* ◇

Proof: Let $x(t)$ be a solution of (4.1) starting in Ω. Since $\dot{V}(x) \le 0$ in Ω, $V(x(t))$ is a decreasing function of t. Since $V(x)$ is continuous on the compact set Ω, it is bounded from below on Ω. Therefore, $V(x(t))$ has a limit a as $t \to \infty$. Note also that the positive limit set L^+ is in Ω because Ω is a closed set. For any $p \in L^+$, there is a sequence t_n with $t_n \to \infty$ and $x(t_n) \to p$ as $n \to \infty$. By continuity of $V(x)$, $V(p) = \lim_{n\to\infty} V(x(t_n)) = a$. Hence, $V(x) = a$ on L^+. Since (by Lemma 4.1) L^+ is an invariant set, $\dot{V}(x) = 0$ on L^+. Thus,

$$L^+ \subset M \subset E \subset \Omega$$

Since $x(t)$ is bounded, $x(t)$ approaches L^+ as $t \to \infty$ (by Lemma 4.1). Hence, $x(t)$ approaches M as $t \to \infty$. □

Unlike Lyapunov's theorem, Theorem 4.4 does not require the function $V(x)$ to be positive definite. Note also that the construction of the set Ω does not have to be tied in with the construction of the function $V(x)$. However, in many applications the construction of $V(x)$ will itself guarantee the existence of a set Ω. In particular, if $\Omega_c = \{x \in R^n \mid V(x) \le c\}$ is bounded and $\dot{V}(x) \le 0$ in Ω_c, then we can take $\Omega = \Omega_c$. When $V(x)$ is positive definite, Ω_c is bounded for sufficiently small $c > 0$. This is not necessarily true when $V(x)$ is not positive definite. For example, if $V(x) = (x_1 - x_2)^2$, the set Ω_c is not bounded no matter how small c is. If $V(x)$ is radially unbounded—that is, $V(x) \to \infty$ as $\|x\| \to \infty$—the set Ω_c is bounded for all values of c. This is true whether or not $V(x)$ is positive definite.

When our interest is in showing that $x(t) \to 0$ as $t \to \infty$, we need to establish that the largest invariant set in E is the origin. This is done by showing that no solution can stay identically in E, other than the trivial solution $x(t) \equiv 0$. Specializing Theorem 4.4 to this case and taking $V(x)$ to be positive definite, we obtain the following two corollaries that extend Theorems 4.1 and 4.2.[9]

Corollary 4.1 *Let $x = 0$ be an equilibrium point for* (4.1). *Let $V : D \to R$ be a continuously differentiable positive definite function on a domain D containing the origin $x = 0$, such that $\dot{V}(x) \le 0$ in D. Let $S = \{x \in D \mid \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution $x(t) \equiv 0$. Then, the origin is asymptotically stable.* ◇

[9] Corollaries 4.1 and 4.2 are known as the theorems of Barbashin and Krasovskii, who proved them before the introduction of LaSalle's invariance principle.

Corollary 4.2 *Let $x = 0$ be an equilibrium point for (4.1). Let $V : R^n \to R$ be a continuously differentiable, radially unbounded, positive definite function such that $\dot{V}(x) \le 0$ for all $x \in R^n$. Let $S = \{x \in R^n \mid \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution $x(t) \equiv 0$. Then, the origin is globally asymptotically stable.* $\diamond$

When $\dot{V}(x)$ is negative definite, $S = \{0\}$. Then, Corollaries 4.1 and 4.2 coincide with Theorems 4.1 and 4.2, respectively.

Example 4.8 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h_1(x_1) - h_2(x_2) \end{aligned}$$

where $h_1(\cdot)$ and $h_2(\cdot)$ are locally Lipschitz and satisfy

$$h_i(0) = 0, \quad y h_i(y) > 0, \quad \forall\ y \ne 0 \text{ and } y \in (-a, a)$$

The system has an isolated equilibrium point at the origin. Depending upon the functions $h_1(\cdot)$ and $h_2(\cdot)$, it might have other equilibrium points. The system can be viewed as a generalized pendulum with $h_2(x_2)$ as the friction term. Therefore, a Lyapunov function candidate may be taken as the energy-like function

$$V(x) = \int_0^{x_1} h_1(y)\ dy \ + \ \tfrac{1}{2} x_2^2$$

Let $D = \{x \in R^2 \mid -a < x_i < a\}$; $V(x)$ is positive definite in D and

$$\dot{V}(x) = h_1(x_1) x_2 + x_2[-h_1(x_1) - h_2(x_2)] = -x_2 h_2(x_2) \le 0$$

is negative semidefinite. To find $S = \{x \in D \mid \dot{V}(x) = 0\}$, note that

$$\dot{V}(x) = 0 \ \Rightarrow\ x_2 h_2(x_2) = 0 \ \Rightarrow\ x_2 = 0, \quad \text{since } -a < x_2 < a$$

Hence, $S = \{x \in D \mid x_2 = 0\}$. Let $x(t)$ be a solution that belongs identically to S:

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow h_1(x_1(t)) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

Therefore, the only solution that can stay identically in S is the trivial solution $x(t) \equiv 0$. Thus, the origin is asymptotically stable. $\triangle$

Example 4.9 Consider again the system of Example 4.8, but this time let $a = \infty$ and assume that $h_1(\cdot)$ satisfies the additional condition:

$$\int_0^y h_1(z)\ dz \ \to\ \infty \ \text{ as } \ |y| \to \infty$$

The Lyapunov function $V(x) = \int_0^{x_1} h_1(y)\ dy + (1/2)x_2^2$ is radially unbounded. Similar to the previous example, it can be shown that $\dot{V}(x) \leq 0$ in R^2, and the set

$$S = \{x \in R^2 \mid \dot{V}(x) = 0\} = \{x \in R^2 \mid x_2 = 0\}$$

contains no solutions other than the trivial solution. Hence, the origin is globally asymptotically stable. △

Not only does LaSalle's theorem relax the negative definiteness requirement of Lyapunov's theorem, but it also extends Lyapunov's theorem in three different directions. First, it gives an estimate of the region of attraction, which is not necessarily of the form $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$. The set Ω of Theorem 4.4 can be any compact positively invariant set. We will use this feature in Section 8.2 to obtain less conservative estimates of the region of attraction. Second, LaSalle's theorem can be used in cases where the system has an equilibrium set, rather than an isolated equilibrium point. This will be illustrated by an application to a simple adaptive control example from Section 1.2.6. Third, the function $V(x)$ does not have to be positive definite. The utility of this feature will be illustrated by an application to the neural network example of Section 1.2.5.

Example 4.10 Consider the first-order system

$$\dot{y} = ay + u$$

together with the adaptive control law

$$u = -ky, \qquad \dot{k} = \gamma y^2, \quad \gamma > 0$$

Taking $x_1 = y$ and $x_2 = k$, the closed-loop system is represented by

$$\begin{aligned} \dot{x}_1 &= -(x_2 - a)x_1 \\ \dot{x}_2 &= \gamma x_1^2 \end{aligned}$$

The line $x_1 = 0$ is an equilibrium set. We want to show that the trajectories approach this equilibrium set as $t \to \infty$, which means that the adaptive controller regulates y to zero. Consider the Lyapunov function candidate

$$V(x) = \tfrac{1}{2}x_1^2 + \frac{1}{2\gamma}(x_2 - b)^2$$

where $b > a$. The derivative of V along the trajectories of the system is given by

$$\dot{V}(x) = x_1\dot{x}_1 + \frac{1}{\gamma}(x_2 - b)\dot{x}_2 = -x_1^2(x_2 - a) + x_1^2(x_2 - b) = -x_1^2(b - a) \leq 0$$

Hence, $\dot{V}(x) \leq 0$. Since $V(x)$ is radially unbounded, the set $\Omega_c = \{x \in R^2 \mid V(x) \leq c\}$ is a compact, positively invariant set. Thus, taking $\Omega = \Omega_c$, all the conditions of Theorem 4.4 are satisfied. The set E is given by $E = \{x \in \Omega_c \mid x_1 = 0\}$. Because any point on the line $x_1 = 0$ is an equilibrium point, E is an invariant set. Therefore, in this example, $M = E$. From Theorem 4.4, we conclude that every trajectory starting in Ω_c approaches E as $t \to \infty$; that is, $x_1(t) \to 0$ as $t \to \infty$. Moreover, since $V(x)$ is radially unbounded, the conclusion is global; that is, it holds for all initial conditions $x(0)$ because for any $x(0)$, the constant c can be chosen large enough that $x(0) \in \Omega_c$. △

Note that the Lyapunov function in Example 4.10 is dependent on a constant b, which is required to satisfy $b > a$. Since in the adaptive control problem the constant a is not known, we may not know the constant b explicitly, but we know that it always exists. This highlights another feature of Lyapunov's method, which we have not seen before; namely, in some situations, we may be able to assert the existence of a Lyapunov function that satisfies the conditions of a certain theorem even though we may not explicitly know that function. In Example 4.10, we can determine the Lyapunov function explicitly if we know some bound on a. For example, if we know that $|a| \leq \alpha$, where the bound α is known, we can choose $b > \alpha$.

Example 4.11 The neural network of Section 1.2.5 is represented by

$$\dot{x}_i = \frac{1}{C_i} h_i(x_i) \left[\sum_j T_{ij} x_j - \frac{1}{R_i} g_i^{-1}(x_i) + I_i \right]$$

for $i = 1, 2, \ldots, n$, where the state variables x_i are the voltages at the amplifier outputs. They can only take values in the set

$$H = \{x \in R^n \mid -V_M < x_i < V_M\}$$

The functions $g_i : R \to (-V_M, V_M)$ are sigmoid functions,

$$h_i(x_i) = \left. \frac{dg_i}{du_i} \right|_{u_i = g_i^{-1}(x_i)} > 0, \quad \forall\ x_i \in (-V_M, V_M)$$

I_i are constant current inputs, $R_i > 0$, and $C_i > 0$. Assume that the symmetry condition $T_{ij} = T_{ji}$ is satisfied. The system may have several equilibrium points in H. We assume that all equilibrium points in H are isolated. Due to the symmetry property $T_{ij} = T_{ji}$, the vector whose ith component is

$$-\left[\sum_j T_{ij} x_j - \frac{1}{R_i} g_i^{-1}(x_i) + I_i \right]$$

is a gradient vector of a scalar function. By integration, similar to what we have done in the variable gradient method, it can be shown that this scalar function is given by

$$V(x) = -\tfrac{1}{2}\sum_i\sum_j T_{ij}x_ix_j + \sum_i \frac{1}{R_i}\int_0^{x_i} g_i^{-1}(y)\ dy - \sum_i I_ix_i$$

This function is continuously differentiable, but (typically) not positive definite. We rewrite the state equations as

$$\dot{x}_i = -\frac{1}{C_i}h_i(x_i)\frac{\partial V}{\partial x_i}$$

Let us now apply Theorem 4.4 with $V(x)$ as a candidate function. The derivative of $V(x)$ along the trajectories of the system is given by

$$\dot{V}(x) = \sum_{i=1}^{n}\frac{\partial V}{\partial x_i}\dot{x}_i = -\sum_{i=1}^{n}\frac{1}{C_i}h_i(x_i)\left(\frac{\partial V}{\partial x_i}\right)^2 \le 0$$

Moreover,

$$\dot{V}(x) = 0 \Rightarrow \frac{\partial V}{\partial x_i} = 0 \Rightarrow \dot{x}_i = 0, \quad \forall\ i$$

Hence, $\dot{V}(x) = 0$ only at equilibrium points. To apply Theorem 4.4, we need to construct a set Ω. Let

$$\Omega(\varepsilon) = \{x \in R^n \mid -(V_M - \varepsilon) \le x_i \le (V_M - \varepsilon)\}$$

where $\varepsilon > 0$ is arbitrarily small. The set $\Omega(\varepsilon)$ is closed and bounded, and $\dot{V}(x) \le 0$ in $\Omega(\varepsilon)$. It remains to show that $\Omega(\varepsilon)$ is a positively invariant set; that is, every trajectory starting in $\Omega(\varepsilon)$ stays for all future time in $\Omega(\varepsilon)$. To simplify the task, we assume a specific form for the sigmoid function $g_i(\cdot)$. Let

$$g_i(u_i) = \frac{2V_M}{\pi}\tan^{-1}\left(\frac{\lambda\pi u_i}{2V_M}\right), \quad \lambda > 0$$

Then,

$$\dot{x}_i = \frac{1}{C_i}h_i(x_i)\left[\sum_j T_{ij}x_j - \frac{2V_M}{\lambda\pi R_i}\tan\left(\frac{\pi x_i}{2V_M}\right) + I_i\right]$$

For $|x_i| \ge V_M - \varepsilon$,

$$\left|\tan\left(\frac{\pi x_i}{2V_M}\right)\right| \ge \tan\left(\frac{\pi(V_M - \varepsilon)}{2V_M}\right) \to \infty \text{ as } \varepsilon \to 0$$

Since x_i and I_i are bounded, ε can be chosen small enough to ensure that

$$x_i\sum_j T_{ij}x_j - \frac{2V_Mx_i}{\lambda\pi R_i}\tan\left(\frac{\pi x_i}{2V_M}\right) + x_iI_i < 0, \quad \text{for } V_M - \varepsilon \le |x_i| < V_M$$

Hence,

$$\frac{d}{dt}\left(x_i^2\right) = 2x_i\dot{x}_i < 0, \quad \text{for } V_M - \varepsilon \le |x_i| < V_M, \ \forall\ i$$

Consequently, trajectories starting in $\Omega(\varepsilon)$ will stay in $\Omega(\varepsilon)$ for all future time. In fact, trajectories starting in $H - \Omega(\varepsilon)$ will converge to $\Omega(\varepsilon)$, implying that all equilibrium points lie in the compact set $\Omega(\varepsilon)$. Hence, there can be only a finite number of isolated equilibrium points. In $\Omega(\varepsilon)$, $E = M =$ the set of equilibrium points inside $\Omega(\varepsilon)$. By Theorem 4.4, we know that every trajectory in $\Omega(\varepsilon)$ approaches M as $t \to \infty$. Since M consists of isolated equilibrium points, it can be shown (Exercise 4.20) that a trajectory approaching M must approach one of these equilibria. Hence, the system will not oscillate. △

4.3 Linear Systems and Linearization

The linear time-invariant system

$$\dot{x} = Ax \tag{4.9}$$

has an equilibrium point at the origin. The equilibrium point is isolated if and only if $\det(A) \neq 0$. If $\det(A) = 0$, the matrix A has a nontrivial null space. Every point in the null space of A is an equilibrium point for the system (4.9). In other words, if $\det(A) = 0$, the system has an equilibrium subspace. Notice that a linear system cannot have multiple isolated equilibrium points. For, if $\bar{x}_1$ and $\bar{x}_2$ are two equilibrium points for (4.9), then by linearity, every point on the line connecting $\bar{x}_1$ and $\bar{x}_2$ is an equilibrium point for the system. Stability properties of the origin can be characterized by the locations of the eigenvalues of the matrix A. Recall from linear system theory[10] that the solution of (4.9) for a given initial state $x(0)$ is given by

$$x(t) = \exp(At)x(0) \tag{4.10}$$

and that for any matrix A there is a nonsingular matrix P (possibly complex) that transforms A into its Jordan form; that is,

$$P^{-1}AP = J = \text{block diag}[J_1, J_2, \ldots, J_r]$$

where J_i is a Jordan block associated with the eigenvalue λ_i of A. A Jordan block of order one takes the form $J_i = \lambda_i$, while a Jordan block of order $m > 1$ takes the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & \ldots & 0 \\ 0 & \lambda_i & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & 0 \\ \vdots & & & & \ddots & 1 \\ 0 & \ldots & \ldots & \ldots & 0 & \lambda_i \end{bmatrix}_{m \times m}$$

[10] See, for example, [9], [35], [81], [94], or [158].

Therefore,

$$\exp(At) = P\exp(Jt)P^{-1} = \sum_{i=1}^{r}\sum_{k=1}^{m_i} t^{k-1}\exp(\lambda_i t)R_{ik} \tag{4.11}$$

where m_i is the order of the Jordan block J_i. If an $n \times n$ matrix A has a repeated eigenvalue λ_i of algebraic multiplicity q_i,[11] then the Jordan blocks associated with λ_i have order one if and only if $\text{rank}(A-\lambda_i I) = n-q_i$. The next theorem characterizes the stability properties of the origin.

Theorem 4.5 *The equilibrium point $x = 0$ of $\dot{x} = Ax$ is stable if and only if all eigenvalues of A satisfy $\text{Re}\lambda_i \leq 0$ and for every eigenvalue with $\text{Re}\lambda_i = 0$ and algebraic multiplicity $q_i \geq 2$, $\text{rank}(A - \lambda_i I) = n - q_i$, where n is the dimension of x. The equilibrium point $x = 0$ is (globally) asymptotically stable if and only if all eigenvalues of A satisfy $\text{Re}\lambda_i < 0$.* ◇

Proof: From (4.10), we can see that the origin is stable if and only if $\exp(At)$ is a bounded function of t for all $t \geq 0$. If one of the eigenvalues of A is in the open right-half complex plane, the corresponding exponential term $\exp(\lambda_i t)$ in (4.11) will grow unbounded as $t \to \infty$. Therefore, we must restrict the eigenvalues to be in the closed left-half complex plane. However, those eigenvalues on the imaginary axis (if any) could give rise to unbounded terms if the order of an associated Jordan block is higher than one, due to the term t^{k-1} in (4.11). Therefore, we must restrict eigenvalues on the imaginary axis to have Jordan blocks of order one, which is equivalent to the rank condition $\text{rank}(A - \lambda_i I) = n - q_i$. Thus, we conclude that the condition for stability is a necessary one. It is clear that the condition is also sufficient to ensure that $\exp(At)$ is bounded. For asymptotic stability of the origin, $\exp(At)$ must approach 0 as $t \to \infty$. From (4.11), this is the case if and only if $\text{Re}\lambda_i < 0$, $\forall\ i$. Since $x(t)$ depends linearly on the initial state $x(0)$, asymptotic stability of the origin is global. □

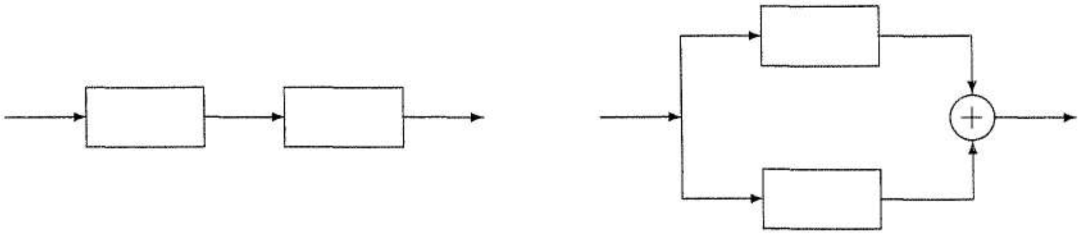
The proof shows, mathematically, why repeated eigenvalues on the imaginary axis must satisfy the rank condition $\text{rank}(A - \lambda_i I) = n - q_i$. The next example may shed some light on the physical meaning of this requirement.

Example 4.12 Figure 4.6 shows a series connection and a parallel connection of two identical systems. Each system is represented by the state model

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned}$$

where u and y are the input and output, respectively. Let A_s and A_p be the matrices of the series and parallel connections, when modeled in the form (4.9) (no driving

[11] Equivalently, q_i is the multiplicity of λ_i as a zero of $\det(\lambda I - A)$.



(a) (b)

Figure 4.6: (a) Series connection; (b) parallel connection.

inputs). Then

$$A_p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad A_s = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

The matrices A_p and A_s have the same eigenvalues on the imaginary axis, $\pm j$ with algebraic multiplicity $q_i = 2$, where $j = \sqrt{-1}$. It can be easily checked that $\text{rank}(A_p - jI) = 2 = n - q_i$, while $\text{rank}(A_s - jI) = 3 \neq n - q_i$. Thus, by Theorem 4.5, the origin of the parallel connection is stable, while the origin of the series connection is unstable. To physically see the difference between the two cases, notice that in the parallel connection, nonzero initial conditions produce sinusoidal oscillations of frequency 1 rad/sec, which are bounded functions of time. The sum of these sinusoidal signals remains bounded. On the other hand, nonzero initial conditions in the first component of the series connection produce a sinusoidal oscillation of frequency 1 rad/sec, which acts as a driving input for the second component. Since the second component has an undamped natural frequency of 1 rad/sec, the driving input causes "resonance" and the response grows unbounded. △

When all eigenvalues of A satisfy $\text{Re}\lambda_i < 0$, A is called a *Hurwitz matrix* or a *stability matrix*. The origin of (4.9) is asymptotically stable if and only if A is Hurwitz. Asymptotic stability of the origin can be also investigated by using Lyapunov's method. Consider a quadratic Lyapunov function candidate

$$V(x) = x^T P x$$

where P is a real symmetric positive definite matrix. The derivative of V along the trajectories of the linear system (4.9) is given by

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x = -x^T Q x$$

where Q is a symmetric matrix defined by

$$PA + A^T P = -Q \tag{4.12}$$

If Q is positive definite, we can conclude by Theorem 4.1 that the origin is asymptotically stable; that is, $\text{Re}\lambda_i < 0$ for all eigenvalues of A. Here we follow the usual procedure of Lyapunov's method, where we choose $V(x)$ to be positive definite and then check the negative definiteness of $\dot{V}(x)$. In the case of linear systems, we can reverse the order of these two steps. Suppose we start by choosing Q as a real symmetric positive definite matrix, and then solve (4.12) for P. If (4.12) has a positive definite solution, then we can again conclude that the origin is asymptotically stable. Equation (4.12) is called the *Lyapunov equation.* The next theorem characterizes asymptotic stability of the origin in terms of the solution of the Lyapunov equation.

Theorem 4.6 *A matrix A is Hurwitz; that is, $\text{Re}\lambda_i < 0$ for all eigenvalues of A, if and only if for any given positive definite symmetric matrix Q there exists a positive definite symmetric matrix P that satisfies the Lyapunov equation* (4.12). *Moreover, if A is Hurwitz, then P is the unique solution of* (4.12). ◇

Proof: Sufficiency follows from Theorem 4.1 with the Lyapunov function $V(x) = x^T Px$, as we have already shown. To prove necessity, assume that all eigenvalues of A satisfy $\text{Re}\lambda_i < 0$ and consider the matrix P, defined by

$$P = \int_0^\infty \exp(A^T t) Q \exp(At) \; dt \tag{4.13}$$

The integrand is a sum of terms of the form $t^{k-1}\exp(\lambda_i t)$, where $\text{Re}\lambda_i < 0$. Therefore, the integral exists. The matrix P is symmetric and positive definite. The fact that it is positive definite can be shown as follows: Supposing it is not so, there is a vector $x \neq 0$ such that $x^T Px = 0$. However,

$$\begin{aligned} x^T Px = 0 \quad &\Rightarrow \quad \int_0^\infty x^T \exp(A^T t) Q \exp(At) x \; dt = 0 \\ &\Rightarrow \quad \exp(At)x \equiv 0, \; \forall \, t \geq 0 \; \Rightarrow \; x = 0 \end{aligned}$$

since $\exp(At)$ is nonsingular for all t. This contradiction shows that P is positive definite. Now, substituting (4.13) in the left-hand side of (4.12) yields

$$\begin{aligned} PA + A^T P &= \int_0^\infty \exp(A^T t) Q \exp(At) A \; dt + \int_0^\infty A^T \exp(A^T t) Q \exp(At) \; dt \\ &= \int_0^\infty \frac{d}{dt} \exp(A^T t) Q \exp(At) \; dt \; = \; \exp(A^T t) Q \exp(At) \big|_0^\infty \; = \; -Q \end{aligned}$$

which shows that P is indeed a solution of (4.12). To show that it is the unique solution, suppose there is another solution $\tilde{P} \neq P$. Then,

$$(P - \tilde{P})A + A^T(P - \tilde{P}) = 0$$

Premultiplying by $\exp(A^T t)$ and postmultiplying by $\exp(At)$, we obtain

$$0 = \exp(A^T t)[(P - \tilde{P})A + A^T(P - \tilde{P})] \exp(At) = \frac{d}{dt} \left\{ \exp(A^T t)(P - \tilde{P}) \exp(At) \right\}$$

Hence,

$$\exp(A^T t)(P - \tilde{P})\exp(At) \equiv \text{a constant } \forall\ t$$

In particular, since $\exp(A0) = I$, we have

$$(P - \tilde{P}) = \exp(A^T t)(P - \tilde{P})\exp(At) \to 0 \text{ as } t \to \infty$$

Therefore, $\tilde{P} = P$. □

The positive definiteness requirement on Q can be relaxed. It is left to the reader (Exercise 4.22) to verify that Q can be taken as a positive semidefinite matrix of the form $Q = C^T C$, where the pair (A, C) is observable.

Equation (4.12) is a linear algebraic equation that can be solved by rearranging it in the form $Mx = y$, where x and y are defined by stacking the elements of P and Q in vectors, as will be illustrated in the next example. There are numerically efficient methods for solving such equations.[12]

Example 4.13 Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and } P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

where, due to symmetry, $p_{12} = p_{21}$. The Lyapunov equation (4.12) can be rewritten as

$$\begin{bmatrix} 0 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

The unique solution of this equation is given by

$$\begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} 1.5 \\ -0.5 \\ 1.0 \end{bmatrix} \quad \Rightarrow \quad P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1.0 \end{bmatrix}$$

The matrix P is positive definite since its leading principal minors (1.5 and 1.25) are positive. Hence, all eigenvalues of A are in the open left-half complex plane. △

The Lyapunov equation can be used to test whether or not a matrix A is Hurwitz, as an alternative to calculating the eigenvalues of A. One starts by choosing a positive definite matrix Q (for example, $Q = I$) and solves the Lyapunov equation (4.12) for P. If the equation has a positive definite solution, we conclude that A is Hurwitz; otherwise, it is not so. However, there is no computational advantage

[12] Consult [67] on numerical methods for solving linear algebraic equations. The Lyapunov equation can also be solved by viewing it as a special case of the Sylvester equation $PA + BP + C = 0$, which is treated in [67]. Almost all commercial software programs for control systems include commands for solving the Lyapunov equation.

in solving the Lyapunov equation over calculating the eigenvalues of A.[13] Besides, the eigenvalues provide more direct information about the response of the linear system. The interest in the Lyapunov equation is not in its use as a stability test for linear systems;[14] rather, it is in the fact that it provides a procedure for finding a Lyapunov function for any linear system $\dot{x} = Ax$ when A is Hurwitz. The mere existence of a Lyapunov function will allow us to draw conclusions about the system when the right-hand side Ax is perturbed, whether such perturbation is a linear perturbation in the coefficients of A or a nonlinear perturbation. This advantage will unfold as we continue our study of Lyapunov's method.

Let us go back to the nonlinear system

$$\dot{x} = f(x) \tag{4.14}$$

where $f : D \to R^n$ is a continuously differentiable map from a domain $D \subset R^n$ into R^n. Suppose the origin $x = 0$ is in D and is an equilibrium point for the system; that is, $f(0) = 0$. By the mean value theorem,

$$f_i(x) = f_i(0) + \frac{\partial f_i}{\partial x}(z_i)\, x$$

where z_i is a point on the line segment connecting x to the origin. The foregoing equality is valid for any point $x \in D$ such that the line segment connecting x to the origin lies entirely in D. Since $f(0) = 0$, we can write

$$f_i(x) = \frac{\partial f_i}{\partial x}(z_i)x = \frac{\partial f_i}{\partial x}(0)x + \left[\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right] x$$

Hence,

$$f(x) = Ax + g(x)$$

where

$$A = \frac{\partial f}{\partial x}(0) \quad \text{and} \quad g_i(x) = \left[\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right] x$$

The function $g_i(x)$ satisfies

$$|g_i(x)| \le \left\| \frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0) \right\| \|x\|$$

By continuity of $[\partial f / \partial x]$, we see that

$$\frac{\|g(x)\|}{\|x\|} \to 0 \ \text{ as } \ \|x\| \to 0$$

[13] A typical procedure for solving the Lyapunov equation, the Bartels–Stewart algorithm [67], starts by transforming A into its real Schur form, which gives the eigenvalues of A. Hence, the computational effort for solving the Lyapunov equation is more than calculating the eigenvalues of A. Other algorithms for solving the Lyapunov equation take an amount of computations comparable to the Bartels–Stewart algorithm.

[14] It might be of interest, however, to know that one can use the Lyapunov equation to derive the classical Routh–Hurwitz criterion. (See [35, pp. 417–419].)

This suggests that in a small neighborhood of the origin we can approximate the nonlinear system (4.14) by its linearization about the origin

$$\dot{x} = Ax, \quad \text{where } A = \frac{\partial f}{\partial x}(0)$$

The next theorem spells out conditions under which we can draw conclusions about the stability of the origin as an equilibrium point for the nonlinear system by investigating its stability as an equilibrium point for the linear system. The theorem is known as *Lyapunov's indirect method.*

Theorem 4.7 *Let $x = 0$ be an equilibrium point for the nonlinear system*

$$\dot{x} = f(x)$$

where $f : D \to R^n$ is continuously differentiable and D is a neighborhood of the origin. Let

$$A = \left.\frac{\partial f}{\partial x}(x)\right|_{x=0}$$

Then,

1. *The origin is asymptotically stable if* $\text{Re}\lambda_i < 0$ *for all eigenvalues of* A.
2. *The origin is unstable if* $\text{Re}\lambda_i > 0$ *for one or more of the eigenvalues of* A.

$\diamond$

Proof: To prove the first part, let A be a Hurwitz matrix. Then, by Theorem 4.6, we know that for any positive definite symmetric matrix Q, the solution P of the Lyapunov equation (4.12) is positive definite. We use $V(x) = x^T P x$ as a Lyapunov function candidate for the nonlinear system. The derivative of $V(x)$ along the trajectories of the system is given by

$$\begin{aligned} \dot{V}(x) &= x^T P f(x) + f^T(x) P x \\ &= x^T P[Ax + g(x)] + [x^T A^T + g^T(x)]Px \\ &= x^T (PA + A^T P)x + 2x^T P g(x) \\ &= -x^T Q x + 2x^T P g(x) \end{aligned}$$

The first term on the right-hand side is negative definite, while the second term is (in general) indefinite. The function $g(x)$ satisfies

$$\frac{\|g(x)\|_2}{\|x\|_2} \to 0 \quad \text{as} \quad \|x\|_2 \to 0$$

Therefore, for any $\gamma > 0$, there exists $r > 0$ such that

$$\|g(x)\|_2 < \gamma \|x\|_2, \quad \forall\ \|x\|_2 < r$$

Hence,

$$\dot{V}(x) < -x^T Q x + 2\gamma \|P\|_2 \|x\|_2^2, \quad \forall\ \|x\|_2 < r$$

But

$$x^T Q x \geq \lambda_{\min}(Q) \|x\|_2^2$$

where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of a matrix. Note that $\lambda_{\min}(Q)$ is real and positive since Q is symmetric and positive definite. Thus,

$$\dot{V}(x) < -[\lambda_{\min}(Q) - 2\gamma \|P\|_2] \|x\|_2^2, \quad \forall\ \|x\|_2 < r$$

Choosing $\gamma < (1/2)\lambda_{\min}(Q)/\|P\|_2$ ensures that $\dot{V}(x)$ is negative definite. By Theorem 4.1, we conclude that the origin is asymptotically stable. To prove the second part of the theorem, let us consider first the special case when A has no eigenvalues on the imaginary axis. If the eigenvalues of A cluster into a group of eigenvalues in the open right-half plane and a group of eigenvalues in the open left-half plane, then there is a nonsingular matrix T such that[15]

$$TAT^{-1} = \begin{bmatrix} -A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

where A_1 and A_2 are Hurwitz matrices. Let

$$z = Tx = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

where the partition of z is compatible with the dimensions of A_1 and A_2. The change of variables $z = Tx$ transforms the system

$$\dot{x} = Ax + g(x)$$

into the form

$$\begin{aligned} \dot{z}_1 &= -A_1 z_1 + g_1(z) \\ \dot{z}_2 &= A_2 z_2 + g_2(z) \end{aligned}$$

where the functions $g_i(z)$ have the property that for any $\gamma > 0$, there exists $r > 0$ such that

$$\|g_i(z)\|_2 < \gamma \|z\|_2, \quad \forall\ \|z\|_2 \leq r, \ i = 1, 2$$

The origin $z = 0$ is an equilibrium point for the system in the z-coordinates. Clearly, any conclusion we arrive at concerning the stability properties of $z = 0$ carries over to the equilibrium point $x = 0$ in the x-coordinates, since T is nonsingular.[16] To show that the origin is unstable, we apply Theorem 4.3. The construction of a

[15]There are several methods for finding the matrix T, one of which is to transform the matrix A into its real Jordan form [67].

[16]See Exercise 4.26 for a general discussion of stability preserving maps.

function $V(z)$ will be done basically as in Example 4.7, except for working with vectors, instead of scalars. Let Q_1 and Q_2 be positive definite symmetric matrices of the dimensions of A_1 and A_2, respectively. Since A_1 and A_2 are Hurwitz, we know from Theorem 4.6 that the Lyapunov equations

$$P_iA_i + A_i^TP_i = -Q_i, \quad i = 1, 2$$

have unique positive definite solutions P_1 and P_2. Let

$$V(z) = z_1^TP_1z_1 - z_2^TP_2z_2 = z^T \begin{bmatrix} P_1 & 0 \\ 0 & -P_2 \end{bmatrix} z$$

In the subspace $z_2 = 0$, $V(z) > 0$ at points arbitrarily close to the origin. Let

$$U = \{z \in R^n \mid \|z\|_2 \le r \text{ and } V(z) > 0\}$$

In U,

$$\begin{aligned}
\dot{V}(z) &= -z_1^T(P_1A_1 + A_1^TP_1)z_1 + 2z_1^TP_1g_1(z) \\
&\quad - z_2^T(P_2A_2 + A_2^TP_2)z_2 - 2z_2^TP_2g_2(z) \\
&= z_1^TQ_1z_1 + z_2^TQ_2z_2 + 2z^T \begin{bmatrix} P_1g_1(z) \\ -P_2g_2(z) \end{bmatrix} \\
&\ge \lambda_{\min}(Q_1)\|z_1\|_2^2 + \lambda_{\min}(Q_2)\|z_2\|_2^2 \\
&\quad - 2\|z\|_2\sqrt{\|P_1\|_2^2\|g_1(z)\|_2^2 + \|P_2\|_2^2\|g_2(z)\|_2^2} \\
&> (\alpha - 2\sqrt{2}\beta\gamma)\|z\|_2^2
\end{aligned}$$

where

$$\alpha = \min\{\lambda_{\min}(Q_1), \lambda_{\min}(Q_2)\} \quad \text{and} \quad \beta = \max\{\|P_1\|_2, \|P_2\|_2\}$$

Thus, choosing $\gamma < \alpha/(2\sqrt{2}\beta)$ ensures that $\dot{V}(z) > 0$ in U. Therefore, by Theorem 4.3, the origin is unstable. Notice that we could have applied Theorem 4.3 in the original coordinates by defining the matrices

$$P = T^T \begin{bmatrix} P_1 & 0 \\ 0 & -P_2 \end{bmatrix} T; \quad Q = T^T \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} T$$

which satisfy the equation

$$PA + A^TP = Q$$

The matrix Q is positive definite, and $V(x) = x^TPx$ is positive for points arbitrarily close to the origin $x = 0$. Let us consider now the general case when A may have eigenvalues on the imaginary axis, in addition to eigenvalues in the open right-half complex plane. We can reduce this case to the special case we have just studied by a simple trick of shifting the imaginary axis. Suppose A has m eigenvalues with

$\text{Re}\lambda_i > \delta > 0$. Then, the matrix $[A - (\delta/2)I]$ has m eigenvalues in the open right-half plane, but no eigenvalues on the imaginary axis. By previous arguments, there exist matrices $P = P^T$ and $Q = Q^T > 0$ such that

$$P\left[A - \frac{\delta}{2}I\right] + \left[A - \frac{\delta}{2}I\right]^T P = Q$$

where $V(x) = x^T P x$ is positive for points arbitrarily close to the origin. The derivative of $V(x)$ along the trajectories of the system is given by

$$\begin{aligned}\dot{V}(x) &= x^T(PA + A^T P)x + 2x^T P g(x) \\ &= x^T\left[P\left(A - \frac{\delta}{2}I\right) + \left(A - \frac{\delta}{2}I\right)^T P\right]x + \delta x^T P x + 2x^T P g(x) \\ &= x^T Q x + \delta V(x) + 2x^T P g(x)\end{aligned}$$

In the set

$$\{x \in R^n \mid \|x\|_2 \leq r \text{ and } V(x) > 0\}$$

where r is chosen such that $\|g(x)\|_2 \leq \gamma\|x\|_2$ for $\|x\|_2 < r$, $\dot{V}(x)$ satisfies

$$\dot{V}(x) \geq \lambda_{\min}(Q)\|x\|_2^2 - 2\|P\|_2\|x\|_2\|g(x)\|_2 \geq (\lambda_{\min}(Q) - 2\gamma\|P\|_2)\,\|x\|_2^2$$

which is positive for $\gamma < (1/2)\lambda_{\min}(Q)/\|P\|_2$. Applying Theorem 4.3 concludes the proof. □

Theorem 4.7 provides us with a simple procedure for determining the stability of an equilibrium point at the origin. We calculate the *Jacobian matrix*

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$$

and test its eigenvalues. If $\text{Re}\lambda_i < 0$ for all i or $\text{Re}\lambda_i > 0$ for some i, we conclude that the origin is asymptotically stable or unstable, respectively. Moreover, the proof of the theorem shows that when $\text{Re}\lambda_i < 0$ for all i, we can also find a Lyapunov function for the system that will work locally in some neighborhood of the origin. The Lyapunov function is the quadratic form $V(x) = x^T P x$, where P is the solution of the Lyapunov equation (4.12) for any positive definite symmetric matrix Q. Note that Theorem 4.7 does not say anything about the case when $\text{Re}\lambda_i \leq 0$ for all i, with $\text{Re}\lambda_i = 0$ for some i. In this case, linearization fails to determine the stability of the equilibrium point.[17]

Example 4.14 Consider the scalar system

$$\dot{x} = ax^3$$

[17]See Section 8.1 for further investigation of the critical case when linearization fails.

Linearizing the system about the origin $x = 0$ yields

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = 3ax^2\big|_{x=0} = 0$$

There is one eigenvalue that lies on the imaginary axis. Hence, linearization fails to determine the stability of the origin. This failure is genuine in the sense that the origin could be asymptotically stable, stable, or unstable, depending on the value of the parameter a. If $a < 0$, the origin is asymptotically stable as can be seen from the Lyapunov function $V(x) = x^4$, whose derivative $\dot{V}(x) = 4ax^6 < 0$ for $x \neq 0$. If $a = 0$, the system is linear and the origin is stable according to Theorem 4.5. If $a > 0$, the origin is unstable as can be seen from Theorem 4.3 and the function $V(x) = x^4$, whose derivative $\dot{V}(x) = 4ax^6 > 0$ for $x \neq 0$. △

Example 4.15 The pendulum equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin x_1 - bx_2 \end{aligned}$$

has two equilibrium points at $(x_1 = 0,\ x_2 = 0)$ and $(x_1 = \pi,\ x_2 = 0)$. Let us investigate stability of both points by using linearization. The Jacobian matrix is given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a\cos x_1 & -b \end{bmatrix}$$

To determine the stability of the origin, we evaluate the Jacobian at $x = 0$:

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}$$

The eigenvalues of A are

$$\lambda_{1,2} = -\tfrac{1}{2}b \pm \tfrac{1}{2}\sqrt{b^2 - 4a}$$

For all $a,\ b > 0$, the eigenvalues satisfy $\text{Re}\lambda_i < 0$. Consequently, the equilibrium point at the origin is asymptotically stable. In the absence of friction ($b = 0$), both eigenvalues are on the imaginary axis. Thus, we cannot determine the stability of the origin through linearization. We have seen in Example 4.3 that, in this case, the origin is a stable equilibrium point as determined by an energy Lyapunov function. To determine the stability of the equilibrium point at $(x_1 = \pi,\ x_2 = 0)$, we evaluate the Jacobian at that point. This is equivalent to performing a change of variables $z_1 = x_1 - \pi,\ z_2 = x_2$ to shift the equilibrium point to the origin, and evaluating the Jacobian $[\partial f/\partial z]$ at $z = 0$:

$$\tilde{A} = \left.\frac{\partial f}{\partial x}\right|_{x_1=\pi, x_2=0} = \begin{bmatrix} 0 & 1 \\ a & -b \end{bmatrix}$$

The eigenvalues of $\tilde{A}$ are

$$\lambda_{1,2} = -\ \tfrac{1}{2}b \pm \tfrac{1}{2}\sqrt{b^2 + 4a}$$

For all $a > 0$ and $b \geq 0$, there is one eigenvalue in the open right-half plane. Hence, the equilibrium point at $(x_1 = \pi,\ x_2 = 0)$ is unstable. △

4.4 Comparison Functions

As we move from autonomous to nonautonomous systems, one degree of difficulty will arise from the fact that the solution of the nonautonomous system $\dot{x} = f(t, x)$, starting at $x(t_0) = x_0$, depends on both t and t_0. To cope with this new situation, we will refine the definitions of stability and asymptotic stability so that they hold uniformly in the initial time t_0. While we can refine Definition 4.1 to achieve the required uniformity, it turns out that there are more transparent definitions which use special comparison functions, known as class $\mathcal{K}$ and class $\mathcal{KL}$ functions.

Definition 4.2 *A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.*

Definition 4.3 *A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.*

Example 4.16

- $\alpha(r) = \tan^{-1}(r)$ is strictly increasing since $\alpha'(r) = 1/(1 + r^2) > 0$. It belongs to class $\mathcal{K}$, but not to class $\mathcal{K}_\infty$ since $\lim_{r\to\infty} \alpha(r) = \pi/2 < \infty$.

- $\alpha(r) = r^c$, for any positive real number c, is strictly increasing since $\alpha'(r) = cr^{c-1} > 0$. Moreover, $\lim_{r\to\infty} \alpha(r) = \infty$; thus, it belongs to class $\mathcal{K}_\infty$.

- $\alpha(r) = \min\{r, r^2\}$ is continuous, strictly increasing, and $\lim_{r\to\infty} \alpha(r) = \infty$. Hence, it belongs to class $\mathcal{K}_\infty$. Notice that $\alpha(r)$ is not continuously differentiable at $r = 1$. Continuous differentiability is not required for a class $\mathcal{K}$ function.

- $\beta(r, s) = r/(ksr + 1)$, for any positive real number k, is strictly increasing in r since
$$\frac{\partial \beta}{\partial r} = \frac{1}{(ksr + 1)^2} > 0$$
and strictly decreasing in s since
$$\frac{\partial \beta}{\partial s} = \frac{-kr^2}{(ksr + 1)^2} < 0$$

Moreover, $\beta(r,s) \to 0$ as $s \to \infty$. Therefore, it belongs to class $\mathcal{KL}$.

- $\beta(r,s) = r^c e^{-s}$, for any positive real number c, belongs to class $\mathcal{KL}$. $\triangle$

The next lemma states some useful properties of class $\mathcal{K}$ and class $\mathcal{KL}$ functions, which will be needed later on. The proof of the lemma is left as an exercise for the reader (Exercise 4.34).

Lemma 4.2 *Let α_1 and α_2 be class $\mathcal{K}$ functions on $[0,a)$, α_3 and α_4 be class $\mathcal{K}_\infty$ functions, and β be a class $\mathcal{KL}$ function. Denote the inverse of α_i by α_i^{-1}. Then,*

- *α_1^{-1} is defined on $[0,\alpha_1(a))$ and belongs to class $\mathcal{K}$.*
- *α_3^{-1} is defined on $[0,\infty)$ and belongs to class $\mathcal{K}_\infty$.*
- *$\alpha_1 \circ \alpha_2$ belongs to class $\mathcal{K}$.*
- *$\alpha_3 \circ \alpha_4$ belongs to class $\mathcal{K}_\infty$.*
- *$\sigma(r,s) = \alpha_1(\beta(\alpha_2(r),s))$ belongs to class $\mathcal{KL}$.* $\diamond$

Class $\mathcal{K}$ and class $\mathcal{KL}$ functions enter into Lyapunov analysis through the next two lemmas.

Lemma 4.3 *Let $V : D \to R$ be a continuous positive definite function defined on a domain $D \subset R^n$ that contains the origin. Let $B_r \subset D$ for some $r > 0$. Then, there exist class $\mathcal{K}$ functions α_1 and α_2, defined on $[0,r]$, such that*

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$$

for all $x \in B_r$. If $D = R^n$, the functions α_1 and α_2 will be defined on $[0,\infty)$ and the foregoing inequality will hold for all $x \in R^n$. Moreover, if $V(x)$ is radially unbounded, then α_1 and α_2 can be chosen to belong to class $\mathcal{K}_\infty$. $\diamond$

Proof: See Appendix C.4.

For a quadratic positive definite function $V(x) = x^T P x$, Lemma 4.3 follows from the inequalities

$$\lambda_{\min}(P)\|x\|_2^2 \le x^T P x \le \lambda_{\max}(P)\|x\|_2^2$$

Lemma 4.4 *Consider the scalar autonomous differential equation*

$$\dot{y} = -\alpha(y), \quad y(t_0) = y_0$$

where α is a locally Lipschitz class $\mathcal{K}$ function defined on $[0,a)$. For all $0 \le y_0 < a$, this equation has a unique solution $y(t)$ defined for all $t \ge t_0$. Moreover,

$$y(t) = \sigma(y_0, t - t_0)$$

where σ is a class $\mathcal{KL}$ function defined on $[0,a) \times [0,\infty)$. $\diamond$

Proof: See Appendix C.5.

We can see that the claim of this lemma is true by examining specific examples, where a closed-form solution of the scalar equation can be found. For example, if $\dot{y} = -ky$, $k > 0$, then the solution is

$$y(t) = y_0 \exp[-k(t - t_0)] \;\Rightarrow\; \sigma(r, s) = r \exp(-ks)$$

As another example, if $\dot{y} = -ky^2$, $k > 0$, then the solution is

$$y(t) = \frac{y_0}{ky_0(t - t_0) + 1} \;\Rightarrow\; \sigma(r, s) = \frac{r}{krs + 1}$$

To see how class $\mathcal{K}$ and class $\mathcal{KL}$ functions enter into Lyapunov analysis, let us see how they could have been used in the proof of Theorem 4.1. In the proof, we wanted to choose β and δ such that $B_\delta \subset \Omega_\beta \subset B_r$. Using the fact that a positive definite function $V(x)$ satisfies

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$$

we can choose $\beta \le \alpha_1(r)$ and $\delta \le \alpha_2^{-1}(\beta)$. This is so because

$$V(x) \le \beta \;\Rightarrow\; \alpha_1(\|x\|) \le \alpha_1(r) \;\Leftrightarrow\; \|x\| \le r$$

and

$$\|x\| \le \delta \;\Rightarrow\; V(x) \le \alpha_2(\delta) \le \beta$$

In the same proof, we wanted to show that when $\dot{V}(x)$ is negative definite, the solution $x(t)$ tends to zero as t tends to infinity. Using Lemma 4.3 we see that there is a class $\mathcal{K}$ function α_3 such that $\dot{V}(x) \le -\alpha_3(\|x\|)$. Hence, V satisfies the differential inequality

$$\dot{V} \le -\alpha_3(\alpha_2^{-1}(V))$$

The comparison lemma (Lemma 3.4) shows that $V(x(t))$ is bounded by the solution of the scalar differential equation

$$\dot{y} = -\alpha_3(\alpha_2^{-1}(y)), \quad y(0) = V(x(0))$$

Lemma 4.2 shows that $\alpha_3 \circ \alpha_2^{-1}$ is a class $\mathcal{K}$ function and Lemma 4.4 shows that the solution of the scalar equation is $y(t) = \beta(y(0), t)$, where β is a class $\mathcal{KL}$ function. Consequently, $V(x(t))$ satisfies the inequality $V(x(t)) \le \beta(V(x(0)), t)$, which shows that $V(x(t))$ tends to zero as t tends to infinity. In fact, we can go beyond the proof of Theorem 4.1 to provide estimates of $\|x(t)\|$ that are not provided in that proof. The inequality $V(x(t)) \le V(x(0))$ implies that

$$\alpha_1(\|x(t)\|) \le V(x(t)) \le V(x(0)) \le \alpha_2(\|x(0)\|)$$

Hence, $\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\|x(0)\|))$, where $\alpha_1^{-1} \circ \alpha_2$ is a class $\mathcal{K}$ function. Similarly, the inequality $V(x(t)) \leq \beta(V(x(0)), t)$ implies that

$$\alpha_1(\|x(t)\|) \leq V(x(t)) \leq \beta(V(x(0)), t) \leq \beta(\alpha_2(\|x(0)\|), t)$$

Therefore, $\|x(t)\| \leq \alpha_1^{-1}(\beta(\alpha_2(\|x(0)\|), t))$, where ${\alpha_1}^{-1}(\beta(\alpha_2(r), t))$ is a class $\mathcal{KL}$ function.

4.5 Nonautonomous Systems

Consider the nonautonomous system

$$\dot{x} = f(t, x) \tag{4.15}$$

where $f : [0, \infty) \times D \to R^n$ is piecewise continuous in t and locally Lipschitz in x on $[0, \infty) \times D$, and $D \subset R^n$ is a domain that contains the origin $x = 0$. The origin is an equilibrium point for (4.15) at $t = 0$ if

$$f(t, 0) = 0, \quad \forall\ t \geq 0$$

An equilibrium point at the origin could be a translation of a nonzero equilibrium point or, more generally, a translation of a nonzero solution of the system. To see the latter point, suppose $\bar{y}(\tau)$ is a solution of the system

$$\frac{dy}{d\tau} = g(\tau, y)$$

defined for all $\tau \geq a$. The change of variables

$$x = y - \bar{y}(\tau); \quad t = \tau - a$$

transforms the system into the form

$$\dot{x} = g(\tau, y) - \dot{\bar{y}}(\tau) = g(t + a, x + \bar{y}(t + a)) - \dot{\bar{y}}(t + a) \stackrel{\text{def}}{=} f(t, x)$$

Since

$$\dot{\bar{y}}(t + a) = g(t + a, \bar{y}(t + a)), \quad \forall\ t \geq 0$$

the origin $x = 0$ is an equilibrium point of the transformed system at $t = 0$. Therefore, by examining the stability behavior of the origin as an equilibrium point for the transformed system, we determine the stability behavior of the solution $\bar{y}(\tau)$ of the original system. Notice that if $\bar{y}(\tau)$ is not constant, the transformed system will be nonautonomous even when the original system is autonomous, that is, even when $g(\tau, y) = g(y)$. This is why studying the stability behavior of solutions in the sense of Lyapunov can be done only in the context of studying the stability behavior of the equilibria of nonautonomous systems.

The notions of stability and asymptotic stability of equilibrium points of nonautonomous systems are basically the same as those introduced in Definition 4.1 for autonomous systems. The new element here is that, while the solution of an autonomous system depends only on $(t-t_0)$, the solution of a nonautonomous system may depend on both t and t_0. Therefore, the stability behavior of the equilibrium point will, in general, be dependent on t_0. The origin $x=0$ is a stable equilibrium point for (4.15) if, for each $\varepsilon > 0$, and any $t_0 \geq 0$ there is $\delta = \delta(\varepsilon, t_0) > 0$ such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall\, t \geq t_0$$

The constant δ is, in general, dependent on the initial time t_0. The existence of δ for every t_0 does not necessarily guarantee that there is one constant δ, dependent only on ε, that would work for all t_0, as illustrated by the next example.

Example 4.17 The linear first-order system

$$\dot{x} = (6t\sin t - 2t)x$$

has the solution

$$\begin{aligned} x(t) &= x(t_0)\exp\left[\int_{t_0}^{t}(6\tau\sin\tau - 2\tau)\; d\tau\right] \\ &= x(t_0)\exp\left[6\sin t - 6t\cos t - t^2 - 6\sin t_0 + 6t_0\cos t_0 + t_0^2\right] \end{aligned}$$

For any t_0, the term $-t^2$ will eventually dominate, which shows that the exponential term is bounded for all $t \geq t_0$ by a constant $c(t_0)$ dependent on t_0. Hence,

$$|x(t)| < |x(t_0)|c(t_0), \quad \forall\, t \geq t_0$$

For any $\varepsilon > 0$, the choice $\delta = \varepsilon/c(t_0)$ shows that the origin is stable. Now, suppose t_0 takes on the successive values $t_0 = 2n\pi$, for $n = 0, 1, 2, \ldots$, and $x(t)$ is evaluated π seconds later in each case. Then,

$$x(t_0 + \pi) = x(t_0)\exp\left[(4n+1)(6-\pi)\pi\right]$$

which implies that, for $x(t_0) \neq 0$,

$$\frac{x(t_0+\pi)}{x(t_0)} \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty$$

Thus, given $\varepsilon > 0$, there is no δ independent of t_0 that would satisfy the stability requirement uniformly in t_0. $\triangle$

Nonuniformity with respect to t_0 could also appear in studying asymptotic stability of the origin, as the next example shows.

Example 4.18 The linear first-order system

$$\dot{x} = -\frac{x}{1+t}$$

has the solution

$$x(t) = x(t_0)\exp\left(\int_{t_0}^{t} \frac{-1}{1+\tau}\, d\tau\right) = x(t_0)\frac{1+t_0}{1+t}$$

Since $|x(t)| \leq |x(t_0)|$, $\forall\ t \geq t_0$, the origin is clearly stable. Actually, given any $\varepsilon > 0$, we can choose δ independent of t_0. It is also clear that

$$x(t) \to 0 \ \text{ as } \ t \to \infty$$

Consequently, according to Definition 4.1, the origin is asymptotically stable. Notice, however, that the convergence of $x(t)$ to the origin is not uniform with respect to the initial time t_0. Recall that convergence of $x(t)$ to the origin is equivalent to saying that, given any $\varepsilon > 0$, there is $T = T(\varepsilon, t_0) > 0$ such that $|x(t)| < \varepsilon$ for all $t \geq t_0 + T$. Although this is true for every t_0, the constant T cannot be chosen independent of t_0. △

As a consequence, we need to refine Definition 4.1 to emphasize the dependence of the stability behavior of the origin on the initial time t_0. We are interested in a refinement that defines stability and asymptotic stability of the origin as uniform properties with respect to the initial time.[18]

Definition 4.4 *The equilibrium point* $x = 0$ *of* (4.15) *is*

- *stable if, for each* $\varepsilon > 0$, *there is* $\delta = \delta(\varepsilon, t_0) > 0$ *such that*

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall\ t \geq t_0 \geq 0 \tag{4.16}$$

- *uniformly stable if, for each* $\varepsilon > 0$, *there is* $\delta = \delta(\varepsilon) > 0$, *independent of* t_0, *such that* (4.16) *is satisfied.*

- *unstable if it is not stable.*

- *asymptotically stable if it is stable and there is a positive constant* $c = c(t_0)$ *such that* $x(t) \to 0$ *as* $t \to \infty$, *for all* $\|x(t_0)\| < c$.

[18] See [72] or [95] for other refinements of Definition 4.1. It is worthwhile to note that, for autonomous systems, the definition of global uniform asymptotic stability given here is equivalent to global asymptotic stability as defined in Section 4.1. In particular, $\delta(\varepsilon)$ can be always chosen such that $\lim_{\varepsilon \to \infty} \delta(\varepsilon) = \infty$. This is shown in the proof of Theorem 4.17. Lemma C.2 shows that, when the origin of an autonomous system is globally asymptotically stable, its solution $x(t)$ satisfies $\|x(t)\| \leq \beta(\|x(t_0)\|, 0)$ for all $x(t_0)$, where $\beta(r, 0)$ is a class $\mathcal{K}_\infty$ function. The function $\delta(\varepsilon)$ can be taken as $\delta(\varepsilon) = \beta^{-1}(\varepsilon, 0)$.

- *uniformly asymptotically stable if it is uniformly stable and there is a positive constant c, independent of t_0, such that for all $\|x(t_0)\| < c$, $x(t) \to 0$ as $t \to \infty$, uniformly in t_0; that is, for each $\eta > 0$, there is $T = T(\eta) > 0$ such that*

$$\|x(t)\| < \eta, \quad \forall\, t \geq t_0 + T(\eta),\ \forall\, \|x(t_0)\| < c \tag{4.17}$$

- *globally uniformly asymptotically stable if it is uniformly stable, $\delta(\varepsilon)$ can be chosen to satisfy $\lim_{\varepsilon\to\infty} \delta(\varepsilon) = \infty$, and, for each pair of positive numbers η and c, there is $T = T(\eta, c) > 0$ such that*

$$\|x(t)\| < \eta, \quad \forall\, t \geq t_0 + T(\eta, c),\ \forall\, \|x(t_0)\| < c \tag{4.18}$$

The next lemma gives equivalent, more transparent, definitions of uniform stability and uniform asymptotic stability by using class $\mathcal{K}$ and class $\mathcal{KL}$ functions.

Lemma 4.5 *The equilibrium point $x = 0$ of (4.15) is*

- *uniformly stable if and only if there exist a class $\mathcal{K}$ function α and a positive constant c, independent of t_0, such that*

$$\|x(t)\| \leq \alpha(\|x(t_0)\|), \quad \forall\, t \geq t_0 \geq 0,\ \forall\, \|x(t_0)\| < c \tag{4.19}$$

- *uniformly asymptotically stable if and only if there exist a class $\mathcal{KL}$ function β and a positive constant c, independent of t_0, such that*

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall\, t \geq t_0 \geq 0,\ \forall\, \|x(t_0)\| < c \tag{4.20}$$

- *globally uniformly asymptotically stable if and only if inequality (4.20) is satisfied for any initial state $x(t_0)$.* ◇

Proof: See Appendix C.6.

As a consequence of Lemma 4.5, we see that in the case of autonomous systems stability and asymptotic stability per Definition 4.1 imply the existence of class $\mathcal{K}$ and class $\mathcal{KL}$ functions that satisfy inequalities (4.19) and (4.20). This is the case because, for autonomous systems, stability and asymptotic stability of the origin are uniform with respect to the initial time t_0.

A special case of uniform asymptotic stability arises when the class $\mathcal{KL}$ function β in (4.20) takes the form $\beta(r, s) = kre^{-\lambda s}$. This case is very important and will be designated as a distinct stability property of equilibrium points.

Definition 4.5 *The equilibrium point $x = 0$ of (4.15) is exponentially stable if there exist positive constants c, k, and λ such that*

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)},\ \forall\, \|x(t_0)\| < c \tag{4.21}$$

and globally exponentially stable if (4.21) is satisfied for any initial state $x(t_0)$.

Lyapunov theory for autonomous systems can be extended to nonautonomous systems. For each of Theorems 4.1 through 4.4, one can state various extensions to nonautonomous systems. We will not document all these extensions here.[19] Instead, we concentrate on uniform stability and uniform asymptotic stability. These are the cases we encounter in most nonautonomous applications of Lyapunov's method.

Theorem 4.8 *Let $x = 0$ be an equilibrium point for (4.15) and $D \subset R^n$ be a domain containing $x = 0$. Let $V : [0, \infty) \times D \to R$ be a continuously differentiable function such that*

$$W_1(x) \le V(t, x) \le W_2(x) \tag{4.22}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le 0 \tag{4.23}$$

$\forall\ t \ge 0$ and $\forall\ x \in D$, where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions on D. Then, $x = 0$ is uniformly stable. $\diamond$

Proof: The derivative of V along the trajectories of (4.15) is given by

$$\dot{V}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \ \le\ 0$$

Choose $r > 0$ and $c > 0$ such that $B_r \subset D$ and $c < \min_{\|x\|=r} W_1(x)$. Then, $\{x \in B_r \mid W_1(x) \le c\}$ is in the interior of B_r. Define a time-dependent set $\Omega_{t,c}$ by

$$\Omega_{t,c} = \{x \in B_r \mid V(t, x) \le c\}$$

The set $\Omega_{t,c}$ contains $\{x \in B_r \mid W_2(x) \le c\}$ since

$$W_2(x) \le c \Rightarrow V(t, x) \le c$$

On the other hand, $\Omega_{t,c}$ is a subset of $\{x \in B_r \mid W_1(x) \le c\}$ since

$$V(t, x) \le c \Rightarrow W_1(x) \le c$$

Thus,

$$\{x \in B_r \mid W_2(x) \le c\} \subset \Omega_{t,c} \subset \{x \in B_r \mid W_1(x) \le c\} \subset B_r \subset D$$

for all $t \ge 0$. These five nested sets are sketched in Figure 4.7. The setup of Figure 4.7 is similar to that of Figure 4.1, except that the surface $V(t, x) = c$ is now dependent on t, and that is why it is surrounded by the time-independent surfaces $W_1(x) = c$ and $W_2(x) = c$.

Since $\dot{V}(t, x) \le 0$ on D, for any $t_0 \ge 0$ and any $x_0 \in \Omega_{t_0,c}$, the solution starting at (t_0, x_0) stays in $\Omega_{t,c}$ for all $t \ge t_0$. Therefore, any solution starting in $\{x \in$

[19] Lyapunov theory for nonautonomous systems is well documented in the literature. Good references on the subject include [72] and [154], while good introductions can be found in [201] and [135].

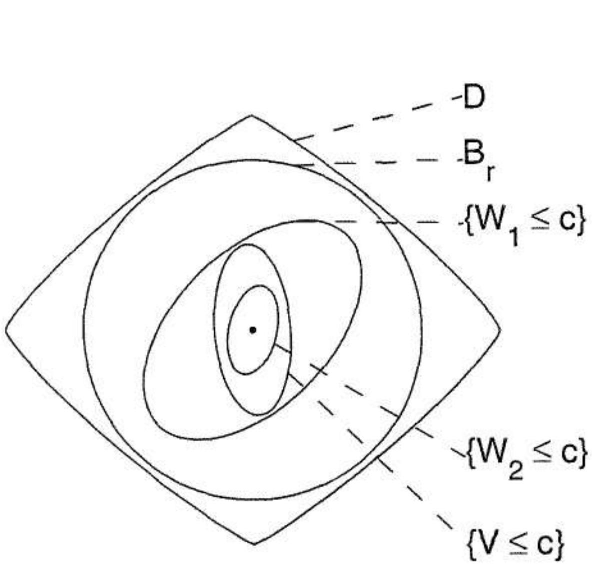


Figure 4.7: Geometric representation of sets in the proof of Theorem 4.8.

$B_r \mid W_2(x) \le c\}$ stays in $\Omega_{t,c}$, and consequently in $\{x \in B_r \mid W_1(x) \le c\}$, for all future time. Hence, the solution is bounded and defined for all $t \ge t_0$. Moreover, since $\dot{V} \le 0$,

$$V(t, x(t)) \le V(t_0, x(t_0)), \quad \forall\ t \ge t_0$$

By Lemma 4.3, there exist class $\mathcal{K}$ functions α_1 and α_2, defined on $[0, r]$, such that

$$\alpha_1(\|x\|) \le W_1(x) \le V(t, x) \le W_2(x) \le \alpha_2(\|x\|)$$

Combining the preceding two inequalities, we see that

$$\|x(t)\| \le \alpha_1^{-1}(V(t, x(t))) \le \alpha_1^{-1}(V(t_0, x(t_0))) \le \alpha_1^{-1}(\alpha_2(\|x(t_0)\|))$$

Since $\alpha_1^{-1} \circ \alpha_2$ is a class $\mathcal{K}$ function (by Lemma 4.2), the inequality $\|x(t)\| \le \alpha_1^{-1}(\alpha_2(\|x(t_0)\|))$ shows that the origin is uniformly stable. □

Theorem 4.9 *Suppose the assumptions of Theorem* 4.8 *are satisfied with inequality* (4.23) *strengthened to*

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le -W_3(x) \tag{4.24}$$

$\forall\ t \ge 0$ *and* $\forall\ x \in D$, *where* $W_3(x)$ *is a continuous positive definite function on* D. *Then,* $x = 0$ *is uniformly asymptotically stable. Moreover, if* r *and* c *are chosen such that* $B_r = \{\|x\| \le r\} \subset D$ *and* $c < \min_{\|x\|=r} W_1(x)$, *then every trajectory starting in* $\{x \in B_r \mid W_2(x) \le c\}$ *satisfies*

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \quad \forall\ t \ge t_0 \ge 0$$

for some class $\mathcal{KL}$ *function* β. *Finally, if* $D = R^n$ *and* $W_1(x)$ *is radially unbounded, then* $x = 0$ *is globally uniformly asymptotically stable.* ◇

Proof: Continuing with the proof of Theorem 4.8, we know that trajectories starting in $\{x \in B_r \mid W_2(x) \leq c\}$ stay in $\{x \in B_r \mid W_1(x) \leq c\}$ for all $t \geq t_0$. By Lemma 4.3, there exists a class $\mathcal{K}$ function α_3, defined on $[0, r]$, such that

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x) \leq -\alpha_3(\|x\|)$$

Using the inequality

$$V \leq \alpha_2(\|x\|) \;\Leftrightarrow\; \alpha_2^{-1}(V) \leq \|x\| \;\Leftrightarrow\; \alpha_3(\alpha_2^{-1}(V)) \leq \alpha_3(\|x\|)$$

we see that V satisfies the differential inequality

$$\dot{V} \leq \; -\alpha_3(\alpha_2^{-1}(V)) \stackrel{\text{def}}{=} -\alpha(V)$$

where $\alpha = \alpha_3 \circ \alpha_2^{-1}$ is a class $\mathcal{K}$ function defined on $[0, r]$. (See Lemma 4.2.) Assume, without loss of generality,[20] that α is locally Lipschitz. Let $y(t)$ satisfy the autonomous first-order differential equation

$$\dot{y} = -\alpha(y), \quad y(t_0) = V(t_0, x(t_0)) \geq 0$$

By (the comparison) Lemma 3.4,

$$V(t, x(t)) \leq y(t), \quad \forall\, t \geq t_0$$

By Lemma 4.4, there exists a class $\mathcal{KL}$ function $\sigma(r, s)$ defined on $[0, r] \times [0, \infty)$ such that

$$V(t, x(t)) \leq \sigma(V(t_0, x(t_0)), t - t_0), \quad \forall\, V(t_0, x(t_0)) \in [0, c]$$

Therefore, any solution starting in $\{x \in B_r \mid W_2(x) \leq c\}$ satisfies the inequality

$$\begin{aligned} \|x(t)\| \;&\leq\; \alpha_1^{-1}(V(t, x(t))) \;\leq\; \alpha_1^{-1}\left(\sigma(V(t_0, x(t_0)), t - t_0)\right) \\ &\leq\; \alpha_1^{-1}\left(\sigma(\alpha_2(\|x(t_0)\|), t - t_0)\right) \;\stackrel{\text{def}}{=}\; \beta(\|x(t_0)\|, t - t_0) \end{aligned}$$

Lemma 4.2 shows that β is a class $\mathcal{KL}$ function. Thus, inequality (4.20) is satisfied, which implies that $x = 0$ is uniformly asymptotically stable. If $D = R^n$, the functions α_1, α_2, and α_3 are defined on $[0, \infty)$. Hence, α, and consequently β, are independent of c. As $W_1(x)$ is radially unbounded, c can be chosen arbitrarily large to include any initial state in $\{W_2(x) \leq c\}$. Thus, (4.20) holds for any initial state, showing that the origin is globally uniformly asymptotically stable. $\square$

[20] If α is not locally Lipschitz, we can choose a locally Lipschitz class $\mathcal{K}$ function β such that $\alpha(r) \geq \beta(r)$ over the domain of interest. Then, $\dot{V} \leq -\beta(V)$, and we can continue the proof with β instead of α. For example, suppose $\alpha(r) = \sqrt{r}$. The function $\sqrt{r}$ is a class $\mathcal{K}$ function, but not locally Lipschitz at $r = 0$. Define β as $\beta(r) = r$, for $r < 1$ and $\beta(r) = \sqrt{r}$, for $r \geq 1$. The function β is class $\mathcal{K}$ and locally Lipschitz. Moreover, $\alpha(r) \geq \beta(r)$ for all $r \geq 0$.

A function $V(t,x)$ is said to be *positive semidefinite* if $V(t,x) \geq 0$. It is said to be *positive definite* if $V(t,x) \geq W_1(x)$ for some positive definite function $W_1(x)$, *radially unbounded* if $W_1(x)$ is so, and *decrescent* if $V(t,x) \leq W_2(x)$. A function $V(t,x)$ is said to be *negative definite (semidefinite)* if $-V(t,x)$ is *positive definite (semidefinite)*. Therefore, Theorems 4.8 and 4.9 say that *the origin is uniformly stable if there is a continuously differentiable, positive definite, decrescent function $V(t,x)$, whose derivative along the trajectories of the system is negative semidefinite. It is uniformly asymptotically stable if the derivative is negative definite, and globally uniformly asymptotically stable if the conditions for uniform asymptotic stability hold globally with a radially unbounded $V(t,x)$.*

Theorem 4.10 *Let $x = 0$ be an equilibrium point for (4.15) and $D \subset R^n$ be a domain containing $x = 0$. Let $V : [0,\infty) \times D \to R$ be a continuously differentiable function such that*

$$k_1\|x\|^a \leq V(t,x) \leq k_2\|x\|^a \tag{4.25}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -k_3\|x\|^a \tag{4.26}$$

$\forall\, t \geq 0$ and $\forall\, x \in D$, where k_1, k_2, k_3, and a are positive constants. Then, $x = 0$ is exponentially stable. If the assumptions hold globally, then $x = 0$ is globally exponentially stable. ◇

Proof: With the help of Figure 4.7, it can be seen that trajectories starting in $\{k_2\|x\|^a \leq c\}$, for sufficiently small c, remain bounded for all $t \geq t_0$. Inequalities (4.25) and (4.26) show that V satisfies the differential inequality

$$\dot{V} \leq -\frac{k_3}{k_2} V$$

By (the comparison) Lemma 3.4,

$$V(t,x(t)) \leq V(t_0,x(t_0))e^{-(k_3/k_2)(t-t_0)}$$

Hence,

$$\begin{aligned}
\|x(t)\| &\leq \left[\frac{V(t,x(t))}{k_1}\right]^{1/a} \leq \left[\frac{V(t_0,x(t_0))e^{-(k_3/k_2)(t-t_0)}}{k_1}\right]^{1/a} \\
&\leq \left[\frac{k_2\|x(t_0)\|^a e^{-(k_3/k_2)(t-t_0)}}{k_1}\right]^{1/a} = \left(\frac{k_2}{k_1}\right)^{1/a} \|x(t_0)\| e^{-(k_3/k_2 a)(t-t_0)}
\end{aligned}$$

Thus, the origin is exponentially stable. If all the assumptions hold globally, c can be chosen arbitrarily large and the foregoing inequality holds for all $x(t_0) \in R^n$. □

Example 4.19 Consider the scalar system

$$\dot{x} = -[1 + g(t)]x^3$$

where $g(t)$ is continuous and $g(t) \geq 0$ for all $t \geq 0$. Using the Lyapunov function candidate $V(x) = x^2/2$, we obtain

$$\dot{V}(t,x) = -[1 + g(t)]x^4 \leq -x^4, \quad \forall\, x \in R,\ \forall\, t \geq 0$$

The assumptions of Theorem 4.9 are satisfied globally with $W_1(x) = W_2(x) = V(x)$ and $W_3(x) = x^4$. Hence, the origin is globally uniformly asymptotically stable. △

Example 4.20 Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 - g(t)x_2 \\ \dot{x}_2 &= x_1 - x_2 \end{aligned}$$

where $g(t)$ is continuously differentiable and satisfies

$$0 \leq g(t) \leq k \ \text{ and } \ \dot{g}(t) \leq g(t), \quad \forall\, t \geq 0$$

Taking $V(t,x) = x_1^2 + [1+g(t)]x_2^2$ as a Lyapunov function candidate, it can be easily seen that

$$x_1^2 + x_2^2 \leq V(t,x) \leq x_1^2 + (1+k)x_2^2, \quad \forall\, x \in R^2$$

Hence, $V(t,x)$ is positive definite, decrescent, and radially unbounded. The derivative of V along the trajectories of the system is given by

$$\dot{V}(t,x) = -2x_1^2 + 2x_1x_2 - [2 + 2g(t) - \dot{g}(t)]x_2^2$$

Using the inequality

$$2 + 2g(t) - \dot{g}(t) \geq 2 + 2g(t) - g(t) \geq 2$$

we obtain

$$\dot{V}(t,x) \leq -2x_1^2 + 2x_1x_2 - 2x_2^2 = -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \stackrel{\text{def}}{=} -x^TQx$$

where Q is positive definite; therefore, $\dot{V}(t,x)$ is negative definite. Thus, all the assumptions of Theorem 4.9 are satisfied globally with positive definite quadratic functions W_1, W_2, and W_3. Recalling that a positive definite quadratic function x^TPx satisfies

$$\lambda_{min}(P)x^Tx \leq x^TPx \leq \lambda_{max}(P)x^Tx$$

we see that the conditions of Theorem 4.10 are satisfied globally with $a = 2$. Hence, the origin is globally exponentially stable. △

Example 4.21 The linear time-varying system

$$\dot{x} = A(t)x \tag{4.27}$$

has an equilibrium point at $x = 0$. Let $A(t)$ be continuous for all $t \geq 0$. Suppose there is a continuously differentiable, symmetric, bounded, positive definite matrix $P(t)$; that is,

$$0 < c_1 I \leq P(t) \leq c_2 I, \quad \forall\, t \geq 0$$

which satisfies the matrix differential equation

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t) \tag{4.28}$$

where $Q(t)$ is continuous, symmetric, and positive definite; that is,

$$Q(t) \geq c_3 I > 0, \quad \forall\, t \geq 0$$

The Lyapunov function candidate

$$V(t,x) = x^T P(t) x$$

satisfies

$$c_1 \|x\|_2^2 \leq V(t,x) \leq c_2 \|x\|_2^2$$

and its derivative along the trajectories of the system (4.27) is given by

$$\begin{aligned} \dot{V}(t,x) &= x^T \dot{P}(t) x + x^T P(t) \dot{x} + \dot{x}^T P(t) x \\ &= x^T [\dot{P}(t) + P(t)A(t) + A^T(t)P(t)] x \;=\; -x^T Q(t) x \;\leq\; -c_3 \|x\|_2^2 \end{aligned}$$

Thus, all the assumptions of Theorem 4.10 are satisfied globally with $a = 2$, and we conclude that the origin is globally exponentially stable. △

4.6 Linear Time-Varying Systems and Linearization

The stability behavior of the origin as an equilibrium point for the linear time-varying system

$$\dot{x}(t) = A(t)x \tag{4.29}$$

can be completely characterized in terms of the state transition matrix of the system. From linear system theory,[21] we know that the solution of (4.29) is given by

$$x(t) = \Phi(t, t_0) x(t_0)$$

where $\Phi(t, t_0)$ is the state transition matrix. The next theorem characterizes uniform asymptotic stability in terms of $\Phi(t, t_0)$.

Theorem 4.11 *The equilibrium point $x = 0$ of (4.29) is (globally) uniformly asymptotically stable if and only if the state transition matrix satisfies the inequality*

$$\|\Phi(t, t_0)\| \leq k e^{-\lambda(t - t_0)}, \quad \forall\, t \geq t_0 \geq 0 \tag{4.30}$$

for some positive constants k and λ. ◇

[21] See, for example, [9], [35], [94], or [158].

Proof: Due to the linear dependence of $x(t)$ on $x(t_0)$, if the origin is uniformly asymptotically stable, it is globally so. Sufficiency of (4.30) is obvious since

$$\|x(t)\| \leq \|\Phi(t,t_0)\| \; \|x(t_0)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}$$

To prove necessity, suppose the origin is uniformly asymptotically stable. Then, there is a class $\mathcal{KL}$ function β such that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t-t_0), \quad \forall\; t \geq t_0, \; \forall\; x(t_0) \in R^n$$

From the definition of an induced matrix norm (Appendix A), we have

$$\|\Phi(t,t_0)\| = \max_{\|x\|=1} \|\Phi(t,t_0)x\| \leq \max_{\|x\|=1} \beta(\|x\|, t-t_0) = \beta(1, t-t_0)$$

Since

$$\beta(1,s) \to 0 \;\; \text{as} \;\; s \to \infty$$

there exists $T > 0$ such that $\beta(1,T) \leq 1/e$. For any $t \geq t_0$, let N be the smallest positive integer such that $t \leq t_0 + NT$. Divide the interval $[t_0, t_0 + (N-1)T]$ into $(N-1)$ equal subintervals of width T each. Using the transition property of $\Phi(t,t_0)$, we can write

$$\Phi(t,t_0) = \Phi(t, t_0 + (N-1)T)\Phi(t_0 + (N-1)T, t_0 + (N-2)T) \cdots \Phi(t_0 + T, t_0)$$

Hence,

$$\begin{aligned}
\|\Phi(t,t_0)\| &\leq \|\Phi(t, t_0 + (N-1)T)\| \prod_{k=1}^{k=N-1} \|\Phi(t_0 + kT, t_0 + (k-1)T)\| \\
&\leq \beta(1,0) \prod_{k=1}^{k=N-1} \frac{1}{e} \; = \; e\beta(1,0)e^{-N} \\
&\leq e\beta(1,0)e^{-(t-t_0)/T} \; = ke^{-\lambda(t-t_0)}
\end{aligned}$$

where $k = e\beta(1,0)$ and $\lambda = 1/T$. □

Theorem 4.11 shows that, for linear systems, uniform asymptotic stability of the origin is equivalent to exponential stability. Although inequality (4.30) characterizes uniform asymptotic stability of the origin without the need to search for a Lyapunov function, it is not as useful as the eigenvalue criterion we have for linear time-invariant systems, because knowledge of the state transition matrix $\Phi(t,t_0)$ requires solving the state equation (4.29). Note that, for linear time-varying systems, uniform asymptotic stability cannot be characterized by the location of the eigenvalues of the matrix A [22] as the following example shows.

[22] There are special cases where uniform asymptotic stability of the origin as an equilibrium point for (4.29) is equivalent to an eigenvalue condition. One case is periodic systems. (See Exercise 4.40 and Example 10.8.) Another case is slowly-varying systems. (See Example 9.9.)

Example 4.22 Consider a second-order linear system with

$$A(t) = \begin{bmatrix} -1 + 1.5\cos^2 t & 1 - 1.5\sin t\cos t \\ -1 - 1.5\sin t\cos t & -1 + 1.5\sin^2 t \end{bmatrix}$$

For each t, the eigenvalues of $A(t)$ are given by $-0.25 \pm 0.25\sqrt{7}j$. Thus, the eigenvalues are independent of t and lie in the open left-half plane. Yet, the origin is unstable. It can be verified that

$$\Phi(t,0) = \begin{bmatrix} e^{0.5t}\cos t & e^{-t}\sin t \\ \\ -e^{0.5t}\sin t & e^{-t}\cos t \end{bmatrix}$$

which shows that there are initial states $x(0)$, arbitrarily close to the origin, for which the solution is unbounded and escapes to infinity. △

Although Theorem 4.11 may not be very helpful as a stability test, we will see that it guarantees the existence of a Lyapunov function for the linear system (4.29). We saw in Example 4.21 that if we can find a positive definite, bounded matrix $P(t)$ that satisfies the differential equation (4.28) for some positive definite $Q(t)$, then $V(t,x) = x^T P(t)x$ is a Lyapunov function for the system. If the matrix $Q(t)$ is chosen to be bounded in addition to being positive definite, that is,

$$0 < c_3 I \leq Q(t) \leq c_4 I, \quad \forall\, t \geq 0$$

and if $A(t)$ is continuous and bounded, then it can be shown that when the origin is exponentially stable, there is a solution of (4.28) that possesses the desired properties.

Theorem 4.12 *Let $x = 0$ be the exponentially stable equilibrium point of* (4.29). *Suppose $A(t)$ is continuous and bounded. Let $Q(t)$ be a continuous, bounded, positive definite, symmetric matrix. Then, there is a continuously differentiable, bounded, positive definite, symmetric matrix $P(t)$ that satisfies* (4.28). *Hence, $V(t,x) = x^T P(t)x$ is a Lyapunov function for the system that satisfies the conditions of Theorem* 4.10. ◇

Proof: Let

$$P(t) = \int_t^\infty \Phi^T(\tau,t)Q(\tau)\Phi(\tau,t)\ d\tau$$

and $\phi(\tau;t,x)$ be the solution of (4.29) that starts at (t,x). Due to linearity, $\phi(\tau;t,x) = \Phi(\tau,t)x$. In view of the definition of $P(t)$, we have

$$x^T P(t)x = \int_t^\infty \phi^T(\tau;t,x)Q(\tau)\phi(\tau;t,x)\ d\tau$$

The use of (4.30) yields

$$\begin{aligned} x^T P(t)x &\leq \int_t^\infty c_4 \|\Phi(\tau,t)\|_2^2 \ \|x\|_2^2 d\tau \\ &\leq \int_t^\infty k^2 e^{-2\lambda(\tau-t)} \ d\tau \ c_4 \|x\|_2^2 \ = \ \frac{k^2 c_4}{2\lambda}\|x\|_2^2 \ \stackrel{\text{def}}{=} \ c_2\|x\|_2^2 \end{aligned}$$

On the other hand, since

$$\|A(t)\|_2 \leq L, \quad \forall \ t \geq 0$$

the solution $\phi(\tau;t,x)$ satisfies the lower bound[23]

$$\|\phi(\tau;t,x)\|_2^2 \geq \|x\|_2^2 e^{-2L(\tau-t)}$$

Hence,

$$\begin{aligned} x^T P(t)x &\geq \int_t^\infty c_3 \|\phi(\tau;t,x)\|_2^2 \ d\tau \\ &\geq \int_t^\infty e^{-2L(\tau-t)} \ d\tau \ c_3\|x\|_2^2 \ = \ \frac{c_3}{2L}\|x\|_2^2 \ \stackrel{\text{def}}{=} \ c_1\|x\|_2^2 \end{aligned}$$

Thus,

$$c_1\|x\|_2^2 \leq x^T P(t)x \leq c_2\|x\|_2^2$$

which shows that $P(t)$ is positive definite and bounded. The definition of $P(t)$ shows that it is symmetric and continuously differentiable. The fact that $P(t)$ satisfies (4.28) can be shown by differentiating $P(t)$ and using the property

$$\frac{\partial}{\partial t}\Phi(\tau,t) = -\Phi(\tau,t)A(t)$$

In particular,

$$\begin{aligned} \dot{P}(t) &= \int_t^\infty \Phi^T(\tau,t)Q(\tau)\frac{\partial}{\partial t}\Phi(\tau,t) \ d\tau \\ &\quad + \int_t^\infty \left[\frac{\partial}{\partial t}\Phi^T(\tau,t)\right] Q(\tau)\Phi(\tau,t) \ d\tau \ - \ Q(t) \\ &= -\int_t^\infty \Phi^T(\tau,t)Q(\tau)\Phi(\tau,t) \ d\tau \ A(t) \\ &\quad -A^T(t)\int_t^\infty \Phi^T(\tau,t)Q(\tau)\Phi(\tau,t) \ d\tau \ - \ Q(t) \\ &= -P(t)A(t) - A^T(t)P(t) - Q(t) \end{aligned}$$

The fact that $V(t,x) = x^T P(t)x$ is a Lyapunov function is shown in Example 4.21. □

[23]See Exercise 3.17.

When the linear system (4.29) is time invariant, that is, when A is constant, the Lyapunov function $V(t,x)$ of Theorem 4.12 can be chosen to be independent of t. Recall that, for linear time-invariant systems,

$$\Phi(\tau,t) = \exp[(\tau - t)A]$$

which satisfies (4.30) when A is Hurwitz. Choosing Q to be a positive definite, symmetric (constant) matrix, the matrix $P(t)$ is given by

$$P = \int_t^\infty \exp[(\tau - t)A^T]Q\exp[(\tau - t)A]\ d\tau = \int_0^\infty \exp[A^T s]Q\exp[As]\ ds$$

which is independent of t. Comparing this expression for P with (4.13) shows that P is the unique solution of the Lyapunov equation (4.12). Thus, the Lyapunov function of Theorem 4.12 reduces to the one we used in Section 4.3.

The existence of Lyapunov functions for linear systems per Theorem 4.12 will now be used to prove a linearization result that extends Theorem 4.7 to the nonautonomous case. Consider the nonlinear nonautonomous system

$$\dot{x} = f(t,x) \tag{4.31}$$

where $f : [0,\infty) \times D \to R^n$ is continuously differentiable and $D = \{x \in R^n \mid \|x\|_2 < r\}$. Suppose the origin $x = 0$ is an equilibrium point for the system at $t = 0$; that is, $f(t,0) = 0$ for all $t \geq 0$. Furthermore, suppose the Jacobian matrix $[\partial f/\partial x]$ is bounded and Lipschitz on D, uniformly in t; thus,

$$\left\| \frac{\partial f_i}{\partial x}(t,x_1) - \frac{\partial f_i}{\partial x}(t,x_2) \right\|_2 \leq L_1\|x_1 - x_2\|_2, \quad \forall\ x_1, x_2 \in D,\ \forall\ t \geq 0$$

for all $1 \leq i \leq n$. By the mean value theorem,

$$f_i(t,x) = f_i(t,0) + \frac{\partial f_i}{\partial x}(t,z_i)\ x$$

where z_i is a point on the line segment connecting x to the origin. Since $f(t,0) = 0$, we can write $f_i(t,x)$ as

$$f_i(t,x) = \frac{\partial f_i}{\partial x}(t,z_i)\ x = \frac{\partial f_i}{\partial x}(t,0)\ x + \left[\frac{\partial f_i}{\partial x}(t,z_i) - \frac{\partial f_i}{\partial x}(t,0)\right] x$$

Hence,

$$f(t,x) = A(t)x + g(t,x)$$

where

$$A(t) = \frac{\partial f}{\partial x}(t,0) \ \text{ and } \ g_i(t,x) = \left[\frac{\partial f_i}{\partial x}(t,z_i) - \frac{\partial f_i}{\partial x}(t,0)\right] x$$

The function $g(t, x)$ satisfies

$$\|g(t,x)\|_2 \leq \left(\sum_{i=1}^{n} \left\| \frac{\partial f_i}{\partial x}(t, z_i) - \frac{\partial f_i}{\partial x}(t, 0) \right\|_2^2 \right)^{1/2} \|x\|_2 \leq L \|x\|_2^2$$

where $L = \sqrt{n} L_1$. Therefore, in a small neighborhood of the origin, we may approximate the nonlinear system (4.31) by its linearization about the origin. The next theorem states Lyapunov's indirect method for showing exponential stability of the origin in the nonautonomous case.

Theorem 4.13 *Let $x = 0$ be an equilibrium point for the nonlinear system*

$$\dot{x} = f(t, x)$$

where $f : [0, \infty) \times D \to R^n$ is continuously differentiable, $D = \{x \in R^n \mid \|x\|_2 < r\}$, and the Jacobian matrix $[\partial f / \partial x]$ is bounded and Lipschitz on D, uniformly in t. Let

$$A(t) = \left. \frac{\partial f}{\partial x}(t, x) \right|_{x=0}$$

Then, the origin is an exponentially stable equilibrium point for the nonlinear system if it is an exponentially stable equilibrium point for the linear system

$$\dot{x} = A(t)x$$

◇

Proof: Since the linear system has an exponentially stable equilibrium point at the origin and $A(t)$ is continuous and bounded, Theorem 4.12 ensures the existence of a continuously differentiable, bounded, positive definite symmetric matrix $P(t)$ that satisfies (4.28), where $Q(t)$ is continuous, positive definite, and symmetric. We use $V(t, x) = x^T P(t) x$ as a Lyapunov function candidate for the nonlinear system. The derivative of $V(t, x)$ along the trajectories of the system is given by

$$\begin{aligned}
\dot{V}(t, x) &= x^T P(t) f(t, x) + f^T(t, x) P(t) x + x^T \dot{P}(t) x \\
&= x^T [P(t) A(t) + A^T(t) P(t) + \dot{P}(t)] x + 2x^T P(t) g(t, x) \\
&= -x^T Q(t) x + 2x^T P(t) g(t, x) \\
&\leq -c_3 \|x\|_2^2 + 2c_2 L \|x\|_2^3 \\
&\leq -(c_3 - 2c_2 L \rho) \|x\|_2^2, \quad \forall \ \|x\|_2 < \rho
\end{aligned}$$

Choosing $\rho < \min\{r, c_3/(2c_2 L)\}$ ensures that $\dot{V}(t, x)$ is negative definite in $\|x\|_2 < \rho$. Therefore, all the conditions of Theorem 4.10 are satisfied in $\|x\|_2 < \rho$, and we conclude that the origin is exponentially stable. □

4.7 Converse Theorems

Theorems 4.9 and 4.10 establish uniform asymptotic stability or exponential stability of the origin by requiring the existence of a Lyapunov function $V(t, x)$ that satisfies certain conditions. Requiring the existence of an auxiliary function $V(t, x)$ that satisfies certain conditions is typical in many theorems of Lyapunov's method. The conditions of these theorems cannot be checked directly on the data of the problem. Instead, one has to search for the auxiliary function. Faced with this searching problem, two questions come to mind. First, is there a function that satisfies the conditions of the theorem? Second, how can we search for such a function? In many cases, Lyapunov theory provides an affirmative answer to the first question. The answer takes the form of a converse Lyapunov theorem, which is the inverse of one of Lyapunov's theorems. For example, a converse theorem for uniform asymptotic stability would confirm that if the origin is uniformly asymptotically stable, then there is a Lyapunov function that satisfies the conditions of Theorem 4.9. Most of these converse theorems are proven by actually constructing auxiliary functions that satisfy the conditions of the respective theorems. Unfortunately, this construction almost always assumes the knowledge of the solution of the differential equation. Therefore, the theorems do not help in the practical search for an auxiliary function. The mere knowledge that a function exists is, however, better than nothing. At least, we know that our search is not hopeless. The theorems are also useful in using Lyapunov theory to draw conceptual conclusions about the behavior of dynamical systems. Theorem 4.15 is an example of such use. Other examples will appear in the following chapters. In this section, we give three converse Lyapunov theorems.[24] The first one is a converse Lyapunov theorem when the origin is exponentially stable and, the second, when it is uniformly asymptotically stable. The third theorem applies to autonomous systems and defines the converse Lyapunov function for the whole region of attraction of an asymptotically stable equilibrium point.

The idea of constructing a converse Lyapunov function is not new to us. We have done it for linear systems in the proof of Theorem 4.12. A careful reading of that proof shows that linearity of the system does not play a crucial role in the proof, except for showing that $V(t, x)$ is quadratic in x. This observation leads to the first of our three converse theorems, whose proof is a simple extension of the proof of Theorem 4.12.

Theorem 4.14 *Let $x = 0$ be an equilibrium point for the nonlinear system*

$$\dot{x} = f(t, x)$$

where $f : [0, \infty) \times D \to R^n$ is continuously differentiable, $D = \{x \in R^n \mid \|x\| < r\}$, and the Jacobian matrix $[\partial f / \partial x]$ is bounded on D, uniformly in t. Let k, λ, and r_0

[24] See [72] or [107] for a comprehensive treatment of converse Lyapunov theorems and [118] and [193] for more recent results.

be positive constants with $r_0 < r/k$. Let $D_0 = \{x \in R^n \mid \|x\| < r_0\}$. Assume that the trajectories of the system satisfy

$$\|x(t)\| \le k\|x(t_0)\|e^{-\lambda(t-t_0)}, \quad \forall\ x(t_0) \in D_0,\ \forall\ t \ge t_0 \ge 0$$

Then, there is a function $V : [0,\infty) \times D_0 \to R$ that satisfies the inequalities

$$c_1\|x\|^2 \le V(t,x) \le c_2\|x\|^2$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -c_3\|x\|^2$$

$$\left\|\frac{\partial V}{\partial x}\right\| \le c_4\|x\|$$

for some positive constants c_1, c_2, c_3, and c_4. Moreover, if $r = \infty$ and the origin is globally exponentially stable, then $V(t,x)$ is defined and satisfies the aforementioned inequalities on R^n. Furthermore, if the system is autonomous, V can be chosen independent of t. ◇

Proof: Due to the equivalence of norms, it is sufficient to prove the theorem for the 2-norm. Let $\phi(\tau; t, x)$ denote the solution of the system that starts at (t,x); that is, $\phi(t;t,x) = x$. For all $x \in D_0$, $\phi(\tau;t,x) \in D$ for all $\tau \ge t$. Let

$$V(t,x) = \int_t^{t+\delta} \phi^T(\tau;t,x)\phi(\tau;t,x)\ d\tau$$

where δ is a positive constant to be chosen. Due to the exponentially decaying bound on the trajectories, we have

$$\begin{aligned} V(t,x) &= \int_t^{t+\delta} \|\phi(\tau;t,x)\|_2^2\ d\tau \\ &\le \int_t^{t+\delta} k^2 e^{-2\lambda(\tau-t)}\ d\tau\ \|x\|_2^2 \ = \ \frac{k^2}{2\lambda}(1 - e^{-2\lambda\delta})\|x\|_2^2 \end{aligned}$$

On the other hand, the Jacobian matrix $[\partial f/\partial x]$ is bounded on D. Let

$$\left\|\frac{\partial f}{\partial x}(t,x)\right\|_2 \le L, \quad \forall\ x \in D$$

Then, $\|f(t,x)\|_2 \le L\|x\|_2$ and $\phi(\tau;t,x)$ satisfies the lower bound[25]

$$\|\phi(\tau;t,x)\|_2^2 \ge \|x\|_2^2 e^{-2L(\tau-t)}$$

Hence,

$$V(t,x) \ge \int_t^{t+\delta} e^{-2L(\tau-t)}\ d\tau\ \|x\|_2^2 = \frac{1}{2L}(1 - e^{-2L\delta})\|x\|_2^2$$

[25] See Exercise 3.17.

Thus, $V(t,x)$ satisfies the first inequality of the theorem with

$$c_1 = \frac{(1 - e^{-2L\delta})}{2L} \quad \text{and} \quad c_2 = \frac{k^2(1 - e^{-2\lambda\delta})}{2\lambda}$$

To calculate the derivative of V along the trajectories of the system, define the sensitivity functions

$$\phi_t(\tau; t, x) = \frac{\partial}{\partial t}\phi(\tau; t, x); \quad \phi_x(\tau; t, x) = \frac{\partial}{\partial x}\phi(\tau; t, x)$$

Then,

$$\begin{aligned}
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) &= \phi^T(t+\delta; t, x)\phi(t+\delta; t, x) - \phi^T(t; t, x)\phi(t; t, x) \\
&\quad + \int_t^{t+\delta} 2\phi^T(\tau; t, x)\phi_t(\tau; t, x)\ d\tau \\
&\quad + \int_t^{t+\delta} 2\phi^T(\tau; t, x)\phi_x(\tau; t, x)\ d\tau f(t,x) \\
&= \phi^T(t+\delta; t, x)\phi(t+\delta; t, x) - \|x\|_2^2 \\
&\quad + \int_t^{t+\delta} 2\phi^T(\tau; t, x)[\phi_t(\tau; t, x) + \phi_x(\tau; t, x) f(t,x)]\ d\tau
\end{aligned}$$

It is not difficult to show that[26]

$$\phi_t(\tau; t, x) + \phi_x(\tau; t, x) f(t,x) \equiv 0, \quad \forall\ \tau \geq t$$

Therefore,

$$\begin{aligned}
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) &= \phi^T(t+\delta; t, x)\phi(t+\delta; t, x) - \|x\|_2^2 \\
&\leq -(1 - k^2 e^{-2\lambda\delta})\|x\|_2^2
\end{aligned}$$

By choosing $\delta = \ln(2k^2)/(2\lambda)$, the second inequality of the theorem is satisfied with $c_3 = 1/2$. To show the last inequality, let us note that $\phi_x(\tau; t, x)$ satisfies the sensitivity equation

$$\frac{\partial}{\partial \tau}\phi_x = \frac{\partial f}{\partial x}(\tau, \phi(\tau; t, x))\ \phi_x, \quad \phi_x(t; t, x) = I$$

Since

$$\left\| \frac{\partial f}{\partial x}(t,x) \right\|_2 \leq L$$

[26] See Exercise 3.30.

on D, ϕ_x satisfies the bound[27]

$$\|\phi_x(\tau; t, x)\|_2 \leq e^{L(\tau - t)}$$

Therefore,

$$\begin{aligned}
\left\| \frac{\partial V}{\partial x} \right\|_2 &= \left\| \int_t^{t+\delta} 2\phi^T(\tau; t, x)\phi_x(\tau; t, x) \, d\tau \right\|_2 \\
&\leq \int_t^{t+\delta} 2\|\phi(\tau; t, x)\|_2 \, \|\phi_x(\tau; t, x)\|_2 \, d\tau \\
&\leq \int_t^{t+\delta} 2ke^{-\lambda(\tau - t)} \, e^{L(\tau - t)} \, d\tau \, \|x\|_2 \\
&= \frac{2k}{(\lambda - L)}[1 - e^{-(\lambda - L)\delta}]\|x\|_2
\end{aligned}$$

Thus, the last inequality of the theorem is satisfied with

$$c_4 = \frac{2k}{(\lambda - L)}[1 - e^{-(\lambda - L)\delta}]$$

If all the assumptions hold globally, then clearly r_0 can be chosen arbitrarily large. If the system is autonomous, then $\phi(\tau; t, x)$ depends only on $(\tau - t)$; that is,

$$\phi(\tau; t, x) = \psi(\tau - t; x)$$

Then,

$$V(t, x) = \int_t^{t+\delta} \psi^T(\tau - t; x)\psi(\tau - t; x) \, d\tau = \int_0^{\delta} \psi^T(s; x)\psi(s; x) \, ds$$

which is independent of t. □

In Theorem 4.13, we saw that if the linearization of a nonlinear system about the origin has an exponentially stable equilibrium point, then the origin is an exponentially stable equilibrium point for the nonlinear system. We will use Theorem 4.14 to prove that exponential stability of the linearization is a necessary and sufficient condition for exponential stability of the origin.

Theorem 4.15 *Let $x = 0$ be an equilibrium point for the nonlinear system*

$$\dot{x} = f(t, x)$$

where $f : [0, \infty) \times D \to R^n$ is continuously differentiable, $D = \{x \in R^n \mid \|x\|_2 < r\}$, and the Jacobian matrix $[\partial f / \partial x]$ is bounded and Lipschitz on D, uniformly in t. Let

$$A(t) = \left. \frac{\partial f}{\partial x}(t, x) \right|_{x=0}$$

[27] See Exercise 3.17.

Then, $x = 0$ is an exponentially stable equilibrium point for the nonlinear system if and only if it is an exponentially stable equilibrium point for the linear system

$$\dot{x} = A(t)x$$

◇

Proof: The "if" part follows from Theorem 4.13. To prove the "only if" part, write the linear system as

$$\dot{x} = f(t,x) - [f(t,x) - A(t)x] = f(t,x) - g(t,x)$$

Recalling the argument preceding Theorem 4.13, we know that

$$\|g(t,x)\|_2 \le L\|x\|_2^2, \quad \forall\; x \in D, \; \forall\; t \ge 0$$

Since the origin is an exponentially stable equilibrium of the nonlinear system, there are positive constants k, λ, and c such that

$$\|x(t)\|_2 \le k\|x(t_0)\|_2 e^{-\lambda(t-t_0)}, \quad \forall\; t \ge t_0 \ge 0, \; \forall\; \|x(t_0)\|_2 < c$$

Choosing $r_0 < \min\{c, r/k\}$, all the conditions of Theorem 4.14 are satisfied. Let $V(t,x)$ be the function provided by Theorem 4.14 and use it as a Lyapunov function candidate for the linear system. Then,

$$\begin{aligned}
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}A(t)x &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x) - \frac{\partial V}{\partial x}g(t,x) \\
&\le -c_3\|x\|_2^2 + c_4 L\|x\|_2^3 \\
&< -(c_3 - c_4 L\rho)\|x\|_2^2, \qquad \forall\; \|x\|_2 < \rho
\end{aligned}$$

The choice $\rho < \min\{r_0, c_3/(c_4 L)\}$ ensures that $\dot{V}(t,x)$ is negative definite in $\|x\|_2 < \rho$. Consequently, all the conditions of Theorem 4.10 are satisfied in $\|x\|_2 < \rho$, and we conclude that the origin is an exponentially stable equilibrium point for the linear system. □

Corollary 4.3 *Let $x = 0$ be an equilibrium point of the nonlinear system $\dot{x} = f(x)$, where $f(x)$ is continuously differentiable in some neighborhood of $x = 0$. Let $A = [\partial f/\partial x](0)$. Then, $x = 0$ is an exponentially stable equilibrium point for the nonlinear system if and only if A is Hurwitz.* ◇

Example 4.23 Consider the first-order system $\dot{x} = -x^3$. We saw in Example 4.14 that the origin is asymptotically stable, but linearization about the origin results in the linear system $\dot{x} = 0$, whose A matrix is not Hurwitz. Using Corollary 4.3, we conclude that the origin is not exponentially stable. △

The following converse Lyapunov theorems (Theorem 4.16 and 4.17) extend Theorem 4.15 in two different directions, but their proofs are more involved. Theorem 4.16 applies to the more general case of uniform asymptotic stability.[28] Theorem 4.17 applies to autonomous systems and produces a Lyapunov function that is defined on the whole region of attraction.

Theorem 4.16 *Let $x = 0$ be an equilibrium point for the nonlinear system*

$$\dot{x} = f(t, x)$$

where $f : [0, \infty) \times D \to R^n$ is continuously differentiable, $D = \{x \in R^n \mid \|x\| < r\}$, and the Jacobian matrix $[\partial f/\partial x]$ is bounded on D, uniformly in t. Let β be a class $\mathcal{KL}$ function and r_0 be a positive constant such that $\beta(r_0, 0) < r$. Let $D_0 = \{x \in R^n \mid \|x\| < r_0\}$. Assume that the trajectory of the system satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall\ x(t_0) \in D_0, \ \forall\ t \geq t_0 \geq 0$$

Then, there is a continuously differentiable function $V : [0, \infty) \times D_0 \to R$ that satisfies the inequalities

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -\alpha_3(\|x\|)$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq \alpha_4(\|x\|)$$

where α_1, α_2, α_3, and α_4 are class $\mathcal{K}$ functions defined on $[0, r_0]$. If the system is autonomous, V can be chosen independent of t. ◇

Proof: See Appendix C.7.

Theorem 4.17 *Let $x = 0$ be an asymptotically stable equilibrium point for the nonlinear system*

$$\dot{x} = f(x)$$

where $f : D \to R^n$ is locally Lipschitz and $D \subset R^n$ is a domain that contains the origin. Let $R_A \subset D$ be the region of attraction of $x = 0$. Then, there is a smooth, positive definite function $V(x)$ and a continuous, positive definite function $W(x)$, both defined for all $x \in R_A$, such that

$$V(x) \to \infty \text{ as } x \to \partial R_A$$

$$\frac{\partial V}{\partial x} f(x) \leq -W(x), \quad \forall\ x \in R_A$$

and for any $c > 0$, $\{V(x) \leq c\}$ is a compact subset of R_A. When $R_A = R^n$, $V(x)$ is radially unbounded. ◇

[28] Theorem 4.16 can be stated for a function $f(t, x)$ that is only locally Lipschitz, rather than continuously differentiable [125, Theorem 14]. It is also possible to state the theorem for the case of global uniform asymptotic stability [125, Theorem 23]

Proof: See Appendix C.8.

An interesting feature of Theorem 4.17 is that any bounded subset S of the region of attraction can be included in a compact set of the form $\{V(x) \leq c\}$ for some constant $c > 0$. This feature is useful because quite often we have to limit our analysis to a positively invariant, compact set of the form $\{V(x) \leq c\}$. With the property $S \subset \{V(x) \leq c\}$, our analysis will be valid for the whole set S. If, on the other hand, all we know is the existence of a Lyapunov function $V_1(x)$ on S, we will have to choose a constant c_1 such that $\{V_1(x) \leq c_1\}$ is compact and included in S; then our analysis will be limited to $\{V_1(x) \leq c_1\}$, which is only a subset of S.

4.8 Boundedness and Ultimate Boundedness

Lyapunov analysis can be used to show boundedness of the solution of the state equation, even when there is no equilibrium point at the origin. To motivate the idea, consider the scalar equation

$$\dot{x} = -x + \delta \sin t, \quad x(t_0) = a, \quad a > \delta > 0$$

which has no equilibrium points and whose solution is given by

$$x(t) = e^{-(t-t_0)}a + \delta \int_{t_0}^{t} e^{-(t-\tau)} \sin \tau \; d\tau$$

The solution satisfies the bound

$$\begin{aligned} |x(t)| &\leq e^{-(t-t_0)}a + \delta \int_{t_0}^{t} e^{-(t-\tau)} \; d\tau \;\; = \;\; e^{-(t-t_0)}a + \delta \left[1 - e^{-(t-t_0)}\right] \\ &\leq a, \;\; \forall \, t \geq t_0 \end{aligned}$$

which shows that the solution is bounded for all $t \geq t_0$, uniformly in t_0, that is, with a bound independent of t_0. While this bound is valid for all $t \geq t_0$, it becomes a conservative estimate of the solution as time progresses, because it does not take into consideration the exponentially decaying term. If, on the other hand, we pick any number b such that $\delta < b < a$, it can be easily seen that

$$|x(t)| \leq b, \quad \forall \, t \geq t_0 + \ln\left(\frac{a-\delta}{b-\delta}\right)$$

The bound b, which again is independent of t_0, gives a better estimate of the solution after a transient period has passed. In this case, the solution is said to be uniformly ultimately bounded and b is called the ultimate bound. Showing that the solution of $\dot{x} = -x + \delta \sin t$ has the uniform boundedness and ultimate boundedness properties can be done via Lyapunov analysis without using the explicit solution of the state equation. Starting with $V(x) = x^2/2$, we calculate the derivative of V along the trajectories of the system, to obtain

$$\dot{V} = x\dot{x} = -x^2 + x\delta \sin t \le -x^2 + \delta|x|$$

The right-hand side of the foregoing inequality is not negative definite because, near the origin, the positive linear term $\delta|x|$ dominates the negative quadratic term $-x^2$. However, $\dot{V}$ is negative outside the set $\{|x| \le \delta\}$. With $c > \delta^2/2$, solutions starting in the set $\{V(x) \le c\}$ will remain therein for all future time since $\dot{V}$ is negative on the boundary $V = c$. Hence, the solutions are uniformly bounded. Moreover, if we pick any number ε such that $(\delta^2/2) < \varepsilon < c$, then $\dot{V}$ will be negative in the set $\{\varepsilon \le V \le c\}$, which shows that, in this set, V will decrease monotonically until the solution enters the set $\{V \le \varepsilon\}$. From that time on, the solution cannot leave the set $\{V \le \varepsilon\}$ because $\dot{V}$ is negative on the boundary $V = \varepsilon$. Thus, we can conclude that the solution is uniformly ultimately bounded with the ultimate bound $|x| \le \sqrt{2\varepsilon}$.

The purpose of this section is to show how Lyapunov analysis can be used to draw similar conclusions for the system

$$\dot{x} = f(t, x) \tag{4.32}$$

where $f : [0, \infty) \times D \to R^n$ is piecewise continuous in t and locally Lipschitz in x on $[0, \infty) \times D$, and $D \subset R^n$ is a domain that contains the origin.

Definition 4.6 *The solutions of* (4.32) *are*

- *uniformly bounded if there exists a positive constant c, independent of $t_0 \ge 0$, and for every $a \in (0, c)$, there is $\beta = \beta(a) > 0$, independent of t_0, such that*

$$\|x(t_0)\| \le a \;\Rightarrow\; \|x(t)\| \le \beta, \;\; \forall\, t \ge t_0 \tag{4.33}$$

- *globally uniformly bounded if* (4.33) *holds for arbitrarily large a.*

- *uniformly ultimately bounded with ultimate bound b if there exist positive constants b and c, independent of $t_0 \ge 0$, and for every $a \in (0, c)$, there is $T = T(a, b) \ge 0$, independent of t_0, such that*

$$\|x(t_0)\| \le a \;\Rightarrow\; \|x(t)\| \le b, \;\; \forall\, t \ge t_0 + T \tag{4.34}$$

- *globally uniformly ultimately bounded if* (4.34) *holds for arbitrarily large a.*

In the case of autonomous systems, we may drop the word "uniformly" since the solution depends only on $t - t_0$.

To see how Lyapunov analysis can be used to study boundedness and ultimate boundedness, consider a continuously differentiable, positive definite function $V(x)$ and suppose that the set $\{V(x) \le c\}$ is compact, for some $c > 0$. Let

$$\Lambda = \{\varepsilon \le V(x) \le c\}$$

for some positive constant $\varepsilon < c$. Suppose the derivative of V along the trajectories of the system $\dot{x} = f(t,x)$ satisfies

$$\dot{V}(t,x) \le -W_3(x), \quad \forall\, x \in \Lambda,\ \forall\, t \ge t_0 \tag{4.35}$$

where $W_3(x)$ is a continuous positive definite function. Inequality (4.35) implies that the sets $\Omega_c = \{V(x) \le c\}$ and $\Omega_\varepsilon = \{V(x) \le \varepsilon\}$ are positively invariant since on the boundaries $\partial\Omega_c$ and $\partial\Omega_\varepsilon$, the derivative $\dot{V}$ is negative. A sketch of the sets Λ, Ω_c, and Ω_ε is shown in Figure 4.8. Since $\dot{V}$ is negative in Λ, a trajectory starting in Λ must move in a direction of decreasing $V(x(t))$. In fact, while in Λ, V satisfies inequalities (4.22) and (4.24) of Theorem 4.9. Therefore, the trajectory behaves as if the origin was uniformly asymptotically stable and satisfies an inequality of the form

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0)$$

for some class $\mathcal{KL}$ function β. The function $V(x(t))$ will continue decreasing until the trajectory enters the set Ω_ε in finite time and stays therein for all future time. The fact that the trajectory enters Ω_ε in finite time can be shown as follows: Let $k = \min_{x\in\Lambda} W_3(x) > 0$. The minimum exists because $W_3(x)$ is continuous and Λ is compact. It is positive since $W_3(x)$ is positive definite. Hence,

$$W_3(x) \ge k, \quad \forall\, x \in \Lambda \tag{4.36}$$

Inequalities (4.35) and (4.36) imply that

$$\dot{V}(t,x) \le -k, \quad \forall\, x \in \Lambda,\ \forall\, t \ge t_0$$

Therefore,

$$V(x(t)) \le V(x(t_0)) - k(t - t_0) \le c - k(t - t_0)$$

which shows that $V(x(t))$ reduces to ε within the time interval $[t_0, t_0 + (c - \varepsilon)/k]$.

In many problems, the inequality $\dot{V} \le -W_3$ is obtained by using norm inequalities. In such cases, it is more likely that we arrive at

$$\dot{V}(t,x) \le -W_3(x), \quad \forall\, \mu \le \|x\| \le r,\ \forall\, t \ge t_0 \tag{4.37}$$

If r is sufficiently larger than μ, we can choose c and ε such that the set Λ is nonempty and contained in $\{\mu \le \|x\| \le r\}$. In particular, let α_1 and α_2 be class $\mathcal{K}$ functions such that [29]

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|) \tag{4.38}$$

From the left inequality of (4.38), we have

$$V(x) \le c \ \Rightarrow\ \alpha_1(\|x\|) \le c \ \Leftrightarrow\ \|x\| \le \alpha_1^{-1}(c)$$

[29] By Lemma 4.3, it is always possible to find such class $\mathcal{K}$ functions.

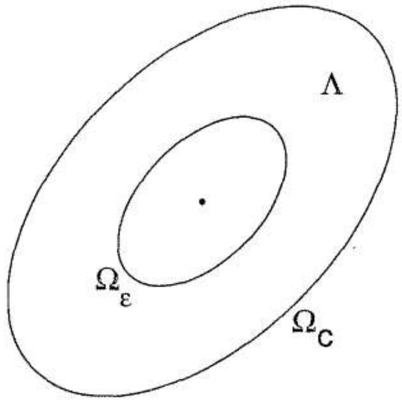


Figure 4.8: Representation of the set Λ, Ω_ε and Ω_c.

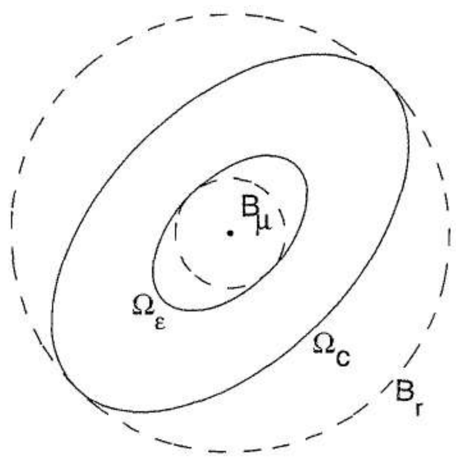


Figure 4.9: Representation of the sets Ω_ε, Ω_c (solid) and B_μ, B_r (dashed).

Therefore, taking $c = \alpha_1(r)$ ensures that $\Omega_c \subset B_r$. On the other hand, from the right inequality of (4.38), we have

$$\|x\| \leq \mu \;\Rightarrow\; V(x) \leq \alpha_2(\mu)$$

Consequently, taking $\varepsilon = \alpha_2(\mu)$ ensures that $B_\mu \subset \Omega_\varepsilon$. To obtain $\varepsilon < c$, we must have $\mu < \alpha_2^{-1}(\alpha_1(r))$. A sketch of the sets Ω_c, Ω_ε, B_r, and B_μ is shown in Figure 4.9.

The foregoing argument shows that all trajectories starting in Ω_c enter Ω_ε within a finite time T.[30] To calculate the ultimate bound on $x(t)$, we use the left inequality of (4.38) to write

$$V(x) \leq \varepsilon \Rightarrow \alpha_1(\|x\|) \leq \varepsilon \Leftrightarrow \|x\| \leq \alpha_1^{-1}(\varepsilon)$$

[30] If the trajectory starts in Ω_ε, $T = 0$.

Recalling that $\varepsilon = \alpha_2(\mu)$, we see that

$$x \in \Omega_\varepsilon \Rightarrow \|x\| \leq \alpha_1^{-1}(\alpha_2(\mu))$$

Therefore, the ultimate bound can be taken as $b = \alpha_1^{-1}(\alpha_2(\mu))$.

The ideas just presented for a continuously differentiable function $V(x)$ can be extended to a continuously differentiable function $V(t, x)$, as long as $V(t, x)$ satisfies inequality (4.38), which leads to the following Lyapunov-like theorem for showing uniform boundedness and ultimate boundedness.

Theorem 4.18 *Let $D \subset R^n$ be a domain that contains the origin and $V : [0, \infty) \times D \to R$ be a continuously differentiable function such that*

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|) \tag{4.39}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x), \quad \forall\ \|x\| \geq \mu > 0 \tag{4.40}$$

$\forall\ t \geq 0$ and $\forall\ x \in D$, where α_1 and α_2 are class $\mathcal{K}$ functions and $W_3(x)$ is a continuous positive definite function. Take $r > 0$ such that $B_r \subset D$ and suppose that

$$\mu < \alpha_2^{-1}(\alpha_1(r)) \tag{4.41}$$

Then, there exists a class $\mathcal{KL}$ function β and for every initial state $x(t_0)$, satisfying $\|x(t_0)\| \leq \alpha_2^{-1}(\alpha_1(r))$, there is $T \geq 0$ (dependent on $x(t_0)$ and μ) such that the solution of (4.32) *satisfies*

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall\ t_0 \leq t \leq t_0 + T \tag{4.42}$$

$$\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\mu)), \quad \forall\ t \geq t_0 + T \tag{4.43}$$

Moreover, if $D = R^n$ and α_1 belongs to class $\mathcal{K}_\infty$, then (4.42) and (4.43) *hold for any initial state $x(t_0)$, with no restriction on how large μ is.* ◇

Proof: See Appendix C.9.

Inequalities (4.42) and (4.43) show that $x(t)$ is uniformly bounded for all $t \geq t_0$ and uniformly ultimately bounded with the ultimate bound $\alpha_1^{-1}(\alpha_2(\mu))$. The ultimate bound is a class $\mathcal{K}$ function of μ; hence, the smaller the value of μ, the smaller the ultimate bound. As $\mu \to 0$, the ultimate bound approaches zero.

The main application of Theorem 4.18 arises in studying the stability of perturbed systems.[31] The next example illustrates the basic idea of that application.

Example 4.24 In Section 1.2.3, we saw that a mass–spring system with a hardening spring, linear viscous damping, and a periodic external force can be represented by the Duffing's equation

$$m\ddot{y} + c\dot{y} + ky + ka^2y^3 = A\cos\omega t$$

[31] See Section 9.2.

Taking $x_1 = y$, $x_2 = \dot{y}$ and assuming certain numerical values for the various constants, the system is represented by the state model

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -(1+x_1^2)x_1 - x_2 + M\cos\omega t\end{aligned}$$

where $M \geq 0$ is proportional to the amplitude of the periodic external force. When $M = 0$, the system has an equilibrium point at the origin. It is shown in Example 4.6 that the origin is globally asymptotically stable and a Lyapunov function can be taken as[32]

$$\begin{aligned}V(x) &= x^T \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + 2\int_0^{x_1} (y+y^3)\; dy \;=\; x^T \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + x_1^2 + \tfrac{1}{2}x_1^4\\ &= x^T \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + \tfrac{1}{2}x_1^4 \stackrel{\text{def}}{=} x^TPx + \tfrac{1}{2}x_1^4\end{aligned}$$

When $M > 0$, we apply Theorem 4.18 with $V(x)$ as a candidate function. The function $V(x)$ is positive definite and radially unbounded; hence, by Lemma 4.3, there exist class $\mathcal{K}_\infty$ functions α_1 and α_2 that satisfy (4.39) globally. The derivative of V along the trajectories of the system is given by

$$\dot{V} = -x_1^2 - x_1^4 - x_2^2 + (x_1 + 2x_2)M\cos\omega t \leq -\|x\|_2^2 - x_1^4 + M\sqrt{5}\|x\|_2$$

where we wrote $(x_1 + 2x_2)$ as y^Tx and used the inequality $y^Tx \leq \|x\|_2\|y\|_2$. To satisfy (4.40), we want to use part of $-\|x\|_2^2$ to dominate $M\sqrt{5}\|x\|_2$ for large $\|x\|$. Towards that end, we rewrite the foregoing inequality as

$$\dot{V} \leq -(1-\theta)\|x\|_2^2 - x_1^4 - \theta\|x\|_2^2 + M\sqrt{5}\|x\|_2$$

where $0 < \theta < 1$. Then,

$$\dot{V} \leq -(1-\theta)\|x\|_2^2 - x_1^4, \quad \forall\ \|x\|_2 \geq \frac{M\sqrt{5}}{\theta}$$

which shows that inequality (4.40) is satisfied globally with $\mu = M\sqrt{5}/\theta$. We conclude that the solutions are globally uniformly ultimately bounded. Suppose we want to go the extra step of calculating the ultimate bound. In this case, we have to find the functions α_1 and α_2. From the inequalities

$$V(x) \geq x^TPx \geq \lambda_{min}(P)\|x\|_2^2$$

$$V(x) \leq x^TPx + \tfrac{1}{2}\|x\|_2^4 \leq \lambda_{max}(P)\|x\|_2^2 + \tfrac{1}{2}\|x\|_2^4$$

[32]The constants δ and k of Example 4.6 are taken as $\delta = 2$ and $k = 1/2$.

we see that α_1 and α_2 can be taken as

$$\alpha_1(r) = \lambda_{min}(P)r^2 \quad \text{and} \quad \alpha_2(r) = \lambda_{max}(P)r^2 + \tfrac{1}{2}r^4$$

Thus, the ultimate bound is given by

$$b = \alpha_1^{-1}(\alpha_2(\mu)) = \sqrt{\frac{\alpha_2(\mu)}{\lambda_{min}(P)}} = \sqrt{\frac{\lambda_{max}(P)\mu^2 + \mu^4/2}{\lambda_{min}(P)}}$$

$\triangle$

4.9 Input-to-State Stability

Consider the system

$$\dot{x} = f(t, x, u) \tag{4.44}$$

where $f : [0, \infty) \times R^n \times R^m \to R^n$ is piecewise continuous in t and locally Lipschitz in x and u. The input $u(t)$ is a piecewise continuous, bounded function of t for all $t \geq 0$. Suppose the unforced system

$$\dot{x} = f(t, x, 0) \tag{4.45}$$

has a globally uniformly asymptotically stable equilibrium point at the origin $x = 0$. What can we say about the behavior of the system (4.44) in the presence of a bounded input $u(t)$? For the linear time-invariant system

$$\dot{x} = Ax + Bu$$

with a Hurwitz matrix A, we can write the solution as

$$x(t) = e^{(t-t_0)A}x(t_0) + \int_{t_0}^{t} e^{(t-\tau)A}Bu(\tau)\, d\tau$$

and use the bound $\|e^{(t-t_0)A}\| \leq ke^{-\lambda(t-t_0)}$ to estimate the solution by

$$\begin{aligned} \|x(t)\| &\leq ke^{-\lambda(t-t_0)}\|x(t_0)\| + \int_{t_0}^{t} ke^{-\lambda(t-\tau)}\|B\|\|u(\tau)\|\, d\tau \\ &\leq ke^{-\lambda(t-t_0)}\|x(t_0)\| + \frac{k\|B\|}{\lambda} \sup_{t_0 \leq \tau \leq t} \|u(\tau)\| \end{aligned}$$

This estimate shows that the zero-input response decays to zero exponentially fast, while the zero-state response is bounded for every bounded input. In fact, the estimate shows more than a bounded-input–bounded-state property. It shows that the bound on the zero-state response is proportional to the bound on the input. How much of this behavior should we expect for the nonlinear system (4.44)? For a general nonlinear system, it should not be surprising that these properties may not hold even when the origin of the unforced system is globally uniformly asymptotically stable. Consider, for example, the scalar system

$$\dot{x} = -3x + (1 + 2x^2)u$$

which has a globally exponentially stable origin when $u = 0$. Yet, when $x(0) = 2$ and $u(t) \equiv 1$, the solution $x(t) = (3 - e^t)/(3 - 2e^t)$ is unbounded; it even has a finite escape time.

Let us view the system (4.44) as a perturbation of the unforced system (4.45). Suppose we have a Lyapunov function $V(t, x)$ for the unforced system and let us calculate the derivative of V in the presence of u. Due to the boundedness of u, it is plausible that in some cases it should be possible to show that $\dot{V}$ is negative outside a ball of radius μ, where μ depends on $\sup \|u\|$. This would be expected, for example, when the function $f(t, x, u)$ satisfies the Lipschitz condition

$$\|f(t, x, u) - f(t, x, 0)\| \leq L\|u\| \tag{4.46}$$

Showing that $\dot{V}$ is negative outside a ball of radius μ would enable us to apply Theorem 4.18 of the previous section to show that $x(t)$ satisfies (4.42) and (4.43). These inequalities show that $\|x(t)\|$ is bounded by a class $\mathcal{KL}$ function $\beta(\|x(t_0)\|, t - t_0)$ over $[t_0, t_0 + T]$ and by a class $\mathcal{K}$ function $\alpha_1^{-1}(\alpha_2(\mu))$ for $t \geq t_0 + T$. Consequently,

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \alpha_1^{-1}(\alpha_2(\mu))$$

is valid for all $t \geq t_0$, which motivates the next definition of *input-to-state stability.*

Definition 4.7 *The system* (4.44) *is said to be input-to-state stable if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for any initial state $x(t_0)$ and any bounded input $u(t)$, the solution $x(t)$ exists for all $t \geq t_0$ and satisfies*

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right) \tag{4.47}$$

Inequality (4.47) guarantees that for any bounded input $u(t)$, the state $x(t)$ will be bounded. Furthermore, as t increases, the state $x(t)$ will be ultimately bounded by a class $\mathcal{K}$ function of $\sup_{t \geq t_0} \|u(t)\|$. We leave it to the reader (Exercise 4.58) to use inequality (4.47) to show that if $u(t)$ converges to zero as $t \to \infty$, so does $x(t)$.[33] Since, with $u(t) \equiv 0$, (4.47) reduces to

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0)$$

input-to-state stability implies that the origin of the unforced system (4.45) is globally uniformly asymptotically stable. The notion of input-to-state stability is defined

[33] Another interesting use of inequality (4.47) will be given shortly in Lemma 4.7.

for the global case where the initial state and the input can be arbitrarily large. A local version of this notion is presented in Exercise 4.60.

The Lyapunov-like theorem that follows gives a sufficient condition for input-to-state stability.[34]

Theorem 4.19 *Let $V : [0, \infty) \times R^n \to R$ be a continuously differentiable function such that*

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|) \tag{4.48}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,u) \leq -W_3(x), \quad \forall\ \|x\| \geq \rho(\|u\|) > 0 \tag{4.49}$$

$\forall\ (t,x,u) \in [0,\infty) \times R^n \times R^m$, where α_1, α_2 are class $\mathcal{K}_\infty$ functions, ρ is a class $\mathcal{K}$ function, and $W_3(x)$ is a continuous positive definite function on R^n. Then, the system (4.44) is input-to-state stable with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$. ◇

Proof: By applying the global version of Theorem 4.18, we find that the solution $x(t)$ exists and satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma\left(\sup_{\tau \geq t_0} \|u(\tau)\|\right), \quad \forall\ t \geq t_0 \tag{4.50}$$

Since $x(t)$ depends only on $u(\tau)$ for $t_0 \leq \tau \leq t$, the supremum on the right-hand side of (4.50) can be taken over $[t_0, t]$, which yields (4.47).[35] □

The next lemma is an immediate consequences of the converse Lyapunov theorem for global exponential stability (Theorem 4.14).

Lemma 4.6 *Suppose $f(t,x,u)$ is continuously differentiable and globally Lipschitz in (x,u), uniformly in t. If the unforced system (4.45) has a globally exponentially stable equilibrium point at the origin $x = 0$, then the system (4.44) is input-to-state stable.* ◇

Proof: View the system (4.44) as a perturbation of the unforced system (4.45). (The converse Lyapunov) Theorem 4.14 shows that the unforced system (4.45) has a Lyapunov function $V(t,x)$ that satisfies the inequality of the theorem globally. Due to the uniform global Lipschitz property of f, the perturbation term satisfies (4.46) for all $t \geq t_0$ and all (x,u). The derivative of V with respect to (4.44) satisfies

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,0) + \frac{\partial V}{\partial x}[f(t,x,u) - f(t,x,0)] \\ &\leq -c_3\|x\|^2 + c_4\|x\| L \|u\| \end{aligned}$$

[34] For autonomous systems, it is shown in [183] that the conditions of Theorem 4.19 are also necessary. In the literature, it is common to abbreviate input-to-state stability as ISS and to call the function V of Theorem 4.19 an ISS-Lyapunov function.

[35] In particular, repeat the aforementioned argument over the period $[0, T]$ to show that

$$\|x(\sigma)\| \leq \beta(\|x(t_0)\|, \sigma - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq T} \|u(\tau)\|\right), \quad \forall\ t_0 \leq \sigma \leq T$$

Then, set $\sigma = T = t$.

To use the term $-c_3\|x\|^2$ to dominate $c_4L\|x\|\|u\|$ for large $\|x\|$, we rewrite the foregoing inequality as

$$\dot{V} \leq -c_3(1-\theta)\|x\|^2 - c_3\theta\|x\|^2 + c_4L\|x\|\|u\|$$

where $0 < \theta < 1$. Then,

$$\dot{V} \leq -c_3(1-\theta)\|x\|^2, \quad \forall\ \|x\| \geq \frac{c_4L\|u\|}{c_3\theta}$$

for all (t, x, u). Hence, the conditions of Theorem 4.19 are satisfied with $\alpha_1(r) = c_1r^2$, $\alpha_2(r) = c_2r^2$, and $\rho(r) = (c_4L/c_3\theta)r$, and we conclude that the system is input-to-state stable with $\gamma(r) = \sqrt{c_2/c_1}(c_4L/c_3\theta)r$. □

Lemma 4.6 requires a globally Lipschitz function f and global exponential stability of the origin of the unforced system to conclude input-to-state stability. It is easy to construct examples where the lemma does not hold in the absence of one of these two conditions. The system $\dot{x} = -3x + (1 + x^2)u$, which we discussed earlier in the section, does not satisfy the global Lipschitz condition. The system

$$\dot{x} = -\frac{x}{1+x^2} + u \stackrel{\text{def}}{=} f(x, u)$$

has a globally Lipschitz f since the partial derivatives of f with respect to x and u are globally bounded. The origin of $\dot{x} = -x/(1 + x^2)$ is globally asymptotically stable, as it can be seen by the Lyapunov function $V(x) = x^2/2$, whose derivative $\dot{V}(x) = -x^2/(1 + x^2)$ is negative definite for all x. It is locally exponentially stable because the linearization at the origin is $\dot{x} = -x$. However, it is not globally exponentially stable. This is easiest seen through the fact that the system is not input-to-state stable. Notice that with $u(t) \equiv 1$, $f(x, u) \geq 1/2$. Hence, $x(t) \geq x(t_0) + t/2$ for all $t \geq 0$, which shows that the solution is unbounded.

In the absence of global exponential stability or globally Lipschitz functions, we may still be able to show input-to-state stability by applying Theorem 4.19. This process is illustrated by the three examples that follow.

Example 4.25 The system

$$\dot{x} = -x^3 + u$$

has a globally asymptotically stable origin when $u = 0$. Taking $V = x^2/2$, the derivative of V along the trajectories of the system is given by

$$\dot{V} = -x^4 + xu = -(1-\theta)x^4 - \theta x^4 + xu \leq -(1-\theta)x^4, \quad \forall\ |x| \geq \left(\frac{|u|}{\theta}\right)^{1/3}$$

where $0 < \theta < 1$. Thus, the system is input-to-state stable with $\gamma(r) = (r/\theta)^{1/3}$. △

Example 4.26 The system

$$\dot{x} = f(x, u) = -x - 2x^3 + (1 + x^2)u^2$$

has a globally exponentially stable origin when $u = 0$, but Lemma 4.6 does not apply since f is not globally Lipschitz. Taking $V = x^2/2$, we obtain

$$\dot{V} = -x^2 - 2x^4 + x(1 + x^2)u^2 \le -x^4, \quad \forall\ |x| \ge u^2$$

Thus, the system is input-to-state stable with $\gamma(r) = r^2$. △

In Examples 4.25 and 4.26, the function $V(x) = x^2/2$ satisfies (4.48) with $\alpha_1(r) = \alpha_2(r) = r^2/2$. Hence, $\alpha_1^{-1}(\alpha_2(r)) = r$ and $\gamma(r)$ reduces to $\rho(r)$. In higher-dimensional systems, the calculation of γ is more involved.

Example 4.27 Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -x_2 + u \end{aligned}$$

We start by setting $u = 0$ and investigate global asymptotic stability of the origin of the unforced system. Using

$$V(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{4}ax_2^4, \quad a > 0$$

as a Lyapunov function candidate, we obtain

$$\dot{V} = -x_1^2 + x_1x_2^2 - ax_2^4 = -(x_1 - \tfrac{1}{2}x_2^2)^2 - \left(a - \tfrac{1}{4}\right)x_2^4$$

Choosing $a > 1/4$ shows that the origin is globally asymptotically stable. Now we allow $u \ne 0$ and use $V(x)$ with $a = 1$ as a candidate function for Theorem 4.19. The derivative $\dot{V}$ is given by

$$\dot{V} = -\tfrac{1}{2}(x_1 - x_2^2)^2 - \tfrac{1}{2}(x_1^2 + x_2^4) + x_2^3u \le -\tfrac{1}{2}(x_1^2 + x_2^4) + |x_2|^3|u|$$

To use the term $-(x_1^2 + x_2^4)/2$ to dominate $|x_2|^3|u|$, we rewrite the foregoing inequality as

$$\dot{V} \le -\tfrac{1}{2}(1 - \theta)(x_1^2 + x_2^4) - \tfrac{1}{2}\theta(x_1^2 + x_2^4) + |x_2|^3|u|$$

where $0 < \theta < 1$. The term

$$-\tfrac{1}{2}\theta(x_1^2 + x_2^4) + |x_2|^3|u|$$

will be ≤ 0 if $|x_2| \ge 2|u|/\theta$ or $|x_2| \le 2|u|/\theta$ and $|x_1| \ge (2|u|/\theta)^2$. This condition is implied by

$$\max\{|x_1|, |x_2|\} \ge \max\left\{\frac{2|u|}{\theta}, \left(\frac{2|u|}{\theta}\right)^2\right\}$$

Using the norm $\|x\|_\infty = \max\{|x_1|, |x_2|\}$ and defining the class $\mathcal{K}$ function ρ by

$$\rho(r) = \max\left\{\frac{2r}{\theta}, \left(\frac{2r}{\theta}\right)^2\right\}$$

we see that inequality (4.49) is satisfied as

$$\dot{V} \le -\tfrac{1}{2}(1-\theta)(x_1^2 + x_2^4), \quad \forall\ \|x\|_\infty \ge \rho(|u|)$$

Inequality (4.48) follows from Lemma 4.3 since $V(x)$ is positive definite and radially unbounded. Hence, the system is input-to-state stable. Suppose we want to find the class $\mathcal{K}$ function γ. In this case, we need to find α_1 and α_2. It is not hard to see that

$$V(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{4}x_2^4 \le \tfrac{1}{2}\|x\|_\infty^2 + \tfrac{1}{4}\|x\|_\infty^4$$

$$V(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{4}x_2^4 \ge \begin{cases} \tfrac{1}{2}|x_1|^2 = \tfrac{1}{2}\|x\|_\infty^2, & \text{if } |x_2| \le |x_1| \\ \\ \tfrac{1}{4}|x_2|^4 = \tfrac{1}{4}\|x\|_\infty^4, & \text{if } |x_2| \ge |x_1| \end{cases}$$

Inequality (4.48) is satisfied with the class $\mathcal{K}_\infty$ functions

$$\alpha_1(r) = \min\left\{\tfrac{1}{2}r^2, \tfrac{1}{4}r^4\right\} \quad \text{and} \quad \alpha_2(r) = \tfrac{1}{2}r^2 + \tfrac{1}{4}r^4$$

Thus, $\gamma(r) = \alpha_1^{-1}(\alpha_2(\rho(r)))$, where

$$\alpha_1^{-1}(s) = \begin{cases} (4s)^{\frac{1}{4}}, & \text{if } s \le 1 \\ \\ \sqrt{2s}, & \text{if } s \ge 1 \end{cases}$$

The function γ depends on the choice of $\|x\|$. Had we chosen another p-norm, we could have ended up with a different γ. $\triangle$

An interesting application of the concept of input-to-state stability arises in the stability analysis of the cascade system

$$\dot{x}_1 = f_1(t, x_1, x_2) \tag{4.51}$$

$$\dot{x}_2 = f_2(t, x_2) \tag{4.52}$$

where $f_1 : [0,\infty) \times R^{n_1} \times R^{n_2} \to R^{n_1}$ and $f_2 : [0,\infty) \times R^{n_2} \to R^{n_2}$ are piecewise continuous in t and locally Lipschitz in $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Suppose both

$$\dot{x}_1 = f_1(t, x_1, 0)$$

and (4.52) have globally uniformly asymptotically stable equilibrium points at their respective origins. Under what condition will the origin $x = 0$ of the cascade system possess the same property? The next lemma shows that this will be the case if (4.51), with x_2 viewed as input, is input-to-state stable.

Lemma 4.7 *Under the stated assumptions, if the system* (4.51), *with x_2 as input, is input-to-state stable and the origin of* (4.52) *is globally uniformly asymptotically stable, then the origin of the cascade system* (4.51) and (4.52) *is globally uniformly asymptotically stable.* ◇

Proof: Let $t_0 \geq 0$ be the initial time. The solutions of (4.51) and (4.52) satisfy

$$\begin{aligned} \|x_1(t)\| &\leq \beta_1(\|x_1(s)\|, t-s) + \gamma_1\left(\sup_{s \leq \tau \leq t} \|x_2(\tau)\|\right) && (4.53) \\ \|x_2(t)\| &\leq \beta_2(\|x_2(s)\|, t-s) && (4.54) \end{aligned}$$

globally, where $t \geq s \geq t_0$, β_1, β_2 are class $\mathcal{KL}$ functions and γ_1 is a class $\mathcal{K}$ function. Apply (4.53) with $s = (t+t_0)/2$ to obtain

$$\|x_1(t)\| \leq \beta_1\left(\left\|x_1\left(\frac{t+t_0}{2}\right)\right\|, \frac{t-t_0}{2}\right) + \gamma_1\left(\sup_{\frac{t+t_0}{2} \leq \tau \leq t} \|x_2(\tau)\|\right) \tag{4.55}$$

To estimate $x_1((t+t_0)/2)$, apply (4.53) with $s = t_0$ and t replaced by $(t+t_0)/2$ to obtain

$$\left\|x_1\left(\frac{t+t_0}{2}\right)\right\| \leq \beta_1\left(\|x_1(t_0)\|, \frac{t-t_0}{2}\right) + \gamma_1\left(\sup_{t_0 \leq \tau \leq \frac{t+t_0}{2}} \|x_2(\tau)\|\right) \tag{4.56}$$

Using (4.54), we obtain

$$\begin{aligned} \sup_{t_0 \leq \tau \leq \frac{t+t_0}{2}} \|x_2(\tau)\| &\leq \beta_2(\|x_2(t_0)\|, 0) && (4.57) \\ \sup_{\frac{t+t_0}{2} \leq \tau \leq t} \|x_2(\tau)\| &\leq \beta_2\left(\|x_2(t_0)\|, \frac{t-t_0}{2}\right) && (4.58) \end{aligned}$$

Substituting (4.56) through (4.58) into (4.55) and using the inequalities

$$\|x_1(t_0)\| \leq \|x(t_0)\|, \quad \|x_2(t_0)\| \leq \|x(t_0)\|, \quad \|x(t)\| \leq \|x_1(t)\| + \|x_2(t)\|$$

yield

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t-t_0)$$

where

$$\beta(r,s) = \beta_1\left(\beta_1\left(r, \frac{s}{2}\right) + \gamma_1\left(\beta_2(r,0)\right), \frac{s}{2}\right) + \gamma_1\left(\beta_2\left(r, \frac{s}{2}\right)\right) + \beta_2(r,s)$$

It can be easily verified that β is a class $\mathcal{KL}$ function for all $r \geq 0$. Hence, the origin of (4.51) and (4.52) is globally uniformly asymptotically stable. □

4.10 Exercises

4.1 Consider a second-order autonomous system. For each of the following types of equilibrium points, classify whether the equilibrium point is stable, unstable, or asymptotically stable:

(1) stable node	(2) unstable node	(3) stable focus
(4) unstable focus	(5) center	(6) saddle

Justify your answer using phase portraits.

4.2 Consider the scalar system $\dot{x} = ax^p + g(x)$, where p is a positive integer and $g(x)$ satisfies $|g(x)| \leq k|x|^{p+1}$ in some neighborhood of the origin $x = 0$. Show that the origin is asymptotically stable if p is odd and $a < 0$. Show that it is unstable if p is odd and $a > 0$ or p is even and $a \neq 0$.

4.3 For each of the following systems, use a quadratic Lyapunov function candidate to show that the origin is asymptotically stable:

$$\begin{array}{lrclrcl} (1) & \dot{x}_1 & = & -x_1 + x_1 x_2, & \dot{x}_2 & = & -x_2 \\ (2) & \dot{x}_1 & = & -x_2 - x_1(1 - x_1^2 - x_2^2), & \dot{x}_2 & = & x_1 - x_2(1 - x_1^2 - x_2^2) \\ (3) & \dot{x}_1 & = & x_2(1 - x_1^2), & \dot{x}_2 & = & -(x_1 + x_2)(1 - x_1^2) \\ (4) & \dot{x}_1 & = & -x_1 - x_2, & \dot{x}_2 & = & 2x_1 - x_2^3 \end{array}$$

Investigate whether the origin is globally asymptotically stable.

4.4 ([151]) Euler equations for a rotating rigid spacecraft are given by

$$\begin{aligned} J_1 \dot{\omega}_1 &= (J_2 - J_3)\omega_2 \omega_3 + u_1 \\ J_2 \dot{\omega}_2 &= (J_3 - J_1)\omega_3 \omega_1 + u_2 \\ J_3 \dot{\omega}_3 &= (J_1 - J_2)\omega_1 \omega_2 + u_3 \end{aligned}$$

where ω_1 to ω_3 are the components of the angular velocity vector ω along the principal axes, u_1 to u_3 are the torque inputs applied about the principal axes, and J_1 to J_3 are the principal moments of inertia.

(a) Show that with $u_1 = u_2 = u_3 = 0$ the origin $\omega = 0$ is stable. Is it asymptotically stable?

(b) Suppose the torque inputs apply the feedback control $u_i = -k_i \omega_i$, where k_1 to k_3 are positive constants. Show that the origin of the closed-loop system is globally asymptotically stable.

4.5 Let $g(x)$ be a map from R^n into R^n. Show that $g(x)$ is the gradient vector of a scalar function $V : R^n \rightarrow R$ if and only if

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \quad \forall\, i, j = 1, 2, \ldots, n$$

4.6 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -(x_1 + x_2) - h(x_1 + x_2)$$

where h is continuously differentiable and $zh(z) > 0$ for all $z \neq 0$. Using the variable gradient method, find a Lyapunov function that shows that the origin is globally asymptotically stable.

4.7 Consider the system $\dot{x} = -Q\phi(x)$, where Q is a symmetric positive definite matrix and $\phi(x)$ is a continuously differentiable function for which the ith component ϕ_i depends only on x_i, that is, $\phi_i(x) = \phi_i(x_i)$. Assume that $\phi_i(0) = 0$ and $y\phi_i(y) > 0$ in some neighborhood of $y = 0$, for all $1 \leq i \leq n$.

(a) Using the variable gradient method, find a Lyapunov function that shows that the origin is asymptotically stable.

(b) Under what conditions will it be globally asymptotically stable?

(c) Apply to the case

$$n = 2, \ \phi_1(x_1) = x_1 - x_1^2, \ \phi_2(x_2) = x_2 + x_2^3, \ Q = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

4.8 ([72]) Consider the second-order system

$$\dot{x}_1 = \frac{-6x_1}{u^2} + 2x_2, \qquad \dot{x}_2 = \frac{-2(x_1 + x_2)}{u^2}$$

where $u = 1 + x_1^2$. Let $V(x) = x_1^2/(1 + x_1^2) + x_2^2$.

(a) Show that $V(x) > 0$ and $\dot{V}(x) < 0$ for all $x \in R^2 - \{0\}$.

(b) Consider the hyperbola $x_2 = 2/(x_1 - \sqrt{2})$. Show, by investigating the vector field on the boundary of this hyperbola, that trajectories to the right of the branch in the first quadrant cannot cross that branch.

(c) Show that the origin is not globally asymptotically stable.

Hint: In part (b), show that $\dot{x}_2/\dot{x}_1 = -1/(1 + 2\sqrt{2}x_1 + 2x_1^2)$ on the hyperbola, and compare with the slope of the tangents to the hyperbola.

4.9 In checking radial unboundedness of a positive definite function $V(x)$, it may appear that it is sufficient to examine $V(x)$ as $\|x\| \to \infty$ along the principal axes. This is not true, as shown by the function

$$V(x) = \frac{(x_1 + x_2)^2}{1 + (x_1 + x_2)^2} + (x_1 - x_2)^2$$

(a) Show that $V(x) \to \infty$ as $\|x\| \to \infty$ along the lines $x_1 = 0$ or $x_2 = 0$.

(b) Show that $V(x)$ is not radially unbounded.

4.10 (Krasovskii's Method) Consider the system $\dot{x} = f(x)$ with $f(0) = 0$. Assume that $f(x)$ is continuously differentiable and its Jacobian $[\partial f/\partial x]$ satisfies

$$P\left[\frac{\partial f}{\partial x}(x)\right] + \left[\frac{\partial f}{\partial x}(x)\right]^T P \leq -I, \quad \forall\, x \in R^n, \quad \text{where } P = P^T > 0$$

(a) Using the representation $f(x) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x)x \; d\sigma$, show that

$$x^T P f(x) + f^T(x) P x \leq -x^T x, \quad \forall\, x \in R^n$$

(b) Show that $V(x) = f^T(x) P f(x)$ is positive definite for all $x \in R^n$ and radially unbounded.

(c) Show that the origin is globally asymptotically stable.

4.11 Using Theorem 4.3, prove Lyapunov's first instability theorem:
For the system (4.1), if a continuously differentiable function $V_1(x)$ can be found in a neighborhood of the origin such that $V_1(0) = 0$, and $\dot{V}_1$ along the trajectories of the system is positive definite, but V_1 itself is not negative definite or negative semidefinite arbitrarily near the origin, then the origin is unstable.

4.12 Using Theorem 4.3, prove Lyapunov's second instability theorem:
For the system (4.1), if in a neighborhood D of the origin, a continuously differentiable function $V_1(x)$ exists such that $V_1(0) = 0$ and $\dot{V}_1$ along the trajectories of the system is of the form $\dot{V}_1 = \lambda V_1 + W(x)$ where $\lambda > 0$ and $W(x) \geq 0$ in D, and if $V_1(x)$ is not negative definite or negative semidefinite arbitrarily near the origin, then the origin is unstable.

4.13 For each of the following systems, show that the origin is unstable:

$$\begin{array}{lrclrcl} (1) & \dot{x}_1 & = & x_1^3 + x_1^2 x_2, & \dot{x}_2 & = & -x_2 + x_2^2 + x_1 x_2 - x_1^3 \\ (2) & \dot{x}_1 & = & -x_1^3 + x_2, & \dot{x}_2 & = & x_1^6 - x_2^3 \end{array}$$

Hint: In part (2), show that $\Gamma = \{0 \leq x_1 \leq 1\} \cap \{x_2 \geq x_1^3\} \cap \{x_2 \leq x_1^2\}$ is a nonempty positively invariant set, and investigate the behavior of the trajectories inside Γ.

4.14 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -g(x_1)(x_1 + x_2)$$

where g is locally Lipschitz and $g(y) \geq 1$ for all $y \in R$. Verify that $V(x) = \int_0^{x_1} y g(y) \; dy + x_1 x_2 + x_2^2$ is positive definite for all $x \in R^2$ and radially unbounded, and use it to show that the equilibrium point $x = 0$ is globally asymptotically stable.

4.15 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h_1(x_1) - x_2 - h_2(x_3), \qquad \dot{x}_3 = x_2 - x_3$$

where h_1 and h_2 are locally Lipschitz functions that satisfy $h_i(0) = 0$ and $yh_i(y) > 0$ for all $y \neq 0$.

(a) Show that the system has a unique equilibrium point at the origin.

(b) Show that $V(x) = \int_0^{x_1} h_1(y)\ dy + x_2^2/2 + \int_0^{x_3} h_2(y)\ dy$ is positive definite for all $x \in R^3$.

(c) Show that the origin is asymptotically stable.

(d) Under what conditions on h_1 and h_2, can you show that the origin is globally asymptotically stable?

4.16 Show that the origin of

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 - x_2^3$$

is globally asymptotically stable.

4.17 ([77]) Consider Liénard's equation

$$\ddot{y} + h(y)\dot{y} + g(y) = 0$$

where g and h are continuously differentiable.

(a) Using $x_1 = y$ and $x_2 = \dot{y}$, write the state equation and find conditions on g and h to ensure that the origin is an isolated equilibrium point.

(b) Using $V(x) = \int_0^{x_1} g(y)\ dy + (1/2)x_2^2$ as a Lyapunov function candidate, find conditions on g and h to ensure that the origin is asymptotically stable.

(c) Repeat part (b) using $V(x) = (1/2)\left[x_2 + \int_0^{x_1} h(y)\ dy\right]^2 + \int_0^{x_1} g(y)\ dy$.

4.18 The mass–spring system of Exercise 1.12 is modeled by

$$M\ddot{y} = Mg - ky - c_1\dot{y} - c_2\dot{y}|\dot{y}|$$

Show that the system has a globally asymptotically stable equilibrium point.

4.19 Consider the equations of motion of an m-link robot, described in Exercise 1.4. Assume that $P(q)$ is a positive definite function of q and $g(q) = 0$ has an isolated root at $q = 0$.

(a) With $u = 0$, use the total energy $V(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$ as a Lyapunov function candidate to show that the origin $(q = 0,\ \dot{q} = 0)$ is stable.

(b) With $u = -K_d\dot{q}$, where K_d is a positive diagonal matrix, show that the origin is asymptotically stable.

(c) With $u = g(q) - K_p(q - q^*) - K_d\dot{q}$, where K_p and K_d are positive diagonal matrices and q^* is a desired robot position in R^m, show that the point $(q = q^*,\ \dot{q} = 0)$ is an asymptotically stable equilibrium point.

4.20 Suppose the set M in LaSalle's theorem consists of a finite number of isolated points. Show that $\lim_{t\to\infty} x(t)$ exists and equals one of these points.

4.21 ([81]) A gradient system is a dynamical system of the form $\dot{x} = -\nabla V(x)$, where $\nabla V(x) = [\partial V/\partial x]^T$ and $V : D \subset R^n \to R$ is twice continuously differentiable.

(a) Show that $\dot{V}(x) \leq 0$ for all $x \in D$, and $\dot{V}(x) = 0$ if and only if x is an equilibrium point.

(b) Take $D = R^n$. Suppose the set $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$ is compact for every $c \in R$. Show that every solution of the system is defined for all $t \geq 0$.

(c) Continuing with part (b), suppose $\nabla V(x) \neq 0$, except for a finite number of points $p_1, \ldots, p_r$. Show that for every solution $x(t)$, $\lim_{t\to\infty} x(t)$ exists and equals one of the points $p_1, \ldots, p_r$.

4.22 Consider the Lyapunov equation $PA + A^TP = -C^TC$, where the pair (A, C) is observable. Show that A is Hurwitz if and only if there exists $P = P^T > 0$ that satisfies the equation. Furthermore, show that if A is Hurwitz, the Lyapunov equation will have a unique solution.
Hint: Apply LaSalle's theorem and recall that for an observable pair (A, C), the vector $C\exp(At)x \equiv 0\ \forall\ t$ if and only if $x = 0$.

4.23 Consider the linear system $\dot{x} = (A - BR^{-1}B^TP)x$, where $P = P^T > 0$ satisfies the Riccati equation

$$PA + A^TP + Q - PBR^{-1}B^TP = 0$$

$R = R^T > 0$, and $Q = Q^T \geq 0$. Using $V(x) = x^TPx$ as a Lyapunov function candidate, show that the origin is globally asymptotically stable when

(1) $Q > 0$.

(2) $Q = C^TC$ and (A, C) is observable; see the hint of Exercise 4.22.

4.24 Consider the system[36]

$$\dot{x} = f(x) - kG(x)R^{-1}(x)G^T(x)\left(\frac{\partial V}{\partial x}\right)^T$$

[36]This is a closed-loop system under optimal stabilizing control. See [172].

where $V(x)$ is a continuously differentiable, positive definite function that satisfies the Hamilton–Jacobi–Bellman equation

$$\frac{\partial V}{\partial x} f(x) + q(x) - \tfrac{1}{4}\frac{\partial V}{\partial x} G(x) R^{-1}(x) G^T(x) \left(\frac{\partial V}{\partial x}\right)^T = 0$$

$q(x)$ is a positive semidefinite function, $R(x)$ is a nonsingular matrix, and k is a positive constant. Using $V(x)$ as a Lyapunov function candidate, show that the origin is asymptotically stable when

(1) $q(x)$ is positive definite and $k \geq 1/4$.

(2) $q(x)$ is positive semidefinite, $k > 1/4$, and the only solution of $\dot{x} = f(x)$ that can stay identically in the set $\{q(x) = 0\}$ is the trivial solution $x(t) \equiv 0$.

When will the origin be globally asymptotically stable?

4.25 Consider the linear system $\dot{x} = Ax + Bu$, where (A, B) is controllable. Let $W = \int_0^\tau e^{-At} BB^T e^{-A^T t}\, dt$ for some $\tau > 0$. Show that W is positive definite and let $K = B^T W^{-1}$. Use $V(x) = x^T W^{-1} x$ as a Lyapunov function candidate for the system $\dot{x} = (A - BK)x$ to show that $(A - BK)$ is Hurwitz.

4.26 Let $\dot{x} = f(x)$, where $f : R^n \to R^n$. Consider the change of variables $z = T(x)$, where $T(0) = 0$ and $T : R^n \to R^n$ is a diffeomorphism in the neighborhood of the origin; that is, the inverse map $T^{-1}(\cdot)$ exists, and both $T(\cdot)$ and $T^{-1}(\cdot)$ are continuously differentiable. The transformed system is

$$\dot{z} = \hat{f}(z), \quad \text{where } \hat{f}(z) = \left.\frac{\partial T}{\partial x} f(x)\right|_{x=T^{-1}(z)}$$

(a) Show that $x = 0$ is an isolated equilibrium point of $\dot{x} = f(x)$ if and only if $z = 0$ is an isolated equilibrium point of $\dot{z} = \hat{f}(z)$.

(b) Show that $x = 0$ is stable (asymptotically stable or unstable) if and only if $z = 0$ is stable (asymptotically stable or unstable).

4.27 Consider the system

$$\dot{x}_1 = -x_2 x_3 + 1, \qquad \dot{x}_2 = x_1 x_3 - x_2, \qquad \dot{x}_3 = x_3^2(1 - x_3)$$

(a) Show that the system has a unique equilibrium point.

(b) Using linearization, show that the equilibrium point asymptotically stable. Is it globally asymptotically stable?

4.28 Consider the system

$$\dot{x}_1 = -x_1, \qquad \dot{x}_2 = (x_1 x_2 - 1)x_2^3 + (x_1 x_2 - 1 + x_1^2)x_2$$

(a) Show that $x = 0$ is the unique equilibrium point.

(b) Show, by using linearization, that $x = 0$ is asymptotically stable.

(c) Show that $\Gamma = \{x \in R^2 \mid x_1 x_2 \geq 2\}$ is a positively invariant set.

(d) Is $x = 0$ globally asymptotically stable?

4.29 Consider the system

$$\dot{x}_1 = x_1 - x_1^3 + x_2, \qquad \dot{x}_2 = 3x_1 - x_2$$

(a) Find all equilibrium point of the system.

(b) Using linearization, study the stability of each equilibrium point.

(c) Using quadratic Lyapunov functions, estimate the region of attraction of each asymptotically stable equilibrium point. Try to make your estimate as large as possible.

(d) Construct the phase portrait of the system and show on it the exact regions of attraction as well as your estimates.

4.30 Repeat the previous exercise for the system

$$\dot{x}_1 = -\tfrac{1}{2}\tan\left(\frac{\pi x_1}{2}\right) + x_2, \qquad \dot{x}_2 = x_1 - \tfrac{1}{2}\tan\left(\frac{\pi x_2}{2}\right)$$

4.31 For each of the systems of Exercise 4.3, use linearization to show that the origin is asymptotically stable.

4.32 For each for the following systems, investigate whether the origin is stable, asymptotically stable, or unstable:

$$(1)\quad \begin{aligned} \dot{x}_1 &= -x_1 + x_1^2 \\ \dot{x}_2 &= -x_2 + x_3^2 \\ \dot{x}_3 &= x_3 - x_1^2 \end{aligned} \qquad\qquad (2)\quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_3 + x_1[-2x_3 - \text{sat}(y)]^2 \\ \dot{x}_3 &= -2x_3 - \text{sat}(y) \\ &\ \text{where } y = -2x_1 - 5x_2 + 2x_3 \end{aligned}$$

$$(3)\quad \begin{aligned} \dot{x}_1 &= -2x_1 + x_1^3 \\ \dot{x}_2 &= -x_2 + x_1^2 \\ \dot{x}_3 &= -x_3 \end{aligned} \qquad\qquad (4)\quad \begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= -x_1 - x_2 - x_3 - x_1 x_3 \\ \dot{x}_3 &= (x_1 + 1)x_2 \end{aligned}$$

4.33 Consider the second-order system $\dot{x} = f(x)$, where $f(0) = 0$ and $f(x)$ is twice continuously differentiable in some neighborhood of the origin. Suppose $[\partial f/\partial x](0) = -B$, where B be Hurwitz. Let P be the positive definite solution of the Lyapunov equation $PB + B^T P = -I$ and take $V(x) = x^T P x$. Show that there exists $c^* > 0$ such that, for every $0 < c < c^*$, the surface $V(x) = c$ is closed and $[\partial V/\partial x]f(x) > 0$ for all $x \in \{V(x) = c\}$.

4.34 Prove Lemma 4.2.

4.35 Let α be a class $\mathcal{K}$ function on $[0, a)$. Show that

$$\alpha(r_1 + r_2) \le \alpha(2r_1) + \alpha(2r_2), \quad \forall\, r_1, r_2 \in [0, a/2)$$

4.36 Is the origin of the scalar system $\dot{x} = -x/(t+1)$, $t \ge 0$, uniformly asymptotically stable?

4.37 For each of the following linear systems, use a quadratic Lyapunov function to show that the origin is exponentially stable:

$$(1)\ \dot{x} = \begin{bmatrix} -1 & \alpha(t) \\ \alpha(t) & -2 \end{bmatrix} x,\ |\alpha(t)| \le 1 \qquad (2)\ \dot{x} = \begin{bmatrix} -1 & \alpha(t) \\ -\alpha(t) & -2 \end{bmatrix} x$$

$$(3)\ \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -\alpha(t) \end{bmatrix} x,\ \alpha(t) \ge 2 \qquad (4)\ \dot{x} = \begin{bmatrix} -1 & 0 \\ \alpha(t) & -2 \end{bmatrix} x$$

In all cases, $\alpha(t)$ is continuous and bounded for all $t \ge 0$.

4.38 (**[95]**) An *RLC* circuit with time-varying elements is represented by

$$\dot{x}_1 = \frac{1}{L(t)} x_2, \qquad \dot{x}_2 = -\frac{1}{C(t)} x_1 - \frac{R(t)}{L(t)} x_2$$

Suppose that $L(t)$, $C(t)$, and $R(t)$ are continuously differentiable and satisfy the inequalities $k_1 \le L(t) \le k_2$, $k_3 \le C(t) \le k_4$, and $k_5 \le R(t) \le k_6$ for all $t \ge 0$, where k_1, k_3, and k_5 are positive. Consider a Lyapunov function candidate

$$V(t, x) = \left[R(t) + \frac{2L(t)}{R(t)C(t)} \right] x_1^2 + 2x_1x_2 + \frac{2}{R(t)} x_2^2$$

(a) Show that $V(t, x)$ is positive definite and decrescent.

(b) Find conditions on $\dot{L}(t)$, $\dot{C}(t)$, and $\dot{R}(t)$ that will ensure exponential stability of the origin.

4.39 (**[154]**) A pendulum with time-varying friction is represented by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - g(t) x_2$$

Suppose that $g(t)$ is continuously differentiable and satisfies

$$0 < a < \alpha \le g(t) \le \beta < \infty \quad \text{and} \quad \dot{g}(t) \le \gamma < 2$$

for all $t \ge 0$. Consider the Lyapunov function candidate

$$V(t, x) = \tfrac{1}{2}(a \sin x_1 + x_2)^2 + [1 + a g(t) - a^2](1 - \cos x_1)$$

(a) Show that $V(t, x)$ is positive definite and decrescent.

(b) Show that $\dot{V} \leq -(\alpha - a)x_2^2 - a(2-\gamma)(1 - \cos x_1) + O(\|x\|^3)$, where $O(\|x\|^3)$ is a term bounded by $k\|x\|^3$ in some neighborhood of the origin.

(c) Show that the origin is uniformly asymptotically stable.

4.40 (Floquet theory) Consider the linear system $\dot{x} = A(t)x$, where $A(t) = A(t+T)$.[37] Let $\Phi(\cdot, \cdot)$ be the state transition matrix. Define a constant matrix B via the equation $\exp(BT) = \Phi(T, 0)$, and let $P(t) = \exp(Bt)\Phi(0, t)$. Show that

(a) $P(t+T) = P(t)$.

(b) $\Phi(t, \tau) = P^{-1}(t) \exp[(t - \tau)B]P(\tau)$.

(c) the origin of $\dot{x} = A(t)x$ is exponentially stable if and only if B is Hurwitz.

4.41 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = 2x_1x_2 + 3t + 2 - 3x_1 - 2(t+1)x_2$$

(a) Verify that $x_1(t) = t$, $x_2(t) = 1$ is a solution.

(b) Show that if $x(0)$ is sufficiently close to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $x(t)$ approaches $\begin{bmatrix} t \\ 1 \end{bmatrix}$ as $t \to \infty$.

4.42 Consider the system

$$\dot{x} = -a[I_n + S(x) + xx^T]x$$

where a is a positive constant, I_n is the $n \times n$ identity matrix, and $S(x)$ is an x-dependent skew symmetric matrix. Show that the origin is globally exponentially stable.

4.43 Consider the system $\dot{x} = f(x) + G(x)u$. Suppose there exist a positive definite symmetric matrix P, a positive semidefinite function $W(x)$, and positive constants γ and σ such that

$$2x^T Pf(x) + \gamma x^T Px + W(x) - 2\sigma x^T PG(x)G^T(x)Px \leq 0, \quad \forall\, x \in R^n$$

Show that with $u = -\sigma G^T(x)Px$ the closed-loop system has a globally exponentially stable equilibrium point at the origin.

4.44 Consider the system

$$\dot{x}_1 = -x_1 + x_2 + (x_1^2 + x_2^2)\sin t, \qquad \dot{x}_2 = -x_1 - x_2 + (x_1^2 + x_2^2)\cos t$$

Show that the origin is exponentially stable and estimate the region of attraction.

[37] See [158] for a comprehensive treatment of Floquet theory.

4.45 Consider the system

$$\dot{x}_1 = h(t)x_2 - g(t)x_1^3, \qquad \dot{x}_2 = -h(t)x_1 - g(t)x_2^3$$

where $h(t)$ and $g(t)$ are bounded, continuously differentiable functions and $g(t) \geq k > 0$, for all $t \geq 0$.

(a) Is the equilibrium point $x = 0$ uniformly asymptotically stable?

(b) Is it exponentially stable?

(c) Is it globally uniformly asymptotically stable?

(d) Is it globally exponentially stable?

4.46 Show that the origin of the system

$$\dot{x}_1 = -x_2 - x_1(1 - x_1^2 - x_2^2), \qquad \dot{x}_2 = x_1 - x_2(1 - x_1^2 - x_2^2)$$

is asymptotically stable. Is it exponentially stable?

4.47 Consider the system

$$\dot{x}_1 = -\phi(t)x_1 + a\phi(t)x_2, \quad \dot{x}_2 = b\phi(t)x_1 - ab\phi(t)x_2 - c\psi(t)x_2^3$$

where a, b, and c are positive constants and $\phi(t)$ and $\psi(t)$ are nonnegative, continuous, bounded functions that satisfy

$$\phi(t) \geq \phi_0 > 0, \quad \psi(t) \geq \psi_0 > 0, \quad \forall\, t \geq 0$$

Show that the origin is globally uniformly asymptotically stable. Is it exponentially stable?

4.48 Consider two systems represented by $\dot{x} = f(x)$ and $\dot{x} = h(x)f(x)$ where $f : R^n \to R^n$ and $h : R^n \to R$ are continuously differentiable, $f(0) = 0$, and $h(0) > 0$. Show that the origin of the first system is exponentially stable if and only if the origin of the second system is exponentially stable.

4.49 Show that the system

$$\dot{x}_1 = -ax_1 + b, \qquad \dot{x}_2 = -cx_2 + x_1(\alpha - \beta x_1 x_2)$$

where all coefficients are positive, has a globally exponentially stable equilibrium point.
Hint: Shift the equilibrium point to the origin and use V of the form $V = k_1 Y_1^2 + k_2 y_2^2 + k_3 y_1^4$, where (y_1, y_2) are the new coordinates.

4.50 Consider the system

$$\dot{x} = f(t,x); \quad f(t,0) = 0$$

where $[\partial f/\partial x]$ is bounded and Lipschitz in x in a neighborhood of the origin, uniformly in t for all $t \geq t_0 \geq 0$. Suppose that the origin of the linearization at $x = 0$ is exponentially stable, and the solutions of the system satisfy

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall\, t \geq t_0 \geq 0, \quad \forall\, \|x(t_0)\| < c \tag{4.59}$$

for some class $\mathcal{KL}$ function β and some positive constant c.

(a) Show that there is a class $\mathcal{K}$ function α and a positive constant γ such that

$$\|x(t)\| \leq \alpha(\|x(t_0)\|) \exp[-\gamma(t - t_0)], \quad \forall\, t \geq t_0, \quad \forall\, \|x(t_0)\| < c$$

(b) Show that there is a positive constant M, possibly dependent on c, such that

$$\|x(t)\| \leq M\|x(t_0)\| \exp[-\gamma(t - t_0)], \quad \forall\, t \geq t_0, \quad \forall\, \|x(t_0)\| < c \tag{4.60}$$

(c) If inequality (4.59) holds globally, can you state inequality (4.60) globally?

4.51 Suppose the assumptions of Theorem 4.18 are satisfied with $\alpha_1(r) = k_1 r^a$, $\alpha_2(r) = k_2 r^a$, and $W(x) \geq k_3\|x\|^a$, for some positive constants k_1, k_2, k_3, and a. Show that (4.42) and (4.43) are satisfied with $\beta(r,s) = kr\exp(-\gamma s)$ and $\alpha_1^{-1}(\alpha_2(\mu)) = k\mu$, where $k = (k_2/k_1)^{1/a}$ and $\gamma = k_3/(k_2 a)$.

4.52 Consider Theorem 4.18 when $V(t,x) = V(x)$ and suppose inequality (4.40) is replaced by

$$\frac{\partial V}{\partial x} f(t,x) \leq -W_3(x), \quad \forall\, W_4(x) \geq \mu > 0$$

for some continuous positive definite functions $W_3(x)$ and $W_4(x)$. Show that (4.42) and (4.43) hold for every initial state $x(t_0) \in \{V(x) \leq c\} \subset D$, provided $\{V(x) \leq c\}$ is compact and $\max_{W_4(x) \leq \mu} V(x) < c$.

4.53 ([72]) Consider the system $\dot{x} = f(t,x)$ and suppose there is a function $V(t,x)$ that satisfies

$$W_1(x) \leq V(t,x) \leq W_2(x), \quad \forall\, \|x\| \geq r > 0$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) < 0, \quad \forall\, \|x\| \geq r_1 \geq r$$

where $W_1(x)$ and $W_2(x)$ are continuous, positive definite functions. Show that the solutions of the system are uniformly bounded.
Hint: Notice that $V(t,x)$ is not necessarily positive definite.

4.54 For each of the following scalar systems, investigate input-to-state stability:

$$(1)\ \dot{x} = -(1+u)x^3 \qquad (2)\ \dot{x} = -(1+u)x^3 - x^5$$

$$(3)\ \dot{x} = -x + x^2 u \qquad (4)\ \dot{x} = x - x^3 + u$$

4.55 For each of the following systems, investigate input-to-state stability:

$$\begin{array}{lrcllrcl}
(1) & \dot{x}_1 & = & -x_1 + x_1^2 x_2, & & \dot{x}_2 & = & -x_1^3 - x_2 + u \\
(2) & \dot{x}_1 & = & -x_1 + x_2, & & \dot{x}_2 & = & -x_1^3 - x_2 + u \\
(3) & \dot{x}_1 & = & x_2, & & \dot{x}_2 & = & -x_1^3 - x_2 + u \\
(4) & \dot{x}_1 & = & (x_1 - x_2 + u)(x_1^2 - 1), & & \dot{x}_2 & = & (x_1 + x_2 + u)(x_1^2 - 1) \\
(5) & \dot{x}_1 & = & -x_1 + x_1^2 x_2, & & \dot{x}_2 & = & -x_2 + x_1 + u \\
(6) & \dot{x}_1 & = & -x_1 - x_2 + u_1, & & \dot{x}_2 & = & x_1 - x_2^3 + u_2 \\
(7) & \dot{x}_1 & = & -x_1 + x_2, & & \dot{x}_2 & = & -x_1 - \sigma(x_1) - x_2 + u
\end{array}$$

where σ is a locally Lipschitz function, $\sigma(0) = 0$, and $y\sigma(y) \geq 0$ for all $y \neq 0$.

4.56 Using Lemma 4.7, show that the origin of the system

$$\dot{x}_1 = -x_1^3 + x_2, \qquad \dot{x}_2 = -x_2^3$$

is globally asymptotically stable.

4.57 Prove another version of Theorem 4.19, where all the assumptions are the same except that inequality (4.49) is replaced by

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, u) \leq -\alpha_3(\|x\|) + \psi(u)$$

where α_3 is a class $\mathcal{K}_\infty$ function and $\psi(u)$ is a continuous function of u with $\psi(0) = 0$.

4.58 Use inequality (4.47) to show that if $u(t)$ converges to zero as $t \to \infty$, so does $x(t)$.

4.59 Consider the scalar system $\dot{x} = -x^3 + e^{-t}$. Show that $x(t) \to 0$ as $t \to \infty$.

4.60 Suppose the assumptions of Theorem 4.19 are satisfied for $\|x\| < r$ and $\|u\| < r_u$ with class $\mathcal{K}$ functions α_1 and α_2 that are not necessarily class $\mathcal{K}_\infty$. Show that there exist positive constants k_1 and k_2 such that inequality (4.47) is satisfied for $\|x(t_0)\| < k_1$ and $\sup_{t \geq t_0} \|u(t)\| < k_2$. In this case, the system is said to be *locally input-to-state stable.*

4.61 Consider the system

$$\dot{x}_1 = x_1 \left\{ \left[\sin\left(\frac{\pi x_2}{2}\right) \right]^2 - 1 \right\}, \qquad \dot{x}_2 = -x_2 + u$$

(a) With $u = 0$, show that the origin is globally asymptotically stable.

(b) Show that for any bounded input $u(t)$, the state $x(t)$ is bounded.

(c) With $u(t) \equiv 1$, $x_1(0) = a$, and $x_2(0) = 1$, show that the solution is $x_1(t) \equiv a$, $x_2(t) \equiv 1$.

(d) Is the system input-to-state stable?

In the next seven exercises, we deal with the discrete-time dynamical system[38]

$$x(k+1) = f(x(k)), \quad f(0) = 0 \tag{4.61}$$

The rate of change of a scalar function $V(x)$ along the motion of (4.61) is defined by

$$\Delta V(x) = V(f(x)) - V(x)$$

4.62 Restate Definition 4.1 for the origin of the discrete-time system (4.61).

4.63 Show that the origin of (4.61) is stable if, in a neighborhood of the origin. there is a continuous positive definite function $V(x)$ so that $\Delta V(x)$ is negative semidefinite. Show that it is asymptotically stable if, in addition, $\Delta V(x)$ is negative definite. Finally, show that the origin is globally asymptotically stable if the conditions for asymptotic stability hold globally and $V(x)$ is radially unbounded.

4.64 Show that the origin of (4.61) is exponentially stable if, in a neighborhood of the origin, there is a continuous positive definite function $V(x)$ such that

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2, \qquad \Delta V(x) \le -c_3\|x\|^2$$

for some positive constants c_1, c_2, and c_3.
Hint: For discrete-time systems, exponential stability is defined by the inequality $\|x(k)\| \le \alpha\|x(0)\|\gamma^k$ for all $k \ge 0$, where $\alpha \ge 1$ and $0 < \gamma < 1$.

4.65 Show that the origin of (4.61) is asymptotically stable if, in a neighborhood of the origin, there is a continuous positive definite function $V(x)$ so that $\Delta V(x)$ is negative semidefinite and $\Delta V(x)$ does not vanish identically for any $x \ne 0$.

4.66 Consider the linear system $x(k+1) = Ax(k)$. Show that the following statements are equivalent:

[38] See [95] for a detailed treatment of Lyapunov stability for discrete-time dynamical systems.

(1) $x = 0$ is asymptotically stable.

(2) $|\lambda_i| < 1$ for all eigenvalues of A.

(3) Given any $Q = Q^T > 0$, there exists $P = P^T > 0$, which is the unique solution of the linear equation $A^T PA - P = -Q$.

4.67 Let A be the linearization of (4.61) at the origin; that is, $A = [\partial f/\partial x](0)$. Show that the origin is asymptotically stable if all the eigenvalues of A have magnitudes less than one.

4.68 Let $x = 0$ be an equilibrium point for the nonlinear discrete-time system $x(k+1) = f(x(k))$, where $f : D \to R^n$ is continuously differentiable and $D = \{x \in R^n \mid \|x\| < r\}$. Let C, $\gamma < 1$, and r_0 be positive constants with $r_0 < r/C$. Let $D_0 = \{x \in R^n \mid \|x\| < r_0\}$. Assume that the solutions of the system satisfy

$$\|x(k)\| \leq C\|x(0)\|\gamma^k, \quad \forall\ x(0) \in D_0,\ \forall\ k \geq 0$$

Show that there is a function $V : D_0 \to R$ that satisfies

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2$$

$$\Delta V(x) = V(f(x)) - V(x) \leq -c_3\|x\|^2$$

$$|V(x) - V(y)| \leq c_4\|x - y\|\,(\|x\| + \|y\|)$$

for all $x, y \in D_0$ for some positive constants c_1, c_2, c_3, and c_4.