4. Dato il sistema dinamico

$$\dot{x} = \underbrace{u}_{f}$$

con condizioni iniziali x_0 fissate e note. Si imposti e si risolva analiticamente un problema di controllo ottimo con tempo iniziale e finale fissati, che porti lo stato del sistema il più vicino possibile all'origine limitando il consumo. Si utilizzi un unico parametro p per controllare il trade-off tra consumo e distanza dall'origine all'istante finale. Si calcoli la soluzione in funzione del parametro p.

CONDIZIONE x_f LIBERO	$[h_x - \lambda^T]_{t_f}\delta x_f + [\cancel{h_t + H}]_{t_f}\delta t_f = 0$	$\gamma = 0$ $\quad x(t_f) = x(t_0)$ SISTEMA FERMO NO DISTANZA ORIGINE	"S, Q" SDP
	$h_x(t_f) \Rightarrow S\,x(t_f) = \lambda(t_f)$	$\gamma \to \infty$ $\quad x(t_f) \to 0$ VALE SOLO ARRIVO CONSUMO TROPPO	"R" DP REALI SIMM

$\dot{x} = Ax + Bu = f \qquad g = x^T Q x + u^T R u$

1) $J = h(t_f) + \int_{t_0}^{t_f} g = \frac{1}{2} x^T S x \big|_{t_f} + \int_{t_0}^{t_f} u^T R u$

$\int h_x \delta x + h_t \delta t + g + \lambda^T f \; ; \; H = g + \lambda^T f = \frac{1}{2}u^2 + \lambda u$

$$\begin{cases} \dot{x} = H_\lambda = u \\ \dot{\lambda} = -H_x = 0 \\ 0 = H_u = u + \lambda \end{cases} \qquad \begin{cases} x(t_f) = x_0 + u(t_f - t_0) \\ \lambda = \text{cost} \\ u = -\lambda \end{cases}$$

$$\begin{cases} x(t_f) = x_0 - \gamma\, x(t_f)(t_f - t_0) \\ \lambda(t_f) = \gamma\, x(t_f) \\ u = -\lambda \end{cases} \qquad \begin{cases} x(t_f) = x_0 \frac{1}{1+\gamma(t-t_0)} \\ u = -\gamma x_0 \frac{1}{1+\gamma(t-t_0)} \end{cases}$$

2) $A = 0 \quad B = 1 \;\big|\; M = B R^{-1} B^T \qquad H = \begin{vmatrix} A & -M \\ -Q & -A^T \end{vmatrix}$

$\;\;\;\; Q = 0 \quad R = 1 \;\big|\; Q = C^T C$

$\begin{vmatrix} \dot{x} \\ \dot{\lambda} \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} x \\ \lambda \end{vmatrix} \Rightarrow \begin{vmatrix} x \\ \lambda \end{vmatrix} = \Psi(t, t_0) \begin{vmatrix} x_0 \\ \lambda_0 \end{vmatrix}$ BELLMAN (t_f, t) $(t_f, 0)$ $(t, 0)$

$\Psi = e^{H} \Rightarrow \begin{vmatrix} 1 & -(t_f - t) \\ 0 & 1 \end{vmatrix} \; ; \quad \Psi_{21} x + \Psi_{22}\lambda = S_{t_f}(\Psi_{11} x + \Psi_{12}\lambda)$

$\lambda = (\Psi_{22} - S\Psi_{12})^{-1}(S\Psi_{11} - \Psi_{21})\,x = [1 - [\gamma][-(t_f - t)]]^{-1}[[\gamma]1 - 0]\,x$

$u = -R^{-1}B^T\lambda = -\lambda = -\gamma x \frac{1}{1+\gamma(t_f - t)}$

3) $\dot{S} + SA + A^T S - SMS + Q = 0$

SOL: $\frac{dS}{dt} = S(t)^2 \rightarrow \frac{dS}{S^2} = 1\,dt$

$\int \frac{1}{S^2}\,dS = \int_t^{t_f} 1\,dt \rightarrow -\frac{1}{S} = t + k$

$S = -\frac{1}{t+k} \rightarrow S_{t_f} = \gamma = -\frac{1}{t_f + k} \rightarrow k = -\frac{1}{\gamma} - t_f$

$S = -\frac{1}{(t - t_f) - \frac{1}{\gamma}} = \gamma \frac{1}{1+\gamma(t_f - t)}$

$\lambda = S x \Rightarrow u = -R^{-1}B^T\lambda = -\lambda$

4. Si consideri il sistema lineare tempo continuo SISO descritto dalle matrici A, B, C. Si trovi, se esiste, al variare di ρ una legge di controllo nella forma $u(t) = K(\rho)x(t)$ che rende minimo il funzionale di costo:

$$J = \int_0^\infty \rho u^2(t)dt$$

nei seguenti tre casi:

(a)

$$A_1 = \begin{pmatrix} -5 & 1 & 2 \\ 0 & -6 & -1 \\ 0 & 0 & -1 \end{pmatrix} \quad B_1 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

(b)

$$A_2 = \begin{pmatrix} -1 & 2 & 5 \\ 0 & -2 & -2 \\ 0 & 0 & 3 \end{pmatrix} \quad B_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

(c)

$$A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \quad B_3 = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

Si discuta nei tre casi la dipendenza da ρ della soluzione.

$J = \frac{1}{2}\int_0^\infty x^TQx + u^TRu \qquad Q = C^TC = 0$

(A,B) ST (NO DIVERGE: $\lambda(\bar{R}) < 0$)
(A,C) RL (Q NASCONDE MODI: $\lambda(\bar{O}) < 0$)
$x(\infty)^T S x(\infty) \to 0$ (SENNÒ DIVERGE)

$H = \begin{vmatrix} A & -M \\ -Q & -A^T \end{vmatrix}$; $\lambda(H) = \lambda(\bar{A}) = -\lambda(\bar{A})$

$\bar{A} = A + BK \qquad \gamma = \frac{2}{\rho}$

$d(H - \lambda I) \triangleq p_d(\lambda) = (\lambda - \lambda_i)^m$ IMPONGO $\lambda < 0$

$\begin{vmatrix} A & -\gamma BB^T \\ 0 & -A^T \end{vmatrix}$ $\quad Q = 0$ SISTEMA NON OSS

a) (TUTTI $\lambda < 0$) SISTEMA AS. $K = [0\ 0\ 0]$ $u = 0$

b) $\lambda_3 = 3 \in \bar{R}$ SISTEMA NO ST

TD $\bar{S} = A^T\bar{S}[I - B[NB+R]^{-1}N]A + Q \qquad N = B^T\bar{S}$
$\bar{S} = S(k+1)$

c1) FCC : $d(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 1 & -1-\lambda \end{vmatrix} = \lambda^3 + \lambda^2 - \lambda - 1$

	1	1	-1	-1
1		1	2	1
	1	2	1	0

$(\lambda - 1)(\lambda^2 + 2\lambda + 1)$ $\to p_d = (\lambda + 1)^3$
$\lambda = 1 ; \lambda_{1,2} = -1$

$A + BK = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1+3k_1 & 1+3k_2 & -1+3k_3 \end{vmatrix}$; $d(\bar{A} - \lambda I) = \lambda^3 + (1 - 3k_3)\lambda^2 + (-1 - 3k_2)\lambda + (-1 - 3k_1)$

$p_d(\lambda) \triangleq \lambda^3 + 3\lambda^2 + 3\lambda + 1$

$\begin{cases} 3 = 1 - 3k_3 \\ 3 = -1 - 3k_2 \\ 1 = -1 - 3k_1 \end{cases} \qquad K = -\frac{1}{3}[2\ \ 4\ \ 2]$

c2) $SA + A^TS + Q - SMS = 0$

$M = \gamma BB^T = \gamma \begin{vmatrix} 0 \\ 0 \\ 3 \end{vmatrix} |0\ 0\ 3| = \gamma \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{vmatrix}$

$K = -R^{-1}B^TS \qquad \begin{matrix} \beta = \hat{S}_1 \\ c = \hat{S}_2 \\ \eta = \hat{S}_3 \end{matrix}$

$-\rho^{-1} |0\ 0\ 3| \begin{vmatrix} s_1 & \beta & c \\ \beta & s_2 & \eta \\ c & \eta & s_3 \end{vmatrix} = -3\rho^{-1} \begin{vmatrix} \hat{S}_2 & \hat{S}_3 & S_3 \end{vmatrix}$

$\begin{vmatrix} s_1 & \hat{s}_1 & \hat{s}_2 \\ \hat{s}_1 & s_2 & \hat{s}_3 \\ \hat{s}_2 & \hat{s}_3 & s_3 \end{vmatrix} \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} \begin{vmatrix} s_1 & \hat{s}_1 & \hat{s}_2 \\ \hat{s}_1 & s_2 & \hat{s}_3 \\ \hat{s}_2 & \hat{s}_3 & s_3 \end{vmatrix} - \gamma \begin{vmatrix} s_1 & \hat{s}_1 & \hat{s}_2 \\ \hat{s}_1 & s_2 & \hat{s}_3 \\ \hat{s}_2 & \hat{s}_3 & s_3 \end{vmatrix} \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{vmatrix} \begin{vmatrix} s_1 & \hat{s}_1 & \hat{s}_2 \\ \hat{s}_1 & s_2 & \hat{s}_3 \\ \hat{s}_2 & \hat{s}_3 & s_3 \end{vmatrix}$

NB ARE

$A = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \quad B = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$

$C = |1\ \ 0| \quad R = \begin{vmatrix} \rho & 0 \\ 0 & 0 \end{vmatrix}$

$S = \begin{vmatrix} s_1 & \hat{s}_1 \\ \hat{s}_1 & s_2 \end{vmatrix} \quad Q = C^TC$

$\dot{S} + A^TS + SA - SMS - Q = 0$

$\begin{vmatrix} -2s_2 & s_1 - s_3 \\ s_1 - s_3 & 2s_2 \end{vmatrix} + \begin{vmatrix} s_2^2k - 1 & s_2 s_3 k \\ s_2 s_3 k & s_3^2 k \end{vmatrix} = 0$

$s_2)$ $s_2^2 k - 2s_2 - 1$
$\hat{s}_1)$ $s_1 - s_3 + s_2 s_3 k$
$\hat{s}_1)$ $s_1 - s_3 + s_2 s_3 k$
$s_2)$ $2s_2 + s_3^2 k$

$S > 0 : \frac{2 \pm \sqrt{4(1+k)}}{2}$
$k = 1 \ \vdots \ \frac{2 \pm 2}{2}$ solo $\frac{2+2}{2}$

4. Dato il sistema dinamico:

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u$$

si calcoli l'ingresso di controllo ottimo $u(t)$ che porta lo stato dal punto $x(t_0 = 0) = [0, 0]$ al punto $x(t_f) = [.25, 0]$ nel tempo $t_f = 1$ minimizzando l'indice di costo:

$$J = \int_{t_0}^{t_f} u^2(\tau) d\tau$$

$$\begin{cases} \dot{x} = H_\lambda = Ax + Bu \\ \dot{\lambda} = -H_x = 0 - A^T\lambda \\ 0 = H_u = u + B^T\lambda \end{cases}$$

$$A = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \qquad B = \begin{vmatrix} 0 \\ -1 \end{vmatrix}$$

$$\begin{cases} x = \cancel{e^{At}x_0} - \underbrace{\int e^{At} B B^T e^{At}}_{GR} \lambda_0 \\ \lambda = e^{-A^T}\lambda_0 \\ u = -B^T\lambda \end{cases}$$

$$x_f = \begin{vmatrix} 1/4 \\ 0 \end{vmatrix} \quad t_f = 1 \quad x_0 = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$$e^{A(t)} = \begin{vmatrix} 1 & t \\ 0 & 1 \end{vmatrix}$$

$$\begin{cases} x_f = -GR\, e^{-A^T(t_f - t_0)} \lambda_0 \\ \lambda_0 = -e^{A^T(t_f + t)} GR^{-1} x_f \end{cases}$$

$$e^{-A^T(t)} = \begin{vmatrix} 1 & -t \\ 0 & 1 \end{vmatrix}$$

DET:

2x2 $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = d(a)\, d(d - c a^{-1} b) = d(d)\, d(a - b d^{-1} c)$

3x3 $\left|\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}\right|\begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$

P(λ):

2x2) $\lambda^2 - (TR)\lambda + (DET)$

3x3) $\lambda^3 - (TR)\lambda^2 + (M)\lambda - (DET)$

TRACCIA: $(11 + 22 + 33)$

M: $\begin{vmatrix} 11 & 12 \\ 21 & 22 \end{vmatrix} + \begin{vmatrix} 22 & 23 \\ 32 & 33 \end{vmatrix} + \begin{vmatrix} 11 & 13 \\ 31 & 33 \end{vmatrix}$

INV: $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

2x2) $\frac{1}{DET}\begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$

3x3)

GAUSS

$\left|\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix}\right|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right|$

$-\left|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{matrix}\right|\begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix}\right|$

$-\left|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix}\right|\begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{matrix}\right|$

$\left|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right|\begin{matrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{matrix}\right|$

INDIAN

$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$ 0 1, 1 0, 0 1 1 0, 1 1 0 1, -1, -1 -1

$A^{-1} = \frac{1}{DET}(ADJ)^T$;

$c_{ij} = -1^{i+j} d(A_{ij})$

SUB MAX CANCELLO RIGA "i" E COLONNA "j"

$x^n = K \to x = \pm\sqrt[n]{K}$ n PARI $K \geq 0$

$\boxed{\begin{matrix} \log_a(x) & a, b > 0 & a \neq 1 \\ e^x & \in [0; \infty] & \end{matrix}}$

$K = \frac{1}{a+1}$

$\alpha \neq -1$

$S(x+y) = Cy\,Sx + Cx\,Sy$

$C(x+y) = Cx\,Cy - Sx\,Sy$

$S(u)|_x = S(u)|_u\, u|_x = C(u)\, u/x = C(u)(1+0)$

$\frac{d}{dx} = |_x$

$u = x + y \quad -1 \leq y \leq 1$

$C(x) = 0 \to x = K\pi - \frac{\pi}{2}$

$S(x) = 0 \to x = K\pi \quad -1 \leq y \leq 1$

$Tg(x) = \frac{S(x)}{C(x)}$; $S^2x = \frac{1 - C2x}{2}$

$C^2x = \frac{1 + C2x}{2}$

$\frac{d}{dt} S^2x = 2SC$

$\frac{d}{dt} C^2x = -2SC$

$S^2 + C^2 = 1$

$C(2x) = C^2 - S^2$

$S(2x) = 2SC$

ROUT NO AS.

$\begin{array}{c|cc} 3 & 1 & 0 \\ 2 & 1 & -1 \\ 1 & \frac{0+1}{1} & \frac{0-0}{1} \\ 0 & \frac{-1}{1} & \end{array}$

AUTOVETT. GEN

NB $G+JW \in K(\bar{A}^2)$ PIANO MA 2?

x COLONNE $\in K(\bar{A})$ RETTA;

x $\in K(\bar{A}^2) \notin K(\bar{A})$ PIANO;

x $\in K(A)$ PEQ

n COLONNE = mg = n = $\rho(K\bar{A})$

$\begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}$

$\lambda = 2$ ma = 3

mg = 2

$\bar{A} = A - 2I = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$ $\rho K = 2 = mg$ RETTA $\to e_1; e_2$

$\bar{A}^2 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$ $\rho K = 3 \to$ PIANO $e_3 \notin K(\bar{A})$

$q^1 = (A - \lambda I) q^2 = (A - 2I) q^2 = e_1$

$Q = (\tilde{q}\, q_1, q_2)$ $\tilde{q}$ INVERTIBILE

$\begin{vmatrix} 4 & -1 & 0 & 1 & -1 \\ 2 & 3 & 0 & 1 & 1 \\ -2 & 2 & 4 & 0 & 0 \\ 2 & 0 & 0 & 4 & 2 \\ 0 & 1 & 0 & -1 & 5 \end{vmatrix}$ $\lambda = 4$, ma = 5, mg = 2, $(A - 4I) \to$ $\begin{vmatrix} 0 & -1 & 0 & 1 & -1 \\ 2 & -1 & 0 & 1 & 1 \\ -2 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 & 1 \end{vmatrix}$ $\rho(K\bar{A}) = 2$

$(e_3; e_1 + e_2 - e_5)$

$\bar{A}^2 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}$ $K(\bar{A}^2) = (e_2; e_3; e_4; e_1 - e_5)$

$\rho = 4$

$\bar{A}^3 = |\bar{0}|$ $\rho = 5$ $(e_1; e_2; e_3; e_4; e_5)$

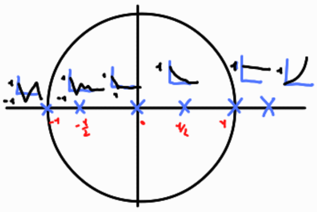
$\left.\begin{matrix} N_3 - N_2 = 5 - 4 = 1 & \text{ORDINE 3} \\ N_2 - N_1 = 4 - 2 = 2 & \text{ORDINE 2} \\ N_1 - 0 = 2 - 0 = 2 & \text{ORDINE 2} \end{matrix}\right\}$ $\to q_1^3 \notin K\bar{A}^2 \in K\bar{A}^3 : e_1$

$\bar{A}(q_1^3) = q_1^2$

$\bar{A}(q_1^2) = q_1^1$

$q_2^2 \in K\bar{A}^2 \notin K\bar{A} = e_2 \to A(q_2^2) = q_2^1$

$Q = [q_1^1\ q_1^2\ q_1^3\ q_2^1\ q_2^2]$



$g = y_1^2 + y_2^2$

$f = y_1 + y_2 - 5 = 0$

$H = g + \lambda f$

$\dot{H} = H_{y_1}\Delta y_1 + H_{y_2}\Delta y_2 + H_\lambda \Delta\lambda$

$\begin{cases} 2y_1 + \lambda = 0 \\ 2y_2 + \lambda = 0 \\ y_1 + y_2 - 5 = 0 \end{cases}$

$\begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix}\begin{vmatrix} y_1 \\ y_2 \\ \lambda \end{vmatrix} = \begin{vmatrix} 5 \\ 0 \\ 0 \end{vmatrix}$

$\begin{vmatrix} y_1 \\ y_2 \\ \lambda \end{vmatrix} = M^{-1}\begin{vmatrix} 5 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 5/2 \\ 5/2 \\ -5 \end{vmatrix}$

$P_d(\lambda) = (\lambda + 1)(\lambda + 2) = \lambda^2 + 3\lambda + 2$

$d(A - \lambda I) = \lambda^2 - \alpha$ $\quad R = \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix}$

$\alpha_D = [3\ 2]$; $\alpha = [0\ -\alpha]$ $\quad U = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

$K = -(\alpha_D - \alpha)(RU)^{-1}$

$= -[3\quad 2+\alpha]\left[\begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix}\middle|\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right]^{-1}$

$= [2+\alpha\quad 3] = [K_1\ K_2]$