Fundamental definitions for Linear Algebra

1 Vector Spaces

A *vector space* V over a field F (such as $\mathbb{R}$ or $\mathbb{C}$) is a set of elements called vectors, where two operations are defined:

- **Vector addition:** For any $u, v \in V$, $u + v \in V$.
- **Scalar multiplication:** For any $\alpha \in F$ and $v \in V$, $\alpha v \in V$.

These operations satisfy certain axioms, such as associativity, commutativity of addition, and distributivity of scalar multiplication.

2 Matrix-Vector Product

Let A be an $m \times n$ matrix, and x a vector in $\mathbb{R}^n$. The matrix-vector product is a vector in $\mathbb{R}^m$ defined as:

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}.$$

This operation is a linear transformation that maps vectors from $\mathbb{R}^n$ to $\mathbb{R}^m$. The vector $A\mathbf{x}$ can be seen also as a linear combination of the columns of A with coefficients the elements of v, i.e.:

$$A\mathbf{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

3 Eigenvalues and Eigenvectors

For a square matrix $A \in \mathbb{R}^{n \times n}$, an eigenvalue $\lambda \in \mathbb{R}$ and a corresponding eigenvector $v \in \mathbb{R}^n$ are defined by the equation:

$$A\mathbf{v} = \lambda\mathbf{v},$$

where $v \neq 0$. In other words, when the matrix A acts on the vector v, the result is a scalar multiple of v. The eigenvalues can be found by solving the characteristic equation:

$$p(\lambda) = \det(A - \lambda I) = 0,$$

where I is the identity matrix. $p(\lambda)$ is the characteristic polynomial of matrix A and the eigenvalues of A are the zeros of such polynomial.

4 Null Space and Image Space

The *null space* (or *kernel*) of a matrix $A \in \mathbb{R}^{m \times n}$ is the set of all vectors $x \in \mathbb{R}^n$ such that:

$$A\mathbf{x} = 0.$$

It represents the set of vectors that are mapped to the zero vector by the matrix A.

The *image space* (or *column space*) of a matrix A is the set of all vectors $b \in \mathbb{R}^m$ such that $Ax = b$ has a solution. In other words, it is the span of the columns of A.

5 Similarity Matrices

Two matrices A and B are said to be *similar* if there exists an invertible matrix T such that:

$$B = TAT^{-1}.$$

Similarity preserves many important properties, such as eigenvalues. Eigenvectors of B are of the form $w = Tv$ where v is an eigenvector of A. If A and B are similar, they represent the same linear transformation in different bases.