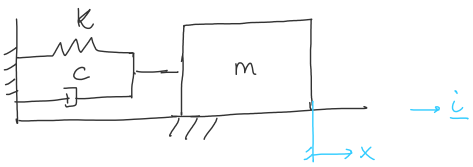
Oscillazioni libere

venerdì 6 dicembre 2024 13:00

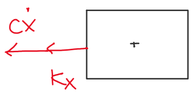


Noto: m, k, c

$$\text{c.i.} \begin{cases} x(0) = x_0 \\ v(0) = \dot{x}(0) = v_0 \end{cases} \Rightarrow x(t)\ ?$$

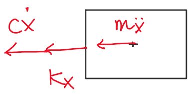
$\Rightarrow$ EQUILIBRIO IN DIREZIONE $\underline{i}$ $\Rightarrow$ DCL

DCL



$x>0$
$\dot{x}>0$

DCL



$x>0$
$\dot{x}>0$
$\ddot{x}>0$

ICD $\quad \underline{R}^{(e)} = m\underline{a}_G$

$$-c\dot{x}\,\underline{i} - kx\,\underline{i} = m\ddot{x}\,\underline{i}$$

$\cdot\underline{i}$) x: $\quad -c\dot{x} - kx = m\ddot{x}$

$$\begin{cases} \boxed{m\ddot{x} + c\dot{x} + kx = 0} \\ x(0) = x_0 \\ \dot{x}(0) = v_0 \end{cases}$$

(1) Eq^{ne} DIFFER., 2° ORDINE, LINEARE, COEFF. COSTANTI OMOGENEA $\Rightarrow$ Eq^{ne} MOTO

Dividiamo (1) per la massa: $\quad \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$

- PULSAZIONE NATURALE O PROPRIA $\quad \omega_n = \sqrt{\frac{k}{m}}$ (rad/s)

- FATTORE DI SMORZAMENTO $\quad \xi = \frac{c}{2m\omega_n} = \frac{c}{c_r} \Rightarrow$ ADIMENSIONALE

$c_r = 2\sqrt{km}$ SMORZAM. CRITICO

$$\ddot{x} + \underbrace{\frac{c}{m}}_{2\xi\omega_n}\dot{x} + \underbrace{\frac{k}{m}}_{\omega_n^2}x = 0 \Rightarrow \boxed{\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 0}$$

$\frac{\cancel{2}\,c}{\cancel{2}m\cancel{\omega_n}}\,\omega_n$

$\hookrightarrow$ RISPOSTA DIPENDE DA 2 FATTORI: ξ, ω_n

$$\begin{cases} x(t) = e^{\mu t} \\ \dot{x}(t) = \mu e^{\mu t} \\ \ddot{x}(t) = \mu^2 e^{\mu t} \end{cases}$$

$x(t) = \mu^{i} e$

$$\mu^2 e^{\mu t} + 2\xi \omega_n \mu e^{\mu t} + \omega_n^2 e^{\mu t} = 0$$

$$\underbrace{(\mu^2 + 2\xi\omega_n\mu + \omega_n^2)}\, e^{\mu t} = 0$$

$\downarrow$

$$\mu^2 + 2\xi\omega_n\mu + \omega_n^2 = 0 \qquad \text{Eq}^{ne} \text{ CARATTERISTICA}$$

$$\mu_{1,2} = \frac{-2\xi\omega_n \pm \sqrt{(2\xi\omega_n)^2 - 4\omega_n^2}}{2} = -\xi\omega_n \pm \underbrace{\sqrt{\xi^2-1}}\,\omega_n$$

OSCILLAZ. SOVRASMORZATE
↑

1) $\xi^2 - 1 > 0 \Rightarrow$ $\xi > 1$, $c > c_r$ $\quad \mu_{1,2} \in \mathbb{R} \rightarrow \mu_{1,2} = -\xi\omega_n \pm \sqrt{\xi^2-1}\,\omega_n \Rightarrow$ OSCILLAZIONI APERIODICHE SMORZATE

2) $\xi^2 - 1 < 0 \Rightarrow$ $\xi < 1$, $c < c_r$ $\quad \mu_{1,2} \in \mathbb{C} \rightarrow \mu_1 = \bar{\mu}_2$

$\mu_{1,2} = -\xi\omega_n \pm i\sqrt{1-\xi^2}\,\omega_n \Rightarrow$ OSCILLAZ. PERIODICHE SMORZATE
↓
OSCILL SOTTOSMORZATE

3) $\xi^2 - 1 = 0 \Rightarrow$ $\xi = 1$, $c = c_r$ $\quad \mu_{1,2} \in \mathbb{R} \longrightarrow \mu_1 = \mu_2 = -\xi\omega_n$

CASO 1 – $\xi > 1$ – OSC. SOVRASM.

$$x(t) = A_1 e^{\mu_1 t} + A_2 e^{\mu_2 t} = e^{-\xi\omega_n t}\left(A_1 e^{\omega_n\sqrt{\xi^2-1}\,t} + A_2 e^{-\omega_n\sqrt{\xi^2-1}\,t}\right)$$

$\Rightarrow A_1$ e A_2 dipendono da c.i $\Rightarrow x(0), \dot{x}(0)$

$\Rightarrow \lim_{t\to+\infty} (x(t)) = 0$

$$\dot{x}(t) = -\xi\omega_n e^{-\xi\omega_n t}\left(A_1 e^{\omega_n\sqrt{\xi^2-1}\,t} + A_2 e^{-\omega_n\sqrt{\xi^2-1}\,t}\right) + e^{-\xi\omega_n t}\left(A_1 \omega_n\sqrt{\xi^2-1}\, e^{\omega_n\sqrt{\xi^2-1}\,t} + A_2(-\omega_n\sqrt{\xi^2-1})e^{-\omega_n\sqrt{\xi^2-1}\,t}\right)$$

$$\begin{cases} x(0) = x_0 \\ \dot{x}(0) = v_0 \end{cases} \Rightarrow \begin{cases} x_0 = A_1 + A_2 \\ v_0 = -\xi\omega_n(\cancel{A_1+A_2}) + (A_1 - A_2)\omega_n\sqrt{\xi^2-1} \end{cases}$$

2 EQ^{NI} + 2 INCOGNITE

es. $x(0) = 0$, $\dot{x}(0) = v_0$

$$\begin{cases} 0 = A_1 + A_2 \\ v_0 = (A_1 - A_2)\,\omega_n\sqrt{\xi^2-1} \end{cases}$$

$$\begin{cases} A_1 = -A_2 \\ A_2 = -\frac{v_0}{} \end{cases}$$

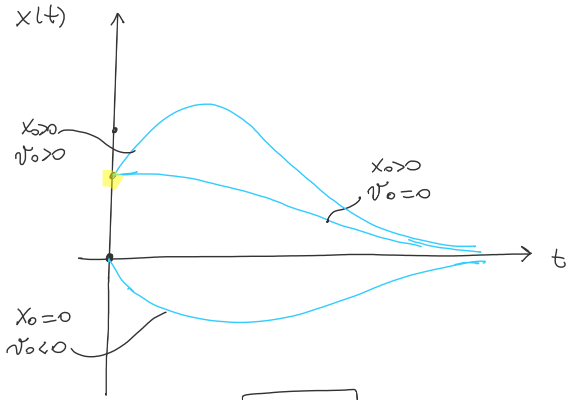
$2\omega_n\sqrt{\xi^2-1}$

es. $x(0)=x_0$
$\dot{x}(0)=0$

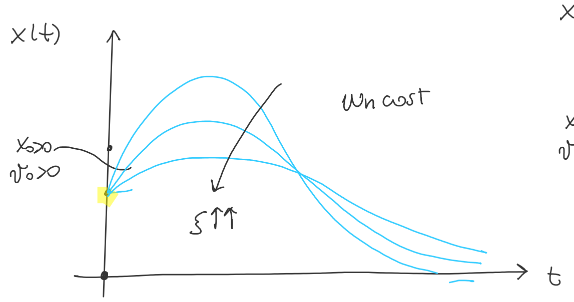
$$\begin{cases} A_1 = \dfrac{x_0\,(\xi+\sqrt{\xi^2-1})}{2\sqrt{\xi^2-1}} \\ A_2 = \dfrac{x_0\,(\sqrt{\xi^2-1}-\xi)}{2\sqrt{\xi^2-1}} \end{cases}$$

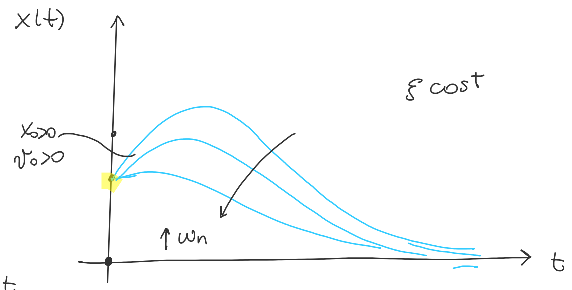
es. $x(0)=0$
$\dot{x}(0)=0$ $\Rightarrow$ $A_1=A_2=0$ $\Rightarrow$ $x(t)=0$ $\Rightarrow$ MASSA FERMA

$$x(t) = e^{-\xi\omega_n t}\left(A_1 e^{\omega_n\sqrt{\xi^2-1}\,t} + A_2 e^{-\omega_n\sqrt{\xi^2-1}\,t}\right)$$



$\boxed{\xi,\ \omega_n}$





2) OSCILL. SOTTOSM. PERIODICHE — $\xi < 1$

$$x(t) = A_1 e^{\mu_1 t} + A_2 e^{\mu_2 t} \qquad \mu_1 = \overline{\mu_2} = -\xi\omega_n \pm i\,\omega_n\sqrt{1-\xi^2}$$

$$= A_1 e^{\mu_1 t} + A_2 e^{\overline{\mu_1} t}$$

$$= e^{-\xi\omega_n t}\left(A_1 e^{i\omega_n\sqrt{1-\xi^2}\,t} + A_2 e^{-i\omega_n\sqrt{1-\xi^2}\,t}\right)$$

$\omega_s = \omega_n\sqrt{1-\xi^2}$ PULSAZ. DELL'OSCILLAZ. SMORZATA

$\omega_s < \omega_n$

$$x(t) = e^{-\xi\omega_n t}\left(A_1 e^{i\omega_s t} + A_2 e^{-i\omega_s t}\right)$$

Affinche $x(t) \in \mathbb{R} \Rightarrow A_1 = \overline{A_2}$

$$A_1 = \frac{C - iD}{2} \qquad A_2 = \frac{C + iD}{2}$$

$$x(t) = e^{-\xi \omega_n t} \left(\frac{C - iD}{2} e^{i\omega_s t} + \frac{C + iD}{2} e^{-i\omega_s t} \right)$$

FORMULE EULERO $\Rightarrow$

$$\sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}$$

$$\cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}$$

$$x(t) = e^{-\xi \omega_n t} \left(C \underbrace{\frac{e^{i\omega_s t} + e^{-i\omega_s t}}{2}}_{\cos(\omega_s t)} - i^2 D \underbrace{\frac{e^{i\omega_s t} - e^{-i\omega_s t}}{2i}}_{\sin(\omega_s t)} \right)$$

$$= e^{-\xi \omega_n t} \left(C \cos(\omega_s t) + D \sin(\omega_s t) \right) \qquad (1) \quad C, D$$

$$= A e^{-\xi \omega_n t} \sin(\omega_s t + \psi) \qquad (2) \quad A, \psi$$

$$= B e^{-\xi \omega_n t} \cos(\omega_s t - \varphi) \qquad (3) \quad B, \varphi$$

2 PARAM + 2 CONDIZ. INIZIALI

$$\begin{cases} A = B = \sqrt{C^2 + D^2} \\ \operatorname{tg}\psi = C/D \\ \operatorname{tg}\varphi = D/C \end{cases}$$

$$\begin{cases} x(t) = A e^{-\xi \omega_n t} \sin(\omega_s t + \psi) \\ \dot{x}(t) = A(-\xi \omega_n) e^{-\xi \omega_n t} \sin(\omega_s t + \psi) + A e^{-\xi \omega_n t} \omega_s \cos(\omega_s t + \psi) \end{cases}$$

$$\begin{cases} x(0) = x_0 \\ \dot{x}(0) = v_0 \end{cases}$$

$$\begin{cases} x_0 = A \sin\psi \\ v_0 = -A \omega_n \xi \sin\psi + A \omega_s \cos\psi \end{cases}$$

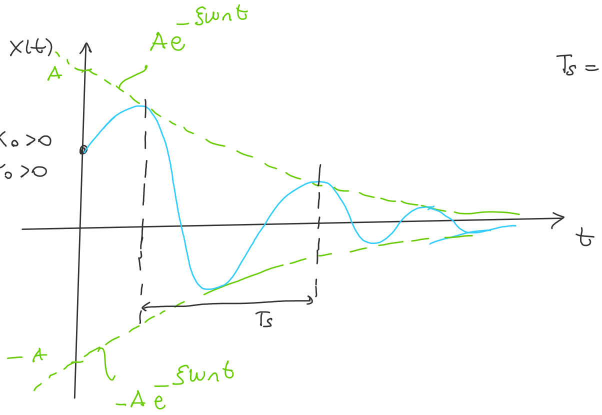
es. $\begin{cases} x(0) = x_0 \\ v(0) = 0 \end{cases} \rightarrow \begin{cases} A = \dfrac{x_0}{\sqrt{1-\xi^2}} \\ \operatorname{tg}\psi = A\sqrt{1-\xi^2} \end{cases}$

es $\begin{cases} x(0) = 0 \\ v(0) = v_0 \end{cases} \longrightarrow \begin{cases} \varphi = 0 \\ A = \dfrac{v_0}{\dots} \end{cases}$

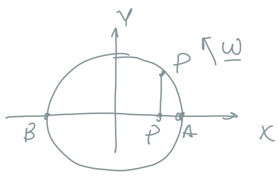
ω_s

$$x(t) = \underbrace{A e^{-\xi \omega_n t}}\ \underbrace{\sin(\omega_s t + \psi)}$$

$$\lim_{t \to +\infty} x(t) = 0$$

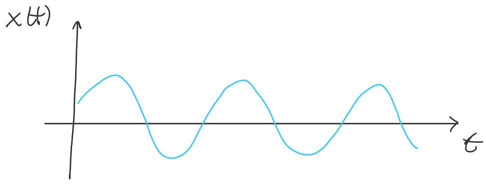


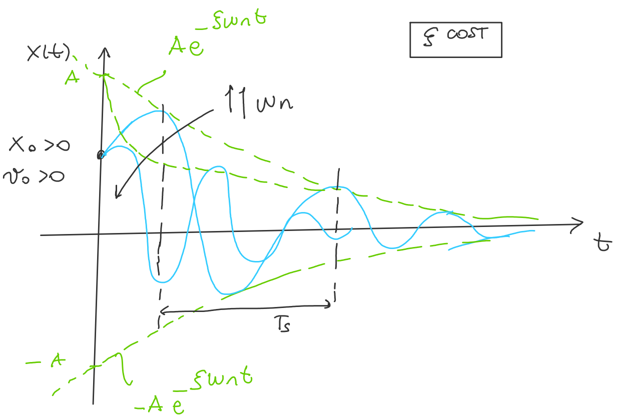
$$T_s = \frac{2\pi}{\omega_s}$$

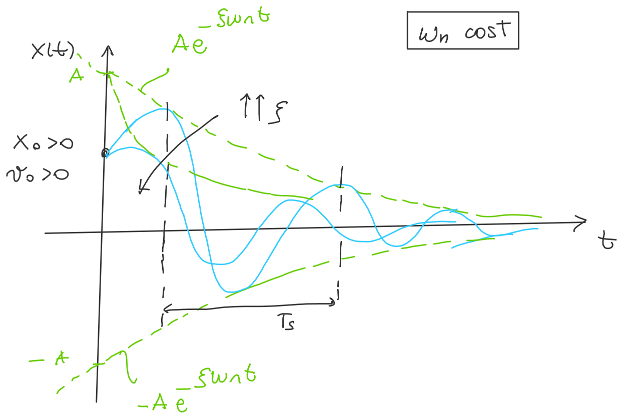


se $\xi = 0 \Rightarrow \ddot{x} + \omega_n^2 x = 0 \quad (*)$

$x(t) = A \sin(\omega_n t + \psi)$ MOTO ARMONICO







$$x(t) = \underbrace{A e^{-\xi \omega_n t}}\ \underbrace{\sin(\overset{\omega_n\sqrt{1-\xi^2}}{\omega_s t} + \psi)}$$