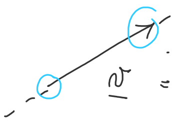
Vettori e algebra vettoriale

sabato 21 settembre 2024 14:49

VETTORE $\underline{v}$

1) MODULO $|\underline{v}|$
2) direzione
3) verso



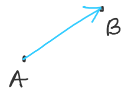
~ RETTA DI AZIONE (RDA)

$\underline{v} = \vec{v} = \mathbf{v}$

1) SEGMENTO ORIENTATO

$\vec{AB} = AB$ VETTORE

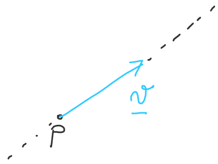
$|\vec{AB}| = \overline{AB}$ MODULO

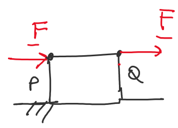


2) VETTORI APPLICATI

$(P, \underline{v})$

→ lo posso spostare solo lungo la RDA



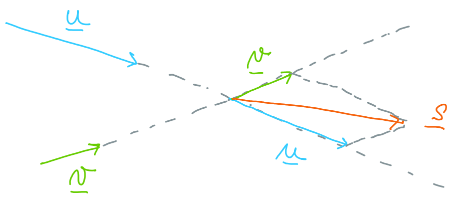


3) VETTORI LIBERI

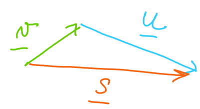
SOMMA DI VETTORI

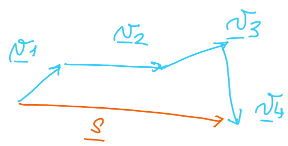
$\underline{s} = \underline{u} + \underline{v}$

⇒ REGOLA PARALLELOGRAMMA



⇒ METODO TESTA-CODA



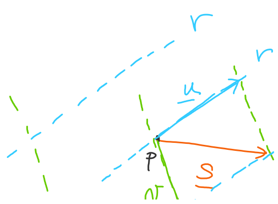


. SOMMA $\underline{s}$ è DETTO RISULTANTE

↕

COMPOSIZIONE

DECOMPOSIZIONE VETTORIALE



1) DECOM. rispetto a 2 direz. non //, complanari ad $\underline{s}$

$\underline{u}$ COMPONENTE DI $\underline{s}$ LUNGO r

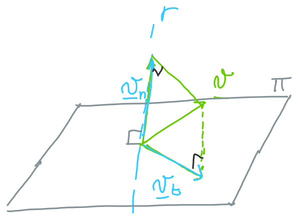
$\underline{v}$ COMPO. DI $\underline{s}$ " t

"IL COMPONENTE" → VETTORE

"LA COMPONENTE" $\longrightarrow$ SCALARE

2) DECOM. rispetto ad asse r e un piano π $\perp$ a r

proiettare $\underline{v}$ su r e su π

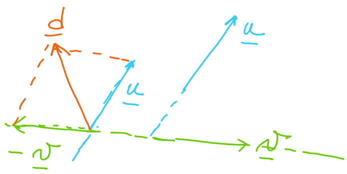


$$\underline{v} = \underline{v}_t + \underline{v}_n$$

DIFFERENZA TRA VETTORI

$$\underline{d} = \underline{u} - \underline{v} = \underline{u} + (-\underline{v})$$

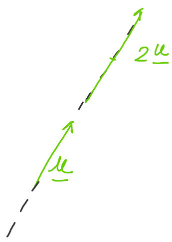
$-\underline{v}$ VETTORE OPPOSTO a $\underline{u}$



PRODOTTO TRA SCALARE e VETTORE

$$\underline{p} = c\,\underline{u}$$

c $\hookrightarrow$ scalare con segno $\begin{cases} c>0 \\ c<0 \end{cases}$



$c = 2$

$\underline{p} = 2\underline{u}$

- •) $\underline{p} \parallel \underline{u}$
- •) $c>0$ concorde $\underline{u}$
  $c<0$ discorde $\underline{u}$
- •) $|c|\,|\underline{u}| = |\underline{p}|$

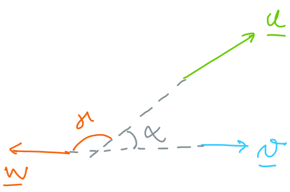
$$\underline{\lambda} = \frac{\underline{u}}{|\underline{u}|}$$ VERSORE •) MODUL. UNITARIO

$\underline{v} \parallel \underline{\lambda}$ $\qquad$ $\underline{v} = c\,\underline{\lambda}$

PRODOTTO SCALARE

$$p_s = \underline{u} \cdot \underline{v} = |\underline{u}|\,|\underline{v}|\cos\alpha$$

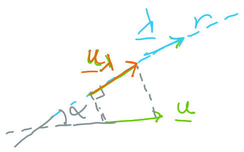
α angolo compreso tra $\underline{u}$ e $\underline{v}$



$\underline{u} \cdot \underline{v} > 0$

$\underline{u} \cdot \underline{w} < 0$

- •) prop. comm. $\quad \underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$
- •) $\underline{u} \perp \underline{v} \qquad p_s = 0$
- •) $\underline{u} \cdot \underline{\lambda} = p_s = |\underline{u}|\,|\underline{\lambda}|\cos\alpha$
  $= |\underline{u}|\cos\alpha$



$u \quad u \quad \lambda$ ... LA COMPONENTE

$u_\lambda = \underline{u} \cdot \underline{\lambda}$ La componente di $\underline{u}$ rispetto ad $\underline{\lambda}$

↓

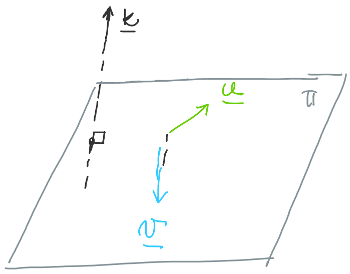
SCALARE CON SEGNO

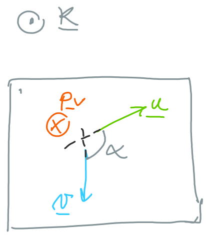
↳ $u_\lambda > 0$ $\underline{u}_\lambda$ concorde λ

$\underline{u}_\lambda = u_\lambda \underline{\lambda}$ IL COMPONENTE

PRODOTTO VETTORIALE

$\underline{p}_v = \underline{u} \wedge \underline{v}$
$\phantom{\underline{p}_v} \overset{!}{=} \underline{u} \times \underline{v}$





$\underline{p}_v$
- direzione $\perp \underline{u}, \perp \underline{v}$
- verso ⇒ REGOLA MANO DX
- $|\underline{p}_v| = |\underline{u}||\underline{v}| \sin\alpha$, $0 \leq \alpha \leq \pi$

- ANTICOM: $\underline{u} \wedge \underline{v} \neq \underline{v} \wedge \underline{u}$
- $\underline{u} \wedge \underline{v} = \underline{0}$ se $\underline{u} \parallel \underline{v}$

PRODOTTO MISTO

$a = \underline{u} \cdot \underline{v} \wedge \underline{w}$

- prop. circolarità: $\underline{u} \cdot \underline{v} \wedge \underline{w} = \underline{w} \cdot \underline{u} \wedge \underline{v}$
- $a = 0$ se $\underline{u}, \underline{v}, \underline{w}$ sono complanari

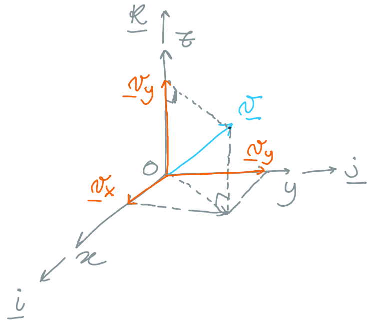
DOPPIO PRODOTTO VETTORIALE

$\underline{b} = (\underline{u} \wedge \underline{v}) \wedge \underline{w} = (\underline{u} \cdot \underline{w})\, \underline{v} - (\underline{v} \cdot \underline{w})\, \underline{u}$

COMPONENTI CARTESIANE DI UN VETTORE

SdR CARTESIANO: $S = \{O; x, y, z\}$

$\underline{k} = \underline{i} \wedge \underline{j}$



$\boxed{\underline{v} = \underline{v}_x + \underline{v}_y + \underline{v}_z}$

$\underline{v}_x$ VETTORE COMPONENTE LUNGO X

v_x LA COMPONENTE (CARTESIANA)

$\begin{cases} v_x = \underline{v} \cdot \underline{i} \\ v_y = \underline{v} \cdot \underline{j} \\ v_z = \underline{v} \cdot \underline{k} \end{cases}$ ⇒ $\underline{v}_x = v_x \underline{i}$

$$\underline{v} = v_x \underline{i} + v_y \underline{j} + v_z \underline{k}$$

$$[\underline{v}] = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = (v_x, v_y, v_z)$$

$$|\underline{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

OPERAZIONI CON LE COMPONENTI

$$\underline{u} = (u_x, u_y, u_z)$$

$$\underline{v} = (v_x, v_y, v_z)$$

•) SOMMA $\quad \underline{s} = \underline{u} + \underline{v} = \begin{bmatrix} u_x + v_x \\ u_y + v_y \\ u_z + v_z \end{bmatrix}$

•) PR. SCAL. $\quad p_s = \underline{u} \cdot \underline{v} = [\underline{u}]^T [\underline{v}] = [u_x \; u_y \; u_z] \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = u_x v_x + u_y v_y + u_z v_z$

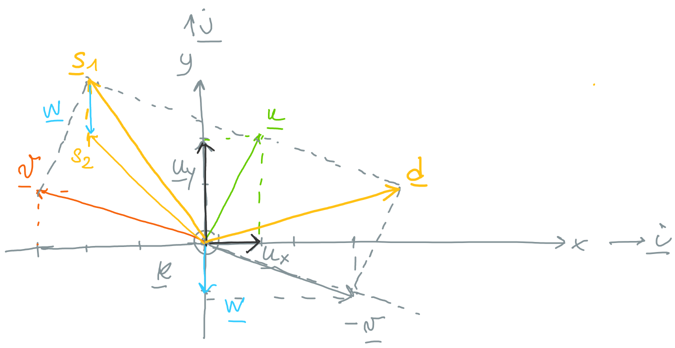
•) PR. VETT. $\quad \underline{p}_v = \underline{u} \wedge \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = (u_y v_z - u_z v_y)\underline{i} + (u_z v_x - u_x v_z)\underline{j} + (u_x v_y - u_y v$

ESERCIZIO

$$\underline{u} = (1, 2, 0)$$

$$\underline{v} = (-3, 1, 0)$$

$$\underline{w} = (0, -1, 0)$$



$$\underline{u} = u_x \underline{i} + u_y \underline{j} + u_z \underline{k}$$

1) $\underline{s}_1 = \underline{u} + \underline{v} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$

2) $\underline{d} = \underline{u} - \underline{v} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$

3) $\underline{s}_2 = \underline{u} + \underline{v} + \underline{w} = (-2, 2, 0)$

4) $c = 2 \qquad c\underline{v} = +2(-3, 1, 0) = (-6, +2, 0)$

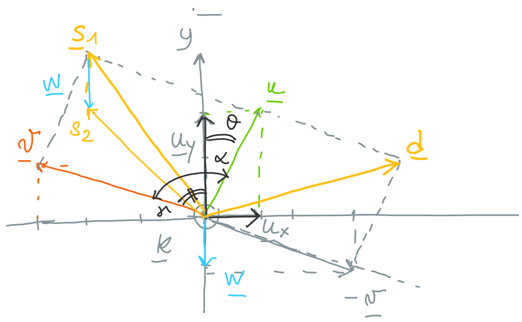
5) $\underline{u} \cdot \underline{v} = (1, 2, 0)^T (-3, 1, 0) = -3 + 2 = -1$

$= |\underline{u}||\underline{v}| \cos\alpha$

$\uparrow \underline{j}$

$|\underline{u}| = \sqrt{1+4} = \sqrt{5}$

$|\underline{v}| = \sqrt{9+1} = \sqrt{10}$



$\alpha = \theta + \gamma$

$\cos(\theta+\gamma) = \cos\theta \cos\gamma - \sin\theta \sin\gamma$

$$\begin{cases} \cos\theta = \dfrac{u_y}{|\underline{u}|} = \dfrac{2}{\sqrt{5}} \\ \sin\theta = \dfrac{u_x}{|\underline{u}|} = \dfrac{1}{\sqrt{5}} \end{cases} \qquad \begin{cases} \cos\gamma = \dfrac{v_y}{|\underline{v}|} = \dfrac{1}{\sqrt{10}} \\ \sin\gamma = \dfrac{|v_x|}{|\underline{v}|} = \dfrac{3}{\sqrt{10}} \end{cases}$$

$$\cos(\alpha) = \cos(\theta+\gamma) = \frac{2}{\sqrt{5}}\frac{1}{\sqrt{10}} - \frac{1}{\sqrt{5}}\cdot\frac{3}{\sqrt{10}} = -\frac{1}{\sqrt{5}\sqrt{10}} = -\frac{1}{5\sqrt{2}}$$

$$\underline{u}\cdot\underline{v} = |\underline{u}|\,|\underline{v}| \cos\alpha = \sqrt{5}\sqrt{10}\left(-\frac{1}{\sqrt{5}\sqrt{10}}\right) = -1$$

6) $$\underline{p}_v = \underline{u} \wedge \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 0 \\ -3 & 1 & 0 \end{vmatrix} = 7\underline{k}$$

•) con la reg. mano dx $\Rightarrow$ $/\!/ \, \underline{k}$ $\Big\}$ $\rightarrow$ $\underline{p}_v = |\underline{p}_v|\,\underline{k}$

•) $|\underline{p}_v| = |\underline{u}|\,|\underline{v}| \sin\alpha$

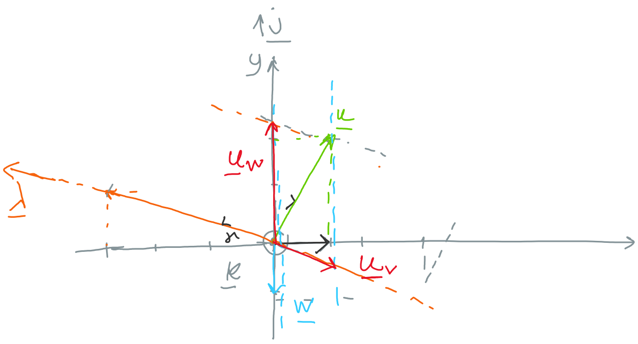
$$\sin\alpha = \sin(\theta+\gamma) = \sin\theta\cos\gamma + \sin\gamma\cos\theta = \frac{1}{\sqrt{5}}\frac{1}{\sqrt{10}} + \frac{3}{\sqrt{10}}\frac{2}{\sqrt{5}} = \frac{7}{\sqrt{5}\sqrt{10}}$$

$$|\underline{p}_v| = \sqrt{5}\sqrt{10}\,\frac{7}{\sqrt{5}\sqrt{10}} = 7$$

7) $\underline{p}_v' = \underline{w} \wedge \underline{u}$ $\quad /\!/ \, \underline{k}$

$\underline{p}_v'' = \underline{w} \wedge \underline{v}$ $\quad /\!/ (-\underline{k})$

$$\underline{p}_v'' = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & -1 & 0 \\ -3 & 1 & 0 \end{vmatrix} = -3\underline{k}$$



8) decomposizione di $\underline{u}$ secondo le direzioni di $\underline{v}$ e $\underline{w}$

$\underline{u} = \underline{u}_w + \underline{u}_v$ $\qquad u_w$? $\; u_v$?

$= u_w \dfrac{\underline{w}}{|\underline{w}|} + u_v \dfrac{\underline{v}}{|\underline{v}|}$ $\qquad$ E_p^{ne} vett.

$\dfrac{\underline{w}}{|\underline{w}|} = -\underline{j}$ $\qquad \dfrac{\underline{v}}{|\underline{v}|} = \underline{\lambda} = \dfrac{1}{\sqrt{10}}(-3\underline{i}$

$$\begin{pmatrix}1\\2\end{pmatrix} = u_w \begin{pmatrix}0\\-1\end{pmatrix} + \frac{u_v}{\sqrt{10}}\begin{pmatrix}-3\\1\end{pmatrix}$$

2 Eq^{ni} scalari

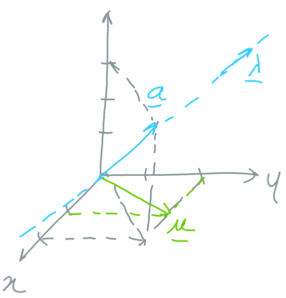
$$\begin{array}{l} \cdot\,\underline{i} \\ \cdot\,\underline{j} \end{array}\quad \begin{cases} 1 = -3\,\dfrac{u_v}{\sqrt{10}} \\ 2 = -u_w + \dfrac{u_v}{\sqrt{10}} \end{cases}$$

$$\begin{cases} u_v = -\dfrac{\sqrt{10}}{3} & <0 \qquad /\!/(-\underline{\lambda}) \\ u_w = -\dfrac{7}{3} & <0 \qquad /\!/\ \underline{j}\,) \end{cases}$$

$$\underline{u}_v = -\frac{\sqrt{10}}{3}\underline{\lambda}$$

$$\underline{u}_w = +\frac{7}{3}\underline{j}$$

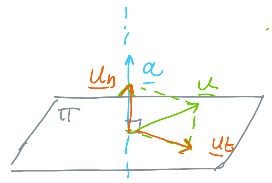
g) Decomposiz. $\underline{u}$ lungo $\underline{a} = (2, 2, 3)$



$$\underline{u} = \underline{u}_n + \underline{u}_t$$

$$\underline{u}_n = \left(\underline{u}\cdot\frac{\underline{a}}{|\underline{a}|}\right)\frac{\underline{a}}{|\underline{a}|} = (\underline{u}\cdot\underline{a})\frac{\underline{a}}{|\underline{a}|^2}$$

$$= (\underline{u}\cdot\underline{\lambda})\,\underline{\lambda}$$



$$|\underline{a}| = \sqrt{4+4+9} = \sqrt{17}$$

$$\underline{u}_n = \left[(1\ \ 2\ \ 0)^T \begin{pmatrix}2\\2\\3\end{pmatrix}\right]\frac{1}{17}\begin{pmatrix}2\\2\\3\end{pmatrix}$$

$$= \frac{6}{17}\begin{pmatrix}2\\2\\3\end{pmatrix}$$

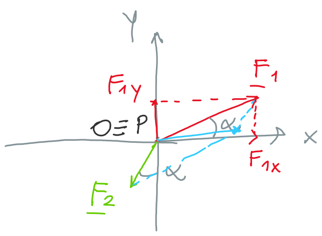
$$\underline{u}_t = \underline{u} - \underline{u}_n = \begin{pmatrix}1\\2\\0\end{pmatrix} - \frac{6}{17}\begin{pmatrix}2\\2\\3\end{pmatrix} = \frac{1}{17}\begin{pmatrix}5\\22\\-18\end{pmatrix}$$

ESERCIZIO

$|\underline{F}_1| = 20\,N \qquad (P, \underline{F}_1)$

$|\underline{F}_2| = 10\,N \qquad (P, \underline{F}_2)$

$\alpha = 30°$



⇒ Valutare $\underline{R} = \underline{F}_1 + \underline{F}_2$

$(P, \underline{R})$

1) soluz. grafica

2) soluz. analitica ⇒ $SDR = \{O;\ x, y, z\}$

$$\underline{F}_1 = \begin{bmatrix}F_{1x}\\F_{1y}\end{bmatrix} = F_{1x}\,\underline{i} + F_{1y}\,\underline{j}$$

$$= \begin{bmatrix}|\underline{F}_1|\cos\alpha\\|\underline{F}_1|\sin\alpha\end{bmatrix} = \begin{bmatrix}20\ \sqrt{3}/2\\20\ \ 1/2\end{bmatrix}$$

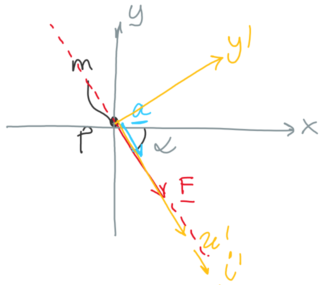
$$\underline{R} = \begin{bmatrix} R_x \\ R_y \end{bmatrix} = \begin{bmatrix} F_{1x} \\ F_{1y} \end{bmatrix} + \begin{bmatrix} F_{2x} \\ F_{2y} \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} 10\sqrt{3} \\ 10 \end{bmatrix} - \begin{bmatrix} 5 \\ 5\sqrt{3} \end{bmatrix} = 5 \begin{bmatrix} 2\sqrt{3} - 1 \\ 2 - \sqrt{3} \end{bmatrix} N$$

$$\stackrel{!}{=} \begin{bmatrix} 10\sqrt{3} \\ 10 \end{bmatrix}$$

$$\underline{F_2} = \begin{bmatrix} -|F_2| \sin\alpha \\ -|F_2| \cos\alpha \end{bmatrix} = -10 \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} = -\begin{bmatrix} 5 \\ 5\sqrt{3} \end{bmatrix}$$

ESERCIZIO



<u>Noto</u>

$m = 10\ kg$

$|\underline{F}| = 100\ N$

$\alpha = 45°$

<u>Valutare</u> $\underline{a}$ — grafica / anali/numer.

$$\underline{F} = m\,\underline{a} \Rightarrow \underline{a} \parallel \underline{F}$$

$$\underline{a} = \frac{1}{m}\underline{F} \rightarrow |\underline{a}| = \frac{1}{m}|\underline{F}| = \frac{100}{10} = 10\ m/s^2$$

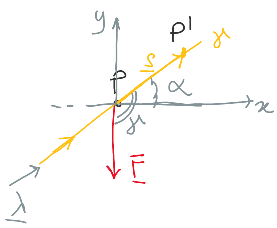
(S) $[\underline{a}] = \frac{1}{m}[\underline{F}] \rightarrow [\underline{F}] = \begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} \underline{F}\cdot\underline{i} \\ \underline{F}\cdot\underline{j} \end{bmatrix} = \begin{bmatrix} |\underline{F}|\cos\alpha \\ -|\underline{F}|\sin\alpha \end{bmatrix} = 100 \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix} = 50\sqrt{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} N$

$$[\underline{a}] = \frac{50\sqrt{2}}{10}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 5\sqrt{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} m/s^2$$

$$|\underline{a}| = \sqrt{(5\sqrt{2})^2 + (5\sqrt{2})^2} = 10\ m/s^2$$

(S') $[\underline{a}] = \begin{bmatrix} 10 \\ 0 \end{bmatrix} m/s^2$

ESERCIZIO



<u>Noto</u>

$(P, \underline{F})$, $\underline{F}$ cost in modulo e direzione

γ traiettoria di P

$s = 0,5\ m \Rightarrow \underline{s} = s\,\underline{\lambda}$

$|\underline{F}| = 50\ N$

$\alpha = 45°$

<u>Trovare</u> Lavoro fatto da $\underline{F}$

1) $\mathcal{L} = \underline{F} \cdot \underline{s}$

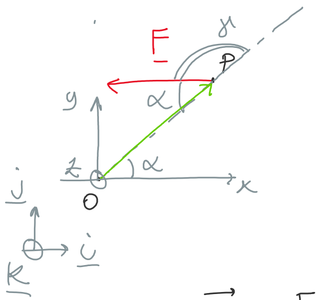
$\stackrel{!}{=} |\underline{F}|\,|\underline{s}|\cos\gamma$

$= \overbrace{\cos(\pi/2 + \alpha)}^{\sin\alpha} = 17.68\ J$

$= 50 \quad 0{,}5 \quad \cos(\pi/2 + \alpha) = \ldots$

2) $L = [\underline{F}]^T [\underline{s}] = (0 \quad -50) \begin{pmatrix} s\cos\alpha \\ s\sin\alpha \end{pmatrix} = -50 \; 0.5 \; \sin\alpha = -17{,}68 \; J$

ESERCIZIO



Noto: $(P, \underline{F})$, $|\underline{F}| = 10 \; N$

$\overline{OP} = 15 \; m$

$\alpha = 45°$

Calcolare $\underline{M}_o$

$\underline{M}_o = \overrightarrow{OP} \wedge \underline{F} \quad \Rightarrow \quad \underline{M}_o \perp \underline{F}, \perp \overrightarrow{OP}$, $\underline{M}_o$ uscente piano

$|\underline{M}_o| = |\overrightarrow{OP}| \, |\underline{F}| \sin\gamma = 15 \; 10 \; \frac{\sqrt{2}}{2} = 75\sqrt{2} \quad Nm$

$$[\underline{M}_o] = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ OP_x & OP_y & \cancel{OP_z} \\ F_x & F_y & \cancel{F_z} \end{vmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 15\sqrt{2}/2 & 15\sqrt{2}/2 & 0 \\ -10 & 0 & 0 \end{vmatrix} = 75\sqrt{2} \; \underline{k} \quad N\,m$$

$$[\overrightarrow{OP}] = \begin{bmatrix} |\overrightarrow{OP}| \cos\alpha \\ |\overrightarrow{OP}| \sin\alpha \end{bmatrix} = \begin{bmatrix} 15 \; \sqrt{2}/2 \\ 15 \; \sqrt{2}/2 \end{bmatrix} m$$

$$[\underline{F}] = \begin{bmatrix} -10 \\ 0 \end{bmatrix} N$$

ANALISI VETTORIALE

$\underline{v}(t)$, SPR $S \equiv \{O; x, y, z\} \longrightarrow \underline{v}(t) = \overset{v_x(t)}{v_x} \underline{i} + \overset{v_y(t)}{v_y} \underline{j} + \overset{v_z(t)}{v_z} \underline{k}$

DERIVATA DI $\underline{v}(t)$

$$\frac{d\underline{v}(t)}{dt} = \frac{dv_x}{dt} \underline{i} + \frac{dv_y}{dt} \underline{j} + \frac{dv_z}{dt} \underline{k}$$

PROPRIETÀ

$$\frac{d}{dt}\left(\underline{v}_1(t) + \underline{v}_2(t)\right) = \frac{d}{dt}\underline{v}_1(t) + \frac{d}{dt}\underline{v}_2(t)$$

$$\frac{d}{dt}\left(f(t) \; \underline{v}(t)\right) = \frac{df(t)}{dt} \underline{v}(t) + f(t) \frac{d\underline{v}(t)}{dt}$$

$$\frac{d}{dt}\left(\underline{v}_1(t) \cdot \underline{v}_2(t)\right) = \frac{d\underline{v}_1(t)}{dt} \cdot \underline{v}_2(t) + \underline{v}_1(t) \cdot \frac{d\underline{v}_2(t)}{dt}$$

$$\frac{d}{dt}\left(\underline{v}_1(t) \wedge \underline{v}_2(t)\right) = \frac{d\underline{v}_1(t)}{dt} \wedge \underline{v}_2(t) + \underline{v}_1(t) \wedge \frac{d\underline{v}_2(t)}{dt}$$

$\overline{dt}$

$s = s(t) \Rightarrow \underline{v}(s(t)) = \underline{v}(s)$

$$\frac{d\underline{v}(t)}{dt} = \frac{d\underline{v}(s)}{ds}\,\frac{ds(t)}{dt}$$

INTEGRALE

$$I = \int \underline{v}(t)\, dt = \int v_x(t)\, dt\, \underline{i} + \int v_y(t)\, dt\, \underline{j} + \int v_z(t)\, dt\, \underline{k}$$