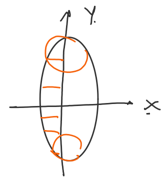
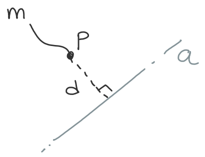
Momenti di inerzia

giovedì 5 dicembre 2024 17:15

QUANTITÀ SCALARE $\Rightarrow$ definita rispetto ad un asse



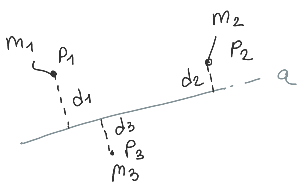
PUNTO MATERIALE



$$J_a = m d^2 \quad \Rightarrow \quad J_a > 0$$

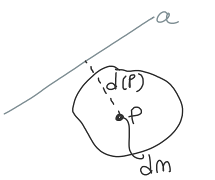
rispetto all'asse a

SISTEMA PUNTI MATERIALI



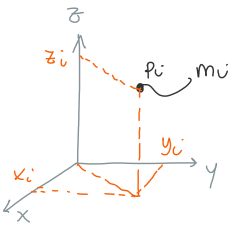
$$J_a = \sum_i m_i d_i^2$$

SISTEMA CONTINUO



$$J_a = \int_M d^2(P)\, dm = \int_V d^2(P)\, \rho \, dV$$

SIST. PUNTI MATER.

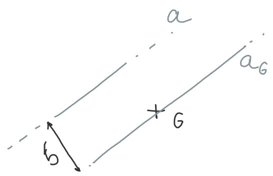


$$\begin{cases} J_x = \sum_i m_i d_{x_i}^2 = \sum_i m_i (y_i^2 + z_i^2) \\ J_y = \sum_i m_i d_{y_i}^2 = \sum_i m_i (x_i^2 + z_i^2) \\ J_z = \sum_i m_i d_{z_i}^2 = \sum_i m_i (x_i^2 + y_i^2) \end{cases}$$

SISTEMA CONTINUO

$$\begin{cases} J_x = \int_V \rho \, (y^2 + z^2)\, dV \\ J_y = \int_V \rho \, (x^2 + z^2)\, dV \\ J_z = \int_V \rho \, (x^2 + y^2)\, dV \end{cases} \xrightarrow[\text{su } V]{\text{se omogeneo } \rho \text{ è cost}} \begin{cases} J_x = \rho \int_V (y^2 + z^2)\, dV \\ J_y = \rho \int_V (x^2 + z^2)\, dV \\ J_z = \rho \int_V (x^2 + y^2)\, dV \end{cases}$$

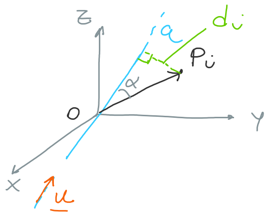
TEOREMA DI HUYGHENS STEINER
(Teo. degli ASSI PARALLELI)



- a_G passa per G
- J_G : M.I. BARICENTRICO
- $a \parallel a_G$

$$J_a = J_{aG} + m b^2$$
$$\downarrow$$
J MINIMO

— VARIAZIONE DEL J RISPETTO AD ASSI CONCORRENTI IN UN PUNTO —



Noto: $\underline{a}$ passa per O

$\underline{u}$ versore di a

$\underline{u} = (u_x, u_y, u_z)$

J_{xx}, J_{yy}, J_{zz}

Trovare: J_a in funzione di $J_{xx,yy,zz}$

$$J_a = \sum_i m_i |\vec{OP_i}|^2 = \sum_i m_i \underbrace{(\underline{u} \wedge \vec{OP_i})^2}_{d_i^2}$$

$$\vec{OP} = (OP_x, OP_y, OP_z)$$

$$\underline{u} \wedge \vec{OP_i} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_x & u_y & u_z \\ OP_x & OP_y & OP_z \end{vmatrix} = \underline{i}\,(u_y OP_z - u_z OP_y) + \underline{j}\,(u_z OP_x - u_x OP_z) + \underline{k}\,(u_x OP_y - u_y OP_x)$$

$$(\underline{u} \wedge \vec{OP_i})^2 = u_y^2 OP_z^2 + u_z^2 OP_y^2 - 2 u_y u_z OP_z OP_y + u_z^2 OP_x^2 + u_x^2 OP_z^2 - 2 u_x u_z OP_x OP_z + u_x^2 OP_y^2 + u_y^2 OP_x^2 - 2 u_x u_y OP_x OP_y =$$

$$= u_x^2 (OP_y^2 + OP_z^2) + u_y^2 (OP_x^2 + OP_z^2) + u_z^2 (OP_x^2 + OP_y^2)^2 + $$
$$- 2 u_y u_z OP_z OP_y - 2 u_x u_z OP_x OP_z - 2 u_x u_y OP_x OP_y$$

$$J_a = \sum_i m_i (\underline{u} \wedge \vec{OP_i})^2 = \sum_i m_i \Big[u_x^2 \overbrace{(OP_y^2 + OP_z^2)}^{J_{xx}} + u_y^2 \overbrace{(OP_x^2 + OP_z^2)}^{J_{yy}} + u_z^2 \overbrace{(OP_x^2 + OP_y^2)^2}^{J_{zz}} + - 2 u_y u_z \underbrace{OP_z OP_y} - 2 u_x u_z \underbrace{OP_x OP_z} - 2 u_x u_y \underbrace{OP_x OP_y} \Big]$$

$$\sum m_i (OP_y^2 + OP_z^2) = \sum m_i (y_i^2 + z_i)^2 = J_{xx}$$

$$\sum_i m_i \, OP_z OP_y = \sum_i m_i \, z_i y_i = J_{xy}$$

MOMENTO CENTRIFUGO o PRODOTTO DI INERZIA $\Rightarrow$

$$\begin{cases} J_{xy} = \sum_i m_i x_i y_i \\ J_{xz} = \sum_i m_i x_i z_i \\ J_{yz} = \sum m_i y_i z_i \end{cases}$$

NB $J_{xy} >=< 0$

$$\boxed{J_a = u_x^2 \, J_{xx} + u_y^2 \, J_{yy} + u_z^2 \, J_{zz} - 2\, u_x u_y \, J_{xy} - 2\, u_x u_z \, J_{xz} - 2\, u_y u_z \, J_{yz}}$$

$$\mathbb{I} = \begin{bmatrix} J_{xx} & -J_{xy} & -J_{xz} \\ -J_{xy} & J_{yy} & J_{yz} \\ -J_{xz} & J_{yz} & J_{zz} \end{bmatrix}$$ TENSORE INERZIA $\Rightarrow$ Matrice simmetrica, Re

$\underline{u} \Rightarrow$ versore dell' a

$$\boxed{J_a = \underline{u}^T \, \mathbb{I} \, \underline{u}}$$

Possiamo diagonalizzare $\mathbb{I}$ $\Rightarrow$

$$\mathbb{I}_p = \begin{bmatrix} J_{xp} & 0 & 0 \\ 0 & J_{yp} & 0 \\ 0 & 0 & J_{zp} \end{bmatrix}$$

TENSORE DI INERZIA PRINCIPALE

sulla \ ci sono gli autovalori di $\mathbb{I}$

1) $J_a = \underline{u}^T \, \mathbb{I}_p \, \underline{u}$

$$\overset{!}{=} J_{xp} \, u_x^2 + J_{yp} \, u_y^2 + J_{zp} \, u_z^2$$

2) $J_{xy} = J_{xz} = J_{yz} = 0$ $\Rightarrow$ vuol dire che la TERNA PRINCIPALE D'INERZIA

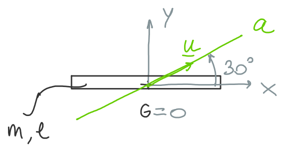
$$S = \{ \underset{G_a}{O} \, ; \, x_p , y_p , z_p \}$$

$\downarrow$

assi sono // agli autovettori di $\mathbb{I}$

ESEMPI

▷ BARRETTA OMOGENEA



$S = \{ \underset{G}{\underset{"}{O}} \, ; \, x, y, z \}$

$\forall$ punto $P \Rightarrow$ $y_p = z_p = 0$

$J_{xx} = \int_V \rho \, (z^2 + y^2) \, dV = 0$ $\Rightarrow$ tutta la massa sta sull'asse x , $\rho = \frac{m}{\ell}$

$$J_{yy} = J_{zz}$$

$$\overset{!}{=} \int \rho \, (x^2 + z^2) \, dV = \rho \int_{-\ell/2}^{\ell/2} x^2 \, dx = \rho \left. \frac{x^3}{3} \right|_{-\ell/2}^{\ell/2} = \frac{m}{\ell} \, 2 \left(\frac{\ell}{2} \right)^3 \frac{1}{3} = \frac{m\ell^2}{12}$$

$J_{xy} = \int \rho \, x \, y \, dV = 0$

$J_{xz} = 0$

$\Big\} \rightarrow$ Terna e' principale di inerzia

$J_{yz} = 0$

$$\mathbb{I}_P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & me^2/12 & 0 \\ 0 & 0 & me^2/12 \end{bmatrix}$$

•) J_a con a passante per G, $\underline{u} = \begin{pmatrix} \cos 30 \\ \sin 30 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \\ 0 \end{pmatrix}$

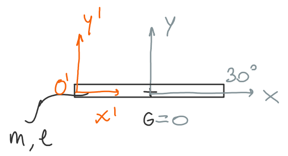
$$J_a = \underline{u}^T \, \mathbb{I}_P \, \underline{u}$$

$$= \left(\sqrt{3}/2 \quad 1/2 \quad 0 \right) \begin{bmatrix} 0 & 0 & 0 \\ 0 & me^2/12 & 0 \\ 0 & 0 & me^2/12 \end{bmatrix} \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \\ 0 \end{pmatrix}$$

$$= \left(\sqrt{3}/2 \quad 1/2 \quad 0 \right) \begin{pmatrix} 0 \\ me^2/12 \cdot 1/2 \\ 0 \end{pmatrix} = \frac{me^2}{48}$$

$$J_a = \underbrace{\cancel{J_{xp}}}_{0} u_x^2 + J_{yp} u_y^2 + \underbrace{J_{zp} \cancel{u_z^2}}_{0} = J_{yp} u_{yp}^2 = \frac{me^2}{12} \cdot \left(\frac{1}{2}\right)^2 = \frac{me^2}{48}$$

•)

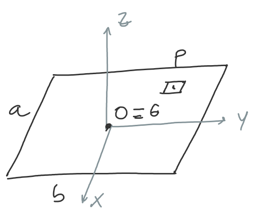


$J_{x'y'} = J_{x'z'} = J_{y'z'} = 0 \Rightarrow$ TERNA PRINC. INERZ.

$$J_{y'y'} = \rho \int_0^e x'^2 \, dx' = \rho \left. \frac{x'^3}{3} \right|_0^e = \frac{me^2}{3}$$

$$J_{y'y'} = \frac{me^2}{3} > J_{yy} = \frac{me^2}{12}$$

RETTANGOLO OMOGENEO



$\forall P \Rightarrow z_P = 0$

$J_{xz} = J_{yz} = 0$

$J_{xy} \neq 0$?

$$J_{xy} = \int \rho \, x y \, dV$$

$$= \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \rho \, x y \, dx \, dy \qquad \rho = \frac{m}{ab}$$

$$= \rho \int_{-b/2}^{b/2} y \, dy \int_{-a/2}^{a/2} x \, dx = 0$$

$\Rightarrow$ TERNA PRINCIPALE DI INERZIA

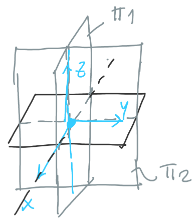
$$J_{xx} = \rho \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} y^2 \, dx \, dy = \frac{m}{ab} \left(\frac{a}{2} - \left(\frac{a}{2}\right) \right) \frac{y^3}{3} \Big|_{b/2}^{b/2} = \frac{mb^2}{12}$$

$$J_{yy} = \frac{ma^2}{12}$$

$$J_{zz} = J_{xx} + J_{yy} = \frac{m}{12}(a^2+b^2)$$

$$= \int_V \rho \, (x^2+y^2) \, dV$$

PROPRIETA'



π_1 e $\pi_2 \Rightarrow$ piani di simmetria ortogonale

$x \perp \pi_2$

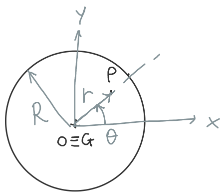
$y \perp \pi_1$

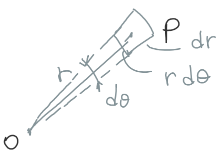
$z \Rightarrow \underline{k} = \underline{i} \wedge \underline{j}$

$\Rightarrow O \in \pi_1 \cap \pi_2$

$S = \{O; x, y, z\}$ e' PRINCIPALE DI INERZIA

- DISCO OMOGENEO





$S = \{O; x, y, z\} \Rightarrow$ TERNA PRINC. DI INERZIA

$P(\theta, r)$

$$J_{xx} = \frac{m}{A} \int_A (y^2 + \cancel{z^2}) \, dA$$

$$= \frac{m}{\pi R^2} \int_0^R \int_0^{2\pi} (r \sin\theta)^2 \, r \, dr \, d\theta$$

$$\int \sin^2\theta \, d\theta = \left(-\frac{\sin\theta}{2} + \frac{\theta}{2}\right)$$

$$= \frac{m}{\cancel{R^2}} \frac{R^{\cancel{4}\,2}}{4} = \frac{mR^2}{4}$$

$$J_{yy} = J_{xx}$$

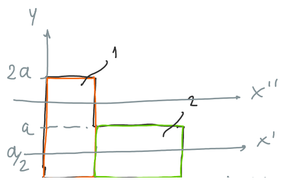
$$J_{zz} = \int_V \rho (x^2+y^2) \, dV = J_{xx} + J_{yy} = 2\frac{mR^2}{4} = \frac{mR^2}{2}$$

$$\boxed{J_{zz} = J_{xx} + J_{yy}} \quad 2D$$

$$J_{zz} = \frac{m}{\pi R^2} \int_0^R \int_0^{2\pi} r^2 \, r \, d\theta \, dr$$

$$= \frac{m}{\pi R} \left(\theta \right)_0^{2\pi} \frac{r^4}{4} \Big|_0^R = \frac{mR^2}{2}$$

- FIGURA A "L"



$2D \Rightarrow J_{xz} = J_{yz} = 0$

$J_{xy} \neq 0$

$$J_{xx} = J_{xx1} + J_{xx2}$$

$$J_{xx1} = \frac{m}{12}(2a)^2$$

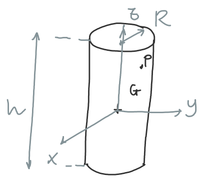
$$J_{xx2} = \frac{m}{12}(a)^2$$

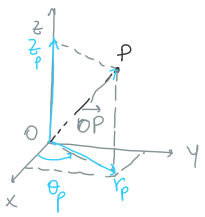
$$J_{xx} = \frac{m}{12}(4a^2 + a^2) = \frac{5}{12}a^2 m$$

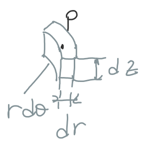
$$J_{xx}' = J_{xx1}' + J_{xx2}'$$

$$J_{xx1} + m_1\left(\frac{a}{2}\right)^2$$

CILINDRO RETTO







$$S_p = \{O;\ x, y, z\}$$

$$J_{xy} = J_{xz} = J_{yz} = 0$$

$$\Rightarrow \ 3D \ \Rightarrow \ J_{zz} \neq J_{xy} + J_{yy}$$

$$P(r, \theta, z) \Rightarrow P(r_p, \theta_p, z_p)$$

$$J_{xx} = \rho \int_V (y^2 + z^2)\, V$$

$$= \frac{m}{\pi R^2 h} \int_{-h/2}^{h/2} \int_0^R \int_0^{2\pi} \left((r\sin\theta)^2 + z^2\right) r\, d\theta\, dr\, dz$$

$$= \frac{m}{\pi R^2 h}\left[\underbrace{h\int_0^R r^3\, dr \int_0^{2\pi} \sin^2\theta\, d\theta}_{1} + \underbrace{\iiint z^2 r\, d\theta\, dr\, dz}_{2}\right]$$

$$1 = h\left[\left(\frac{r^4}{4}\Big|_0^R\right)\left(\left(-\frac{1}{2}\sin\theta + \frac{\theta}{2}\right)\Big|_0^{2\pi}\right)\right] = h\frac{R^4}{4}\pi$$

$$2 = 2\pi \int_0^R r\, dr \int_{-h/2}^{h/2} z^2\, dz = 2\pi \frac{R^2}{2}\frac{z^3}{3}\Big|_{-h/2}^{h/2} = \frac{\pi}{12}R^2 h^3$$

$$J_{xx} = \frac{m}{4}\left(R^2 + \frac{h^2}{3}\right) = J_{yy}$$

$$J_{zz} = \int_V \rho\, (x^2 + y^2)\, dV$$

$$= \frac{m}{\pi R^2 h}\iiint r^2 r\, dr\, d\theta\, dz$$

$$= \frac{m}{\pi R^2 h}\, 2\pi\, h\, \frac{r^4}{4}\Big|_0^R = \frac{mR^2}{2}$$