2.1)

a) $\dot{\vec{x}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \vec{x} = A_1 \vec{x}$

$\tilde{A}_1$: char. Poly:
$\det(A_1 - \lambda E)$

$\det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 0 & 2-\lambda \end{pmatrix} \rightarrow p(\lambda) = (1-\lambda)^2(2-\lambda)$

$\lambda_1 = 1 \quad n_1 = 2 \quad g_1 = n - \text{rang}(A - \lambda_1 E) = 2$
$\lambda_2 = 2 \quad n_2 = 1 \quad g_2 = 1$

$\lambda_1 = 1$: $\begin{pmatrix} 1-1 & 0 & 0 \\ 0 & 1-1 & -1 \\ 0 & 0 & 2-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{0} \rightarrow \underline{x_3 = 0}$

$x_1 = t, \; x_2 = s$

$\vec{x}_1 = \begin{pmatrix} t=1 \\ s=0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ; \; \vec{x}_2 = \begin{pmatrix} t=0 \\ s=1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 2$: $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{0} \rightarrow \underline{x_1 = 0} \quad x_3 = t \quad x_2 = -t$

$\begin{pmatrix} 0 \\ -t \\ t \end{pmatrix} \quad t = 1 \rightarrow \vec{x}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$V = \begin{pmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

V^{-1}: $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$ ⊕

$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$

$\tilde{A} = V^{-1} A V \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

$\tilde{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & & \\ & \lambda_1 & \\ & & \lambda_2 \end{pmatrix}$

= Jordansche Normalform

b)

$$\dot{\vec{x}} = \underbrace{\begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}}_{A_2} \vec{x}$$

Step 1: EW: $\det(A - E\lambda)$

$p = (1-\lambda)^2(2-\lambda)$ $\quad \lambda_1 = 1 \quad \lambda_2 = 2$

$n_1 = 2 \quad n_2 = 1$

$g_2 = 1 \quad g_2 = 1$

$g_2: n - \text{rang}(A - \lambda_1 E)$

$$\Rightarrow 3 - \text{rang}\begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} = 1$$

EV: λ_1: $\begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \vec{v}_1 = \vec{0} \rightarrow \underline{v_3 = 0} \quad v_1 = t \quad v_2 = 0$

$$\vec{v}_1 = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} \quad t = 1 \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \checkmark$$

λ_2: $\begin{pmatrix} -1 & 2 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \vec{v}_2 = \vec{0} \rightarrow \quad v_3 = t \quad v_2 = -t \quad v_1 = -2t + 2t = 0$

$$\vec{v}_2 = \begin{pmatrix} 0 \\ -t \\ t \end{pmatrix} \quad t = 1 \quad \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Hauptvektor $\vec{h}_1$: $\begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \vec{h}_1 = \vec{v}_1 \quad h_2 = \frac{1}{2} \quad ; h_3 = 0 \quad , h_1 = t$

$$\vec{h}_1 = \begin{pmatrix} t \\ \frac{1}{2} \\ 0 \end{pmatrix} \quad t = 1 \quad \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

$$V = (v_1 \; h_1 \; v_2) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \frac{1}{2} & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$V^{-1}: \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\tilde{A}_2 = V^{-1} A_2 V \Rightarrow \begin{pmatrix} 1 & -2 & -2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow$$

$$V^{-1} = \begin{pmatrix} 1 & -2 & -2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

$$\rightarrow \begin{pmatrix} 1 & -2 & -2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & \frac{1}{2} & -1 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & \frac{1}{2} & -1 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Transitionsmatrix: $\tilde{\Phi} = e^{\tilde{A}_2 t} = \sum_{k=0}^{\infty} A^k \cdot \frac{t^k}{k!}$

Einfache Methode: $\tilde{A}_2 = \underbrace{D}_{\text{Diagonal}} + \underbrace{N}_{\text{Nilpotent}} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\tilde{\Phi} = e^{\tilde{A}_2 t} = e^{(D+N)t} = e^{Dt} e^{Nt} = \sum_{k=0}^{\infty} D^k \cdot \frac{t^k}{k!} \cdot \sum_{k=0}^{\infty} N^k \cdot \frac{t^k}{k!} =$$

NR $N^2 = \begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} = \underline{\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}}$

Wenn Diag. dann $e^{Dt} = \begin{pmatrix} e^{d_1 t} & \cdot & \cdot \\ \cdot & e^{d_2 t} & \cdot \\ \cdot & \cdot & e^{d_3 t} \end{pmatrix}$

$$\begin{pmatrix} e^t & & \\ & e^t & \\ & & e^{2t} \end{pmatrix} \cdot \sum_{k=0}^{\infty} N^k \cdot \frac{t^k}{k!} = \begin{pmatrix} e^t & & \\ & e^t & \\ & & e^{2t} \end{pmatrix} \cdot \left(E + \begin{pmatrix} \cdot & t & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \right)$$

$$\Rightarrow \begin{pmatrix} e^t & \cdot & \cdot \\ \cdot & e^t & \cdot \\ \cdot & \cdot & e^{2t} \end{pmatrix} \cdot \begin{pmatrix} 1 & t & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} = \underline{\begin{pmatrix} e^t & e^t t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}}$$