Recall. Every infinite subset of the real numbers is either countable or bijective with the whole of $\mathbb{R}$.

Equivalently: $2^{\aleph_0} = \aleph_1$ i.e. $\mathcal{P}(\mathbb{N})$ has size the first uncountable cardinal.

CH was first advanced by Cantor 1878

ZFC $\nvdash \neg\,$CH (Gödel 1940)

ZFC $\nvdash$ CH (Cohen 1963)

Cannot do: ZFC $\vdash$ "$\underset{\sim}{\text{ZFC}} \nvdash \text{CH}$" $\longrightarrow$ find a model for ZFC $+\neg\,$CH $\iff$ Con(ZFC $+\neg\,$CH)

By Gödel's Second Incompleteness Theorem: If ZFC is consistent then ZFC $\nvdash$ Con(ZFC).

instead: Con(ZFC) $\implies$ Con(ZFC $+\neg\,$CH) which we will show using ZFC and

Con(ZFC) $\implies$ Con(ZFC $+$ CH)

Definition. The language of set theory (LST) is the first order predicate language $\mathcal{L}_\in$ which consists of the logical connectives and quantifiers ($\neg$, $\implies$, $\vee, \wedge, \exists, \forall$), parentheses, variable symbols $v_0, v_1, \ldots$, a logical binary predicates "$=$" abd a single non-logical binary predicate "$\in$".

Con(ZFC) $\implies ZFC$ is satisfiable, i.e. there is an $\mathcal{L}_\in$-structure $(X, \underbrace{E}_{\subseteq X \times X})$ s.t. $(X, E) \vDash$ ZFC. We want a "nicer" model of ZFC.

Method 1. *Assume the existance of an inaccessible cardinal κ. Then $(V_\kappa, \in) \vDash$ ZFC.*

Method 2. *To show* ZFC $+\neg\,$CH *is consistent, it is enough to show every finite subset is consistent. For every $\Sigma \subset$ ZFC finite find $(X, \in) \vDash \Sigma +$ enough of ZFC to prove what we need (where X is transitive).*
Transform this into a model of $\Sigma + \neg\,$CH.

For a set X and $E \subseteq X \times X$, φ a sentence of LST we know frmo logic how to define the satisfaction relation $(X, E) \vDash \varphi$. To define this we used:

- Symbol set of the language

- define by recursion terms and formulae, free variables etc. / satisfaction

- set D, relations / functions on D, elements of D

- interpretation function maps from non-logical symbols of $\mathcal{L}$ to $D/\mathcal{P}(D \times D)$

- Variable assignment from symbols to D

Definition. We define the reletivization of formulae of LST to a set W by recursion on formulae.

1. $(x \in y)_W := x \in y$ $\qquad\qquad\qquad\qquad\qquad (x = y)_W := x = y$

2. $(\neg\varphi)_W := \neg\varphi_W$ $\qquad\qquad\qquad\qquad\qquad (\varphi \wedge \psi)_W := \varphi_W \wedge \psi_W$

3. $(\exists x : \varphi)_W := \exists x : (x \in W \wedge \varphi_W)$

$$(\forall x : \varphi)_W \iff (\neg \exists x : \neg \varphi)_W$$
$$= \neg(\exists x : \neg \varphi)_W$$
$$= \neg \exists x : (x \in W \wedge (\neg \varphi)_W)$$
$$= \neg \exists x : (x \in W \wedge \neg \varphi_W)$$
$$= \forall x : \neg(x \in W \wedge \neg \varphi_W)$$
$$= \forall x : \neg x \in W \vee \varphi_W = \forall x : x \in W \longrightarrow \varphi_W$$

We claim that $(W, \in \restriction W \times W) \vDash \varphi$ iff φ_W. We write $(W, \in)$ for $(W, \in \restriction W \times W)$.

Example. Extensionality: $(W, \in) \vDash$ Axiom of Extensionality:

$$(\text{Ax. Ext. })_W = (\forall x : \forall y : (\forall z : z \in x \longleftrightarrow z \in y) \longrightarrow x = y)_W$$
$$= \forall x \in W : \forall y \in W : (\forall z \in W : z \in x \longleftrightarrow z \in y) \longrightarrow x = y$$

If $W = \{\emptyset, \{\{\emptyset\}\}\}$ then $(W, \in) \nvDash$ Ax. Ext.

Proposition. *If W is transitive then $(W, \in) \vDash$ Ax. Ext.*

Proof. Let W be transitive, $x, y \in W$ with $x \neq y$. Then there is some set z with $z \in x \leftrightarrow z \in y$. Wlog assume $z \in x, z \notin y$. Then $z \in W$ by transitivity of W. $\square$

Example. Axiom Empty set:

$$(W, \in) \vDash \text{Ax. Exmpty set} \iff (\exists x : x = \emptyset)_W$$
$$\iff (\exists x : \forall y : y \notin x)_W$$
$$\iff \exists x \in W : \forall y \in W : y \notin x \iff \exists x \in W : x \cap W = \emptyset$$

Example. $W = \{\{\emptyset\}\}$ Then $(W, \in) \vDash$ Ax. Empty set even though $\emptyset \notin W$.

Proposition. *If $W \neq \emptyset$ then $(W, \in) \vDash$ Ax. Empty Set.*

Proof. By the Axiom of Foundation there is $y \in W$ with $y \cap W = \emptyset$. $\square$

Definition. We say a formula φ is absolute for W if $\text{FV}(\varphi) \subseteq \{x_1, \ldots, x_n\}$ and

$$\forall x_1 \in W, \ldots, \forall x_n \in W : (\varphi_W \leftrightarrow \varphi)$$

Lemma. *The following formule are absolute for any transitive set W*

1. $z = \emptyset$

2. $x \subseteq y$

3. $z = x \cap y$

4. $z = x \cup y$

5. $z = \bigcup x$

6. $z = \{x, y\}$

7. $z = (x, y)$

Proof. Let W be transitive

1. Let $z \in W$

$$z = \emptyset \iff \forall x : x \notin z \implies \forall x \in W : x \notin z \iff (z = \emptyset)_W$$

Also $x \in z \implies x \in W$ as W is transitive so we must also have " $\impliedby$ "

2.

3. Let $x, y, z \in W$

$$z = x \cap y \iff \forall u : u \in z \leftrightarrow (u \in x \wedge u \in y) \implies \forall u \in W : u \in z \leftrightarrow (u \in x \wedge u \in y) \iff (z = x \cap y)_W$$

" $\impliedby$ " follows as $u \in z \implies u \in W, u \in x \wedge u \in y \implies u \in W$

4.

5.

6. Let $x, y, z \in W$
$$z = \{x, y\} \iff \forall u (u \in z \leftrightarrow (u = x \vee u = y))$$
$$\implies \forall u \in W (u \in z \leftrightarrow (u = x \vee u = y)) \iff (z = \{x, y\})_W$$
" $\impliedby$ " follows as $u \in z \implies u \in W, u = x \vee u = y \implies u \in W$

7. Let $x, y, z \in W$

$$z = (x, y) \iff \exists u, v : z = \{x, y\} \wedge u = \{x\} \wedge v = \{x, y\}$$

$$\overset{W \text{trans.}}{\implies} \exists u \in W : \exists v \in W : z = \{x, y\} \wedge u = \{x\} \wedge v = \{x, y\} \overset{6.}{\iff} (z = (x, y))_W$$

" $\impliedby$ " follows simply.

$\square$

Example. $z = \mathcal{P}(x)$ abd $z = x \times y$ are <u>not absolute</u> for transitive sets. Let $W = \omega, 2 \in \omega$.
$(\omega, \in) \vDash 3 = \mathcal{P}(2)$
$x \times y = \{(a, b) \mid a \in x, b \in y\}$. W transitive then $x, y \in W \implies$ such $a, b \in W, z = (a, b)$
is absolute for W but not all such ordered pairs may be in W.
If $z = W \cap (x \times y), z \in W$ then $(W, \in) \vDash z = x \times y$

Lemma. *If W is transitive and non-empty then it satisfies the axioms of extensionality, empty set and foundation.*

Proof. We just have to show

$$(\text{Ax. Foundation})_W \iff (\forall x : x \neq \emptyset \longrightarrow \exists y : (y \in x \land x \cap y = \emptyset))_W$$
$$\iff \forall x \in W : (x \neq \emptyset \longrightarrow \exists y \in W : (y \in x \land x \cap y = \emptyset))$$

using the absoluteness of Empty set and $x \cap y$. Let $x \in W, x \neq \emptyset$. Then by the Axiom of Foundation $\exists y : y \in x \land x \cap y = \emptyset$ so $y \in W$ by transitivity so $\exists y \in W (y \in x \land x \cap y = \emptyset)$ $\square$

Lemma. *For a limit ordinal α, $(V_\alpha, \in)$ satisfies the axioms of pairing, union, powerset and the axiom schema of separation. If additionally $\alpha > \omega$, V_α satisfies the axiom of infinity.*

Proof. <u>Pairing:</u> $(\forall x : \forall y : \exists z : z = \{x, y\})_{V_\alpha}$ iff $\forall x \in V_\alpha : \forall y \in V_\alpha : \exists z \in V_\alpha : z = \{x, y\}$. Let $x, y \in V_\alpha, \text{rk}(\{x, y\}) = \max(\text{rk}(x), rk(y)) + 1 < \alpha$ so $\{x, y\} \in V_\alpha$
<u>Union:</u> $(\forall x : \exists z : z = \bigcup x)_{V_\alpha}$ iff $\forall x \in V_\alpha : \exists z \in V_\alpha : z = \bigcup x$ so we just need $x \in V_\alpha \implies \bigcup x \in V_\alpha$ which is true as $\text{rk}(\bigcup x) \leq \text{rk}(x) < \alpha$.
<u>Powerset:</u> $(\forall x : \exists y : \forall z : (z \in y \longleftrightarrow z \subseteq x))_{V_\alpha}$ iff $\forall x \in V_\alpha : \exists y \in V_\alpha : \forall z \in V_\alpha : (z \in y \longleftrightarrow z \subseteq x)$. $\text{rk}(\mathcal{P}(x)) = \text{rk}(x) + 1$ so $x \in V_\alpha \implies \mathcal{P}(x) \in V_\alpha$. Let $x \in V_\alpha, y = \mathcal{P}(x) \in V_\alpha$ then $\forall z : z \in y \longleftrightarrow z \subseteq x$ so in particular $\forall z \in V_\alpha : (z \in y \leftrightarrow z \subseteq x)$ i.e. $(y = \mathcal{P}(x))_{V_\alpha}$.
<u>Infinity:</u> Let $\alpha > \omega$. $(\exists x : (\emptyset \in x \land \forall y : (y \in x \longrightarrow y \cup \{y\} \in x)))_{V_\alpha}$ iff $\exists x \in V_\alpha : (\emptyset \in x \land \forall y \in V_\alpha : (y \in x \longrightarrow y \cup \{y\} \in x))$. Let $x = \omega$ then $x \in V_\alpha, \emptyset \in V_\alpha, \forall y : (y \in x \longrightarrow y \cup \{y\} \in x)$ so in particular $\forall y \in V_\alpha : (y \in x \longrightarrow y \cup \{y\} \in x)$
<u>Separation:</u> Let φ be a formula of LST with $x, y \notin \text{FV}(\varphi)$. Then

$$(\forall x : \exists y : y = x \cap \{z : \varphi\})_{V_\alpha} \iff (\forall x : \exists y : \forall z : z \in y \longleftrightarrow (\varphi \land z \in x))_{V_\alpha}$$
$$\iff \forall x \in V_\alpha : \exists y \in V_\alpha : \forall z \in V_\alpha : z \in y \longleftrightarrow \underbrace{(\varphi_{V_\alpha} \land z \in x)}_{\psi}$$

Let $x \in V_\alpha$. By the Axiom of Separation for ψ $\exists y : y = x \cap \{z : \varphi_{V_\alpha}\}$. Then $y \subseteq x$ so $\text{rk}(y) \leq \text{rk}(x) \implies y \in V_\alpha$ i.e. $\exists y \in V_\alpha : \forall z \in V_\alpha : (z \in y \longleftrightarrow (\varphi_{V_\alpha} \land z \in x))$ $\square$

Proof. Axiom of Choice for a limit ordinal α.
TODO

$\square$

Theorem. *For any inaccessible cardinal κ $(V_\kappa, \in) \vDash \text{ZFC}$.*

Proof. We've shown that for any limit ordinal $\alpha > \omega$ $(V_\alpha, \in) \vDash$ All Axioms of ZFC except Replacement. So we need to show $(V_\kappa, \in) \vDash$ Ax. of Replacement.

Proposition (1). *If κ is inaccessible $x \in V_\kappa \implies |x| < \kappa$.*

Proposition (2). *If κ is regular $x \subseteq V_\kappa \land |x| < \kappa \implies x \in V_\kappa$.*

Recall that the Axiom of Replacement is actually an axiom schema, so fix $\varphi \in \text{LST}$ with $x, y \in \text{FV}(\varphi)$. We need to show ... TODO $\qquad\square$

Definition. Let $W \subseteq Z$ be sets or class terms and $\varphi \in \text{Fml}_{\mathcal{L}_\in}$ with $\text{FV}(\varphi) = \{x_1, \ldots, x_n\}$. We say φ is upward (downward) absolute betwwen W, Z if $\forall x_1, \ldots, x_n \in W : (\varphi_W \longleftrightarrow \varphi_Z)$

Compare in model theory if $\mathcal{N}, \mathcal{M}$ are $\mathcal{L}$-structures $\mathcal{N}$ is a substructure of $\mathcal{M}, \mathcal{N} \subseteq \mathcal{M}$ iff $N \subseteq M$ and the interpretation function for $\mathcal{N}$ is the restriction of the interpretation function of $\mathcal{M}$.
For $\varphi \in \mathcal{L}, \text{FV}(\varphi) = \{x_1, \ldots, x_n\}$ $\mathcal{N} \preceq_\varphi \mathcal{M}$ if $\forall a_1, \ldots, a_m \in N : \mathcal{N} \vDash \varphi(a_1, \ldots, a_n) \iff \mathcal{M} \vDash \varphi(a_1, \ldots, a_n)$

Definition. $\mathcal{N} \preceq \mathcal{M}$ is an elementary substructure of $\mathcal{M}$ iff for all $\varphi \in \mathcal{L} : \mathcal{N} \preceq_\varphi \mathcal{M}$
In other context this says $(W, \in) \preceq (Z, \in)$ iff all $\varphi \in \text{Lst}$ are absolute between W, Z.

Definition. A list of formulae $\varphi_0, \varphi_1, \ldots$ is <u>subformula closed</u> if every subformula of a formula on the list is on the list.

Lemma (1)**.** *Let $\vec{\varphi}$ be a subformula closed list, $W \subseteq Z$. The following are equivalent:*

(i) $\vec{\varphi}$ are absolute for W, Z

(ii) Whenever φ_i is of the form $\exists x : \varphi_j(x, \vec{y})$ with $\text{FV}(\varphi_j) \subseteq \{\vec{y}\}$ then $\forall \vec{y} \in W : (\exists x \in Z : \varphi_j(x, \vec{y})_Z \longrightarrow \exists x \in W : \varphi_j(x, \vec{y})_W)$ i.e. φ_i is downward absolute between W, Z.

Proof. $(i) \implies (ii)$ is clear: Fix $\vec{y} \in W$ and assume $\varphi_i(\vec{y})_Z$ i.e. $(\exists x : \varphi_j(x, \vec{y}))_Z \iff \exists x \in Z : \varphi_j(x, \vec{y})$. Then by absoluteness of $\varphi_i, (\varphi_i(\vec{y}))_W$ so $\exists x \in W : (\varphi_j(x, \vec{y})_W)$.
$(ii) \implies (i)$ by induction on the length of φ_i: so we assume as the lefthand side that absoluteness holds for subformulae.

- φ_i atomic - by definition of absoltue

- $\varphi_i = \varphi_j \wedge \varphi_k$ by IH φ_j, φ_k are absolute so by relativization φ_i is absolute. $\varphi_i = \neg\varphi_j$ similarly.

- $\varphi_i = \exists x : \varphi_j(x, \vec{y})$. Fix $\vec{y} \in W$.

  $\varphi_i(\vec{y})_W \overset{def}{\iff} \exists x \in W : \varphi_j(x, \vec{y})_W \overset{IH}{\iff} \exists x \in W : \varphi_j(x, \vec{y})_Z \iff \exists x \in Z : \varphi_j(x, \vec{y})_Z \overset{def}{\iff} \varphi_i(\vec{y})_Z$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition. A formula of $\mathcal{L}_\in$ is Δ_0 iff it only uses bounded quantifiers i.e.

- $x \in y, x = y$ are Δ_0

- If φ, ψ are Δ_0 so are $\neg\varphi, \varphi \wedge \psi$

- If φ is Δ_0 so is $\exists x \in y : \varphi$

Lemma (2). *Let W be a transitive set. Then any Δ_0 formula is absolute for W.*

Proof. By induction on the length of formulae using lemma 1. We just need to show that if φ is of the form $\exists x : (x \in a \wedge \psi(x, \vec{y}, a))$ then $\forall a \in W : \forall \vec{y} \in W : (\exists x : (x \in a \wedge \psi(x, \vec{y}, a)) \longrightarrow \exists x \in W : (x \in a \wedge \psi(x, \vec{y}, a)))_W$

So let $a, \vec{y} \in W$ and suppose $\exists x \in a : \psi(x, \vec{y}, a)$. ψ is Δ_0 as φ is Δ_0 and of length less than φ so by IH ψ is absolute for W i.e. $\psi(x, \vec{y}, a) \longrightarrow \psi(x, \vec{y}, a)_W$. Further as $a \in W, a \subseteq W$ so $x \in a \longleftrightarrow x \in W \cap a$. Thus $\exists x \in a : \psi(x, \vec{y}, a) \implies \exists x \in W : (x \in a \wedge \psi(x, \vec{y}, a))$ □

Theorem (Downward Löwenheim-Skolem Theorem). *Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure. Fix a cardinal κ with $\max(|\mathcal{L}|, \aleph_0) \leq \kappa \leq |M|$ and let $S \subseteq M$ with $|S| \leq \kappa$. Then there is an $\mathcal{N} \preceq \mathcal{M}$ s.t. $S \subseteq N$ and $|N| = \kappa$.*

Proof. We prove this for $\mathcal{L} = \mathcal{L}_\in$ and $\in$-models.

Fix M an infinite set and $\kappa \leq |M|$ and infinite cardinal and $S \subseteq M$ with $|S| \leq \kappa$. If $|S| < \kappa$ let $S' \subseteq M$ with $S \subseteq S'$ and $|S'| = \kappa$.

We use Lemma 1 to build up $S' \subseteq N \subseteq M$ with $(N, \in) \preceq (M, \in)$. Clearly, a list of all formulae in $\mathcal{L}_\in$ is subformula closed. Let R be a well-ordering on M. For any existential formula $\varphi = \exists x : \psi$ let n_φ be $|\operatorname{FV}(\varphi)|$. We define a <u>Skolem-function</u> $f_\varphi : M^{n_\varphi} \to M$ as follows.

For $\vec{y} \in M^{n_\varphi}$, if $(M, \in) \vDash \exists x : \psi(x, \vec{y})$ then let $f_\varphi(\vec{y})$ be the R-least in M s.t. $(M, \in) \vDash \psi(f_\varphi(\vec{y}), \vec{y})$. If $(M, \in) \nvDash \exists x : \psi(x, \vec{y})$ set $f_\varphi(\vec{y}) = 0$. Now set $N_0 = S'$ and define by recursion $N_{i+1} = N_i \cup \left\{ f_\varphi'' N_i^{n_\varphi} \mid \varphi \in \mathcal{L}_\in \text{ s.t. } \varphi = \exists x : \psi \text{ for some } \psi \right\}$. Set $N = \bigcup_{i \in \omega} N_i$

<u>Claim 1</u>: $|N| = \kappa$. Clearly $|N| \geq \kappa$ as $|N_0| = \kappa$.

$$|N_1| \leq |N_0| \oplus \sup_{n \in \omega} |N_0^n| \otimes \aleph_0 \leq \kappa \otimes (\underbrace{\kappa \otimes \ldots \kappa}_{n}) \otimes \aleph_0$$

Similarly $|N_i| = |N_0| = \kappa$. Thus $|N| \leq |N_0| \oplus |N_1| \oplus \cdots = \kappa \otimes \aleph_0 = \kappa$

<u>Claim 2</u>: $(N, \in) \preceq (M, \in)$: By Lemma 1 we just need to show that for any $\varphi = \exists x : \psi$ $\forall y_1, \ldots, y_{n_\varphi} \in N : (\exists x \in M : \psi(x, \vec{y}))_M \longrightarrow \exists x \in N : \psi(x, \vec{y}_N)$. By induction on the length of formulae assume that ψ is absolute for M, N so let $\vec{y} \in N$ ans assume $\exists x \in M : \psi(x, \vec{y})_M$. Then there must be some i s.t. $\vec{y} \in N_i$. Thus $f_\varphi(\vec{y}) \in N_{i+1} \subseteq N$ so $\psi(f_\varphi(\vec{y}), \vec{y})_M$ and by IH $\psi(f_\varphi(\vec{y}), \vec{y})_N$ Thus $\exists x \in N : \psi(x, \vec{y}_N)$. Such an N is called a <u>Skolem Hull</u> of M. N is not transitive. □

Definition. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $X \subseteq M^n$ is <u>definable in $\mathcal{M}$</u> iff there is an $\mathcal{L}$-formula φ with n free variables s.t. for $a_1, \ldots, a_n \in M$ $\mathcal{M} \vDash \varphi(a_1, \ldots, a_n) \iff (a_1, \ldots, a_n) \in X$

$\{\emptyset\}$ is definable in V_λ; any hereditary finite set is definable in V_λ

Theorem (Montague-Levy Reflection Theorem). *Let $\varphi_1, \ldots, \varphi_n$ be any finite list of formulae. Then $\mathrm{ZFC} \vdash \forall \alpha : \exists \beta > \alpha (\vec{\varphi} \text{ are absolute for } V_\beta)$*

This is a theorem schema

If $\vec{\varphi}$ are all axioms of ZFC then $\mathrm{ZFC} \vdash \forall \alpha : \exists \beta > \alpha : (\wedge \vec{\varphi})_{V_\beta}$

Proof. By lengthening the list if necessary assume it is subformula closed. For each $i \leq n$ s.t. $\varphi_i = \exists x : \varphi_j$ define

$$F_i : V \to \mathrm{On}$$

$$F_i(\vec{y}) = \begin{cases} 0 & \text{if } \neg\exists x : \varphi_j(x, \vec{y}) \\ \eta & \text{where } \eta \text{ is least s.t. } \exists x \in V_\eta : \varphi_i(x, \vec{y}) \end{cases}$$

Now define $G_i : \mathrm{On} \to \mathrm{On}$ by $G_i(\gamma) = \sup \{F_i(\vec{y}) \mid \vec{y} \in V_\gamma\}$. G_i is well-defined by the Axiom of replacement $F_i'' V_\gamma$ is a set of ordinals so its supremum is an ordinal.

<u>Claim</u>: $\forall\alpha : \exists\beta > \alpha : \lim(\beta) \wedge \forall\gamma < \beta : \forall i \leq n : G_i(\gamma) \leq \beta$.

Define by recursion $\lambda_0 = \alpha, \lambda_{k+1} = \max \{\lambda_k + 1, G_1(\lambda_k), \ldots, G_n(\lambda_k)\}, \beta := \sup \{\lambda_k \mid k \in \omega\}$. Then $\lim(\beta)$ and if $\gamma < \beta$ then $\gamma < \lambda_k$ for some $k \in \omega$. Hence $G_i(\gamma) \leq G_i(\lambda_k) \leq \lambda_{k+1} < \beta$. Then apply Lemma 1 to V_β. $\qquad\square$

Theorem (Mostowski-Shepherdson Collapsing Lemma)**.** *Let W be a set and $R \subseteq W \times W$ be wellfounded s.t. $(W, R) \vDash$ Ax. of Ext. i.e. $u, v \in W, u \neq W \longrightarrow \exists z \in W : zRu \longleftrightarrow \neg zRv$*

Then there is a unique transitive set M and unique isomorphism $\pi : (W, R) \cong (M, \in)$

Additionally, if $Z \subseteq W$ with $R \restriction W \times Z = \in \restriction W \times Z$ and $v \in Z, u \in W$ with $uRv \longrightarrow u \in Z$ and Z is transitive then $\pi \restriction Z = id \restriction Z$

Proof. <u>Claim 1</u>: If π exists, it is unique.

Suppose we have π, M as above. Let $u, v \in W$. If uRv then $\pi(u) \in \pi(v)$ as π is an isomorphism. Then $\{\pi(u) \mid u \in W, uRv\} \subseteq \pi(v)$.

Further if $z \in \pi(v)$ then $z \in M$ by transitivitiy hence $z = \pi(x)$ for some $x \in W$ with xRv thus $\pi(v) \subseteq \{\pi(u) \mid u \in W, uRv\}$. Taken together $\pi(v) = \{\pi(u) \mid u \in W, uRv\}$, so if the isomrphism exists it must take this form.

<u>Claim 2</u>: Such a π exists.

Define by recursion on R

$$\pi(v) = \{\pi(u) \mid u \in W, uRv\} \tag{1}$$

Whis is this well defined? A R is wellfounded there is $x \in W$ s.t. $\forall y \in W : \neg yRx$. For such an x we have $\pi(x) = \emptyset$. Further for $W \setminus \{x\}$ there is $x' \in W \setminus \{x\}$ with $\forall y \in W \setminus \{x\} : \neg yRx'$ so $\pi(x') = \{\pi(u) \mid u \in W, uRx'\} = \{\pi(u) \mid u = x, uRx'\}$.

π is clearly surjective.

<u>Claim 3</u>: π is injective:

Assume not and take $t \in$-minimal s.t. there are $u \neq v \in W$ with $t = \pi(u) = \pi(v)$. As $u \neq v$ there is some $x \in W$ with $xRu \longleftrightarrow \neg xRv$. Wlog assume $xRu \wedge \neg xRv$. Then $\pi(x) \in \pi(u) = t$. So we must have that for some y with yRv $\pi(y) = \pi(x)$. But this contradicts the minimality of t.

<u>Claim 4</u>: π is orderpreserving

" $\Longrightarrow$ " By (**??**) if uRv then $\pi(u) \in \pi(v)$.

" $\Longleftarrow$ " If $\pi(u) \in \pi(v)$ then for some $z \in W$ with zRv $\pi(z) = \pi(u)$. As π is injective $z = u$.

<u>Claim 5</u> (Also): Suppose for some $u \in X$ $\pi(u) \neq u$ and take $v \in$-minimal such. Then

$$
\begin{aligned}
\pi(v) &= \{\pi(u) \mid u \in W, uRv\} \\
&= \{\pi(u) \mid u \in W, u \in v\} \\
&= \{u \mid u \in v\} = v
\end{aligned}
$$

$\square$

Example. $(N, \in) \preceq (V_\kappa, \in)$ with κ inaccessible, N countable. M is the collapse of $(N, \in)$. $\pi(\omega_1) = \omega_1^{ck}$ the least non-definable countable ordinal.
$P(\omega) \in N : \pi(\mathcal{P}(\omega)) = \{\pi(y) \mid y \in \mathcal{P}(\omega) \cap N\} = \mathcal{P}(\omega) \cap N$

Example. If X, R is a wellorder then its transitive collapse is $\mathrm{ot}((X, R))$

Example. We can code V_ω as a subset of ω i.e. there is $E \subseteq \omega^2$ s.t. $(\omega, E) \cong (V_\omega, \in)$
$M \vDash x = \mathcal{P}(\omega)$, $x = \mathcal{P}(\omega)^M$, $M \vDash x$ is uncountable $\implies \pi(y) = y$.
Let $o(m)$ denote $\mathrm{Ord} \cap M$ then $o(M) \in \mathrm{Ord}$ is countable.

Here: $(M, \in)$ countable transitive model of ZFC (ctm). Assume $M \vDash$ CH

Definition (Poset). A <u>forcing Poset</u> is a triple $(\mathbb{P}, \leq, \mathbb{1})$ where $\mathbb{P}$ is a set, $\leq$ is a pre-order on $\mathbb{P}$ i.e. transitive and reflexive and $\mathbb{1} \in \mathbb{P}$ is a largest element: $\forall p \in \mathbb{P} : p \leq \mathbb{1}$.
Elements of $\mathbb{P}$ are called <u>forcing conditions</u>.
$p \leq q$: "p extends q"
Abuse notation: and use $\mathbb{P}$ to refer to $(\mathbb{P}, \leq, \mathbb{1})$.

Example. Normally we use partial orders $\mathbb{P}$. $\mathbb{1}$ ensures the ordering is connected. If it is a partial ordering then $\mathbb{1}$ is unique.

Definition. $p, q \in \mathbb{P}$ are compatible ($p \not\perp q$) if $\exists r \in \mathbb{P}$ s.t. $r \leq p \wedge r \leq q$ i.e. they have a common extension.
$p, q \in \mathbb{P}$ are incompatible ($p \perp q$) iff they are not compatible.

Example. In a tree any two elements not along a branch are incompatible. In trees $(p \not\leq q \wedge q \not\leq p) \implies p \perp q$. This is not true in general.

Example. $(\mathcal{P}(\omega), \supseteq, \emptyset)$

Example. (Infinite subsetsets of $\omega, \subseteq^*, \omega$) with $p \subseteq^* q : \iff p \setminus q$ is finite.
This is not a partial order e.g. $\omega \subseteq^* \{42, 43, \dots\}$ and $\omega \supseteq^* \{42, 43, \dots\}$.
$p \perp q$ iff $p \cap q$ is finite
There is no least element.

Definition. $Fn(I, J)$ for sets I, J is the set of all finite partial functions from I to J i.e. $\{p \subseteq I \times J \mid |p| < \omega, \mathrm{func}(p)\}$
The associated forcing poset has the order $\supseteq$ and $\mathbb{1} = \emptyset$. $p \leq q \iff p \supseteq q$ i.e. p extends q as a function.

$p \not\perp q$ iff they have a common extension iff they are the same on $\operatorname{dom}(p) \cap \operatorname{dom}(q)$ iff $p \cup q \in Fn(I, J)$ and if so $p \cup q \leq p, p \cup q \leq q$.

$p \in Fn(I, J) \longrightarrow$ a finite approximation of $f : I \to J$, p is a condition such an f must satisfy.

Definition. Let $\mathbb{P}$ be a forcing poset. Then $D \subseteq \mathbb{P}$ is <u>dense</u> in $\mathbb{P}$ if $\forall p \in \mathbb{P} : \exists q \in D : q \leq p$.

Example. Let I be infinite, $J \neq \emptyset$ then for $i \in I$ then $D_i := \{q \in Fn(I, J) \mid i \in \operatorname{dom}(q)\}$ is dense.
Let $p \in Fn(I, J), i \in I$. If $i \in \operatorname{dom}(p)$ then $p \in D_i$ otherwise set $q = p \cup \{(i, j)\}$ where $j \in J$ is arbitrary. Then $q \in Fn(I, J)$ and $q \leq p$ and $q \in D_i$.
Also for $j \in J$ the set $\{q \in Fn(I, J) \mid j \in \operatorname{ran}(q)\}$ is dense.
<u>not dense</u>: $L = \{q \in Fn(I, J) \mid (i, j) \in q\}$ if $|J| \geq 2$. $j' \neq j, p = \{(i, j')\} \in Fn(I, J)$ and $\neg \exists q : q \leq p$ and $q \in L$.

Definition. Let $\mathbb{P}$ be a forcing psoet. Then $G \subseteq \mathbb{P}$ is a <u>filter</u> on $\mathbb{P}$ iff

1. $\mathbb{1} \in G$

2. $\forall p, q \in G : \exists r \in G : r \leq p \wedge r \leq q$

3. $\forall p \in G : \forall q \in \mathbb{P} : p \leq q \implies q \in G$

Note: If $G \neq \emptyset$ then $3 \implies 1$.

Example. $\{\mathbb{1}\}$ is a filter if $\mathbb{P}$ is a partial order

Example. $A \neq \emptyset, \mathbb{P} = \mathcal{P}(A) \setminus \{\emptyset\}, \leq = \subseteq$. $\mathbb{1} = A$.
$F \subseteq A$ is a filter:

1. $\iff A \in F$

2. $\iff F$ closed under intersections

3. $\iff F$ closed under superset

Example. For a tree, a filter is a line from the root that stops somewhere

Example. $\mathbb{P} = Fn(I, J), I$ infinite, $J \neq \emptyset$. If G is a filter on $\mathcal{P}$ then any $p, q \in G$ agree on $\operatorname{dom}(p) \cap \operatorname{dom}(q)$ so setting $f_G := \bigcup G$ is a a function with $\operatorname{dom}(f_G) \subseteq I, \operatorname{ran}(f_G) \subseteq J$. $D_i = \{p \in Fn(I, J) \mid i \in \operatorname{dom}(p)\}$ is dense in $\mathbb{P}$ so if $\forall i \in I : G \cap D_i \neq \emptyset$ then $f_G : I \to J$.

Notation. M is ctm (of ZFC), $\mathbb{P} \in M$ we write to mean $(\mathbb{P}, \leq, \mathbb{1}) \in M$.

Example. If $I, J \in M, M$ transitive and $M \vDash$ ZFC then $Fn(I, J) \in M$ as well as the ordering on this poset.

Definition. For a forcing poset $\mathbb{P}, G$ is <u>$\mathbb{P}$-generic over M</u> iff G is a filter on $\mathbb{P}$ and $G \cap D \neq \emptyset$ for all dense $D \subseteq \mathbb{P}$ s.t. $D \in M$.

Example. $Fn(\omega, 2), M$ a ctm for ZFC. Suppose G is $\mathbb{P}$-generic for M: $\omega \in M, 2 \in M, Fn(\omega, 2) \in M$. $D_i \in M$ for $i \in \omega$ thus $f_G : \omega \to 2$.

For each $h \in M$ with $h : \omega \to 2$ let $E_h = \{q \in \mathbb{P} \mid \exists n \in \omega \cap \mathrm{dom}(q) : q(n) \neq h(n)\}$. Then E_h is dense: Let $p \in \mathbb{P}$ with $h \upharpoonright \mathrm{dom}(p) = p$. $\mathrm{dom}(p)$ is finite, so we can pick $n \notin \mathrm{dom}(p)$ and set $q = p \cup \{(n, \neg h(n))\}$. Then $q \leq p$ and $q \in E_h$. Also $E_h \in M$ so $G \cap E_h \neq \emptyset$. Thus $f_G \neq h$. Thus $f_G \notin M$.

Lemma (Generic Filter existance). *Let $\mathbb{P}$ be any forcing poset, let $\{D_i \mid i \in \omega\}$ be a countable family of dense subsets of $\mathbb{P}$ and $p \in \mathbb{P}$. Then there is a filter G on $\mathbb{P}$ s.t. $p \in G$ and $G \cap D_i \neq \emptyset$ for $i \in \omega$.*

Thus for any ctm M with $\mathbb{P} \in M$ and any $p \in \mathbb{P}$ there is a $\mathbb{P}$-generic filter G over M with $p \in G$.

Proof. By recurion and AC for each $n \in \omega$ choose $r_{n+1} \in \mathbb{P}$ s.t. $r_0 = p$ and $r_{n+1} \leq r_n$ and $r_{n+1} \in D_n$. At each step such r_{n+1} must exists as D_n is dense.

Set $G = \{q \in \mathbb{P} \mid \exists n \in \omega \, r_n \leq q\}$ then G is a filter: 1,3 are easy. For 2 let $p, q \in G$ then $\exists n, m \in \omega : p \geq r_n, q \geq r_m$ so $p \geq r_{max\{m,n\}} \in G$, $q \geq r_{max\{m,n\}} \in G$. Also $p \in G$, each $r_n \in G$ so $r_{n+1} \in G \cap D_n \neq \emptyset$. $\square$

Definition. $r \in \mathbb{P}$ is an atom of $\mathbb{P}$ there are no $p, q \in r$ s.t. $p \perp q$. $\mathbb{P}$ is atomless iff there are no atoms.

Example. $r \in \mathbb{P}$ with no extension is an atom

$(\mathcal{P}(\omega), \subseteq, \omega)$ every element is an atom

linear order: all elements are atoms

A tree with branches at every node is atomless

$Fn(I, J), J \neq \emptyset$ if I is finite has atoms. If I is infinite and $|J| \geq 2$ then $Fn(I, J)$ is atomless

Lemma. *If $\mathbb{P} \in M, M$ a ctm of ZFC, $\mathbb{P}$ atomless and G is $\mathbb{P}$-generic over M then $G \notin M$.*

Proof. Let $D = \mathbb{P} \setminus G$. Then D is dense as if $r \in \mathbb{P}$ there are $p, q \leq r : p \perp q$. Then at least one of these must be in D.

If $G \in M$ then $D \in M$ by the axiom of separation in M. But as G is $\mathbb{P}$-generic over M we would have $D \cap G \neq \emptyset$. $\square$

Definition. τ is a <u>$\mathbb{P}$-name</u> iff τ is a set of ordered pairs s.t. $\forall (\sigma, p) \in \tau$ we have that σ is a $\mathbb{P}$-name and $p \in \mathbb{P}$ (Definition by recursion). $V^{\mathbb{P}}$ denotes the class of all $\mathbb{P}$-names.

$\emptyset$ is a $\mathbb{P}$-name, $\{(\emptyset, p), (\emptyset, 1)\}$ is a $\mathbb{P}$-name $(p, q \in \mathbb{P})$

For any set x look at the associated graph. Label each node in the tree with some element.

If M is a transitive model of ZFC with $\mathbb{P} \in M$ then $M^{\mathbb{P}} := V^{\mathbb{P}} \cap M = \{\tau \in M \mid (\tau \text{ is a } \mathbb{P}\text{-name})_M\}$

Definition. For a $\mathbb{P}$-name τ and $G \subseteq \mathbb{P}$ define by recursion

$$val(\tau, G) = \tau_G := \{val(\sigma, G) \mid \exists p \in G : (\sigma, p) \in \tau\}$$

Then $M[G] := \{\tau_G \mid \tau \in M^{\mathbb{P}}\}$ for a ctm M with $\mathbb{P} \in M$.

Example. $\emptyset_G = \emptyset$

If G is a filter, $\mathbb{1} \in G$, let $\tau = \{(\sigma, \mathbb{1}), (\theta, \mathbb{1})\}$ then $\tau_G = \{val(\sigma, G), val(\theta, G)\} = \{\sigma_G, \theta_G\}$.

$\tau = \{(\emptyset, p), (\{\emptyset, q\}, r)\}, p, r \in G, q \notin G$ then $\tau_G = \{\emptyset_G, \{(\emptyset, q)\}_G\} = \{\emptyset, \emptyset\} = \{\emptyset\}$

$p, r \notin G, q$ does not matter: $\tau_G = \emptyset$

$p \notin G, r, q \in G$: $\tau_G = \{\{\emptyset\}\}$

We want to show that $M[G]$ is the desired extension of M which contains G and everything else needed to ensure $M[G] \vDash$ ZFC. More precisely we will show:

(1) $M \subseteq M[G]$

(2) $G \in M[G]$

(3) $M[G]$ is transitive and countable

(4) $M[G] \vDash$ ZFC

(5) For any N with the above properties $N \supseteq M[G]$

For (i) we find for each $x \in M$ a name $\tau \in M^{\mathbb{P}}$, which for any filter G on $\mathbb{P}$ satisfies $\tau_G = x$

Definition. For a forcing poset $(\mathbb{P}, \leq, \mathbb{1})$ and any set x define $\check{x} := \{(\check{y}, \mathbb{1}) \mid y \in x\}$ (definition by recursion)

Clearly $\hat{x}$ is a $\mathbb{P}$-name

$\check{2} = \{(\check{0}, \mathbb{1}), (\check{1}, \mathbb{1})\} = \{(\emptyset, \mathbb{1}), (\{(\emptyset, \mathbb{1})\}, \mathbb{1})\}$

Lemma (1). *Let G be a filter on $\mathbb{1}$. Then*

  (i) $\forall x \in M : \check{x} \in M^{\mathbb{P}}$ and $val(\check{x}, G) = x$

  (ii) $M[G] \supseteq M$

Proof. (ii) follows by definition from (i)

For (i) $\check{x} \in M$ by absoluteness and $val(\check{x}, G) = x$ by induction as $val(\check{x}, G) = \{val(\check{y}, G) \mid y \in x\}$

$\square$

Lemma (3). *For any filter G on $\mathbb{P}$, $M[G]$ is transitive and countable.*

Proof. Transitive: $x \in M[G]$ then $x = \tau_G$ for some $\tau \in M^{\mathbb{P}}$ so $\forall y \in x : y = \sigma_G$ for some $\sigma \in M^{\mathbb{P}}$ thus $y \in M^{\mathbb{P}}$

Countable: $|M[G]| \leq |M^{\mathbb{P}}| \leq |M| \leq |M[G]|$

$\square$

Definition. Given $\mathbb{P}$, define $\Gamma := \{(\check{p}, p) \mid p \in \mathbb{P}\}$

$\Gamma_G = \{\check{p} \mid p \in G\} \overset{\text{lem 1}}{=} \{p \mid p \in G\} = G$

Hence we have

Lemma (2). *$\Gamma \in M$ is a $\mathbb{P}$-name and $\Gamma_G = G$ hence $G \in M[G]$.*

Proof. Γ is definable within M and clearly a $\mathbb{P}$-name.

$\square$

Definition (Names for unordered and ordered pairs). $up(\sigma, \tau) := \{(\sigma, \mathbb{1}), (\tau, \mathbb{1})\}$
$op(\sigma, \tau) := up(up(\sigma, \sigma), up(\sigma, \tau))$
Observe: $\sigma, \tau \in M^{\mathbb{P}} \implies up(\sigma, \tau), op(\sigma, \tau) \in M^{\mathbb{P}}$

Lemma (4). *Let $M, \mathbb{P}$ be as above and G a filter on $\mathbb{P}$. Then $M[G]$ is a ctm of the axioms of Extenionality, Emptyset, Foundation, Pairing, Infinity and Union.*

Proof. We have seen all transitive non-empty sets satisfy Emptyset, Extensionality and Foundation.
$M[G]$ satisfies pairing by observing as $a, b \in M[G] \implies$ there is $\sigma, \tau \in M^{\mathbb{P}}$ s.t. $\tau_G = a, \sigma_G = b$. Then by above $up(\sigma, \tau) \in M^{\mathbb{P}}$ and $(up(\sigma, \tau))_G = \{\sigma_G, \tau_G\} = \{a, b\}$.
$M[G]$ satisfies the Axiom of Infinity as $\omega \in M[G]$ and absoluteness.
Union: Let $a \in M[G]$. Fix $\tau \in M^{\mathbb{P}}$ s.t. $a = \tau_G$.
set $\pi = \{(\theta, p) \mid \exists (\sigma, q) \in \tau, \exists r \in \mathbb{P} : ((\theta, r) \in \sigma \wedge p \leq r \wedge p \leq q)\}$
Claim. $\bigcup a = b := \pi_G$
$\subseteq$: Let $d \in \bigcup a$ then $\exists q \in G : \exists \sigma : (\sigma, q) \in \tau$ and $d \in \sigma_G$. Thus there is also $r \in G, \theta$ s.t. $(\theta, r) \in \sigma$ and $\theta_G = d \in \pi_G$
$\supseteq$: Let $d \in \pi_G$. Then $\exists p \in G, \exists \theta$ s.t. $(\theta, p) \in \pi$ and $\theta_G = d, p \leq r, p \leq q \implies \theta_G \in \sigma_G$ and $\sigma_G \in \tau_G = a \implies \theta_G \in \bigcup a$ $\square$

Example. $\mathbb{P} = Fn(I, J), I, J \in M, G$ a filter then $\bigcup G$ is a function. We have a name Γ for G so we cann write a name for $\bigcup G$ as above. But there is a much more natural name for $\bigcup G$.

$$\mathring{f} = \left\{ (op((\check{i}), (\check{j})), p) \mid p \in \mathbb{P} \wedge (i, j) \in p \right\}$$

We have seen $P, G \in M[G]$. What about $\mathbb{P} \setminus G =: C$?
Candidate name $\mathring{C} = \{(\check{p}, q) \mid p, q \in \mathbb{P}, p \perp q\}$
$\mathring{C}_G = \{p \in \mathbb{P} \mid \exists q \in G : p \perp q\}$. As all elements of a filter aare compatible: $\mathring{C}_G \cap G = \emptyset$
Do we have $G \cup \mathring{C}_G = \mathbb{P}$?
Let $p \in \mathbb{P}$ set $D_p = \{q \in \mathbb{P} \mid p \perp q \vee q \leq p\}$. $D_p \in M$ and D_p is dense ($r \in \mathbb{P}, r \perp p \implies r \in D_p$ otherwise $\exists q \in \mathbb{P} : q \leq r, q \leq p \implies q \in D_p$)
If G is generic we find $q \in G \cap D_p \implies q \in D_p$
If $q \perp p$ then $p \in \mathring{C}_G$. If $q \leq p$ then $p \in G \longrightarrow$ is complement.

Lemma (5). *Let N be a transitive model of ZFC with $M \subseteq N$ and $G \in N$. Then $M[G] \subseteq N$.*

Proof. The definition of $val(\tau, G) = \tau_G$ is absolute between transitive models of ZFC so if $M^{\mathbb{P}} \subseteq M \subseteq N$ and $G \in N$ then $val(\tau, G) \in N$ for any $\tau \in M^{\mathbb{P}}$. Thus $M[G] \subseteq N$. $\square$

Lemma (6). *For all $\tau \in M^{\mathbb{P}}$*

(1) $rk(\tau_G) \leq rk(\tau)$

(2) $o(M[G]) = o(M)$ i.e. $On \cap M[G] = On \cap M$

Proof. (1) By $\in$-induction on τ:

$$\begin{aligned}
rk(\tau_G) &= \sup\{rk(x) + 1 \mid x \in \tau_G\} \\
&= \sup\{rk(\sigma_G) + 1 \mid (\sigma, p) \in \tau \wedge p \in G\} \\
&\leq \sup\{rk(\sigma_G) + 1 \mid \sigma \in \mathrm{dom}(\tau)\} \\
&\overset{IH}{\leq} \sup\{rk(\sigma) + 1 \mid \sigma \in \mathrm{dom}(\tau)\} \\
&\leq rk(\tau)
\end{aligned}$$

(2) $\alpha \in M[G] \cap \mathrm{On} \implies \alpha = \tau_G$ for some $\tau \in M^{\mathbb{P}}$ but $\alpha = rk(\alpha) = rk(\tau_G) \leq rk(\tau) \in M \cap \mathrm{On}$ as $\tau \in M \vDash \mathrm{ZFC}$ so $\alpha \in M$. Thus $o(M[G]) \subseteq o(M)$ but as $M \subseteq M[G]$ we also have $o(M) \subseteq o(M[G])$.

$\square$

We need to show $M[G] \vDash$ Axiom of Separation i.e. for φ a formula of $\mathcal{L}_\in, \vec{a}, b \in M[G]$ we have $\exists s \in M[G]$ s.t. $s = \{z \mid z \in b \wedge \varphi(z, veca)_{M[G]}\}$

Let's look at a simpler case $b = \omega, \varphi = z \in a, , a = \tau_G$ so we need to find a name $\sigma \in M^{\mathbb{P}}$ s.t. $\sigma_G = \tau_G \cap \omega$. Why not take $\tau_G \tilde{\cap} \omega$? If $\tau_G \notin M$ then we can't assume $\tau_G \cap \omega \in M$. If $\tau = \{(\check{x}_i, p_i) \mid i \in I\}$ then $\sigma = \{(\check{x}_i, p_i) \mid i \in I \wedge x_i \in \omega\}$. This works but in general we cannot tell for $\pi \in \mathrm{dom}(\tau)$ if $\pi_G \in \omega$ or not without knowing what G is. For example, suppose $(\pi, p) \in \tau, \pi = \{(\check{n}, p_n) \mid n \in \omega\}$. Then $\pi_G \in \omega$ iff there is some $n \in \omega$ s.t. $\forall m \leq n : p_m \in G$ and $\forall m > n : p_m \notin G$. Thus if $q \in \mathbb{P}$, new $p \geq q, \forall m \leq n : p_m \geq q$ and $\forall m > n : p_m \perp q$ then $q \in G \implies \pi_G \in \tau_G \cap \omega$. So we could put (π, q) in σ.

Definition. $\mathcal{FL}_{M,\mathbb{P}}$ is the collection of all formulae in the language $\mathcal{L}_\in$ extended with constant symbols for each name in $M^{\mathbb{P}}$.

Example. If φ is a formula of Le with two free variables then $\varphi(\check{n}, \tau)$ is a sentence in $\mathcal{FL}_{M,\mathbb{P}}$

Definition. Let ψ be a sentence of $\mathcal{FL}_{M,\mathbb{P}}$. Then $M[G] \vDash \psi$ has the usual model theoretic meaning with $\in$ interpreted as membership andd any constant symbol or P-name τ is interpreted as τ_G.

Definition. For a sentence ψ of $\mathcal{FL}_{M,\mathbb{P}}$ we say $p \Vdash_{\mathbb{P},M} \psi$ if for all $\mathbb{P}$-generic filter G over M with $p \in G$ we have $M[G] \vDash \psi$.

We omit subscripts $\mathbb{P}, M$ when this is clear from the context.

Example. (example from above): $q \Vdash \pi \in \tau \cap \check{\omega}$

Example. Let $\Gamma : \{(\check{p}, p) \mid p \in \mathbb{P}\}$ be the name for the generic.

If $p \leq q$ then $p \Vdash \check{q} \in \Gamma$ i.e. $\forall G$ s.t. $p \in G$ we have $M[G] \vDash \check{q} \in \Gamma$ i.e. $M[G] \vDash \check{q}_G \in \Gamma_G$ i.e. $M[G] \vDash q \in G$ i.e. $q \in G$.

$\mathbb{1} \Vdash \psi$ iff $M[G] \vDash \psi$ for all $\mathbb{P}$-generic filters G over M e.g. $\mathbb{1} \Vdash$ Axiom of Union by Lemma 4.

If $\mathbb{P}$ is atomless and $p \perp q \in \mathbb{P}$ then $p \Vdash \check{q} \notin \Gamma$.

$\mathbb{1} \nVdash \check{p} \in \Gamma$ and $\mathbb{1} \nVdash \check{p} \notin \Gamma$

We may have that $p \nVdash \psi$ and $p \nVdash \neg\psi$

$\mathbb{1} \Vdash \check{\omega} \in \omega \check{+} 1$

Lemma. *If $p \Vdash \varphi$ and $q \leq p$ then $q \Vdash \varphi$*

Proof. If $q \leq p$ and $q \in G$ for a filter G then $p \in G$ by upwards closure. $\qquad\square$

Back to the example: Our proposal is $\sigma = \{(\check{n}, p) \mid n \in \omega \wedge p \in \mathbb{P} \wedge p \Vdash \check{n} \in \tau\}$. σ is a name but is it definable in M i.e. is $\sigma \in M^{\mathbb{P}}$?

<u>Problem 1</u>: We defined $\Vdash$ using generics for M and these are generally not in M, so there is no obvious reason to think that $\sigma \in M$.

<u>Problem 2</u>: How can we be sure $\sigma_G = \omega \cap \tau_G$. $\sigma_G \subseteq \omega \cap \tau_G$ as if G is a generic with $p \in G$ then $n \in \sigma \iff p \Vdash \check{n} \in \tau \implies M[G] \vDash \check{n} \in \tau \implies n \in \tau_G \cap \omega$.

But how do we know for every $n \in \omega \cap \tau_G$ there is some $p \in \mathbb{P}$ s.t. $p \Vdash \check{n} \in \tau$?

Lemma (Truth Lemma). *Let ψ be a sentence in $\mathcal{FL}_{M,\mathbb{P}}, G$ $\mathbb{P}$-generic over M then $M[G] \vDash \psi$ iff there is a $p \in G$ s.t. $p \Vdash \psi$.*

Note: $\impliedby$ is obvious from the definition of $\Vdash$.

Lemma (Definability Lemma). *Let φ be a formula of $\mathcal{L}_\in$ with $FV(\varphi) = \{x_1, \ldots, x_n\}$. Then $\big\{(p, \mathbb{P}, \leq, \mathbb{1}, \theta_1, \ldots, \theta_n) \mid (\mathbb{P}, \leq, \mathbb{1})$ is a forcing poset in $M, p \in \mathbb{P}, \theta_1, \ldots, \theta_n \in M^{\mathbb{P}}, p \Vdash \varphi(\theta_1, \ldots, \theta_n)\big\}$ is definable (without parameters) in M.*

Using these, we see σ works:

Definability Lemma for $\varphi = x \in y \implies \sigma \in M^{\mathbb{P}}$

Truth Lemma: $(n \in \tau_G \cap \omega)_{M[G]}$ i.e. $M[G] \vDash \check{n} \in \tau \cap \check{\omega} \implies \exists p \in G : p \Vdash \check{n} \in \tau \cap \check{\omega}$ So $p \Vdash \check{n} \in \tau \implies (\check{n}, p) \in \sigma_G$

Lemma (1). *For any $\mathbb{P} \in M$ and sentencess $\varphi, \psi \in \mathcal{FL}_{M,\mathbb{P}}$*

1. *No p forces both φ and $\neg\varphi$*

2. *If φ and ψ are logically equivalent then $p \Vdash \varphi$ iff $p \Vdash \psi$*

3. *$p \Vdash \varphi$ and $q \leq p$ then $q \Vdash \varphi$*

4. *$p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi \wedge p \Vdash \psi$*

Lemma. *$M[G] \vDash$ Axiom of Separation*

Proof. Fix a formula φ of $\mathcal{L}_\in$ with $y \notin FV(\varphi)$. We need to show that $\forall z : \forall v_1, \ldots, v_n \in M[G] : \exists y \in M[G] : \forall x \in M[G] : (x \in y \leftrightarrow (x \in z \wedge \varphi(x, z, \vec{v})_{M[G]}))$

Fix $\pi, \sigma^1, \ldots \sigma^n \in M^{\mathbb{P}}$ i.e. $\pi_G, \sigma_G^1, \ldots, \sigma_G^n \in M[G]$.

Set $S := \big\{x \in \pi_G \mid \varphi(x, \pi_G, \vec{\sigma_G})_{M[G]}\big\}$. We show $S \in M[G]$ so we need $\tau \in M^{\mathbb{P}}$ s.t. $\tau_G = S$.

Let $\tilde{\varphi}(x)$ denote $\varphi(x, \pi, \vec{\sigma}) \in \mathcal{FL}_{M,\mathbb{P}}$.

Now $S = \{\theta_G \mid \theta \in \mathrm{dom}(\pi) \wedge M[G] \vDash (\theta \in \pi \wedge \vec{\varphi}(\theta))\}$. Define $\tau := \{(\theta, p) \mid \theta \in \mathrm{dom}(\pi) \wedge p \in \mathbb{P} \wedge p \Vdash \vec{\varphi}(\theta)\}$.

Then $\tau \in M^{\mathbb{P}}$ by the Definability Lemma and $\tau_G = \{\theta_G \mid \theta \in \operatorname{dom}(\pi) \wedge \exists p \in G : p \Vdash (\theta \in \pi \wedge \vec{\varphi}(\theta))\}$. By definition of forcing $\tau_G \subseteq S$. To prove $S \subseteq \tau_G$ fix $\theta_G \in S$ with $\theta \in \operatorname{dom}(\pi)$ then $M[G] \vDash \in \pi \wedge \vec{\varphi}(\theta)$ so by the Truth Lemma we can find $p \in G$ s.t. $p \Vdash \theta \in \pi \wedge \vec{\varphi}(\theta)$. Thus $(\theta, p) \in \tau$ so $\theta_G \in \tau_G$ thus $S \in M[G]$. $\qquad\square$

Theorem. $M[G] \vDash \mathrm{ZFC}$

Proof. <u>Powerset</u>: It is enough to prove that if $a \in M[G]$ then $\exists b \in M[G]$ s.t. $\mathcal{P}(a) \cap M[G] \subseteq b$. Then by separation $\mathcal{P}(a) \cap M[G] \in M[G]$.

Fix $\tau \in M^{\mathbb{P}}$ s.t. $\tau_G = a$. Let $Q = (\mathcal{P}(\operatorname{dom}(\tau) \times \mathbb{P}))_M$ i.e. all names $\theta \in m^{\mathbb{P}}$ s.t. $\theta \subseteq \operatorname{dom}(\tau)$.

Let $\pi = Q \times \{\mathbb{1}\}$ and $b = \pi_G = \{\theta_G \mid \theta \in Q\}$. Now let $c \in \mathcal{P}(a) \cap M[G]$, we show $c \in b$. Fix $\gamma \in M^{\mathbb{P}}$ s.t. $\gamma_G = c$ and let $\theta = \{(\sigma, p) \mid \sigma \in \operatorname{dom}\tau \wedge p \Vdash \sigma \in \gamma \Vdash \sigma \in \gamma\}$
$\theta \in M$ by the Definability Lemma. Since $\theta \in Q$, we're done if we can show $\theta_G = c$. $\theta_G \subseteq c$ as $\theta_G = \{\sigma_G \mid \sigma \in \operatorname{dom}(\tau), \exists p \in G : p \Vdash \sigma \in \gamma\}$ and by definition of $\Vdash$ such $\sigma_G \in \gamma_G = c$. Since $c \subseteq a = \tau_G$ each element of c is some σ_G for some $\sigma \in \operatorname{dom}(\tau)$. Since $\sigma_G \in c = \gamma_G$ by the Truth Lemma there is $p \in G$ s.t. $p \Vdash (\sigma, p) \in \theta$ and $\sigma_G \in \theta_G$, so $c \supseteq \theta_G$.

<u>Replacement</u>: Let $a \in M[G]$ $(\forall x \in a : \exists y : \tilde{\varphi}(x,y))_{M[G]}$ and find $b \in M[G]$ s.t. $(\forall x \in a : \exists y \in b : \tilde{\varphi}(x,y))_{M[G]}$ so $\vec{\varphi} \in \mathcal{FL}_{M,\mathbb{P}}$ is an abbreviation for φ with names/constants as parameters as in the proof of the Axiom of Separation.

Say $a = \tau_G$ so $M[G] \vDash \forall x \in \tau : \exists y : \vec{\varphi}(x,y)$. Working in M using the Definability Lemma let Q be a set of names s.t. $\forall p \in \mathbb{P} : \forall \sigma \in \operatorname{dom}\tau$ if there is θ s.t. $p \Vdash \sigma, \vec{p}$ then at least one such $\theta \in Q$.

How do we know such a set (not a proper class) Q exists? By the Montague-Levy Reflection Theorem in $M : \forall p : \forall \sigma : \exists \theta$ s.t. $p \Vdash \varphi(\sigma, \theta)$ holdss in M then it holds restricted to some $(V_\sigma)_M$ s.t. $\mathbb{P}, \tau \in (V_\sigma)_M$ so we could take $Q = M^{\mathbb{P}} \cap (V_\sigma)_M$.

Let $\pi = Q \times \{\mathbb{1}\}$ and $b = \pi_G = \{\theta_G \mid \theta \in G\}$. Fix $x \in a$ so $x = \sigma_G$ for some $\sigma \in \operatorname{dom}(\tau)$. Then $M[G] \vDash \tilde{\varphi}(\sigma, \theta)$ and then by the Truth Lemma there is $p \in G$ s.t. $p \Vdash \vec{\varphi}(\sigma, \theta)$. Then there is a $\theta \in G$ s.t. $p \Vdash \tilde{\varphi}(\sigma, \theta)$ if $y = \theta_G$ then $y \in b$ and $\tilde{\varphi}(x,y)_{M[G]}$. $\qquad\square$

Lemma (1 (cont'd)). *5.* $p \Vdash \neg\varphi$ *iff* $\neg\exists q \leq p : q \Vdash \varphi$ *and* $p \Vdash \varphi$ *iff* $\neg\exists q \leq p : q \Vdash \neg\varphi$

6. $p \Vdash \varphi \longrightarrow \psi$ *iff* $\neg\exists q \leq p : (q \Vdash \varphi \wedge q \Vdash \neg\psi)$

7. $p \Vdash \varphi \wedge \psi$ *iff* $\{q \leq p \mid q \Vdash \varphi \text{ or } q \Vdash \psi\}$ *is dense below* p.

Using the Truth Lemma. 5. $(\longrightarrow)$ is clear.

$(\longleftarrow)$ Assume $p \nVdash \neg\varphi$ then we can find some generic G with $p \in G$ and $M[G] \vDash \varphi$. By the Truth Lemma there is $r \in G$ s.t. $r \Vdash \varphi$. Since G is a filter we can find $q \in G$ s.t. $q \leq r$ and $q \leq p$ then $q \Vdash \varphi$ by (3). So $p \nVdash \neg\varphi$.

6. follows from (5) and (4).

7.

$$p \nVdash \neg\varphi \longrightarrow \psi \overset{(6)}{\Longleftrightarrow} \exists r \leq p : (r \Vdash \neg\varphi \wedge r \Vdash \neg\psi)$$
$$\overset{(5)}{\Longleftrightarrow} \exists r \leq p : \forall q \leq r : q \Vdash \varphi \wedge q \nVdash \psi$$

so $p \Vdash \varphi \vee \psi \iff \forall r \leq p : \exists q \leq r : (q \Vdash \varphi \vee q \Vdash \psi)$

Observe $M[G] \Vdash \varphi$ and $p \in G$ then there is a $q \in G$ with $q \leq p$ and $q \Vdash \varphi$.

$\square$

Lemma. *Let $\varphi(x) \in \mathcal{FL}_{M,\mathbb{P}}$ with only one free variable x.*

1. $p \Vdash \forall x : \varphi(x)$ iff $p \Vdash \varphi(\tau)$ for all $\tau \in M^{\mathbb{P}}$

2. $p \Vdash \exists x : \varphi(x)$ iff $\{q \leq p \mid \exists \tau \in M^{\mathbb{P}} : q \Vdash \varphi(\tau)\}$ is dense below p.

Proof. 1. $M[G] \vDash \forall x : \varphi(x)$ iff $M[G] \vDash \varphi(\tau)$ for all $\tau \in M^{\mathbb{P}}$ so $p \Vdash \forall x : \varphi(x)$ iff for all $\tau \in M^{\mathbb{P}}$ and all generics G with $p \in G$ $M[G] \vDash \varphi(\tau)$ iff $\forall \tau \in M^{\mathbb{P}} : p \Vdash \varphi(\tau)$.

2. By Lemma 1(5) and Lemma 2(1) $p \Vdash \neg \forall x : \neg \varphi(x) \iff$ for all $r \leq p : r \nVdash \neg \varphi(x)$ iff by (1) $\forall r \leq p : \exists \tau \in M^{\mathbb{P}} : r \nVdash \neg \varphi(\tau)$ iff by (5) $\forall r \leq p : \exists \tau \in M^{\mathbb{P}} : \exists q \leq r : q \Vdash \varphi(\tau)$
$\square$

Definition. Define by recursion on the complexity of formulae

1. $p \Vdash^* \varphi \wedge \psi$ iff $p \Vdash^* \varphi$ and $p \Vdash^* \psi$

2. $p \Vdash^* \varphi \wedge \psi$ iff $\neg \exists q \leq p : q \Vdash^* \varphi$

3. $p \Vdash^* \varphi \longrightarrow \psi$ iff $\neg \exists q \leq p : (q \Vdash^* \varphi \wedge q \Vdash^* \neg \psi)$

4. $p \Vdash^* \varphi \vee \psi$ iff $\{q \mid q \Vdash^* \varphi \vee q \Vdash^* \psi\}$ is dense below p

5. $p \Vdash^* \forall x : \varphi(x)$ iff $\forall \tau \in M^{\mathbb{P}} : p \Vdash^* \varphi(\tau)$

6. $p \Vdash^* \exists x : \varphi(x)$ iff $\{q \mid \exists \tau \in M^{\mathbb{P}} : p \Vdash^* \varphi(\tau)\}$

Definition. For a forcing poset $\mathbb{P}$ and names $\tau, \theta, \pi \in M^{\mathbb{P}}$:

1. $p \Vdash^* \tau = \theta$ iff $\forall \sigma \in \mathrm{dom}(\tau) \cup \mathrm{dom}(\theta)$: $\forall q \leq p : q \Vdash^* \sigma \in \tau \longleftrightarrow q \Vdash^* \sigma \in \theta$

2. $p \Vdash^* \pi \in \tau$ iff $\{q \leq p \mid \exists (\sigma, r) \in \tau : (q \leq r \wedge q \Vdash^* \pi = \sigma)\}$ is dense below p

Example. $p \Vdash^* \tau = \tau$ holds for all $\tau \in M^{\mathbb{P}}$ trivially.
If $(\pi, r') \in \tau$ and $p \leq r'$ then $p \Vdash^* \pi \in \tau$ because $q \Vdash^* \pi = \pi$ for all q

Is this well defined? Defined by recursion on the rank of names.
For names of rank 0 (i.e. $\emptyset$) then 1. is vacuously true and 2. is trivially false
Suppose 1 and 2 are defined for names of rank $< \alpha$ and $\max\{rk(\tau), rk(\theta)\} = \alpha$. To evaluate 1 we need to evaluate $q \Vdash^* \sigma \in \tau/\theta$ where $rk(\sigma) < \alpha$ so we use 2. We need to evaluate $q \Vdash^* \sigma = \pi$ where $rk(\pi) < \alpha$
To evaluate 2 with $rk(\pi) < \alpha$ as above.
To evaluate 2 with $rk(\pi) = \alpha, rk(\tau) \leq \alpha$. We need to evaluate $q \Vdash^* \pi = \sigma$ where $rk(\sigma) < \alpha$ so use 1.
As we've just shown 1 is well-defined on names of rank α. Thus $\Vdash^*$ is well-defined

Lemma (1). *For $\varphi \in \mathcal{FL}_{M,\mathbb{P}}$ a sentence*

(a) If $p \Vdash^* \varphi$ and $p_1 \leq p$ then $p_1 \Vdash^* \varphi$

(b) $p \Vdash^* \varphi$ iff $\{p_1 \leq p \mid p_1 \Vdash^* \varphi\}$ is dense below p

(c) $p \Vdash^* \varphi$ iff $\neg\exists q \leq p : q \Vdash^* \neg\varphi$

Proof. (c) follows from (b): If $p \Vdash^* \varphi$ and $q \leq p, q \Vdash^* \neg\varphi$ then $\neg\exists r \leq q : r \Vdash^* \varphi \implies$ $\{p_1 \leq p \mid p_1 \Vdash^* \varphi\}$ is dense below p.
(a) For atomic φ this follows from the definition. For a composed formula this is an easy induction.
(b) ($\implies$) follows from (a)
Note: If $A \subseteq \mathbb{P}$ with $\{q \mid A$ is dense below $q\}$ dense below p then A is dense below p.
($\impliedby$) When φ is $\pi \in \tau$. If the right hand side holds, by 2 of the definition $A = \{q \leq p \mid \exists(\sigma, r) \in \tau : q \leq r \wedge q \Vdash^* \pi = \sigma\}$ is dense below each $p_1 \leq p$ s.t. $p_1 \Vdash^* \pi \in \tau$, so A is dense below p by the note and thus $p \Vdash^* \pi \in \tau$ by definition.
If φ is $\tau = \theta$ and the right hand side holds then
$\forall\sigma \in \mathrm{dom}(\tau)\cup\mathrm{dom}(\theta) : \{p_1 \leq p \mid p_1 \Vdash^* \sigma \in \tau \longleftrightarrow p_1 \Vdash^* \sigma \in \theta\}$ dense below p and hence also dense below any $q \leq p$ thus if $q \Vdash^* \sigma \in \tau$ by (a) $\{p_1 \leq q \mid p_1 \Vdash^* \sigma \in \tau \wedge p_1 \Vdash^* \sigma \in \theta\}$ is dense below q so by (b) $q \Vdash^* \sigma \in \theta$. Similarly if $q \Vdash^* \sigma \in \theta$ then $q \Vdash *\sigma \in \tau$. Thus $p \Vdash^* \tau = \theta$.
For compound φ we will just look at 2 examples:
1. $\{p_1 \leq p \mid p_1 \Vdash^* \varphi \wedge \psi\}$ is dense below p then $\{p_1 \leq p \mid p_1 \Vdash^* \varphi \wedge p_1 \Vdash^* \psi\}$ is dense below p. By the inductive hypothesis we get $p \Vdash^* \varphi$ and $p \Vdash^* \psi$ so $p \Vdash^* \varphi \wedge \psi$.
6. $\{p_1 \leq p \mid p_1 \Vdash *\exists x : \varphi(x)\}$ dense below p then $A = \{q \leq p \mid \exists\tau \in M^{\mathbb{P}} : q \Vdash^* \varphi(\tau)\}$ is dense below each p_1 above. So by the note A is dense below p so $p \Vdash^* \exists x : \varphi(x)$ $\qquad\square$

Lemma (2). *Let $D \subseteq \mathbb{P}$ with $D \in M$ be dense below $p \in \mathbb{P}$. If G is $\mathbb{P}$-generic over M and $p \in G$ then $G \cap D = \emptyset$.*

Proof. Exercise $\qquad\square$

Lemma (3). *Let M be a ctm with $\mathbb{P} \in M$ and G be $\mathbb{P}$-generic over M. Fix $\varphi \in \mathcal{FL}_{M,\mathbb{P}}$ a sentence. Then*

(a) If $p \in G$ and $p \Vdash^* \varphi$ then $M[G] \vDash \varphi$

(b) If $M[G] \vDash \varphi$ then there is some $p \in G$ s.t. $p \Vdash^* \varphi$

Corollary. *Let $M, \mathbb{P}, \varphi$ be as in Lemma 3. Then $p \Vdash \varphi$ iff $p \Vdash^* \varphi$*

of corollary. $p \Vdash^* \varphi \overset{\text{Lem 3a}}{\implies}$ for any generic $G \ni p$: $M[G] \vDash \varphi \overset{\text{def}}{\iff} p \Vdash \varphi$
Suppose $p \Vdash \varphi$ but $p \not\Vdash^* \varphi$. By Lemma 1c) $\exists q \leq p : q \Vdash^* \neg\varphi$. Then for such a q by definition of $\Vdash^*$: $\neg\exists r \leq q : r \Vdash^* \varphi$.
Let G be generic with $q \in G$ then $p \in G$ so $M[G] \vDash \varphi$ by definition of $\Vdash$. By lemma 3b) $s \in G$ s.t. $s \Vdash^* \varphi$. Since $q, s \in G$ $\exists r \in G : r \leq q, r \leq s$. Then $r \Vdash^* \varphi$ by Lemma 1a) $\qquad\square$

of Lemma 3. We start with the case φ atomic and prove (a) and (b) by induction on the rank of names. Inductive hypothesis: Assume claim (a/b) is true for sentences of the form $\theta = \tau$ where $rk(\theta), rk(\tau) < \alpha$

1. Using the inductive hypothesis, show true for $\varphi = $ "$\pi \in \tau$" where $rk(\pi) < \alpha, rk(\tau) \leq \alpha$

2. Using 1 show true for $\varphi = $ "$\theta = \tau$" with $rk(\theta), rk(\tau) \leq \alpha$

3. Using 1 show true for $\varphi = $ "$\theta \in \tau$" where $rk(\pi), rk(\tau) \leq \alpha$

Note: We do not need to do step 3 seperately as it follows from step 1 and the inductive hypothesis for $\alpha + 1$

(a) Fix $p \in G$. If $p \Vdash^* \pi \in \tau$ then $D := \{q \leq p \mid \exists (\sigma, r) \in \tau : (q \leq r \wedge q \Vdash^* \pi \in \sigma)\}$ is dense below p and $D \in M$, so by Lemma 2 fix $q \in G \cap D$ and then fix $(\sigma, r) \in \tau$ with $q \leq r$ and $q \Vdash^* \pi = \sigma$. By IH(a) $M[G] \vDash \pi = \sigma$ i.e. $\pi_G = \sigma_G$. Also $r \in G$ as G is a filter so $\sigma_G \in \tau_G$. Thus $M[G] \vDash \pi \in \tau$.

If $p \Vdash^* \tau = \theta$ we need to show $\tau_G = \theta_G$. We prove $\tau_G \subseteq \theta_G$, the other way is symmetric. Fix an element σ_G of τ_G with $(\sigma, r) \in \tau$ and $r \in G$. We show $\sigma_G \in \theta_G$. Fix $q \in G$ s.t. $q \leq p$ and $q \leq r$. Then $q \Vdash^* \sigma \in \tau$ so $q \Vdash^* \sigma \in \theta$ by definition of $\Vdash^*$. By IH(a) $M[G] \vDash \sigma \in \theta$ so $\sigma_G \in \theta_G$.

(b) If $M[G] \vDash \pi \in \tau$ we must find $p \in G$ s.t. $D := \{q \leq p \mid \exists (\sigma, r) \in \tau : q \leq r \wedge q \Vdash^* \pi = \sigma\}$ is dense below p. By definition of τ_G fix $(\sigma, r) \in \tau$ with $r \in G$ s.t. $\pi_G = \sigma_G$. By IH(b) fix $p \in G$ s.t. $p \Vdash^* \pi = \sigma$. Since G is.

Lemma 1a) implies we may assume $p \leq r$ then lemma 1a implies $\forall q \leq p : q \in D$ so D is dense below p. When φ is $\tau = \theta$ let D be the set of all $p \in \mathbb{P}$ s.t. at least one off the following holds:

1. $p \Vdash^* \tau = \theta$

2. For some $\sigma \in dom(\tau) \cup dom(\theta) : p \Vdash^* \sigma \in \tau$ and $p \Vdash^* \sigma \notin \tau$

3. For some $\sigma \in dom(\tau) \cup dom(\theta) : p \Vdash^* \sigma \notin \tau$ and $p \Vdash^* \sigma \in \tau$

Then $D \in M$ and D is dense: Suppose $q \in \mathbb{P}$ with $\neg q \Vdash^* \tau = \theta$ then $\exists \sigma \in dom(\tau) \cup dom(\theta) : \exists q' \leq q : q' \Vdash^* \sigma \in \tau \leftrightarrow q' \Vdash^* \sigma \in \theta$ Wlog suppose $q' \Vdash^* \sigma \in \tau$ and $\neg q' \Vdash^* \sigma \in \theta$. Then by lemma 1c) $\exists p \leq q' : p \Vdash^* \sigma \notin \theta$ and by lemma 1a $p \Vdash^* \sigma \in \tau$. Thus p satisfies 2 so $p \in D$ and $p \leq q$.

If $M[G] \vDash \tau = \theta$ fix $p \in G \cap D$. If 1 holds we're done. If 2 holds then $p \Vdash^* \sigma \in \tau$ so $M[G] \vDash \sigma \in \tau$ so $\sigma_G \in \tau_G$ so by IH(b) fix $q \in G$ s.t. $q \Vdash^* \sigma \in \theta$ then fix $r \in G$ with $r \leq p$ and $r \leq q$. Then $r \Vdash^* \sigma \in \theta$ by lemma 1a and $r \Vdash^* \sigma \notin \theta$.

For (3) we argue symmetricly.

For compound formulae φ we prove a and b together by induction on the length of φ. There are 6 cases but we just do two here.

$\neg\varphi$: suppose a and b hold for φ. For a if $p \in G$ and $p \Vdash^* \neg\varphi$ and $M[G] \vDash \varphi$ then by IH(b) find $r \in G$ s.t. $r \Vdash^* \varphi$ and $q \in G$ with $q < p$ and $q < r$. Then $q \Vdash^* \varphi$ by lemma 1a and $\neg q \Vdash^* \varphi$ by the definition of $p \Vdash^* \neg\varphi$.

For (b) note that $D := \{p \in \mathbb{P} \mid p \Vdash^* \varphi \vee p \Vdash^* \neg\varphi\} \in M$ and D is dense by definition of $p \Vdash^* \neg\varphi$. Assume $M[G] \vDash \neg\varphi$ and fix $p \in G \cap D$. $p \Vdash^* \varphi$ would contradict IH(a) so we must have $p \Vdash^* \neg\varphi$.

$\forall x : \varphi(x)$: Suppose $\forall \tau \in M^{\mathbb{P}}$ (a) and (b) hold for $\varphi(\tau)$. For (a) If $p \in G$ and $p \Vdash^* \forall x : \varphi(x)$ Then by the definition of $\forall \tau \in M^{\mathbb{P}}$ $p \Vdash^* \varphi(\tau)$ so by IH $\forall \tau \in M^{\mathbb{P}}$: $M[G] \vDash \varphi(\tau)$ so $M[G] \vDash \varphi(\tau_G)$ thus $M[G] \vDash \forall x : \varphi(x)$

For (b) suppose $M[G] \vDash \forall x : \varphi(x)$ thus $\forall \tau \in M^{\mathbb{P}} : M[G] \vDash \varphi(\tau)$

$D' = \{q \mid q \Vdash^* \forall x : \varphi(x) \vee q \Vdash^* \exists \neg\varphi(x)\} \in M$ and D' is dense as it is just the set D replacing φ with $\forall x \varphi(x)$ here.

Let $p \in D \cap G$. If $p \Vdash^* \forall x \varphi(x)$ otherwise $p \Vdash^* \exists x \neg\varphi(x)$ so by definition $\{q \mid \exists \tau \in M^{\mathbb{P}} : q \Vdash^* \neg\varphi(\tau)\}$ is dense below p. So by lemma 2 we can find $q \in G, \tau \in M^{\mathbb{P}}$ s.t. $q \Vdash^* \neg\varphi(\tau), q \leq p$. But then by IH(a) $M[G] \vDash \neg\varphi(\tau)$.

$\square$

Let $M \vDash$ ZFC be a ctm and $\mathbb{P} \in M$ an atomless forcing poset, G $\mathbb{P}$-generic over M. Then (1) $M[G] \vDash$ ZFC is a ctm, (2) $G \in M[G] \setminus M$ and $M[G]$ is minimal s.t. (1) and (2) hold.
We want to show that $\mathrm{Con}(\mathrm{ZFC}) \implies \mathrm{Con}(\mathrm{ZFC} + \neg\,\mathrm{CH})$.
If $M \vDash \mathrm{ZFC} + \neg\,\mathrm{CH}$ we're done so we assume $M \vDash \mathrm{ZFC} + \mathrm{CH}$ and we want to choose $\mathbb{P}$ s.t. for any $\mathbb{P}$-generic G $M[G] \vDash \neg\,\mathrm{CH}$.
<u>Recall</u>: $Fn(I, J)$, I infinite, $J \neq \emptyset$, $I, J \in M$. If G is $\mathbb{P}$-generic over M then $\bigcup G =: f_G$ is a function with $\mathrm{dom}(f_G) = I, f_G \notin M$.

Example. $Fn(\omega, 2)$. If G is a generic over M then $\bigcup G$ is a characteristic function for a new subset of ω.
Let $\chi \in M$ with $(\kappa = \omega_2)_M$ i.e. $\kappa = \omega_2^M$. Let $f_G = \bigcup G$ then $f_G : \kappa \times \omega \to 2$ as above. We can think of f_G as coding κ-many functions $h_\alpha : \omega \to 2$ for $\alpha < \kappa$ where $h_\alpha(n) = f_G(\alpha, n)$.
Then $E_{\alpha,\beta} = \{p \in \mathbb{P} \mid \exists n \in \omega [(\alpha, n), (\beta, n) \in \mathrm{dom}(p) \wedge p(\alpha, n) \neq p(\beta, n)]\}$ is dense for $\alpha < \beta < \kappa$:
For $q \in \mathbb{P}$ we extend q to $p := q \cup \{((\alpha, n), 0), ((\beta, n), 1)\}$ where $n > \max\{m \mid \exists \gamma < \kappa (\gamma, m) \in \mathrm{dom}(q)\}$. Then $p \in E_{\alpha,\beta}$. Then G must meet each $E_{\alpha,\beta}$ anch therefore $h_\alpha \neq h_\beta$ for $\alpha \neq \beta$.