Recall. Every infinite subset of the real numbers is either countable or bijective with the whole of $\mathbb{R}$.
Equivalently: $2^{\aleph_0} = \aleph_1$ i.e. $\mathcal{P}(\mathbb{N})$ has size the first uncountable cardinal.

CH was first advanced by Cantor 1878
ZFC $\nvdash \neg$ CH (Gödel 1940)
ZFC $\nvdash$ CH (Cohen 1963)

Cannot do: ZFC $\vdash$ "ZFC $\nvdash$ CH" $\longrightarrow$ find a model for ZFC $+\neg$ CH $\iff$ Con(ZFC $+\neg$ CH)
By Gödel's Second Incompleteness Theorem: If ZFC is consistent then ZFC $\nvdash$ Con(ZFC).
instead: Con(ZFC) $\implies$ Con(ZFC $+\neg$ CH) which we will show using ZFC and
Con(ZFC) $\implies$ Con(ZFC $+$ CH)

Definition. The language of set theory (LST) is the first order predicate language $\mathcal{L}_\in$ which consists of the logical connectives and quantifiers ($\neg, \implies, \vee, \wedge, \exists, \forall$), parentheses, variable symbols $v_0, v_1, \ldots,$ a logical binary predicates "=" abd a single non-logical binary predicate "$\in$".

Con(ZFC) $\implies$ ZFC is satisfiable, i.e. there is an $\mathcal{L}_\in$-structure $(X, \underbrace{E}_{\subseteq X \times X})$ s.t. $(X, E) \vDash$ ZFC. We want a "nicer" model of ZFC.

Method 1. *Assume the existance of an inaccessible cardinal κ. Then $(V_\kappa, \in) \vDash$ ZFC.*

Method 2. *To show* ZFC $+\neg$ CH *is consistent, it is enough to show every finite subset is consistent. For every $\Sigma \subset$ ZFC finite find $(X, \in) \vDash \Sigma$ + enough of ZFC to prove what we need (where X is transitive).*
Transform this into a model of $\Sigma + \neg$ CH.

For a set X and $E \subseteq X \times X$, φ a sentence of LST we know frmo logic how to define the satisfaction relation $(X, E) \vDash \varphi$. To define this we used:

- Symbol set of the language
- define by recursion terms and formulae, free variables etc. / satisfaction
- set D, relations / functions on D, elements of D
- interpretation function maps from non-logical symbols of $\mathcal{L}$ to $D/\mathcal{P}(D \times D)$
- Variable assignment from symbols to D

Definition. We define the reletivization of formulae of LST to a set W by recursion on formulae.

1. $(x \in y)_W := x \in y$ $\qquad$ $(x = y)_W := x = y$
2. $(\neg\varphi)_W := \neg\varphi_W$ $\qquad$ $(\varphi \wedge \psi)_W := \varphi_W \wedge \psi_W$
3. $(\exists x : \varphi)_W := \exists x : (x \in W \wedge \varphi_W)$

$$
\begin{aligned}
(\forall x : \varphi)_W \iff & (\neg \exists x : \neg\varphi)_W \\
& = \neg(\exists x : \neg\varphi)_W \\
& = \neg\exists x : (x \in W \wedge (\neg\varphi)_W) \\
& = \neg\exists x : (x \in W \wedge \neg\varphi_W) \\
& = \forall x : \neg(x \in W \wedge \neg\varphi_W) \\
& = \forall x : \neg x \in W \vee \varphi_W = \forall x : x \in W \longrightarrow \varphi_W
\end{aligned}
$$

We claim that $(W, \in\restriction W \times W) \vDash \varphi$ iff φ_W. We write $(W, \in)$ for $(W, \in\restriction W \times W)$.

Example. Extensionality: $(W, \in) \vDash$ Axiom of Extensionality:

$$
\begin{aligned}
(\text{Ax. Ext. })_W &= (\forall x : \forall y : (\forall z : z \in x \longleftrightarrow z \in y) \longrightarrow x = y)_W \\
&= \forall x \in W : \forall y \in W : (\forall z \in W : z \in x \longleftrightarrow z \in y) \longrightarrow x = y
\end{aligned}
$$

If $W = \{\emptyset, \{\{\emptyset\}\}\}$ then $(W, \in) \nvDash$ Ax. Ext.

Proposition. *If W is transitive then $(W, \in) \vDash$ Ax. Ext.*

Proof. Let W be transitive, $x, y \in W$ with $x \neq y$. Then there is some set z with $z \in x \nleftrightarrow z \in y$. Wlog assume $z \in x, z \notin y$. Then $z \in W$ by transitivity of W. $\square$

Example. Axiom Empty set:

$$
\begin{aligned}
(W, \in) \vDash \text{Ax. Exmpty set} &\iff (\exists x : x = \emptyset)_W \\
&\iff (\exists x : \forall y : y \notin x)_W \\
&\iff \exists x \in W : \forall y \in W : y \notin x \iff \exists x \in W : x \cap W = \emptyset
\end{aligned}
$$

Example. $W = \{\{\emptyset\}\}$ Then $(W, \in) \vDash$ Ax. Empty set even though $\emptyset \notin W$.

Proposition. *If $W \neq \emptyset$ then $(W, \in) \vDash$ Ax. Empty Set.*

Proof. By the Axiom of Foundation there is $y \in W$ with $y \cap W = \emptyset$. $\square$

Definition. We say a formula φ is absolute for W if $\mathrm{FV}(\varphi) \subseteq \{x_1, \dots, x_n\}$ and

$$
\forall x_1 \in W, \dots, \forall x_n \in W : (\varphi_W \leftrightarrow \varphi)
$$

Lemma. *The following formule are absolute for any transitive set W*

1. $z = \emptyset$

2. $x \subseteq y$

3. $z = x \cap y$

4. $z = x \cup y$

5. $z = \bigcup x$

6. $z = \{x, y\}$

7. $z = (x, y)$

Proof. Let W be transitive

1. Let $z \in W$

$$z = \emptyset \iff \forall x : x \notin z \implies \forall x \in W : x \notin z \iff (z = \emptyset)_W$$

Also $x \in z \implies x \in W$ as W is transitive so we must also have "$\impliedby$"

2.

3. Let $x, y, z \in W$

$$z = x \cap y \iff \forall u : u \in z \leftrightarrow (u \in x \wedge u \in y) \implies \forall u \in W : u \in z \leftrightarrow (u \in x \wedge u \in y) \iff (z = x \cap y)_W$$

"$\impliedby$" follows as $u \in z \implies u \in W, u \in x \wedge u \in y \implies u \in W$

4.

5.

6. Let $x, y, z \in W$

$$z = \{x, y\} \iff \forall u (u \in z \leftrightarrow (u = x \vee u = y))$$
$$\implies \forall u \in W (u \in z \leftrightarrow (u = x \vee u = y)) \iff (z = \{x, y\})_W$$

"$\impliedby$" follows as $u \in z \implies u \in W, u = x \vee u = y \implies u \in W$

7. Let $x, y, z \in W$

$$z = (x, y) \iff \exists u, v : z = \{x, y\} \wedge u = \{x\} \wedge v = \{x, y\}$$

$$\overset{W \text{trans.}}{\implies} \exists u \in W : \exists v \in W : z = \{x, y\} \wedge u = \{x\} \wedge v = \{x, y\} \overset{6.}{\iff} (z = (x, y))_W$$

"$\impliedby$" follows simply.

$\square$

Example. $z = \mathcal{P}(x)$ abd $z = x \times y$ are <u>not absolute</u> for transitive sets. Let $W = \omega, 2 \in \omega$. $(\omega, \in) \vDash 3 = \mathcal{P}(2)$
$x \times y = \{(a, b) \mid a \in x, b \in y\}$. W transitive then $x, y \in W \implies$ such $a, b \in W$, $z = (a, b)$ is absolute for W but not all such ordered pairs may be in W.
If $z = W \cap (x \times y)$, $z \in W$ then $(W, \in) \vDash z = x \times y$

Lemma. *If W is transitive and non-empty then it satisfies the axioms of extensionality, empty set and foundation.*

Proof. We just have to show

$$\begin{aligned}(\text{Ax. Foundation})_W &\iff (\forall x : x \neq \emptyset \longrightarrow \exists y : (y \in x \wedge x \cap y = \emptyset))_W \\ &\iff \forall x \in W : (x \neq \emptyset \longrightarrow \exists y \in W : (y \in x \wedge x \cap y = \emptyset))\end{aligned}$$

using the absoluteness of Empty set and $x \cap y$. Let $x \in W, x \neq \emptyset$. Then by the Axiom of Foundation $\exists y : y \in x \wedge x \cap y = \emptyset$ so $y \in W$ by transitivity so $\exists y \in W(y \in x \wedge x \cap y = \emptyset)$ □

Lemma. *For a limit ordinal $\alpha, (V_\alpha, \in)$ satisfies the axioms of pairing, union, powerset and the axiom schema of separation. If additionally $\alpha > \omega, V_\alpha$ satisfies the axiom of infinity.*

Proof. Pairing: $(\forall x : \forall y : \exists z : z = \{x, y\})_{V_\alpha}$ iff $\forall x \in V_\alpha : \forall y \in V_\alpha : \exists z \in V_\alpha : z = \{x, y\}$. Let $x, y \in V_\alpha, \text{rk}(\{x, y\}) = \max(\text{rk}(x), rk(y)) + 1 < \alpha$ so $\{x, y\} \in V_\alpha$
Union: $(\forall x : \exists z : z = \bigcup x)_{V_\alpha}$ iff $\forall x \in V_\alpha : \exists z \in V_\alpha : z = \bigcup x$ so we just need $x \in V_\alpha \implies \bigcup x \in V_\alpha$ which is true as $\text{rk}(\bigcup x) \leq \text{rk}(x) < \alpha$.
Powerset: $(\forall x : \exists y : \forall z : (z \in y \longleftrightarrow z \subseteq x))_{V_\alpha}$ iff $\forall x \in V_\alpha : \exists y \in V_\alpha : \forall z \in V_\alpha : (z \in y \longleftrightarrow z \subseteq x)$. $\text{rk}(\mathcal{P}(x)) = \text{rk}(x) + 1$ so $x \in V_\alpha \implies \mathcal{P}(x) \in V_\alpha$. Let $x \in V_\alpha, y = \mathcal{P}(x) \in V_\alpha$ then $\forall z : z \in y \longleftrightarrow z \subseteq x$ so in particular $\forall z \in V_\alpha : (z \in y \leftrightarrow z \subseteq x)$ i.e. $(y = \mathcal{P}(x))_{V_\alpha}$.
Infinity: Let $\alpha > \omega$. $(\exists x : (\emptyset \in x \wedge \forall y : (y \in x \longrightarrow y \cup \{y\} \in x)))_{V_\alpha}$ iff $\exists x \in V_\alpha : (\emptyset \in x \wedge \forall y \in V_\alpha : (y \in x \longrightarrow y \cup \{y\} \in x))$. Let $x = \omega$ then $x \in V_\alpha, \emptyset \in V_\alpha, \forall y : (y \in x \longrightarrow y \cup \{y\} \in x)$ so in particular $\forall y \in V_\alpha : (y \in x \longrightarrow y \cup \{y\} \in x)$
Separation: Let φ be a formula of LST with $x, y \notin \text{FV}(\varphi)$. Then

$$\begin{aligned}(\forall x : \exists y : y = x \cap \{z : \varphi\})_{V_\alpha} &\iff (\forall x : \exists y : \forall z : z \in y \longleftrightarrow (\varphi \wedge z \in x))_{V_\alpha} \\ &\iff \forall x \in V_\alpha : \exists y \in V_\alpha : \forall z \in V_\alpha : z \in y \longleftrightarrow (\underbrace{\varphi_{V_\alpha}}_{\psi} \wedge z \in x)\end{aligned}$$

Let $x \in V_\alpha$. By the Axiom of Separation for ψ $\exists y : y = x \cap \{z : \varphi_{V_\alpha}\}$. Then $y \subseteq x$ so $\text{rk}(y) \leq \text{rk}(x) \implies y \in V_\alpha$ i.e. $\exists y \in V_\alpha : \forall z \in V_\alpha : (z \in y \longleftrightarrow (\varphi_{V_\alpha} \wedge z \in x))$ □

Proof. Axiom of Choice for a limit ordinal α.
TODO

□

Theorem. *For any inaccessiable cardinal κ $(V_\kappa, \in) \vDash$ ZFC.*

Proof. We've shown that for any limit ordinal $\alpha > \omega$ $(V_\alpha, \in) \vDash$ All Axioms of ZFC except Replacement. So we need to show $(V_\kappa, \in) \vDash$ Ax. of Replacement.

Proposition (1)**.** *If κ is inaccessible $x \in V_\kappa \implies |x| < \kappa$.*

Proposition (2)**.** *If κ is regular $x \subseteq V_\kappa \wedge |x| < \kappa \implies x \in V_\kappa$.*

Recall that the Axiom of Replacement is actually an axiom schema, so fix $\varphi \in \text{LST}$ with $x, y \in \text{FV}(\varphi)$. We need to show ... TODO $\square$

Definition. Let $W \subseteq Z$ be sets or class terms and $\varphi \in \text{Fml}_{\mathcal{L}_\in}$ with $\text{FV}(\varphi) = \{x_1, \dots, x_n\}$. We say φ is upward (downward) absolute betwwen W, Z if $\forall x_1, \dots, x_n \in W : (\varphi_W \longleftrightarrow \varphi_Z)$

Compare in model theory if $\mathcal{N}, \mathcal{M}$ are $\mathcal{L}$-structures $\mathcal{N}$ is a substructure of $\mathcal{M}, \mathcal{N} \subseteq \mathcal{M}$ iff $N \subseteq M$ and the interpretation function for $\mathcal{N}$ is the restriction of the interpretation function of $\mathcal{M}$.
For $\varphi \in \mathcal{L}, \text{FV}(\varphi) = \{x_1, \dots, x_n\}$ $\mathcal{N} \preceq_\varphi \mathcal{M}$ if $\forall a_1, \dots, a_m \in N : \mathcal{N} \models \varphi(a_1, \dots, a_n) \iff \mathcal{M} \models \varphi(a_1, \dots, a_n)$

Definition. $\mathcal{N} \preceq \mathcal{M}$ is an elementary substructure of $\mathcal{M}$ iff for all $\varphi \in \mathcal{L} : \mathcal{N} \preceq_\varphi \mathcal{M}$
In other context this says $(W, \in) \preceq (Z, \in)$ iff all $\varphi \in \text{Lst}$ are absolute between W, Z.

Definition. A list of formulae $\varphi_0, \varphi_1, \dots$ is <u>subformula closed</u> if every subformula of a formula on the list is on the list.

Lemma (1). *Let $\vec{\varphi}$ be a subformula closed list, $W \subseteq Z$. The following are equivalent:*

(i) $\vec{\varphi}$ are absolute for W, Z

(ii) Whenever φ_i is of the form $\exists x : \varphi_j(x, \vec{y})$ with $\text{FV}(\varphi_j) \subseteq \{\vec{y}\}$ then $\forall \vec{y} \in W : (\exists x \in Z : \varphi_j(x, \vec{y})_Z \longrightarrow \exists x \in W : \varphi_j(x, \vec{y})_W)$ i.e. φ_i is downward absolute between W, Z.

Proof. <u>$(i) \implies (ii)$</u> is clear: Fix $\vec{y} \in W$ and assume $\varphi_i(\vec{y})_Z$ i.e. $(\exists x : \varphi_j(x, \vec{y}))_Z \iff \exists x \in Z : \varphi_j(x, \vec{y})$. Then by absoluteness of $\varphi_i, (\varphi_i(\vec{y}))_W$ so $\exists x \in W : (\varphi_j(x, \vec{y})_W)$.
<u>$(ii) \implies (i)$</u> by induction on the length of φ_i: so we assume as the lefthand side that absoluteness holds for subformulae.

- φ_i atomic - by definition of absoltue
- $\varphi_i = \varphi_j \wedge \varphi_k$ by IH φ_j, φ_k are absolute so by relativization φ_i is absolute. $\varphi_i = \neg\varphi_j$ similarly.
- $\varphi_i = \exists x : \varphi_j(x, \vec{y})$. Fix $\vec{y} \in W$.

  $\varphi_i(\vec{y})_W \overset{def}{\iff} \exists x \in W : \varphi_j(x, \vec{y})_W \overset{IH}{\iff} \exists x \in W : \varphi_j(x, \vec{y})_Z \iff \exists x \in Z : \varphi_j(x, \vec{y})_Z \overset{def}{\iff} \varphi_i(\vec{y})_Z$

$\square$

Definition. A formula of $\mathcal{L}_\in$ is Δ_0 iff it only uses bounded quantifiers i.e.

- $x \in y, x = y$ are Δ_0
- If φ, ψ are Δ_0 so are $\neg\varphi, \varphi \wedge \psi$
- If φ is Δ_0 so is $\exists x \in y : \varphi$

Lemma (2)**.** *Let W be a transitive set. Then any Δ_0 formula is absolute for W.*

Proof. By induction on the length of formulae using lemma 1. We just need to show that if φ is of the form $\exists x : (x \in a \wedge \psi(x, \vec{y}, a))$ then $\forall a \in W : \forall \vec{y} \in W : (\exists x : (x \in a \wedge \psi(x, \vec{y}, a)) \longrightarrow \exists x \in W : (x \in a \wedge \psi(x, \vec{y}, a)))_W$
So let $a, \vec{y} \in W$ and suppose $\exists x \in a : \psi(x, \vec{y}, a)$. ψ is Δ_0 as φ is Δ_0 and of length less than φ so by IH ψ is absolute for W i.e. $\psi(x, \vec{y}, a) \longrightarrow \psi(x, \vec{y}, a)_W$. Further as $a \in W, a \subseteq W$ so $x \in a \longleftrightarrow x \in W \cap a$. Thus $\exists x \in a : \psi(x, \vec{y}, a) \implies \exists x \in W : (x \in a \wedge \psi(x, \vec{y}, a))$ $\square$

Theorem (Downward Löwenheim-Skolem Theorem)**.** *Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure. Fix a cardinal κ with $\max(|\mathcal{L}|, \aleph_0) \leq \kappa \leq |M|$ and let $S \subseteq M$ with $|S| \leq \kappa$.*
Then there is an $\mathcal{N} \preceq \mathcal{M}$ s.t. $S \subseteq N$ and $|N| = \kappa$.

Proof. We prove this for $\mathcal{L} = \mathcal{L}_\in$ and $\in$-models.
Fix M an infinite set and $\kappa \leq |M|$ and infinite cardinal and $S \subseteq M$ with $|S| \leq \kappa$. If $|S| < \kappa$ let $S' \subseteq M$ with $S \subseteq S'$ and $|S'| = \kappa$.
We use Lemma 1 to build up $S' \subseteq N \subseteq M$ with $(N, \in) \preceq (M, \in)$. Clearly, a list of all formulae in $\mathcal{L}_\in$ is subformula closed. Let R be a well-ordering on M. For any existential formula $\varphi = \exists x : \psi$ let n_φ be $|\operatorname{FV}(\varphi)|$. We define a <u>Skolem-function</u> $f_\varphi : M^{n_\varphi} \to M$ as follows.
For $\vec{y} \in M^{n_\varphi}$, if $(M, \in) \vDash \exists x : \psi(x, \vec{y})$ then let $f_\varphi(\vec{y})$ be the R-least in M s.t. $(M, \in) \vDash \psi(f_\varphi(\vec{y}), \vec{y})$. If $(M, \in) \nvDash \exists x : \psi(x, \vec{y})$ set $f_\varphi(\vec{y}) = 0$. Now set $N_0 = S'$ and define by recursion $N_{i+1} = N_i \cup \left\{ f_\varphi'' N_i^{n_\varphi} \mid \varphi \in \mathcal{L}_\in \text{ s.t. } \varphi = \exists x : \psi \text{ for some } \psi \right\}$. Set $N = \bigcup_{i \in \omega} N_i$
<u>Claim 1</u>: $|N| = \kappa$. Clearly $|N| \geq \kappa$ as $|N_0| = \kappa$.

$$|N_1| \leq |N_0| \oplus \sup_{n \in \omega} |N_0^n| \otimes \aleph_0 \leq \kappa \otimes (\underbrace{\kappa \otimes \ldots \kappa}_{n}) \otimes \aleph_0$$

Similarly $|N_i| = |N_0| = \kappa$. Thus $|N| \leq |N_0| \oplus |N_1| \oplus \cdots = \kappa \otimes \aleph_0 = \kappa$
<u>Claim 2</u>: $(N, \in) \preceq (M, \in)$: By Lemma 1 we just need to show that for any $\varphi = \exists x : \psi$ $\forall y_1, \ldots, y_{n_\varphi} \in N : (\exists x \in M : \psi(x, \vec{y}))_M \longrightarrow \exists x \in N : \psi(x, \vec{y}_N)$. By induction on the length of formulae assume that ψ is absolute for M, N so let $\vec{y} \in N$ ans assume $\exists x \in M : \psi(x, \vec{y})_M$. Then there must be some i s.t. $\vec{y} \in N_i$. Thus $f_\varphi(\vec{y}) \in N_{i+1} \subseteq N$ so $\psi(f_\varphi(\vec{y}), \vec{y})_M$ and by IH $\psi(f_\varphi(\vec{y}), \vec{y})_N$ Thus $\exists x \in N : \psi(x, \vec{y})_N$. Such an N is called a <u>Skolem Hull</u> of M. N is not transitive. $\square$

Definition. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $X \subseteq M^n$ is <u>definable in $\mathcal{M}$</u> iff there is an $\mathcal{L}$-formula φ with n free variables s.t. for $a_1, \ldots, a_n \in M$ $\mathcal{M} \vDash \varphi(a_1, \ldots, a_n) \iff (a_1, \ldots, a_n) \in X$

$\{\emptyset\}$ is definable in V_λ; any hereditary finite set is definable in V_λ

Theorem (Montague-Levy Reflection Theorem)**.** *Let $\varphi_1, \ldots, \varphi_n$ be any finite list of formulae. Then* $\mathrm{ZFC} \vdash \forall \alpha : \exists \beta > \alpha$*($\vec{\varphi}$ are absolute for V_β)*
This is a theorem schema
If $\vec{\varphi}$ are all axioms of ZFC *then* $\mathrm{ZFC} \vdash \forall \alpha : \exists \beta > \alpha : (\wedge \vec{\varphi})_{V_\beta}$

Proof. By lengthening the list if necessary assume it is subformula closed. For each $i \leq n$ s.t. $\varphi_i = \exists x : \varphi_j$ define

$$F_i : V \to \mathrm{On}$$
$$F_i(\vec{y}) = \begin{cases} 0 & \text{if } \neg\exists x : \varphi_j(x, \vec{y}) \\ \eta & \text{where } \eta \text{ is least s.t. } \exists x \in V_\eta : \varphi_i(x, \vec{y}) \end{cases}$$

Now define $G_i : \mathrm{On} \to \mathrm{On}$ by $G_i(\gamma) = \sup\{F_i(\vec{y}) \mid \vec{y} \in V_\gamma\}$. G_i is well-defined by the Axiom of replacement $F_i''V_\gamma$ is a set of ordinals so its supremum is an ordinal.
Claim: $\forall \alpha : \exists \beta > \alpha : \lim(\beta) \wedge \forall \gamma < \beta : \forall i \leq n : G_i(\gamma) \leq \beta$.
Define by recursion $\lambda_0 = \alpha, \lambda_{k+1} = \max\{\lambda_k + 1, G_1(\lambda_k), \ldots, G_n(\lambda_k)\}$, $\beta := \sup\{\lambda_k \mid k \in \omega\}$. Then $\lim(\beta)$ and if $\gamma < \beta$ then $\gamma < \lambda_k$ for some $k \in \omega$. Hence $G_i(\gamma) \leq G_i(\lambda_k) \leq \lambda_{k+1} < \beta$. Then apply Lemma 1 to V_β. □

Theorem (Mostowski-Shepherdson Collapsing Lemma). *Let W be a set and $R \subseteq W \times W$ be wellfounded s.t. $(W, R) \vDash$ Ax. of Ext. i.e. $u, v \in W, u \neq W \longrightarrow \exists z \in W : zRu \longleftrightarrow \neg zRv$*
Then there is a unique transitive set M and unique isomorphism $\pi : (W, R) \cong (M, \in)$
Additionally, if $Z \subseteq W$ with $R \upharpoonright W \times Z = \in \upharpoonright W \times Z$ and $v \in Z, u \in W$ with $uRv \longrightarrow u \in Z$ and Z is transitive then $\pi \upharpoonright Z = id \upharpoonright Z$

Proof. Claim 1: If π exists, it is unique.
Suppose we have π, M as above. Let $u, v \in W$. If uRv then $\pi(u) \in \pi(v)$ as π is an isomorphism. Then $\{\pi(u) \mid u \in W, uRv\} \subseteq \pi(v)$.
Further if $z \in \pi(v)$ then $z \in M$ by transitivitiy hence $z = \pi(x)$ for some $x \in W$ with xRv thus $\pi(v) \subseteq \{\pi(u) \mid u \in W, uRv\}$. Taken together $\pi(v) = \{\pi(u) \mid u \in W, uRv\}$, so if the isomrphism exists it must take this form.
Claim 2: Such a π exists.
Define by recursion on R

$$\pi(v) = \{\pi(u) \mid u \in W, uRv\} \tag{1}$$

Whis is this well defined? A R is wellfounded there is $x \in W$ s.t. $\forall y \in W : \neg yRx$. For such an x we have $\pi(x) = \emptyset$. Further for $W \setminus \{x\}$ there is $x' \in W \setminus \{x\}$ with $\forall y \in W \setminus \{x\} : \neg yRx'$ so $\pi(x') = \{\pi(u) \mid u \in W, uRx'\} = \{\pi(u) \mid u = x, uRx'\}$.
π is clearly surjective.
Claim 3: π is injective:
Assume not and take t $\in$-minimal s.t. there are $u \neq v \in W$ with $t = \pi(u) = \pi(v)$. As $u \neq v$ there is some $x \in W$ with $xRu \longleftrightarrow \neg xRv$. Wlog assume $xRu \wedge \neg xRv$. Then $\pi(x) \in \pi(u) = t$. So we must have that for some y with yRv $\pi(y) = \pi(x)$. But this contradicts the minimality of t.
Claim 4: π is orderpreserving
"$\Longrightarrow$" By (1) if uRv then $\pi(u) \in \pi(v)$.
"$\Longleftarrow$" If $\pi(u) \in \pi(v)$ then for some $z \in W$ with zRv $\pi(z) = \pi(u)$. As π is injective $z = u$.

Claim 5 (Also): Suppose for some $u \in X$ $\pi(u) \neq u$ and take v $\in$-minimal such. Then

$$\begin{aligned}\pi(v) &= \{\pi(u) \mid u \in W, uRv\} \\ &= \{\pi(u) \mid u \in W, u \in v\} \\ &= \{u \mid u \in v\} = v\end{aligned}$$

$\square$

Example. $(N, \in) \preceq (V_\kappa, \in)$ with κ inaccessible, N countable. M is the collapse of $(N, \in)$. $\pi(\omega_1) = \omega_1^{ck}$ the least non-definable countable ordinal.
$P(\omega) \in N : \pi(\mathcal{P}(\omega)) = \{\pi(y) \mid y \in \mathcal{P}(\omega) \cap N\} = \mathcal{P}(\omega) \cap N$

Example. If X, R is a wellorder then its transitive collapse is $\mathrm{ot}((X, R))$

Example. We can code V_ω as a subset of ω i.e. there is $E \subseteq \omega^2$ s.t. $(\omega, E) \cong (V_\omega, \in)$
$M \vDash x = \mathcal{P}(\omega)$, $x = \mathcal{P}(\omega)^M$, $M \vDash x$ is uncountable $\implies \pi(y) = y$.
Let $o(m)$ denote $\mathrm{Ord} \cap M$ then $o(M) \in \mathrm{Ord}$ is countable.

Here: $(M, \in)$ countable transitive model of ZFC (ctm). Assume $M \vDash \mathrm{CH}$

Definition (Poset)**.** A forcing Poset is a triple $(\mathbb{P}, \leq, \mathbb{1})$ where $\mathbb{P}$ is a set, $\leq$ is a pre-order on $\mathbb{P}$ i.e. transitive and reflexive and $\mathbb{1} \in \mathbb{P}$ is a largest element: $\forall p \in \mathbb{P} : p \leq \mathbb{1}$.
Elements of $\mathbb{P}$ are called forcing conditions.
$p \leq q$: "p extends q"
Abuse notation: and use $\mathbb{P}$ to refer to $(\mathbb{P}, \leq, \mathbb{1})$.

Example. Normally we use partial orders $\mathbb{P}$. $\mathbb{1}$ ensures the ordering is connected. If it is a partial ordering then $\mathbb{1}$ is unique.

Definition. $p, q \in \mathbb{P}$ are compatible ($p \not\perp q$) if $\exists r \in \mathbb{P}$ s.t. $r \leq p \wedge r \leq q$ i.e. they have a common extension.
$p, q \in \mathbb{P}$ are incompatible ($p \perp q$) iff they are not compatible.

Example. In a tree any two elements not along a branch are incompatible. In trees $(p \not\leq q \wedge q \not\leq p) \implies p \perp q$. This is not true in general.

Example. $(\mathcal{P}(\omega), \supseteq, \emptyset)$

Example. (Infinite subsetsets of $\omega, \subseteq^*, \omega$) with $p \subseteq^* q :\iff p \setminus q$ is finite.
This is not a partial order e.g. $\omega \subseteq^* \{42, 43, \dots\}$ and $\omega \supseteq^* \{42, 43, \dots\}$.
$p \perp q$ iff $p \cap q$ is finite
There is no least element.

Definition. $Fn(I, J)$ for sets I, J is the set of all finite partial functions from I to J i.e. $\{p \subseteq I \times J \mid |p| < \omega, \mathrm{func}(p)\}$
The associated forcing poset has the order $\supseteq$ and $\mathbb{1} = \emptyset$. $p \leq q \iff p \supseteq q$ i.e. p extends q as a function.

$p \not\perp q$ iff they have a common extension iff they are the same on $\mathrm{dom}(p) \cap \mathrm{dom}(q)$ iff $p \cup q \in Fn(I, J)$ and if so $p \cup q \leq p, p \cup q \leq q$.
$p \in Fn(I, J) \longrightarrow$ a finite approximation of $f : I \to J$, p is a condition such an f must satisfy.

Definition. Let $\mathbb{P}$ be a forcing poset. Then $D \subseteq \mathbb{P}$ is <u>dense</u> in $\mathbb{P}$ if $\forall p \in \mathbb{P} : \exists q \in D : q \leq p$.

Example. Let I be infinite, $J \neq \emptyset$ then for $i \in I$ then $D_i := \{q \in Fn(I, J) \mid i \in \mathrm{dom}(q)\}$ is dense.
Let $p \in Fn(I, J), i \in I$. If $i \in \mathrm{dom}(p)$ then $p \in D_i$ otherwise set $q = p \cup \{(i, j)\}$ where $j \in J$ is arbitrary. Then $q \in Fn(I, J)$ and $q \leq p$ and $q \in D_i$.
Also for $j \in J$ the set $\{q \in Fn(I, J) \mid j \in \mathrm{ran}(q)\}$ is dense.
<u>not dense</u>: $L = \{q \in Fn(I, J) \mid (i, j) \in q\}$ if $|J| \geq 2$. $j' \neq j, p = \{(i, j')\} \in Fn(I, J)$ and $\neg \exists q : q \leq p$ and $q \in L$.

Definition. Let $\mathbb{P}$ be a forcing psoet. Then $G \subseteq \mathbb{P}$ is a <u>filter</u> on $\mathbb{P}$ iff

1. $\mathbb{1} \in G$
2. $\forall p, q \in G : \exists r \in G : r \leq p \wedge r \leq q$
3. $\forall p \in G : \forall q \in \mathbb{P} : p \leq q \implies q \in G$

Note: If $G \neq \emptyset$ then $3 \implies 1$.

Example. $\{\mathbb{1}\}$ is a filter if $\mathbb{P}$ is a partial order

Example. $A \neq \emptyset, \mathbb{P} = \mathcal{P}(A) \setminus \{\emptyset\}, \leq = \subseteq$. $\mathbb{1} = A$.
$F \subseteq A$ is a filter:

1. $\iff A \in F$
2. $\iff F$ closed under intersections
3. $\iff F$ closed under superset

Example. For a tree, a filter is a line from the root that stops somewhere

Example. $\mathbb{P} = Fn(I, J), I$ infinite, $J \neq \emptyset$. If G is a filter on $\mathcal{P}$ then any $p, q \in G$ agree on $\mathrm{dom}(p) \cap \mathrm{dom}(q)$ so setting $f_G := \bigcup G$ is a a function with $\mathrm{dom}(f_G) \subseteq I, \mathrm{ran}(f_G) \subseteq J$.
$D_i = \{p \in Fn(I, J) \mid i \in \mathrm{dom}(p)\}$ is dense in $\mathbb{P}$ so if $\forall i \in I : G \cap D_i \neq \emptyset$ then $f_G : I \to J$.

Notation. M is ctm (of ZFC), $\mathbb{P} \in M$ we write to mean $(\mathbb{P}, \leq, \mathbb{1}) \in M$.

Example. If $I, J \in M, M$ transitive and $M \models \mathrm{ZFC}$ then $Fn(I, J) \in M$ as well as the ordering on this poset.

Definition. For a forcing poset $\mathbb{P}, G$ is <u>$\mathbb{P}$-generic over M</u> iff G is a filter on $\mathbb{P}$ and $G \cap D \neq \emptyset$ for all dense $D \subseteq \mathbb{P}$ s.t. $D \in M$.

Example. $Fn(\omega,2), M$ a ctm for ZFC. Suppose G is $\mathbb{P}$-generic for M: $\omega \in M, 2 \in M, Fn(\omega,2) \in M$. $D_i \in M$ for $i \in \omega$ thus $f_G : \omega \to 2$.
For each $h \in M$ with $h : \omega \to 2$ let $E_h = \{q \in \mathbb{P} \mid \exists n \in \omega \cap \operatorname{dom}(q) : q(n) \neq h(n)\}$. Then E_h is dense: Let $p \in \mathbb{P}$ with $h \restriction \operatorname{dom}(p) = p$. $\operatorname{dom}(p)$ is finite, so we can pick $n \notin \operatorname{dom}(p)$ and set $q = p \cup \{(n, \neg h(n))\}$. Then $q \leq p$ and $q \in E_h$. Also $E_h \in M$ so $G \cap E_h \neq \emptyset$. Thus $f_G \neq h$. Thus $f_G \notin M$.

Lemma (Generic Filter existance). *Let $\mathbb{P}$ be any forcing poset, let $\{D_i \mid i \in \omega\}$ be a countable family of dense subsets of $\mathbb{P}$ and $p \in \mathbb{P}$. Then there is a filter G on $\mathbb{P}$ s.t. $p \in G$ and $G \cap D_i \neq \emptyset$ for $i \in \omega$.*
Thus for any ctm M with $\mathbb{P} \in M$ and any $p \in \mathbb{P}$ there is a $\mathbb{P}$-generic filter G over M with $p \in G$.

Proof. By recurion and AC for each $n \in \omega$ choose $r_{n+1} \in \mathbb{P}$ s.t. $r_0 = p$ and $r_{n+1} \leq r_n$ and $r_{n+1} \in D_n$. At each step such r_{n+1} must exists as D_n is dense.
Set $G = \{q \in \mathbb{P} \mid \exists n \in \omega r_n \leq q\}$ then G is a filter: 1,3 are easy. For 2 let $p, q \in G$ then $\exists n, m \in \omega : p \geq r_n, q \geq r_m$ so $p \geq r_{max\{m,n\}} \in G$, $q \geq r_{max\{m,n\}} \in G$. Also $p \in G$, each $r_n \in G$ so $r_{n+1} \in G \cap D_n \neq \emptyset$. ☐

Definition. $r \in \mathbb{P}$ is an atom of $\mathbb{P}$ there are no $p, q \in r$ s.t. $p \perp q$. $\mathbb{P}$ is atomless iff there are no atoms.

Example. $r \in \mathbb{P}$ with no extension is an atom
$(\mathcal{P}(\omega), \subseteq, \omega)$ every element is an atom
linear order: all elements are atoms
A tree with branches at every node is atomless
$Fn(I,J), J \neq \emptyset$ if I is finite has atoms. If I is infinite and $|J| \geq 2$ then $Fn(I,J)$ is atomless

Lemma. *If $\mathbb{P} \in M, M$ a ctm of ZFC, $\mathbb{P}$ atomless and G is $\mathbb{P}$-generic over M then $G \notin M$.*

Proof. Let $D = \mathbb{P} \setminus G$. Then D is dense as if $r \in \mathbb{P}$ there are $p, q \leq r : p \perp q$. Then at least one of these must be in D.
If $G \in M$ then $D \in M$ by the axiom of separation in M. But as G is $\mathbb{P}$-generic over M we would have $D \cap G \neq \emptyset$. ☐

Definition. τ is a <u>$\mathbb{P}$-name</u> iff τ is a set of ordered pairs s.t. $\forall (\sigma, p) \in \tau$ we have that σ is a $\mathbb{P}$-name and $p \in \mathbb{P}$ (Definition by recursion). $V^{\mathbb{P}}$ denotes the class of all $\mathbb{P}$-names.

$\emptyset$ is a $\mathbb{P}$-name, $\{(\emptyset, p), (\emptyset, 1)\}$ is a $\mathbb{P}$-name $(p, q \in \mathbb{P})$
For any set x look at the associated graph. Label each node in the tree with some element.
If M is a transitive model of ZFC with $\mathbb{P} \in M$ then $M^{\mathbb{P}} := V^{\mathbb{P}} \cap M = \{\tau \in M \mid (\tau \text{ is a } \mathbb{P}\text{-name})_M\}$

Definition. For a $\mathbb{P}$-name τ and $G \subseteq \mathbb{P}$ define by recursion

$$val(\tau, G) = \tau_G := \{val(\sigma, G) \mid \exists p \in G : (\sigma, p) \in \tau\}$$

Then $M[G] := \left\{\tau_G \mid \tau \in M^{\mathbb{P}}\right\}$ for a ctm M with $\mathbb{P} \in M$.

Example. $\emptyset_G = \emptyset$
If G is a filter, $\mathbb{1} \in G$, let $\tau = \{(\sigma, \mathbb{1}), (\theta, \mathbb{1})\}$ then $\tau_G = \{val(\sigma, G), val(\theta, G)\} = \{\sigma_G, \theta_G\}$.
$\tau = \{(\emptyset, p), (\{\emptyset, q\}, r)\}, p, r \in G, q \notin G$ then $\tau_G = \{\emptyset_G, \{(\emptyset, q)\}_G\} = \{\emptyset, \emptyset\} = \{\emptyset\}$
$p, r \notin G, q$ does not matter: $\tau_G = \emptyset$
$p \notin G, r, q \in G$: $\tau_G = \{\{\emptyset\}\}$

We want to show that $M[G]$ is the desired extension of M which contains G and everything else needed to ensure $M[G] \vDash$ ZFC. More precisely we will show:

(1) $M \subseteq M[G]$

(2) $G \in M[G]$

(3) $M[G]$ is transitive and countable

(4) $M[G] \vDash$ ZFC

(5) For any N with the above properties $N \supseteq M[G]$

For (i) we find for each $x \in M$ a name $\tau \in M^{\mathbb{P}}$, which for any filter G on $\mathbb{P}$ satisfies $\tau_G = x$

Definition. For a forcing poset $(\mathbb{P}, \leq, \mathbb{1})$ and any set x define $\check{x} := \{(\check{y}, \mathbb{1}) \mid y \in x\}$ (definition by recursion)
Clearly $\hat{x}$ is a $\mathbb{P}$-name
$\check{2} = \{(\check{0}, \mathbb{1}), (\check{1}, \mathbb{1})\} = \{(\emptyset, \mathbb{1}), (\{(\emptyset, \mathbb{1})\}, \mathbb{1})\}$

Lemma (1). *Let G be a filter on $\mathbb{1}$. Then*

(i) $\forall x \in M : \check{x} \in M^{\mathbb{P}}$ *and* $val(\check{x}, G) = x$

(ii) $M[G] \supseteq M$

Proof. (ii) follows by definition from (i)
For (i) $\check{x} \in M$ by absoluteness and $val(\check{x}, G) = x$ by induction as $val(\check{x}, G) = \{val(\check{y}, G) \mid y \in x\}$
□

Lemma (3). *For any filter G on $\mathbb{P}$, $M[G]$ is transitive and countable.*

Proof. Transitive: $x \in M[G]$ then $x = \tau_G$ for some $\tau \in M^{\mathbb{P}}$ so $\forall y \in x : y = \sigma_G$ for some $\sigma \in M^{\mathbb{P}}$ thus $y \in M^{\mathbb{P}}$
Countable: $|M[G]| \leq |M^{\mathbb{P}}| \leq |M| \leq |M[G]|$ □

Definition. Given $\mathbb{P}$, define $\Gamma := \{(\check{p}, p) \mid p \in \mathbb{P}\}$
$\Gamma_G = \{\check{p} \mid p \in G\} \overset{\text{lem 1}}{=} \{p \mid p \in G\} = G$
Hence we have

Lemma (2). *$\Gamma \in M$ is a $\mathbb{P}$-name and $\Gamma_G = G$ hence $G \in M[G]$.*

Proof. Γ is definable within M and clearly a $\mathbb{P}$-name. □

Definition (Names for unordered and ordered pairs)**.** $up(\sigma, \tau) := \{(\sigma, \mathbb{1}), (\tau, \mathbb{1})\}$
$op(\sigma, \tau) := up(up(\sigma, \sigma), up(\sigma, \tau))$
Observe: $\sigma, \tau \in M^{\mathbb{P}} \implies up(\sigma, \tau), op(\sigma, \tau) \in M^{\mathbb{P}}$

Lemma (4)**.** *Let $M, \mathbb{P}$ be as above and G a filter on $\mathbb{P}$. Then $M[G]$ is a ctm of the axioms of Extenionality, Emptyset, Foundation, Pairing, Infinity and Union.*

Proof. We have seen all transitive non-empty sets satisfy Emptyset, Extensionality and Foundation.
$M[G]$ satisfies pairing by observing as $a, b \in M[G] \implies$ there is $\sigma, \tau \in M^{\mathbb{P}}$ s.t. $\tau_G = a, \sigma_G = b$. Then by above $up(\sigma, \tau) \in M^{\mathbb{P}}$ and $(up(\sigma, \tau))_G = \{\sigma_G, \tau_G\} = \{a, b\}$.
$M[G]$ satisfies the Axiom of Infinity as $\omega \in M[G]$ and absoluteness.
Union: Let $a \in M[G]$. Fix $\tau \in M^{\mathbb{P}}$ s.t. $a = \tau_G$.
set $\pi = \{(\theta, p) \mid \exists(\sigma, q) \in \tau, \exists r \in \mathbb{P} : ((\theta, r) \in \sigma \wedge p \leq r \wedge p \leq q)\}$
Claim. $\bigcup a = b := \pi_G$
$\subseteq$: Let $d \in \bigcup a$ then $\exists q \in G : \exists \sigma : (\sigma, q) \in \tau$ and $d \in \sigma_G$. Thus there is also $r \in G, \theta$ s.t. $(\theta, r) \in \sigma$ and $\theta_G = d \in \pi_G$
$\supseteq$: Let $d \in \pi_G$. Then $\exists p \in G, \exists \theta$ s.t. $(\theta, p) \in \pi$ and $\theta_G = d, p \leq r, p \leq q \implies \theta_G \in \sigma_G$ and $\sigma_G \in \tau_G = a \implies \theta_G \in \bigcup a$ $\square$

Example. $\mathbb{P} = Fn(I, J), I, J \in M, G$ a filter then $\bigcup G$ is a function. We have a name Γ for G so we cann write a name for $\bigcup G$ as above. But there is a much more natural name for $\bigcup G$.

$$\mathring{f} = \left\{(op(\check{(i)}, \check{(j)}), p) \mid p \in \mathbb{P} \wedge (i, j) \in p\right\}$$

We have seen $P, G \in M[G]$. What about $\mathbb{P} \setminus G =: C$?
Candidate name $\mathring{C} = \{(\check{p}, q) \mid p, q \in \mathbb{P}, p \perp q\}$
$\mathring{C}_G = \{p \in \mathbb{P} \mid \exists q \in G : p \perp q\}$. As all elements of a filter aare compatible: $\mathring{C}_G \cap G = \emptyset$
Do we have $G \cup \mathring{C}_G = \mathbb{P}$?
Let $p \in \mathbb{P}$ set $D_p = \{q \in \mathbb{P} \mid p \perp q \vee q \leq p\}$. $D_p \in M$ and D_p is dense ($r \in \mathbb{P}, r \perp p \implies r \in D_p$ otherwise $\exists q \in \mathbb{P} : q \leq r, q \leq p \implies q \in D_p$)
If G is generic we find $q \in G \cap D_p \implies q \in D_p$
If $q \perp p$ then $p \in \mathring{C}_G$. If $q \leq p$ then $p \in G \longrightarrow$ is complement.