

Recall. Every infinite subset of the real numbers is either countable or bijective with the whole of \mathbb{R} .

Equivalently: $2^{\aleph_0} = \aleph_1$ i.e. $\mathcal{P}(\mathbb{N})$ has size the first uncountable cardinal.

CH was first advanced by Cantor 1878

$\text{ZFC} \not\models \neg \text{CH}$ (Gödel 1940)

$\text{ZFC} \not\models \text{CH}$ (Cohen 1963)

Cannot do: $\text{ZFC} \vdash \text{“}\text{ZFC} \not\models \text{CH}\text{”} \longrightarrow$ find a model for $\text{ZFC} + \neg \text{CH} \iff \text{Con}(\text{ZFC} + \neg \text{CH})$

By Gödel’s Second Incompleteness Theorem: If ZFC is consistent then $\text{ZFC} \not\models \text{Con}(\text{ZFC})$.

instead: $\text{Con}(\text{ZFC}) \implies \text{Con}(\text{ZFC} + \neg \text{CH})$ which we will show using ZFC and

$\text{Con}(\text{ZFC}) \implies \text{Con}(\text{ZFC} + \text{CH})$

Definition. The language of set theory (LST) is the first order predicate language \mathcal{L}_\in which consists of the logical connectives and quantifiers ($\neg, \implies, \vee, \wedge, \exists, \forall$), parentheses, variable symbols v_0, v_1, \dots , a logical binary predicates “=” and a single non-logical binary predicate “ \in ”.

$\text{Con}(\text{ZFC}) \implies \text{ZFC}$ is satisfiable, i.e. there is an \mathcal{L}_\in -structure $(X, \underbrace{E}_{\subseteq X \times X})$ s.t. $(X, E) \models$

ZFC. We want a “nicer” model of ZFC.

Method 1. Assume the existence of an inaccessible cardinal κ . Then $(V_\kappa, \in) \models \text{ZFC}$.

Method 2. To show $\text{ZFC} + \neg \text{CH}$ is consistent, it is enough to show every finite subset is consistent. For every $\Sigma \subset \text{ZFC}$ finite find $(X, \in) \models \Sigma$ + enough of ZFC to prove what we need (where X is transitive).

Transform this into a model of $\Sigma + \neg \text{CH}$.

For a set X and $E \subseteq X \times X$, φ a sentence of LST we know from logic how to define the satisfaction relation $(X, E) \models \varphi$. To define this we used:

- Symbol set of the language
- define by recursion terms and formulae, free variables etc. / satisfaction
- set D , relations / functions on D , elements of D
- interpretation function maps from non-logical symbols of \mathcal{L} to $D/\mathcal{P}(D \times D)$
- Variable assignment from symbols to D

Definition. We define the relativization of formulae of LST to a set W by recursion on formulae.

1. $(x \in y)_W := x \in y$ $(x = y)_W := x = y$
2. $(\neg \varphi)_W := \neg \varphi_W$ $(\varphi \wedge \psi)_W := \varphi_W \wedge \psi_W$
3. $(\exists x : \varphi)_W := \exists x : (x \in W \wedge \varphi_W)$

$$\begin{aligned}
(\forall x : \varphi)_W &\iff (\neg \exists x : \neg \varphi)_W \\
&= \neg(\exists x : \neg \varphi)_W \\
&= \neg \exists x : (x \in W \wedge (\neg \varphi)_W) \\
&= \neg \exists x : (x \in W \wedge \neg \varphi_W) \\
&= \forall x : \neg(x \in W \wedge \neg \varphi_W) \\
&= \forall x : \neg x \in W \vee \varphi_W = \forall x : x \in W \longrightarrow \varphi_W
\end{aligned}$$

We claim that $(W, \in \upharpoonright W \times W) \models \varphi$ iff φ_W . We write (W, \in) for $(W, \in \upharpoonright W \times W)$.

Example. Extensionality: $(W, \in) \models \text{Axiom of Extensionality}$:

$$\begin{aligned}
(\text{Ax. Ext.})_W &= (\forall x : \forall y : (\forall z : z \in x \longleftrightarrow z \in y) \longrightarrow x = y)_W \\
&= \forall x \in W : \forall y \in W : (\forall z \in W : z \in x \longleftrightarrow z \in y) \longrightarrow x = y
\end{aligned}$$

If $W = \{\emptyset, \{\{\emptyset\}\}\}$ then $(W, \in) \not\models \text{Ax. Ext.}$

Proposition. *If W is transitive then $(W, \in) \models \text{Ax. Ext.}$*

Proof. Let W be transitive, $x, y \in W$ with $x \neq y$. Then there is some set z with $z \in x \leftrightarrow z \in y$. Wlog assume $z \in x, z \notin y$. Then $z \in W$ by transitivity of W . \square

Example. Axiom Empty set:

$$\begin{aligned}
(W, \in) \models \text{Ax. Empty set} &\iff (\exists x : x = \emptyset)_W \\
&\iff (\exists x : \forall y : y \notin x)_W \\
&\iff \exists x \in W : \forall y \in W : y \notin x \iff \exists x \in W : x \cap W = \emptyset
\end{aligned}$$

Example. $W = \{\{\emptyset\}\}$ Then $(W, \in) \models \text{Ax. Empty set}$ even though $\emptyset \notin W$.

Proposition. *If $W \neq \emptyset$ then $(W, \in) \models \text{Ax. Empty Set}$.*

Proof. By the Axiom of Foundation there is $y \in W$ with $y \cap W = \emptyset$. \square

Definition. We say a formula φ is absolute for W if $\text{FV}(\varphi) \subseteq \{x_1, \dots, x_n\}$ and

$$\forall x_1 \in W, \dots, \forall x_n \in W : (\varphi_W \leftrightarrow \varphi)$$

Lemma. *The following formule are absolute for any transitive set W*

1. $z = \emptyset$
2. $x \subseteq y$
3. $z = x \cap y$

4. $z = x \cup y$

5. $z = \bigcup x$

6. $z = \{x, y\}$

7. $z = (x, y)$

Proof. Let W be transitive

1. Let $z \in W$

$$z = \emptyset \iff \forall x : x \notin z \implies \forall x \in W : x \notin z \iff (z = \emptyset)_W$$

Also $x \in z \implies x \in W$ as W is transitive so we must also have “ \Leftarrow ”

2.

3. Let $x, y, z \in W$

$$z = x \cap y \iff \forall u : u \in z \leftrightarrow (u \in x \wedge u \in y) \implies \forall u \in W : u \in z \leftrightarrow (u \in x \wedge u \in y) \iff (z = x \cap y)_W$$

“ \Leftarrow ” follows as $u \in z \implies u \in W, u \in x \wedge u \in y \implies u \in W$

4.

5.

6. Let $x, y, z \in W$

$$\begin{aligned} z = \{x, y\} &\iff \forall u (u \in z \leftrightarrow (u = x \vee u = y)) \\ &\implies \forall u \in W (u \in z \leftrightarrow (u = x \vee u = y)) \iff (z = \{x, y\})_W \end{aligned}$$

“ \Leftarrow ” follows as $u \in z \implies u \in W, u = x \vee u = y \implies u \in W$

7. Let $x, y, z \in W$

$$z = (x, y) \iff \exists u, v : z = \{x, y\} \wedge u = \{x\} \wedge v = \{x, y\}$$

$$\xrightarrow{W_{\text{trans.}}} \exists u \in W : \exists v \in W : z = \{x, y\} \wedge u = \{x\} \wedge v = \{x, y\} \xrightarrow{6.} (z = (x, y))_W$$

“ \Leftarrow ” follows simply.

□

Example. $z = \mathcal{P}(x)$ and $z = x \times y$ are not absolute for transitive sets. Let $W = \omega, 2 \in \omega$.
 $(\omega, \in) \models 3 = \mathcal{P}(2)$

$x \times y = \{(a, b) \mid a \in x, b \in y\}$. W transitive then $x, y \in W \implies$ such $a, b \in W, z = (a, b)$ is absolute for W but not all such ordered pairs may be in W .

If $z = W \cap (x \times y), z \in W$ then $(W, \in) \models z = x \times y$

Lemma. *If W is transitive and non-empty then it satisfies the axioms of extensionality, empty set and foundation.*

Proof. We just have to show

$$\begin{aligned} (\text{Ax. Foundation})_W &\iff (\forall x : x \neq \emptyset \longrightarrow \exists y : (y \in x \wedge x \cap y = \emptyset))_W \\ &\iff \forall x \in W : (x \neq \emptyset \longrightarrow \exists y \in W : (y \in x \wedge x \cap y = \emptyset)) \end{aligned}$$

using the absoluteness of Empty set and $x \cap y$. Let $x \in W, x \neq \emptyset$. Then by the Axiom of Foundation $\exists y : y \in x \wedge x \cap y = \emptyset$ so $y \in W$ by transitivity so $\exists y \in W (y \in x \wedge x \cap y = \emptyset)$ \square

Lemma. *For a limit ordinal α , (V_α, \in) satisfies the axioms of pairing, union, powerset and the axiom schema of separation. If additionally $\alpha > \omega$, V_α satisfies the axiom of infinity.*

Proof. Pairing: $(\forall x : \forall y : \exists z : z = \{x, y\})_{V_\alpha}$ iff $\forall x \in V_\alpha : \forall y \in V_\alpha : \exists z \in V_\alpha : z = \{x, y\}$.

Let $x, y \in V_\alpha, \text{rk}(\{x, y\}) = \max(\text{rk}(x), \text{rk}(y)) + 1 < \alpha$ so $\{x, y\} \in V_\alpha$

Union: $(\forall x : \exists z : z = \bigcup x)_{V_\alpha}$ iff $\forall x \in V_\alpha : \exists z \in V_\alpha : z = \bigcup x$ so we just need $x \in V_\alpha \implies \bigcup x \in V_\alpha$ which is true as $\text{rk}(\bigcup x) \leq \text{rk}(x) < \alpha$.

Powerset: $(\forall x : \exists y : \forall z : (z \in y \iff z \subseteq x))_{V_\alpha}$ iff $\forall x \in V_\alpha : \exists y \in V_\alpha : \forall z \in V_\alpha : (z \in y \iff z \subseteq x)$. $\text{rk}(\mathcal{P}(x)) = \text{rk}(x) + 1$ so $x \in V_\alpha \implies \mathcal{P}(x) \in V_\alpha$. Let $x \in V_\alpha, y = \mathcal{P}(x) \in V_\alpha$ then $\forall z : z \in y \iff z \subseteq x$ so in particular $\forall z \in V_\alpha : (z \in y \iff z \subseteq x)$ i.e. $(y = \mathcal{P}(x))_{V_\alpha}$.

Infinity: Let $\alpha > \omega$. $(\exists x : (\emptyset \in x \wedge \forall y : (y \in x \longrightarrow y \cup \{y\} \in x)))_{V_\alpha}$ iff $\exists x \in V_\alpha : (\emptyset \in x \wedge \forall y \in V_\alpha : (y \in x \longrightarrow y \cup \{y\} \in x))$. Let $x = \omega$ then $x \in V_\alpha, \emptyset \in V_\alpha, \forall y : (y \in x \longrightarrow y \cup \{y\} \in x)$ so in particular $\forall y \in V_\alpha : (y \in x \longrightarrow y \cup \{y\} \in x)$

Separation: Let φ be a formula of LST with $x, y \notin \text{FV}(\varphi)$. Then

$$\begin{aligned} (\forall x : \exists y : y = x \cap \{z : \varphi\})_{V_\alpha} &\iff (\forall x : \exists y : \forall z : z \in y \iff (\varphi \wedge z \in x))_{V_\alpha} \\ &\iff \forall x \in V_\alpha : \exists y \in V_\alpha : \forall z \in V_\alpha : z \in y \iff (\underbrace{\varphi_{V_\alpha}}_{\psi} \wedge z \in x) \end{aligned}$$

Let $x \in V_\alpha$. By the Axiom of Separation for $\psi \exists y : y = x \cap \{z : \varphi_{V_\alpha}\}$. Then $y \subseteq x$ so $\text{rk}(y) \leq \text{rk}(x) \implies y \in V_\alpha$ i.e. $\exists y \in V_\alpha : \forall z \in V_\alpha : (z \in y \iff (\varphi_{V_\alpha} \wedge z \in x))$ \square

Proof. Axiom of Choice for a limit ordinal α .

TODO

\square

Theorem. *For any inaccessible cardinal κ $(V_\kappa, \in) \models \text{ZFC}$.*

Proof. We've shown that for any limit ordinal $\alpha > \omega$ $(V_\alpha, \in) \models$ All Axioms of ZFC except Replacement. So we need to show $(V_\kappa, \in) \models \text{Ax. of Replacement}$.

Proposition (1). *If κ is inaccessible $x \in V_\kappa \implies |x| < \kappa$.*

Proposition (2). *If κ is regular $x \subseteq V_\kappa \wedge |x| < \kappa \implies x \in V_\kappa$.*

Recall that the Axiom of Replacement is actually an axiom schema, so fix $\varphi \in \text{LST}$ with $x, y \in \text{FV}(\varphi)$. We need to show ... TODO \square

Definition. Let $W \subseteq Z$ be sets or class terms and $\varphi \in \text{Fml}_{\mathcal{L}_\in}$ with $\text{FV}(\varphi) = \{x_1, \dots, x_n\}$. We say φ is upward (downward) absolute between W, Z if $\forall x_1, \dots, x_n \in W : (\varphi_W \longleftrightarrow \varphi_Z)$

Compare in model theory if \mathcal{N}, \mathcal{M} are \mathcal{L} -structures \mathcal{N} is a substructure of $\mathcal{M}, \mathcal{N} \subseteq \mathcal{M}$ iff $N \subseteq M$ and the interpretation function for \mathcal{N} is the restriction of the interpretation function of \mathcal{M} .

For $\varphi \in \mathcal{L}, \text{FV}(\varphi) = \{x_1, \dots, x_n\}$ $\mathcal{N} \preceq_\varphi \mathcal{M}$ if $\forall a_1, \dots, a_n \in N : \mathcal{N} \models \varphi(a_1, \dots, a_n) \iff \mathcal{M} \models \varphi(a_1, \dots, a_n)$

Definition. $\mathcal{N} \preceq \mathcal{M}$ is an elementary substructure of \mathcal{M} iff for all $\varphi \in \mathcal{L} : \mathcal{N} \preceq_\varphi \mathcal{M}$
In other context this says $(W, \in) \preceq (Z, \in)$ iff all $\varphi \in \text{Lst}$ are absolute between W, Z .

Definition. A list of formulae $\varphi_0, \varphi_1, \dots$ is subformula closed if every subformula of a formula on the list is on the list.

Lemma (1). Let $\vec{\varphi}$ be a subformula closed list, $W \subseteq Z$. The following are equivalent:

- (i) $\vec{\varphi}$ are absolute for W, Z
- (ii) Whenever φ_i is of the form $\exists x : \varphi_j(x, \vec{y})$ with $\text{FV}(\varphi_j) \subseteq \{\vec{y}\}$ then $\forall \vec{y} \in W : (\exists x \in Z : \varphi_j(x, \vec{y})_Z \longrightarrow \exists x \in W : \varphi_j(x, \vec{y})_W)$ i.e. φ_i is downward absolute between W, Z .

Proof. (i) \implies (ii) is clear: Fix $\vec{y} \in W$ and assume $\varphi_i(\vec{y})_Z$ i.e. $(\exists x : \varphi_j(x, \vec{y}))_Z \iff \exists x \in Z : \varphi_j(x, \vec{y})$. Then by absoluteness of $\varphi_i, (\varphi_i(\vec{y}))_W$ so $\exists x \in W : (\varphi_j(x, \vec{y}))_W$.

(ii) \implies (i) by induction on the length of φ_i : so we assume as the lefthand side that absoluteness holds for subformulae.

- φ_i atomic - by definition of absolute
- $\varphi_i = \varphi_j \wedge \varphi_k$ by IH φ_j, φ_k are absolute so by relativization φ_i is absolute. $\varphi_i = \neg \varphi_j$ similarly.
- $\varphi_i = \exists x : \varphi_j(x, \vec{y})$. Fix $\vec{y} \in W$.

$$\begin{aligned} \varphi_i(\vec{y})_W &\stackrel{\text{def}}{\iff} \exists x \in W : \varphi_j(x, \vec{y})_W \stackrel{\text{IH}}{\iff} \exists x \in W : \varphi_j(x, \vec{y})_Z \iff \exists x \in Z : \\ &\varphi_j(x, \vec{y})_Z \stackrel{\text{def}}{\iff} \varphi_i(\vec{y})_Z \end{aligned}$$

\square

Definition. A formula of \mathcal{L}_\in is Δ_0 iff it only uses bounded quantifiers i.e.

- $x \in y, x = y$ are Δ_0
- If φ, ψ are Δ_0 so are $\neg \varphi, \varphi \wedge \psi$
- If φ is Δ_0 so is $\exists x \in y : \varphi$

Lemma (2). *Let W be a transitive set. Then any Δ_0 formula is absolute for W .*

Proof. By induction on the length of formulae using lemma 1. We just need to show that if φ is of the form $\exists x : (x \in a \wedge \psi(x, \vec{y}, a))$ then $\forall a \in W : \forall \vec{y} \in W : (\exists x : (x \in a \wedge \psi(x, \vec{y}, a)) \longrightarrow \exists x \in W : (x \in a \wedge \psi(x, \vec{y}, a)))_W$

So let $a, \vec{y} \in W$ and suppose $\exists x \in a : \psi(x, \vec{y}, a)$. ψ is Δ_0 as φ is Δ_0 and of length less than φ so by IH ψ is absolute for W i.e. $\psi(x, \vec{y}, a) \longrightarrow \psi(x, \vec{y}, a)_W$. Further as $a \in W, a \subseteq W$ so $x \in a \longleftrightarrow x \in W \cap a$. Thus $\exists x \in a : \psi(x, \vec{y}, a) \implies \exists x \in W : (x \in a \wedge \psi(x, \vec{y}, a))$ \square

Theorem (Downward Löwenheim-Skolem Theorem). *Let \mathcal{M} be an infinite \mathcal{L} -structure. Fix a cardinal κ with $\max(|\mathcal{L}|, \aleph_0) \leq \kappa \leq |M|$ and let $S \subseteq M$ with $|S| \leq \kappa$. Then there is an $\mathcal{N} \preceq \mathcal{M}$ s.t. $S \subseteq N$ and $|N| = \kappa$.*

Proof. We prove this for $\mathcal{L} = \mathcal{L}_\in$ and \in -models.

Fix M an infinite set and $\kappa \leq |M|$ and infinite cardinal and $S \subseteq M$ with $|S| \leq \kappa$. If $|S| < \kappa$ let $S' \subseteq M$ with $S \subseteq S'$ and $|S'| = \kappa$.

We use Lemma 1 to build up $S' \subseteq N \subseteq M$ with $(N, \in) \preceq (M, \in)$. Clearly, a list of all formulae in \mathcal{L}_\in is subformula closed. Let R be a well-ordering on M . For any existential formula $\varphi = \exists x : \psi$ let n_φ be $|\text{FV}(\varphi)|$. We define a Skolem-function $f_\varphi : M^{n_\varphi} \rightarrow M$ as follows.

For $\vec{y} \in M^{n_\varphi}$, if $(M, \in) \models \exists x : \psi(x, \vec{y})$ then let $f_\varphi(\vec{y})$ be the R -least in M s.t. $(M, \in) \models \psi(f_\varphi(\vec{y}), \vec{y})$. If $(M, \in) \not\models \exists x : \psi(x, \vec{y})$ set $f_\varphi(\vec{y}) = 0$. Now set $N_0 = S'$ and define by recursion $N_{i+1} = N_i \cup \{f_\varphi'' N_i^{n_\varphi} \mid \varphi \in \mathcal{L}_\in \text{ s.t. } \varphi = \exists x : \psi \text{ for some } \psi\}$. Set $N = \bigcup_{i \in \omega} N_i$
Claim 1: $|N| = \kappa$. Clearly $|N| \geq \kappa$ as $|N_0| = \kappa$.

$$|N_1| \leq |N_0| \oplus \sup_{n \in \omega} |N_0^n| \otimes \aleph_0 \leq \kappa \otimes \underbrace{(\kappa \otimes \dots \otimes \kappa)}_n \otimes \aleph_0$$

Similarly $|N_i| = |N_0| = \kappa$. Thus $|N| \leq |N_0| \oplus |N_1| \oplus \dots = \kappa \otimes \aleph_0 = \kappa$

Claim 2: $(N, \in) \preceq (M, \in)$: By Lemma 1 we just need to show that for any $\varphi = \exists x : \psi$ $\forall y_1, \dots, y_{n_\varphi} \in N : (\exists x \in M : \psi(x, \vec{y}))_M \longrightarrow \exists x \in N : \psi(x, \vec{y}_N)$. By induction on the length of formulae assume that ψ is absolute for M, N so let $\vec{y} \in N$ and assume $\exists x \in M : \psi(x, \vec{y})_M$. Then there must be some i s.t. $\vec{y} \in N_i$. Thus $f_\varphi(\vec{y}) \in N_{i+1} \subseteq N$ so $\psi(f_\varphi(\vec{y}), \vec{y})_M$ and by IH $\psi(f_\varphi(\vec{y}), \vec{y})_N$. Thus $\exists x \in N : \psi(x, \vec{y})_N$. Such an N is called a Skolem Hull of M . N is not transitive. \square

Definition. Let \mathcal{M} be an \mathcal{L} -structure, $X \subseteq M^n$ is definable in \mathcal{M} iff there is an \mathcal{L} -formula φ with n free variables s.t. for $a_1, \dots, a_n \in M$ $\mathcal{M} \models \varphi(a_1, \dots, a_n) \iff (a_1, \dots, a_n) \in X$

$\{\emptyset\}$ is definable in V_λ ; any hereditary finite set is definable in V_λ

Theorem (Montague-Levy Reflection Theorem). *Let $\varphi_1, \dots, \varphi_n$ be any finite list of formulae. Then $\text{ZFC} \vdash \forall \alpha : \exists \beta > \alpha (\vec{\varphi} \text{ are absolute for } V_\beta)$*

This is a theorem schema

If $\vec{\varphi}$ are all axioms of ZFC then $\text{ZFC} \vdash \forall \alpha : \exists \beta > \alpha : (\wedge \vec{\varphi})_{V_\beta}$

Proof. By lengthening the list if necessary assume it is subformula closed. For each $i \leq n$ s.t. $\varphi_i = \exists x : \varphi_j$ define

$$F_i : V \rightarrow \text{On}$$

$$F_i(\vec{y}) = \begin{cases} 0 & \text{if } \neg \exists x : \varphi_j(x, \vec{y}) \\ \eta & \text{where } \eta \text{ is least s.t. } \exists x \in V_\eta : \varphi_i(x, \vec{y}) \end{cases}$$

Now define $G_i : \text{On} \rightarrow \text{On}$ by $G_i(\gamma) = \sup \{F_i(\vec{y}) \mid \vec{y} \in V_\gamma\}$. G_i is well-defined by the Axiom of replacement $F_i''V_\gamma$ is a set of ordinals so its supremum is an ordinal.

Claim: $\forall \alpha : \exists \beta > \alpha : \lim(\beta) \wedge \forall \gamma < \beta : \forall i \leq n : G_i(\gamma) \leq \beta$.

Define by recursion $\lambda_0 = \alpha$, $\lambda_{k+1} = \max \{\lambda_k + 1, G_1(\lambda_k), \dots, G_n(\lambda_k)\}$, $\beta := \sup \{\lambda_k \mid k \in \omega\}$. Then $\lim(\beta)$ and if $\gamma < \beta$ then $\gamma < \lambda_k$ for some $k \in \omega$. Hence $G_i(\gamma) \leq G_i(\lambda_k) \leq \lambda_{k+1} < \beta$. Then apply Lemma 1 to V_β . \square

Theorem (Mostowski-Shepherdson Collapsing Lemma). *Let W be a set and $R \subseteq W \times W$ be wellfounded s.t. $(W, R) \models \text{Ax. of Ext. i.e. } u, v \in W, u \neq v \longrightarrow \exists z \in W : zRu \longleftrightarrow \neg zRv$*

Then there is a unique transitive set M and unique isomorphism $\pi : (W, R) \cong (M, \in)$ Additionally, if $Z \subseteq W$ with $R \upharpoonright W \times Z = \in \upharpoonright W \times Z$ and $v \in Z, u \in W$ with $uRv \longrightarrow u \in Z$ and Z is transitive then $\pi \upharpoonright Z = \text{id} \upharpoonright Z$

Proof. Claim 1: If π exists, it is unique.

Suppose we have π, M as above. Let $u, v \in W$. If uRv then $\pi(u) \in \pi(v)$ as π is an isomorphism. Then $\{\pi(u) \mid u \in W, uRv\} \subseteq \pi(v)$.

Further if $z \in \pi(v)$ then $z \in M$ by transitivity hence $z = \pi(x)$ for some $x \in W$ with xRv thus $\pi(v) \subseteq \{\pi(u) \mid u \in W, uRv\}$. Taken together $\pi(v) = \{\pi(u) \mid u \in W, uRv\}$, so if the isomorphism exists it must take this form.

Claim 2: Such a π exists.

Define by recursion on R

$$\pi(v) = \{\pi(u) \mid u \in W, uRv\} \tag{1}$$

Whis is this well defined? A R is wellfounded there is $x \in W$ s.t. $\forall y \in W : \neg yRx$. For such an x we have $\pi(x) = \emptyset$. Further for $W \setminus \{x\}$ there is $x' \in W \setminus \{x\}$ with $\forall y \in W \setminus \{x\} : \neg yRx'$ so $\pi(x') = \{\pi(u) \mid u \in W, uRx'\} = \{\pi(u) \mid u = x, uRx'\}$.

π is clearly surjective.

Claim 3: π is injective:

Assume not and take $t \in$ -minimal s.t. there are $u \neq v \in W$ with $t = \pi(u) = \pi(v)$. As $u \neq v$ there is some $x \in W$ with $xRu \longleftrightarrow \neg xRv$. Wlog assume $xRu \wedge \neg xRv$. Then $\pi(x) \in \pi(u) = t$. So we must have that for some y with yRv $\pi(y) = \pi(x)$. But this contradicts the minimality of t .

Claim 4: π is orderpreserving

“ \implies ” By (??) if uRv then $\pi(u) \in \pi(v)$.

“ \impliedby ” If $\pi(u) \in \pi(v)$ then for some $z \in W$ with zRv $\pi(z) = \pi(u)$. As π is injective $z = u$.

Claim 5 (Also): Suppose for some $u \in X$ $\pi(u) \neq u$ and take $v \in$ -minimal such. Then

$$\begin{aligned}\pi(v) &= \{\pi(u) \mid u \in W, uRv\} \\ &= \{\pi(u) \mid u \in W, u \in v\} \\ &= \{u \mid u \in v\} = v\end{aligned}$$

□

Example. $(N, \in) \preceq (V_\kappa, \in)$ with κ inaccessible, N countable. M is the collapse of (N, \in) . $\pi(\omega_1) = \omega_1^{ck}$ the least non-definable countable ordinal.

$P(\omega) \in N : \pi(\mathcal{P}(\omega)) = \{\pi(y) \mid y \in \mathcal{P}(\omega) \cap N\} = \mathcal{P}(\omega) \cap N$

Example. If X, R is a wellorder then its transitive collapse is $\text{ot}((X, R))$

Example. We can code V_ω as a subset of ω i.e. there is $E \subseteq \omega^2$ s.t. $(\omega, E) \cong (V_\omega, \in)$
 $M \models x = \mathcal{P}(\omega)$, $x = \mathcal{P}(\omega)^M$, $M \models x$ is uncountable $\implies \pi(y) = y$.

Let $o(m)$ denote $\text{Ord} \cap M$ then $o(M) \in \text{Ord}$ is countable.

Here: (M, \in) countable transitive model of ZFC (ctm). Assume $M \models \text{CH}$

Definition (Poset). A forcing Poset is a triple $(\mathbb{P}, \leq, \mathbb{1})$ where \mathbb{P} is a set, \leq is a pre-order on \mathbb{P} i.e. transitive and reflexive and $\mathbb{1} \in \mathbb{P}$ is a largest element: $\forall p \in \mathbb{P} : p \leq \mathbb{1}$.

Elements of \mathbb{P} are called forcing conditions.

$p \leq q$: “ p extends q ”

Abuse notation: and use \mathbb{P} to refer to $(\mathbb{P}, \leq, \mathbb{1})$.

Example. Normally we use partial orders \mathbb{P} . $\mathbb{1}$ ensures the ordering is connected. If it is a partial ordering then $\mathbb{1}$ is unique.

Definition. $p, q \in \mathbb{P}$ are compatible ($p \not\perp q$) if $\exists r \in \mathbb{P}$ s.t. $r \leq p \wedge r \leq q$ i.e. they have a common extension.

$p, q \in \mathbb{P}$ are incompatible ($p \perp q$) iff they are not compatible.

Example. In a tree any two elements not along a branch are incompatible. In trees $(p \not\leq q \wedge q \not\leq p) \implies p \perp q$. This is not true in general.

Example. $(\mathcal{P}(\omega), \supseteq, \emptyset)$

Example. (Infinite subsetsets of $\omega, \subseteq^*, \omega$) with $p \subseteq^* q : \iff p \setminus q$ is finite.

This is not a partial order e.g. $\omega \subseteq^* \{42, 43, \dots\}$ and $\omega \supseteq^* \{42, 43, \dots\}$.

$p \perp q$ iff $p \cap q$ is finite

There is no least element.

Definition. $Fn(I, J)$ for sets I, J is the set of all finite partial functions from I to J i.e. $\{p \subseteq I \times J \mid |p| < \omega, \text{func}(p)\}$

The associated forcing poset has the order \supseteq and $\mathbb{1} = \emptyset$. $p \leq q \iff p \supseteq q$ i.e. p extends q as a function.

$p \not\leq q$ iff they have a common extension iff they are the same on $\text{dom}(p) \cap \text{dom}(q)$ iff $p \cup q \in Fn(I, J)$ and if so $p \cup q \leq p, p \cup q \leq q$.

$p \in Fn(I, J) \longrightarrow$ a finite approximation of $f : I \rightarrow J$, p is a condition such an f must satisfy.

Definition. Let \mathbb{P} be a forcing poset. Then $D \subseteq \mathbb{P}$ is dense in \mathbb{P} if $\forall p \in \mathbb{P} : \exists q \in D : q \leq p$.

Example. Let I be infinite, $J \neq \emptyset$ then for $i \in I$ then $D_i := \{q \in Fn(I, J) \mid i \in \text{dom}(q)\}$ is dense.

Let $p \in Fn(I, J), i \in I$. If $i \in \text{dom}(p)$ then $p \in D_i$ otherwise set $q = p \cup \{(i, j)\}$ where $j \in J$ is arbitrary. Then $q \in Fn(I, J)$ and $q \leq p$ and $q \in D_i$.

Also for $j \in J$ the set $\{q \in Fn(I, J) \mid j \in \text{ran}(q)\}$ is dense.

not dense: $L = \{q \in Fn(I, J) \mid (i, j) \in q\}$ if $|J| \geq 2$. $j' \neq j, p = \{(i, j')\} \in Fn(I, J)$ and $\neg \exists q : q \leq p$ and $q \in L$.

Definition. Let \mathbb{P} be a forcing poset. Then $G \subseteq \mathbb{P}$ is a filter on \mathbb{P} iff

1. $1 \in G$
2. $\forall p, q \in G : \exists r \in G : r \leq p \wedge r \leq q$
3. $\forall p \in G : \forall q \in \mathbb{P} : p \leq q \implies q \in G$

Note: If $G \neq \emptyset$ then $3 \implies 1$.

Example. $\{1\}$ is a filter if \mathbb{P} is a partial order

Example. $A \neq \emptyset, \mathbb{P} = \mathcal{P}(A) \setminus \{\emptyset\}, \leq = \subseteq. 1 = A$.

$F \subseteq A$ is a filter:

1. $\iff A \in F$
2. $\iff F$ closed under intersections
3. $\iff F$ closed under superset

Example. For a tree, a filter is a line from the root that stops somewhere

Example. $\mathbb{P} = Fn(I, J), I$ infinite, $J \neq \emptyset$. If G is a filter on \mathbb{P} then any $p, q \in G$ agree on $\text{dom}(p) \cap \text{dom}(q)$ so setting $f_G := \bigcup G$ is a function with $\text{dom}(f_G) \subseteq I, \text{ran}(f_G) \subseteq J$. $D_i = \{p \in Fn(I, J) \mid i \in \text{dom}(p)\}$ is dense in \mathbb{P} so if $\forall i \in I : G \cap D_i \neq \emptyset$ then $f_G : I \rightarrow J$.

Notation. M is ctm (of ZFC), $\mathbb{P} \in M$ we write to mean $(\mathbb{P}, \leq, 1) \in M$.

Example. If $I, J \in M, M$ transitive and $M \models \text{ZFC}$ then $Fn(I, J) \in M$ as well as the ordering on this poset.

Definition. For a forcing poset \mathbb{P}, G is \mathbb{P} -generic over M iff G is a filter on \mathbb{P} and $G \cap D \neq \emptyset$ for all dense $D \subseteq \mathbb{P}$ s.t. $D \in M$.

Example. $Fn(\omega, 2), M$ a ctm for ZFC. Suppose G is \mathbb{P} -generic for M : $\omega \in M, 2 \in M, Fn(\omega, 2) \in M$. $D_i \in M$ for $i \in \omega$ thus $f_G : \omega \rightarrow 2$.

For each $h \in M$ with $h : \omega \rightarrow 2$ let $E_h = \{q \in \mathbb{P} \mid \exists n \in \omega \cap \text{dom}(q) : q(n) \neq h(n)\}$. Then E_h is dense: Let $p \in \mathbb{P}$ with $h \upharpoonright \text{dom}(p) = p$. $\text{dom}(p)$ is finite, so we can pick $n \notin \text{dom}(p)$ and set $q = p \cup \{(n, \neg h(n))\}$. Then $q \leq p$ and $q \in E_h$. Also $E_h \in M$ so $G \cap E_h \neq \emptyset$. Thus $f_G \neq h$. Thus $f_G \notin M$.

Lemma (Generic Filter existence). *Let \mathbb{P} be any forcing poset, let $\{D_i \mid i \in \omega\}$ be a countable family of dense subsets of \mathbb{P} and $p \in \mathbb{P}$. Then there is a filter G on \mathbb{P} s.t. $p \in G$ and $G \cap D_i \neq \emptyset$ for $i \in \omega$.*

Thus for any ctm M with $\mathbb{P} \in M$ and any $p \in \mathbb{P}$ there is a \mathbb{P} -generic filter G over M with $p \in G$.

Proof. By recursion and AC for each $n \in \omega$ choose $r_{n+1} \in \mathbb{P}$ s.t. $r_0 = p$ and $r_{n+1} \leq r_n$ and $r_{n+1} \in D_n$. At each step such r_{n+1} must exist as D_n is dense.

Set $G = \{q \in \mathbb{P} \mid \exists n \in \omega r_n \leq q\}$ then G is a filter: 1,3 are easy. For 2 let $p, q \in G$ then $\exists n, m \in \omega : p \geq r_n, q \geq r_m$ so $p \geq r_{\max\{m, n\}} \in G, q \geq r_{\max\{m, n\}} \in G$. Also $p \in G$, each $r_n \in G$ so $r_{n+1} \in G \cap D_n \neq \emptyset$. \square

Definition. $r \in \mathbb{P}$ is an atom of \mathbb{P} there are no $p, q \in r$ s.t. $p \perp q$. \mathbb{P} is atomless iff there are no atoms.

Example. $r \in \mathbb{P}$ with no extension is an atom

$(\mathcal{P}(\omega), \subseteq, \omega)$ every element is an atom

linear order: all elements are atoms

A tree with branches at every node is atomless

$Fn(I, J), J \neq \emptyset$ if I is finite has atoms. If I is infinite and $|J| \geq 2$ then $Fn(I, J)$ is atomless

Lemma. *If $\mathbb{P} \in M, M$ a ctm of ZFC, \mathbb{P} atomless and G is \mathbb{P} -generic over M then $G \notin M$.*

Proof. Let $D = \mathbb{P} \setminus G$. Then D is dense as if $r \in \mathbb{P}$ there are $p, q \leq r : p \perp q$. Then at least one of these must be in D .

If $G \in M$ then $D \in M$ by the axiom of separation in M . But as G is \mathbb{P} -generic over M we would have $D \cap G \neq \emptyset$. \square

Definition. τ is a \mathbb{P} -name iff τ is a set of ordered pairs s.t. $\forall (\sigma, p) \in \tau$ we have that σ is a \mathbb{P} -name and $p \in \mathbb{P}$ (Definition by recursion). $V^{\mathbb{P}}$ denotes the class of all \mathbb{P} -names.

\emptyset is a \mathbb{P} -name, $\{(\emptyset, p), (\emptyset, 1)\}$ is a \mathbb{P} -name ($p, q \in \mathbb{P}$)

For any set x look at the associated graph. Label each node in the tree with some element.

If M is a transitive model of ZFC with $\mathbb{P} \in M$ then $M^{\mathbb{P}} := V^{\mathbb{P}} \cap M = \{\tau \in M \mid (\tau \text{ is a } \mathbb{P}\text{-name})_M\}$

Definition. For a \mathbb{P} -name τ and $G \subseteq \mathbb{P}$ define by recursion

$$val(\tau, G) = \tau_G := \{val(\sigma, G) \mid \exists p \in G : (\sigma, p) \in \tau\}$$

Then $M[G] := \{\tau_G \mid \tau \in M^{\mathbb{P}}\}$ for a ctm M with $\mathbb{P} \in M$.

Example. $\emptyset_G = \emptyset$

If G is a filter, $1 \in G$, let $\tau = \{(\sigma, 1), (\theta, 1)\}$ then $\tau_G = \{val(\sigma, G), val(\theta, G)\} = \{\sigma_G, \theta_G\}$.

$\tau = \{(\emptyset, p), (\{\emptyset, q\}, r)\}$, $p, r \in G, q \notin G$ then $\tau_G = \{\emptyset_G, \{\{\emptyset, q\}_G\} = \{\emptyset, \emptyset\} = \{\emptyset\}$

$p, r \notin G, q$ does not matter: $\tau_G = \emptyset$

$p \notin G, r, q \in G$: $\tau_G = \{\{\emptyset\}\}$

We want to show that $M[G]$ is the desired extension of M which contains G and everything else needed to ensure $M[G] \models \text{ZFC}$. More precisely we will show:

- (1) $M \subseteq M[G]$
- (2) $G \in M[G]$
- (3) $M[G]$ is transitive and countable
- (4) $M[G] \models \text{ZFC}$
- (5) For any N with the above properties $N \supseteq M[G]$

For (i) we find for each $x \in M$ a name $\tau \in M^{\mathbb{P}}$, which for any filter G on \mathbb{P} satisfies $\tau_G = x$

Definition. For a forcing poset $(\mathbb{P}, \leq, 1)$ and any set x define $\check{x} := \{(\check{y}, 1) \mid y \in x\}$ (definition by recursion)

Clearly \hat{x} is a \mathbb{P} -name

$$\check{2} = \{(\check{0}, 1), (\check{1}, 1)\} = \{(\emptyset, 1), (\{\{\emptyset, 1\}\}, 1)\}$$

Lemma (1). *Let G be a filter on 1 . Then*

- (i) $\forall x \in M : \check{x} \in M^{\mathbb{P}}$ and $val(\check{x}, G) = x$
- (ii) $M[G] \supseteq M$

Proof. (ii) follows by definition from (i)

For (i) $\check{x} \in M$ by absoluteness and $val(\check{x}, G) = x$ by induction as $val(\check{x}, G) = \{val(\check{y}, G) \mid y \in x\}$ \square

Lemma (3). *For any filter G on \mathbb{P} , $M[G]$ is transitive and countable.*

Proof. Transitive: $x \in M[G]$ then $x = \tau_G$ for some $\tau \in M^{\mathbb{P}}$ so $\forall y \in x : y = \sigma_G$ for some $\sigma \in M^{\mathbb{P}}$ thus $y \in M^{\mathbb{P}}$

Countable: $|M[G]| \leq |M^{\mathbb{P}}| \leq |M| \leq |M[G]|$ \square

Definition. Given \mathbb{P} , define $\Gamma := \{(\check{p}, p) \mid p \in \mathbb{P}\}$

$$\Gamma_G = \{\check{p} \mid p \in G\} \stackrel{\text{lem 1}}{=} \{p \mid p \in G\} = G$$

Hence we have

Lemma (2). $\Gamma \in M$ is a \mathbb{P} -name and $\Gamma_G = G$ hence $G \in M[G]$.

Proof. Γ is definable within M and clearly a \mathbb{P} -name. \square

Definition (Names for unordered and ordered pairs). $up(\sigma, \tau) := \{(\sigma, \mathbb{1}), (\tau, \mathbb{1})\}$
 $op(\sigma, \tau) := up(up(\sigma, \sigma), up(\sigma, \tau))$
Observe: $\sigma, \tau \in M^{\mathbb{P}} \implies up(\sigma, \tau), op(\sigma, \tau) \in M^{\mathbb{P}}$

Lemma (4). *Let M, \mathbb{P} be as above and G a filter on \mathbb{P} . Then $M[G]$ is a ctm of the axioms of Extensionality, Emptyset, Foundation, Pairing, Infinity and Union.*

Proof. We have seen all transitive non-empty sets satisfy Emptyset, Extensionality and Foundation.

$M[G]$ satisfies pairing by observing as $a, b \in M[G] \implies$ there is $\sigma, \tau \in M^{\mathbb{P}}$ s.t. $\tau_G = a, \sigma_G = b$. Then by above $up(\sigma, \tau) \in M^{\mathbb{P}}$ and $(up(\sigma, \tau))_G = \{\sigma_G, \tau_G\} = \{a, b\}$.

$M[G]$ satisfies the Axiom of Infinity as $\omega \in M[G]$ and absoluteness.

Union: Let $a \in M[G]$. Fix $\tau \in M^{\mathbb{P}}$ s.t. $a = \tau_G$.

set $\pi = \{(\theta, p) \mid \exists(\sigma, q) \in \tau, \exists r \in \mathbb{P} : ((\theta, r) \in \sigma \wedge p \leq r \wedge p \leq q)\}$

Claim. $\bigcup a = b := \pi_G$

\subseteq : Let $d \in \bigcup a$ then $\exists q \in G : \exists \sigma : (\sigma, q) \in \tau$ and $d \in \sigma_G$. Thus there is also $r \in G, \theta$ s.t. $(\theta, r) \in \sigma$ and $\theta_G = d \in \pi_G$

\supseteq : Let $d \in \pi_G$. Then $\exists p \in G, \exists \theta$ s.t. $(\theta, p) \in \pi$ and $\theta_G = d, p \leq r, p \leq q \implies \theta_G \in \sigma_G$ and $\sigma_G \in \tau_G = a \implies \theta_G \in \bigcup a$ \square

Example. $\mathbb{P} = Fn(I, J), I, J \in M, G$ a filter then $\bigcup G$ is a function. We have a name Γ for G so we can write a name for $\bigcup G$ as above. But there is a much more natural name for $\bigcup G$.

$$\check{f} = \left\{ (op(\check{i}, \check{j}), p) \mid p \in \mathbb{P} \wedge (i, j) \in p \right\}$$

We have seen $P, G \in M[G]$. What about $\mathbb{P} \setminus G =: C$?

Candidate name $\check{C} = \{(\check{p}, q) \mid p, q \in \mathbb{P}, p \perp q\}$

$\check{C}_G = \{p \in \mathbb{P} \mid \exists q \in G : p \perp q\}$. As all elements of a filter are compatible: $\check{C}_G \cap G = \emptyset$

Do we have $G \cup \check{C}_G = \mathbb{P}$?

Let $p \in \mathbb{P}$ set $D_p = \{q \in \mathbb{P} \mid p \perp q \vee q \leq p\}$. $D_p \in M$ and D_p is dense ($r \in \mathbb{P}, r \perp p \implies r \in D_p$ otherwise $\exists q \in \mathbb{P} : q \leq r, q \leq p \implies q \in D_p$)

If G is generic we find $q \in G \cap D_p \implies q \in D_p$

If $q \perp p$ then $p \in \check{C}_G$. If $q \leq p$ then $p \in G \implies$ is complement.

Lemma (5). *Let N be a transitive model of ZFC with $M \subseteq N$ and $G \in N$. Then $M[G] \subseteq N$.*

Proof. The definition of $val(\tau, G) = \tau_G$ is absolute between transitive models of ZFC so if $M^{\mathbb{P}} \subseteq M \subseteq N$ and $G \in N$ then $val(\tau, G) \in N$ for any $\tau \in M^{\mathbb{P}}$. Thus $M[G] \subseteq N$. \square

Lemma (6). *For all $\tau \in M^{\mathbb{P}}$*

$$(1) \ rk(\tau_G) \leq rk(\tau)$$

$$(2) \ o(M[G]) = o(M) \text{ i.e. } On \cap M[G] = On \cap M$$

Proof. (1) By \in -induction on τ :

$$\begin{aligned}
rk(\tau_G) &= \sup \{rk(x) + 1 \mid x \in \tau_G\} \\
&= \sup \{rk(\sigma_G) + 1 \mid (\sigma, p) \in \tau \wedge p \in G\} \\
&\leq \sup \{rk(\sigma_G) + 1 \mid \sigma \in \text{dom}(\tau)\} \\
&\stackrel{IH}{\leq} \sup \{rk(\sigma) + 1 \mid \sigma \in \text{dom}(\tau)\} \\
&\leq rk(\tau)
\end{aligned}$$

(2) $\alpha \in M[G] \cap \text{On} \implies \alpha = \tau_G$ for some $\tau \in M^{\mathbb{P}}$ but $\alpha = rk(\alpha) = rk(\tau_G) \leq rk(\tau) \in M \cap \text{On}$ as $\tau \in M \models \text{ZFC}$ so $\alpha \in M$. Thus $o(M[G]) \subseteq o(M)$ but as $M \subseteq M[G]$ we also have $o(M) \subseteq o(M[G])$. \square

We need to show $M[G] \models \text{Axiom of Separation}$ i.e. for φ a formula of \mathcal{L}_{\in} , $\vec{a}, b \in M[G]$ we have $\exists s \in M[G]$ s.t. $s = \{z \mid z \in b \wedge \varphi(z, \text{veca})_{M[G]}\}$

Let's look at a simpler case $b = \omega, \varphi = z \in a, a = \tau_G$ so we need to find a name $\sigma \in M^{\mathbb{P}}$ s.t. $\sigma_G = \tau_G \cap \omega$. Why not take $\tau_G \restriction \omega$? If $\tau_G \notin M$ then we can't assume $\tau_G \cap \omega \in M$. If $\tau = \{(\check{x}_i, p_i) \mid i \in I\}$ then $\sigma = \{(\check{x}_i, p_i) \mid i \in I \wedge x_i \in \omega\}$. This works but in general we cannot tell for $\pi \in \text{dom}(\tau)$ if $\pi_G \in \omega$ or not without knowing what G is. For example, suppose $(\pi, p) \in \tau, \pi = \{(\check{n}, p_n) \mid n \in \omega\}$. Then $\pi_G \in \omega$ iff there is some $n \in \omega$ s.t. $\forall m \leq n : p_m \in G$ and $\forall m > n : p_m \notin G$. Thus if $q \in \mathbb{P}$, new $p \geq q, \forall m \leq n : p_m \geq q$ and $\forall m > n : p_m \perp q$ then $q \in G \implies \pi_G \in \tau_G \cap \omega$. So we could put (π, q) in σ .

Definition. $\mathcal{FL}_{M, \mathbb{P}}$ is the collection of all formulae in the language \mathcal{L}_{\in} extended with constant symbols for each name in $M^{\mathbb{P}}$.

Example. If φ is a formula of \mathcal{L}_{\in} with two free variables then $\varphi(\check{n}, \tau)$ is a sentence in $\mathcal{FL}_{M, \mathbb{P}}$

Definition. Let ψ be a sentence of $\mathcal{FL}_{M, \mathbb{P}}$. Then $M[G] \models \psi$ has the usual model theoretic meaning with \in interpreted as membership and any constant symbol or P -name τ is interpreted as τ_G .

Definition. For a sentence ψ of $\mathcal{FL}_{M, \mathbb{P}}$ we say $p \Vdash_{\mathbb{P}, M} \psi$ if for all \mathbb{P} -generic filter G over M with $p \in G$ we have $M[G] \models \psi$.

We omit subscripts \mathbb{P}, M when this is clear from the context.

Example. (example from above): $q \Vdash \pi \in \tau \cap \check{\omega}$

Example. Let $\Gamma : \{(\check{p}, p) \mid p \in \mathbb{P}\}$ be the name for the generic.

If $p \leq q$ then $p \Vdash \check{q} \in \Gamma$ i.e. $\forall G$ s.t. $p \in G$ we have $M[G] \models \check{q} \in \Gamma$ i.e. $M[G] \models \check{q}_G \in \Gamma_G$ i.e. $M[G] \models q \in G$ i.e. $q \in G$.

$\mathbb{1} \Vdash \psi$ iff $M[G] \models \psi$ for all \mathbb{P} -generic filters G over M e.g. $\mathbb{1} \Vdash \text{Axiom of Union}$ by Lemma 4.

If \mathbb{P} is atomless and $p \perp q \in \mathbb{P}$ then $p \Vdash \check{q} \notin \Gamma$.

$\mathbb{1} \Vdash \check{p} \in \Gamma$ and $\mathbb{1} \nVdash \check{p} \notin \Gamma$

We may have that $p \nVdash \psi$ and $p \nVdash \neg\psi$

$\mathbb{1} \Vdash \check{\omega} \in \omega \dot{+} 1$

Lemma. *If $p \Vdash \varphi$ and $q \leq p$ then $q \Vdash \varphi$*

Proof. If $q \leq p$ and $q \in G$ for a filter G then $p \in G$ by upwards closure. \square

Back to the example: Our proposal is $\sigma = \{(\check{n}, p) \mid n \in \omega \wedge p \in \mathbb{P} \wedge p \Vdash \check{n} \in \tau\}$. σ is a name but is it definable in M i.e. is $\sigma \in M^{\mathbb{P}}$?

Problem 1: We defined \Vdash using generics for M and these are generally not in M , so there is no obvious reason to think that $\sigma \in M$.

Problem 2: How can we be sure $\sigma_G = \omega \cap \tau_G$. $\sigma_G \subseteq \omega \cap \tau_G$ as if G is a generic with $p \in G$ then $n \in \sigma \iff p \Vdash \check{n} \in \tau \implies M[G] \models \check{n} \in \tau \implies n \in \tau_G \cap \omega$.

But how do we know for every $n \in \omega \cap \tau_G$ there is some $p \in \mathbb{P}$ s.t. $p \Vdash \check{n} \in \tau$?

Lemma (Truth Lemma). *Let ψ be a sentence in $\mathcal{FL}_{M, \mathbb{P}, G}$ \mathbb{P} -generic over M then $M[G] \models \psi$ iff there is a $p \in G$ s.t. $p \Vdash \psi$.*

Note: \Leftarrow is obvious from the definition of \Vdash .

Lemma (Definability Lemma). *Let φ be a formula of \mathcal{L}_{\in} with $FV(\varphi) = \{x_1, \dots, x_n\}$. Then $\{(p, \mathbb{P}, \leq, \mathbb{1}, \theta_1, \dots, \theta_n) \mid (\mathbb{P}, \leq, \mathbb{1}) \text{ is a forcing poset in } M, p \in \mathbb{P}, \theta_1, \dots, \theta_n \in M^{\mathbb{P}}, p \Vdash \varphi(\theta_1, \dots, \theta_n)\}$ is definable (without parameters) in M .*

Using these, we see σ works:

Definability Lemma for $\varphi = x \in y \implies \sigma \in M^{\mathbb{P}}$

Truth Lemma: $(n \in \tau_G \cap \omega)_{M[G]}$ i.e. $M[G] \models \check{n} \in \tau \cap \check{\omega} \implies \exists p \in G : p \Vdash \check{n} \in \tau \cap \check{\omega}$ So $p \Vdash \check{n} \in \tau \implies (\check{n}, p) \in \sigma_G$

Lemma (1). *For any $\mathbb{P} \in M$ and sentences $\varphi, \psi \in \mathcal{FL}_{M, \mathbb{P}}$*

1. *No p forces both φ and $\neg\varphi$*
2. *If φ and ψ are logically equivalent then $p \Vdash \varphi$ iff $p \Vdash \psi$*
3. *$p \Vdash \varphi$ and $q \leq p$ then $q \Vdash \varphi$*
4. *$p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi \wedge p \Vdash \psi$*

Lemma. $M[G] \models$ *Axiom of Separation*

Proof. Fix a formula φ of \mathcal{L}_{\in} with $y \notin FV(\varphi)$. We need to show that $\forall z : \forall v_1, \dots, v_n \in M[G] : \exists y \in M[G] : \forall x \in M[G] : (x \in y \leftrightarrow (x \in z \wedge \varphi(x, z, \vec{v}))_{M[G]})$

Fix $\pi, \sigma^1, \dots, \sigma^n \in M^{\mathbb{P}}$ i.e. $\pi_G, \sigma_G^1, \dots, \sigma_G^n \in M[G]$.

Set $S := \{x \in \pi_G \mid \varphi(x, \pi_G, \vec{\sigma}_G)_{M[G]}\}$. We show $S \in M[G]$ so we need $\tau \in M^{\mathbb{P}}$ s.t. $\tau_G = S$.

Let $\vec{\varphi}(x)$ denote $\varphi(x, \pi, \vec{\sigma}) \in \mathcal{FL}_{M, \mathbb{P}}$.

Now $S = \{\theta_G \mid \theta \in \text{dom}(\pi) \wedge M[G] \models (\theta \in \pi \wedge \vec{\varphi}(\theta))\}$. Define $\tau := \{(\theta, p) \mid \theta \in \text{dom}(\pi) \wedge p \in \mathbb{P} \wedge p \Vdash \vec{\varphi}(\theta)\}$.

Then $\tau \in M^{\mathbb{P}}$ by the Definability Lemma and $\tau_G = \{\theta_G \mid \theta \in \text{dom}(\pi) \wedge \exists p \in G : p \Vdash (\theta \in \pi \wedge \vec{\varphi}(\theta))\}$. By definition of forcing $\tau_G \subseteq S$. To prove $S \subseteq \tau_G$ fix $\theta_G \in S$ with $\theta \in \text{dom}(\pi)$ then $M[G] \models \pi \wedge \vec{\varphi}(\theta)$ so by the Truth Lemma we can find $p \in G$ s.t. $p \Vdash \theta \in \pi \wedge \vec{\varphi}(\theta)$. Thus $(\theta, p) \in \tau$ so $\theta_G \in \tau_G$ thus $S \in M[G]$. \square

Theorem. $M[G] \models \text{ZFC}$

Proof. Powerset: It is enough to prove that if $a \in M[G]$ then $\exists b \in M[G]$ s.t. $\mathcal{P}(a) \cap M[G] \subseteq b$. Then by separation $\mathcal{P}(a) \cap M[G] \in M[G]$.

Fix $\tau \in M^{\mathbb{P}}$ s.t. $\tau_G = a$. Let $Q = (\mathcal{P}(\text{dom}(\tau) \times \mathbb{P}))_M$ i.e. all names $\theta \in m^{\mathbb{P}}$ s.t. $\theta \subseteq \text{dom}(\tau)$.

Let $\pi = Q \times \{1\}$ and $b = \pi_G = \{\theta_G \mid \theta \in Q\}$. Now let $c \in \mathcal{P}(a) \cap M[G]$, we show $c \in b$. Fix $\gamma \in M^{\mathbb{P}}$ s.t. $\gamma_G = c$ and let $\theta = \{(\sigma, p) \mid \sigma \in \text{dom} \tau \wedge p \Vdash \sigma \in \gamma\}$

$\theta \in M$ by the Definability Lemma. Since $\theta \in Q$, we're done if we can show $\theta_G = c$. $\theta_G \subseteq c$ as $\theta_G = \{\sigma_G \mid \sigma \in \text{dom}(\tau), \exists p \in G : p \Vdash \sigma \in \gamma\}$ and by definition of \Vdash such $\sigma_G \in \gamma_G = c$. Since $c \subseteq a = \tau_G$ each element of c is some σ_G for some $\sigma \in \text{dom}(\tau)$. Since $\sigma_G \in c = \gamma_G$ by the Truth Lemma there is $p \in G$ s.t. $p \Vdash (\sigma, p) \in \theta$ and $\sigma_G \in \theta_G$, so $c \supseteq \theta_G$.

Replacement: Let $a \in M[G]$ $(\forall x \in a : \exists y : \vec{\varphi}(x, y))_{M[G]}$ and find $b \in M[G]$ s.t. $(\forall x \in a : \exists y \in b : \vec{\varphi}(x, y))_{M[G]}$ so $\vec{\varphi} \in \mathcal{FL}_{M, \mathbb{P}}$ is an abbreviation for φ with names/constants as parameters as in the proof of the Axiom of Separation.

Say $a = \tau_G$ so $M[G] \models \forall x \in \tau : \exists y : \vec{\varphi}(x, y)$. Working in M using the Definability Lemma let Q be a set of names s.t. $\forall p \in \mathbb{P} : \forall \sigma \in \text{dom} \tau$ if there is θ s.t. $p \Vdash \sigma, \vec{p}$ then at least one such $\theta \in Q$.

How do we know such a set (not a proper class) Q exists? By the Montague-Levy Reflection Theorem in $M : \forall p : \forall \sigma : \exists \theta$ s.t. $p \Vdash \varphi(\sigma, \theta)$ holds in M then it holds restricted to some $(V_\sigma)_M$ s.t. $\mathbb{P}, \tau \in (V_\sigma)_M$ so we could take $Q = M^{\mathbb{P}} \cap (V_\sigma)_M$.

Let $\pi = Q \times \{1\}$ and $b = \pi_G = \{\theta_G \mid \theta \in Q\}$. Fix $x \in a$ so $x = \sigma_G$ for some $\sigma \in \text{dom}(\tau)$. Then $M[G] \models \vec{\varphi}(\sigma, \theta)$ and then by the Truth Lemma there is $p \in G$ s.t. $p \Vdash \vec{\varphi}(\sigma, \theta)$. Then there is a $\theta \in G$ s.t. $p \Vdash \vec{\varphi}(\sigma, \theta)$ if $y = \theta_G$ then $y \in b$ and $\vec{\varphi}(x, y)_{M[G]}$. \square

Lemma (1 (cont'd)). 5. $p \Vdash \neg \varphi$ iff $\neg \exists q \leq p : q \Vdash \varphi$ and $p \Vdash \varphi$ iff $\neg \exists q \leq p : q \Vdash \neg \varphi$

6. $p \Vdash \varphi \longrightarrow \psi$ iff $\neg \exists q \leq p : (q \Vdash \varphi \wedge q \Vdash \neg \psi)$

7. $p \Vdash \varphi \wedge \psi$ iff $\{q \leq p \mid q \Vdash \varphi \text{ or } q \Vdash \psi\}$ is dense below p .

Using the Truth Lemma. 5. (\longrightarrow) is clear.

(\longleftarrow) Assume $p \nVdash \neg \varphi$ then we can find some generic G with $p \in G$ and $M[G] \models \varphi$.

By the Truth Lemma there is $r \in G$ s.t. $r \Vdash \varphi$. Since G is a filter we can find $q \in G$ s.t. $q \leq r$ and $q \leq p$ then $q \Vdash \varphi$ by (3). So $p \Vdash \neg \varphi$.

6. follows from (5) and (4).

7.

$$\begin{aligned} p \nVdash \neg \varphi \longrightarrow \psi &\stackrel{(6)}{\iff} \exists r \leq p : (r \Vdash \neg \varphi \wedge r \Vdash \neg \psi) \\ &\stackrel{(5)}{\iff} \exists r \leq p : \forall q \leq r : q \Vdash \varphi \wedge q \nVdash \psi \end{aligned}$$

so $p \Vdash \varphi \vee \psi \iff \forall r \leq p : \exists q \leq r : (q \Vdash \varphi \vee q \Vdash \psi)$

Observe $M[G] \Vdash \varphi$ and $p \in G$ then there is a $q \in G$ with $q \leq p$ and $q \Vdash \varphi$.

□

Lemma. Let $\varphi(x) \in \mathcal{FL}_{M, \mathbb{P}}$ with only one free variable x .

1. $p \Vdash \forall x : \varphi(x)$ iff $p \Vdash \varphi(\tau)$ for all $\tau \in M^{\mathbb{P}}$
2. $p \Vdash \exists x : \varphi(x)$ iff $\{q \leq p \mid \exists \tau \in M^{\mathbb{P}} : q \Vdash \varphi(\tau)\}$ is dense below p .

Proof. 1. $M[G] \models \forall x : \varphi(x)$ iff $M[G] \models \varphi(\tau)$ for all $\tau \in M^{\mathbb{P}}$ so $p \Vdash \forall x : \varphi(x)$ iff for all $\tau \in M^{\mathbb{P}}$ and all generics G with $p \in G$ $M[G] \models \varphi(\tau)$ iff $\forall \tau \in M^{\mathbb{P}} : p \Vdash \varphi(\tau)$.

2. By Lemma 1(5) and Lemma 2(1) $p \Vdash \neg \forall x : \neg \varphi(x) \iff$ for all $r \leq p : r \nVdash \neg \varphi(x)$ iff by (1) $\forall r \leq p : \exists \tau \in M^{\mathbb{P}} : r \nVdash \neg \varphi(\tau)$ iff by (5) $\forall r \leq p : \exists \tau \in M^{\mathbb{P}} : \exists q \leq r : q \Vdash \varphi(\tau)$

□

Definition. Define by recursion on the complexity of formulae

1. $p \Vdash^* \varphi \wedge \psi$ iff $p \Vdash^* \varphi$ and $p \Vdash^* \psi$
2. $p \Vdash^* \varphi \wedge \psi$ iff $\neg \exists q \leq p : q \Vdash^* \varphi$
3. $p \Vdash^* \varphi \longrightarrow \psi$ iff $\neg \exists q \leq p : (q \Vdash^* \varphi \wedge q \Vdash^* \neg \psi)$
4. $p \Vdash^* \varphi \vee \psi$ iff $\{q \mid q \Vdash^* \varphi \vee q \Vdash^* \psi\}$ is dense below p
5. $p \Vdash^* \forall x : \varphi(x)$ iff $\forall \tau \in M^{\mathbb{P}} : p \Vdash^* \varphi(\tau)$
6. $p \Vdash^* \exists x : \varphi(x)$ iff $\{q \mid \exists \tau \in M^{\mathbb{P}} : p \Vdash^* \varphi(\tau)\}$

Definition. For a forcing poset \mathbb{P} and names $\tau, \theta, \pi \in M^{\mathbb{P}}$:

1. $p \Vdash^* \tau = \theta$ iff $\forall \sigma \in \text{dom}(\tau) \cup \text{dom}(\theta) : \forall q \leq p : q \Vdash^* \sigma \in \tau \iff q \Vdash^* \sigma \in \theta$
2. $p \Vdash^* \pi \in \tau$ iff $\{q \leq p \mid \exists (\sigma, r) \in \tau : (q \leq r \wedge q \Vdash^* \pi = \sigma)\}$ is dense below p

Example. $p \Vdash^* \tau = \tau$ holds for all $\tau \in M^{\mathbb{P}}$ trivially.

If $(\pi, r') \in \tau$ and $p \leq r'$ then $p \Vdash^* \pi \in \tau$ because $q \Vdash^* \pi = \pi$ for all q

Is this well defined? Defined by recursion on the rank of names.

For names of rank 0 (i.e. \emptyset) then 1. is vacuously true and 2. is trivially false

Suppose 1 and 2 are defined for names of rank $< \alpha$ and $\max\{rk(\tau), rk(\theta)\} = \alpha$. To evaluate 1 we need to evaluate $q \Vdash^* \sigma \in \tau/\theta$ where $rk(\sigma) < \alpha$ so we use 2. We need to evaluate $q \Vdash^* \sigma = \pi$ where $rk(\pi) < \alpha$

To evaluate 2 with $rk(\pi) < \alpha$ as above.

To evaluate 2 with $rk(\pi) = \alpha, rk(\tau) \leq \alpha$. We need to evaluate $q \Vdash^* \pi = \sigma$ where $rk(\sigma) < \alpha$ so use 1.

As we've just shown 1 is well-defined on names of rank α . Thus \Vdash^* is well-defined

Lemma (1). For $\varphi \in \mathcal{FL}_{M,\mathbb{P}}$ a sentence

- (a) If $p \Vdash^* \varphi$ and $p_1 \leq p$ then $p_1 \Vdash^* \varphi$
- (b) $p \Vdash^* \varphi$ iff $\{p_1 \leq p \mid p_1 \Vdash^* \varphi\}$ is dense below p
- (c) $p \Vdash^* \varphi$ iff $\neg \exists q \leq p : q \Vdash^* \neg \varphi$

Proof. (c) follows from (b): If $p \Vdash^* \varphi$ and $q \leq p, q \Vdash^* \neg \varphi$ then $\neg \exists r \leq q : r \Vdash^* \varphi \implies \{p_1 \leq p \mid p_1 \Vdash^* \varphi\}$ is dense below p .

(a) For atomic φ this follows from the definition. For a composed formula this is an easy induction.

(b) (\implies) follows from (a)

Note: If $A \subseteq \mathbb{P}$ with $\{q \mid A \text{ is dense below } q\}$ dense below p then A is dense below p .

(\impliedby) When φ is $\pi \in \tau$. If the right hand side holds, by 2 of the definition $A = \{q \leq p \mid \exists (\sigma, r) \in \tau : q \leq r \wedge q \Vdash^* \pi = \sigma\}$ is dense below each $p_1 \leq p$ s.t. $p_1 \Vdash^* \pi \in \tau$, so A is dense below p by the note and thus $p \Vdash^* \pi \in \tau$ by definition.

If φ is $\tau = \theta$ and the right hand side holds then

$\forall \sigma \in \text{dom}(\tau) \cup \text{dom}(\theta) : \{p_1 \leq p \mid p_1 \Vdash^* \sigma \in \tau \iff p_1 \Vdash^* \sigma \in \theta\}$ dense below p and hence also dense below any $q \leq p$ thus if $q \Vdash^* \sigma \in \tau$ by (a) $\{p_1 \leq q \mid p_1 \Vdash^* \sigma \in \tau \wedge p_1 \Vdash^* \sigma \in \theta\}$ is dense below q so by (b) $q \Vdash^* \sigma \in \theta$. Similarly if $q \Vdash^* \sigma \in \theta$ then $q \Vdash^* \sigma \in \tau$. Thus $p \Vdash^* \tau = \theta$.

For compound φ we will just look at 2 examples:

1. $\{p_1 \leq p \mid p_1 \Vdash^* \varphi \wedge \psi\}$ is dense below p then $\{p_1 \leq p \mid p_1 \Vdash^* \varphi \wedge p_1 \Vdash^* \psi\}$ is dense below p . By the inductive hypothesis we get $p \Vdash^* \varphi$ and $p \Vdash^* \psi$ so $p \Vdash^* \varphi \wedge \psi$.
6. $\{p_1 \leq p \mid p_1 \Vdash^* \exists x : \varphi(x)\}$ dense below p then $A = \{q \leq p \mid \exists \tau \in M^{\mathbb{P}} : q \Vdash^* \varphi(\tau)\}$ is dense below each p_1 above. So by the note A is dense below p so $p \Vdash^* \exists x : \varphi(x)$ \square

Lemma (2). Let $D \subseteq \mathbb{P}$ with $D \in M$ be dense below $p \in \mathbb{P}$. If G is \mathbb{P} -generic over M and $p \in G$ then $G \cap D = \emptyset$.

Proof. Exercise \square

Lemma (3). Let M be a ctm with $\mathbb{P} \in M$ and G be \mathbb{P} -generic over M . Fix $\varphi \in \mathcal{FL}_{M,\mathbb{P}}$ a sentence. Then

- (a) If $p \in G$ and $p \Vdash^* \varphi$ then $M[G] \models \varphi$
- (b) If $M[G] \models \varphi$ then there is some $p \in G$ s.t. $p \Vdash^* \varphi$

Corollary. Let M, \mathbb{P}, φ be as in Lemma 3. Then $p \Vdash \varphi$ iff $p \Vdash^* \varphi$

of corollary. $p \Vdash^* \varphi \xrightarrow{\text{Lem 3a}} \text{for any generic } G \ni p: M[G] \models \varphi \xleftrightarrow{\text{def}} p \Vdash \varphi$

Suppose $p \Vdash \varphi$ but $p \nVdash^* \varphi$. By Lemma 1c) $\exists q \leq p : q \Vdash^* \neg \varphi$. Then for such a q by definition of \Vdash^* : $\neg \exists r \leq q : r \Vdash^* \varphi$.

Let G be generic with $q \in G$ then $p \in G$ so $M[G] \models \varphi$ by definition of \Vdash . By lemma 3b) $s \in G$ s.t. $s \Vdash^* \varphi$. Since $q, s \in G \exists r \in G : r \leq q, r \leq s$. Then $r \Vdash^* \varphi$ by Lemma 1a) \square