

*Recall.* Every infinite subset of the real numbers is either countable or bijective with the whole of  $\mathbb{R}$ .

Equivalently:  $2^{\aleph_0} = \aleph_1$  i.e.  $\mathcal{P}(\mathbb{N})$  has size the first uncountable cardinal.

CH was first advanced by Cantor 1878

ZFC  $\not\models \neg$ CH (Gödel 1940)

ZFC  $\not\models$  CH (Cohen 1963)

Cannot do:  $\text{ZFC} \vdash \text{“ZFC} \not\models \text{CH”} \longrightarrow \text{find a model for } \text{ZFC} + \neg \text{CH} \iff \text{Con}(\text{ZFC} + \neg \text{CH})$

By Gödel’s Second Incompleteness Theorem: If ZFC is consistent then  $\text{ZFC} \not\models \text{Con}(\text{ZFC})$ .

instead:  $\text{Con}(\text{ZFC}) \implies \text{Con}(\text{ZFC} + \neg \text{CH})$  which we will show using ZFC and

$\text{Con}(\text{ZFC}) \implies \text{Con}(\text{ZFC} + \text{CH})$

**Definition.** The language of set theory (LST) is the first order predicate language  $\mathcal{L}_\in$  which consists of the logical connectives and quantifiers ( $\neg, \implies, \vee, \wedge, \exists, \forall$ ), parentheses, variable symbols  $v_0, v_1, \dots$ , a logical binary predicates “=” and a single non-logical binary predicate “ $\in$ ”.

$\text{Con}(\text{ZFC}) \implies \text{ZFC}$  is satisfiable, i.e. there is an  $\mathcal{L}_\in$ -structure  $(X, \underbrace{E}_{\subseteq X \times X})$  s.t.  $(X, E) \models$

ZFC. We want a “nicer” model of ZFC.

**Method 1.** Assume the existence of an inaccessible cardinal  $\kappa$ . Then  $(V_\kappa, \in) \models \text{ZFC}$ .

**Method 2.** To show  $\text{ZFC} + \neg \text{CH}$  is consistent, it is enough to show every finite subset is consistent. For every  $\Sigma \subset \text{ZFC}$  finite find  $(X, \in) \models \Sigma$  + enough of ZFC to prove what we need (where  $X$  is transitive).

Transform this into a model of  $\Sigma + \neg \text{CH}$ .

For a set  $X$  and  $E \subseteq X \times X$ ,  $\varphi$  a sentence of LST we know from logic how to define the satisfaction relation  $(X, E) \models \varphi$ . To define this we used:

- Symbol set of the language
- define by recursion terms and formulae, free variables etc. / satisfaction
- set  $D$ , relations / functions on  $D$ , elements of  $D$
- interpretation function maps from non-logical symbols of  $\mathcal{L}$  to  $D/\mathcal{P}(D \times D)$
- Variable assignment from symbols to  $D$

**Definition.** We define the relativization of formulae of LST to a set  $W$  by recursion on formulae.

1.  $(x \in y)_W := x \in y$   $(x = y)_W := x = y$
2.  $(\neg \varphi)_W := \neg \varphi_W$   $(\varphi \wedge \psi)_W := \varphi_W \wedge \psi_W$
3.  $(\exists x : \varphi)_W := \exists x : (x \in W \wedge \varphi_W)$

$$\begin{aligned}
(\forall x : \varphi)_W &\iff (\neg \exists x : \neg \varphi)_W \\
&= \neg(\exists x : \neg \varphi)_W \\
&= \neg \exists x : (x \in W \wedge (\neg \varphi)_W) \\
&= \neg \exists x : (x \in W \wedge \neg \varphi_W) \\
&= \forall x : \neg(x \in W \wedge \neg \varphi_W) \\
&= \forall x : \neg x \in W \vee \varphi_W = \forall x : x \in W \longrightarrow \varphi_W
\end{aligned}$$

We claim that  $(W, \in \upharpoonright W \times W) \models \varphi$  iff  $\varphi_W$ . We write  $(W, \in)$  for  $(W, \in \upharpoonright W \times W)$ .

**Example.** Extensionality:  $(W, \in) \models \text{Axiom of Extensionality}$ :

$$\begin{aligned}
(\text{Ax. Ext.})_W &= (\forall x : \forall y : (\forall z : z \in x \longleftrightarrow z \in y) \longrightarrow x = y)_W \\
&= \forall x \in W : \forall y \in W : (\forall z \in W : z \in x \longleftrightarrow z \in y) \longrightarrow x = y
\end{aligned}$$

If  $W = \{\emptyset, \{\{\emptyset\}\}\}$  then  $(W, \in) \not\models \text{Ax. Ext.}$

**Proposition.** If  $W$  is transitive then  $(W, \in) \models \text{Ax. Ext.}$

*Proof.* Let  $W$  be transitive,  $x, y \in W$  with  $x \neq y$ . Then there is some set  $z$  with  $z \in x \leftrightarrow z \in y$ . Wlog assume  $z \in x, z \notin y$ . Then  $z \in W$  by transitivity of  $W$ .  $\square$

**Example.** Axiom Empty set:

$$\begin{aligned}
(W, \in) \models \text{Ax. Empty set} &\iff (\exists x : x = \emptyset)_W \\
&\iff (\exists x : \forall y : y \notin x)_W \\
&\iff \exists x \in W : \forall y \in W : y \notin x \iff \exists x \in W : x \cap W = \emptyset
\end{aligned}$$

**Example.**  $W = \{\{\emptyset\}\}$  Then  $(W, \in) \models \text{Ax. Empty set}$  even though  $\emptyset \notin W$ .

**Proposition.** If  $W \neq \emptyset$  then  $(W, \in) \models \text{Ax. Empty Set}$ .

*Proof.* By the Axiom of Foundation there is  $y \in W$  with  $y \cap W = \emptyset$ .  $\square$

**Definition.** We say a formula  $\varphi$  is absolute for  $W$  if  $\text{FV}(\varphi) \subseteq \{x_1, \dots, x_n\}$  and

$$\forall x_1 \in W, \dots, \forall x_n \in W : (\varphi_W \leftrightarrow \varphi)$$

**Lemma.** The following formule are absolute for any transitive set  $W$

1.  $z = \emptyset$
2.  $x \subseteq y$
3.  $z = x \cap y$

$$4. z = x \cup y$$

$$5. z = \bigcup x$$

$$6. z = \{x, y\}$$

$$7. z = (x, y)$$

*Proof.* Let  $W$  be transitive

1. Let  $z \in W$

$$z = \emptyset \iff \forall x : x \notin z \implies \forall x \in W : x \notin z \iff (z = \emptyset)_W$$

Also  $x \in z \implies x \in W$  as  $W$  is transitive so we must also have “ $\Leftarrow$ ”

2.

3. Let  $x, y, z \in W$

$$z = x \cap y \iff \forall u : u \in z \leftrightarrow (u \in x \wedge u \in y) \implies \forall u \in W : u \in z \leftrightarrow (u \in x \wedge u \in y) \iff (z = x \cap y)_W$$

“ $\Leftarrow$ ” follows as  $u \in z \implies u \in W, u \in x \wedge u \in y \implies u \in W$

4.

5.

6. Let  $x, y, z \in W$

$$\begin{aligned} z = \{x, y\} &\iff \forall u (u \in z \leftrightarrow (u = x \vee u = y)) \\ &\implies \forall u \in W (u \in z \leftrightarrow (u = x \vee u = y)) \iff (z = \{x, y\})_W \end{aligned}$$

“ $\Leftarrow$ ” follows as  $u \in z \implies u \in W, u = x \vee u = y \implies u \in W$

7. Let  $x, y, z \in W$

$$z = (x, y) \iff \exists u, v : z = \{x, y\} \wedge u = \{x\} \wedge v = \{x, y\}$$

$$\xrightarrow{W_{\text{trans.}}} \exists u \in W : \exists v \in W : z = \{x, y\} \wedge u = \{x\} \wedge v = \{x, y\} \xrightarrow{6.} (z = (x, y))_W$$

“ $\Leftarrow$ ” follows simply.

□

**Example.**  $z = \mathcal{P}(x)$  and  $z = x \times y$  are not absolute for transitive sets. Let  $W = \omega, 2 \in \omega$ .  
 $(\omega, \in) \models 3 = \mathcal{P}(2)$

$x \times y = \{(a, b) \mid a \in x, b \in y\}$ .  $W$  transitive then  $x, y \in W \implies$  such  $a, b \in W, z = (a, b)$  is absolute for  $W$  but not all such ordered pairs may be in  $W$ .

If  $z = W \cap (x \times y), z \in W$  then  $(W, \in) \models z = x \times y$

**Lemma.** *If  $W$  is transitive and non-empty then it satisfies the axioms of extensionality, empty set and foundation.*

*Proof.* We just have to show

$$\begin{aligned} (\text{Ax. Foundation})_W &\iff (\forall x : x \neq \emptyset \longrightarrow \exists y : (y \in x \wedge x \cap y = \emptyset))_W \\ &\iff \forall x \in W : (x \neq \emptyset \longrightarrow \exists y \in W : (y \in x \wedge x \cap y = \emptyset)) \end{aligned}$$

using the absoluteness of Empty set and  $x \cap y$ . Let  $x \in W, x \neq \emptyset$ . Then by the Axiom of Foundation  $\exists y : y \in x \wedge x \cap y = \emptyset$  so  $y \in W$  by transitivity so  $\exists y \in W (y \in x \wedge x \cap y = \emptyset)$   $\square$

**Lemma.** *For a limit ordinal  $\alpha$ ,  $(V_\alpha, \in)$  satisfies the axioms of pairing, union, powerset and the axiom schema of separation. If additionally  $\alpha > \omega$ ,  $V_\alpha$  satisfies the axiom of infinity.*

*Proof.* Pairing:  $(\forall x : \forall y : \exists z : z = \{x, y\})_{V_\alpha}$  iff  $\forall x \in V_\alpha : \forall y \in V_\alpha : \exists z \in V_\alpha : z = \{x, y\}$ .

Let  $x, y \in V_\alpha$ ,  $\text{rk}(\{x, y\}) = \max(\text{rk}(x), \text{rk}(y)) + 1 < \alpha$  so  $\{x, y\} \in V_\alpha$

Union:  $(\forall x : \exists z : z = \bigcup x)_{V_\alpha}$  iff  $\forall x \in V_\alpha : \exists z \in V_\alpha : z = \bigcup x$  so we just need  $x \in V_\alpha \implies \bigcup x \in V_\alpha$  which is true as  $\text{rk}(\bigcup x) \leq \text{rk}(x) < \alpha$ .

Powerset:  $(\forall x : \exists y : \forall z : (z \in y \iff z \subseteq x))_{V_\alpha}$  iff  $\forall x \in V_\alpha : \exists y \in V_\alpha : \forall z \in V_\alpha : (z \in y \iff z \subseteq x)$ .  $\text{rk}(\mathcal{P}(x)) = \text{rk}(x) + 1$  so  $x \in V_\alpha \implies \mathcal{P}(x) \in V_\alpha$ . Let  $x \in V_\alpha, y = \mathcal{P}(x) \in V_\alpha$  then  $\forall z : z \in y \iff z \subseteq x$  so in particular  $\forall z \in V_\alpha : (z \in y \iff z \subseteq x)$  i.e.  $(y = \mathcal{P}(x))_{V_\alpha}$ .

Infinity: Let  $\alpha > \omega$ .  $(\exists x : (\emptyset \in x \wedge \forall y : (y \in x \longrightarrow y \cup \{y\} \in x)))_{V_\alpha}$  iff  $\exists x \in V_\alpha : (\emptyset \in x \wedge \forall y \in V_\alpha : (y \in x \longrightarrow y \cup \{y\} \in x))$ . Let  $x = \omega$  then  $x \in V_\alpha, \emptyset \in V_\alpha, \forall y : (y \in x \longrightarrow y \cup \{y\} \in x)$  so in particular  $\forall y \in V_\alpha : (y \in x \longrightarrow y \cup \{y\} \in x)$

Separation: Let  $\varphi$  be a formula of LST with  $x, y \notin \text{FV}(\varphi)$ . Then

$$\begin{aligned} (\forall x : \exists y : y = x \cap \{z : \varphi\})_{V_\alpha} &\iff (\forall x : \exists y : \forall z : z \in y \iff (\varphi \wedge z \in x))_{V_\alpha} \\ &\iff \forall x \in V_\alpha : \exists y \in V_\alpha : \forall z \in V_\alpha : z \in y \iff (\underbrace{\varphi_{V_\alpha}}_{\psi} \wedge z \in x) \end{aligned}$$

Let  $x \in V_\alpha$ . By the Axiom of Separation for  $\psi$   $\exists y : y = x \cap \{z : \varphi_{V_\alpha}\}$ . Then  $y \subseteq x$  so  $\text{rk}(y) \leq \text{rk}(x) \implies y \in V_\alpha$  i.e.  $\exists y \in V_\alpha : \forall z \in V_\alpha : (z \in y \iff (\varphi_{V_\alpha} \wedge z \in x))$   $\square$

*Proof.* Axiom of Choice for a limit ordinal  $\alpha$ .

TODO

$\square$

**Theorem.** *For any inaccessible cardinal  $\kappa$   $(V_\kappa, \in) \models \text{ZFC}$ .*

*Proof.* We've shown that for any limit ordinal  $\alpha > \omega$   $(V_\alpha, \in) \models$  All Axioms of ZFC except Replacement. So we need to show  $(V_\kappa, \in) \models \text{Ax. of Replacement}$ .

**Proposition (1).** *If  $\kappa$  is inaccessible  $x \in V_\kappa \implies |x| < \kappa$ .*

**Proposition (2).** *If  $\kappa$  is regular  $x \subseteq V_\kappa \wedge |x| < \kappa \implies x \in V_\kappa$ .*

Recall that the Axiom of Replacement is actually an axiom schema, so fix  $\varphi \in \text{LST}$  with  $x, y \in \text{FV}(\varphi)$ . We need to show ... TODO  $\square$

**Definition.** Let  $W \subseteq Z$  be sets or class terms and  $\varphi \in \text{Fml}_{\mathcal{L}_\in}$  with  $\text{FV}(\varphi) = \{x_1, \dots, x_n\}$ . We say  $\varphi$  is upward (downward) absolute between  $W, Z$  if  $\forall x_1, \dots, x_n \in W : (\varphi_W \longleftrightarrow \varphi_Z)$

Compare in model theory if  $\mathcal{N}, \mathcal{M}$  are  $\mathcal{L}$ -structures  $\mathcal{N}$  is a substructure of  $\mathcal{M}, \mathcal{N} \subseteq \mathcal{M}$  iff  $N \subseteq M$  and the interpretation function for  $\mathcal{N}$  is the restriction of the interpretation function of  $\mathcal{M}$ .

For  $\varphi \in \mathcal{L}, \text{FV}(\varphi) = \{x_1, \dots, x_n\}$   $\mathcal{N} \preceq_\varphi \mathcal{M}$  if  $\forall a_1, \dots, a_n \in N : \mathcal{N} \models \varphi(a_1, \dots, a_n) \iff \mathcal{M} \models \varphi(a_1, \dots, a_n)$

**Definition.**  $\mathcal{N} \preceq \mathcal{M}$  is an elementary substructure of  $\mathcal{M}$  iff for all  $\varphi \in \mathcal{L} : \mathcal{N} \preceq_\varphi \mathcal{M}$ . In other context this says  $(W, \in) \preceq (Z, \in)$  iff all  $\varphi \in \text{Lst}$  are absolute between  $W, Z$ .

**Definition.** A list of formulae  $\varphi_0, \varphi_1, \dots$  is subformula closed if every subformula of a formula on the list is on the list.

**Lemma (1).** Let  $\vec{\varphi}$  be a subformula closed list,  $W \subseteq Z$ . The following are equivalent:

- (i)  $\vec{\varphi}$  are absolute for  $W, Z$
- (ii) Whenever  $\varphi_i$  is of the form  $\exists x : \varphi_j(x, \vec{y})$  with  $\text{FV}(\varphi_j) \subseteq \{\vec{y}\}$  then  $\forall \vec{y} \in W : (\exists x \in Z : \varphi_j(x, \vec{y})_Z \longrightarrow \exists x \in W : \varphi_j(x, \vec{y})_W)$  i.e.  $\varphi_i$  is downward absolute between  $W, Z$ .

*Proof.* (i)  $\implies$  (ii) is clear: Fix  $\vec{y} \in W$  and assume  $\varphi_i(\vec{y})_Z$  i.e.  $(\exists x : \varphi_j(x, \vec{y}))_Z \iff \exists x \in Z : \varphi_j(x, \vec{y})$ . Then by absoluteness of  $\varphi_i, (\varphi_i(\vec{y}))_W$  so  $\exists x \in W : (\varphi_j(x, \vec{y}))_W$ .

(ii)  $\implies$  (i) by induction on the length of  $\varphi_i$ : so we assume as the lefthand side that absoluteness holds for subformulae.

- $\varphi_i$  atomic - by definition of absolute
- $\varphi_i = \varphi_j \wedge \varphi_k$  by IH  $\varphi_j, \varphi_k$  are absolute so by relativization  $\varphi_i$  is absolute.  $\varphi_i = \neg \varphi_j$  similarly.
- $\varphi_i = \exists x : \varphi_j(x, \vec{y})$ . Fix  $\vec{y} \in W$ .  

$$\begin{aligned} \varphi_i(\vec{y})_W &\stackrel{\text{def}}{\iff} \exists x \in W : \varphi_j(x, \vec{y})_W \stackrel{\text{IH}}{\iff} \exists x \in W : \varphi_j(x, \vec{y})_Z \iff \exists x \in Z : \\ &\varphi_j(x, \vec{y})_Z \stackrel{\text{def}}{\iff} \varphi_i(\vec{y})_Z \end{aligned}$$

$\square$

**Definition.** A formula of  $\mathcal{L}_\in$  is  $\Delta_0$  iff it only uses bounded quantifiers i.e.

- $x \in y, x = y$  are  $\Delta_0$
- If  $\varphi, \psi$  are  $\Delta_0$  so are  $\neg \varphi, \varphi \wedge \psi$
- If  $\varphi$  is  $\Delta_0$  so is  $\exists x \in y : \varphi$

**Lemma (2).** *Let  $W$  be a transitive set. Then any  $\Delta_0$  formula is absolute for  $W$ .*

*Proof.* By induction on the length of formulae using lemma 1. We just need to show that if  $\varphi$  is of the form  $\exists x : (x \in a \wedge \psi(x, \vec{y}, a))$  then  $\forall a \in W : \forall \vec{y} \in W : (\exists x : (x \in a \wedge \psi(x, \vec{y}, a)) \longrightarrow \exists x \in W : (x \in a \wedge \psi(x, \vec{y}, a)))_W$

So let  $a, \vec{y} \in W$  and suppose  $\exists x \in a : \psi(x, \vec{y}, a)$ .  $\psi$  is  $\Delta_0$  as  $\varphi$  is  $\Delta_0$  and of length less than  $\varphi$  so by IH  $\psi$  is absolute for  $W$  i.e.  $\psi(x, \vec{y}, a) \longrightarrow \psi(x, \vec{y}, a)_W$ . Further as  $a \in W, a \subseteq W$  so  $x \in a \longleftrightarrow x \in W \cap a$ . Thus  $\exists x \in a : \psi(x, \vec{y}, a) \implies \exists x \in W : (x \in a \wedge \psi(x, \vec{y}, a))$   $\square$

**Theorem** (Downward Löwenheim-Skolem Theorem). *Let  $\mathcal{M}$  be an infinite  $\mathcal{L}$ -structure. Fix a cardinal  $\kappa$  with  $\max(|\mathcal{L}|, \aleph_0) \leq \kappa \leq |M|$  and let  $S \subseteq M$  with  $|S| \leq \kappa$ . Then there is an  $\mathcal{N} \preceq \mathcal{M}$  s.t.  $S \subseteq N$  and  $|N| = \kappa$ .*

*Proof.* We prove this for  $\mathcal{L} = \mathcal{L}_\in$  and  $\in$ -models.

Fix  $M$  an infinite set and  $\kappa \leq |M|$  and infinite cardinal and  $S \subseteq M$  with  $|S| \leq \kappa$ . If  $|S| < \kappa$  let  $S' \subseteq M$  with  $S \subseteq S'$  and  $|S'| = \kappa$ .

We use Lemma 1 to build up  $S' \subseteq N \subseteq M$  with  $(N, \in) \preceq (M, \in)$ . Clearly, a list of all formulae in  $\mathcal{L}_\in$  is subformula closed. Let  $R$  be a well-ordering on  $M$ . For any existential formula  $\varphi = \exists x : \psi$  let  $n_\varphi$  be  $|\text{FV}(\varphi)|$ . We define a Skolem-function  $f_\varphi : M^{n_\varphi} \rightarrow M$  as follows.

For  $\vec{y} \in M^{n_\varphi}$ , if  $(M, \in) \models \exists x : \psi(x, \vec{y})$  then let  $f_\varphi(\vec{y})$  be the  $R$ -least in  $M$  s.t.  $(M, \in) \models \psi(f_\varphi(\vec{y}), \vec{y})$ . If  $(M, \in) \not\models \exists x : \psi(x, \vec{y})$  set  $f_\varphi(\vec{y}) = 0$ . Now set  $N_0 = S'$  and define by recursion  $N_{i+1} = N_i \cup \{f''_\varphi N_i^{n_\varphi} \mid \varphi \in \mathcal{L}_\in \text{ s.t. } \varphi = \exists x : \psi \text{ for some } \psi\}$ . Set  $N = \bigcup_{i \in \omega} N_i$   
Claim 1:  $|N| = \kappa$ . Clearly  $|N| \geq \kappa$  as  $|N_0| = \kappa$ .

$$|N_1| \leq |N_0| \oplus \sup_{n \in \omega} |N_0^n| \otimes \aleph_0 \leq \kappa \otimes \underbrace{(\kappa \otimes \dots \otimes \kappa)}_n \otimes \aleph_0$$

Similarly  $|N_i| = |N_0| = \kappa$ . Thus  $|N| \leq |N_0| \oplus |N_1| \oplus \dots = \kappa \otimes \aleph_0 = \kappa$

Claim 2:  $(N, \in) \preceq (M, \in)$ : By Lemma 1 we just need to show that for any  $\varphi = \exists x : \psi$   $\forall y_1, \dots, y_{n_\varphi} \in N : (\exists x \in M : \psi(x, \vec{y}))_M \longrightarrow \exists x \in N : \psi(x, \vec{y}_N)$ . By induction on the length of formulae assume that  $\psi$  is absolute for  $M, N$  so let  $\vec{y} \in N$  and assume  $\exists x \in M : \psi(x, \vec{y})_M$ . Then there must be some  $i$  s.t.  $\vec{y} \in N_i$ . Thus  $f_\varphi(\vec{y}) \in N_{i+1} \subseteq N$  so  $\psi(f_\varphi(\vec{y}), \vec{y})_M$  and by IH  $\psi(f_\varphi(\vec{y}), \vec{y})_N$ . Thus  $\exists x \in N : \psi(x, \vec{y})_N$ . Such an  $N$  is called a Skolem Hull of  $M$ .  $N$  is not transitive.  $\square$

**Definition.** Let  $\mathcal{M}$  be an  $\mathcal{L}$ -structure,  $X \subseteq M^n$  is definable in  $\mathcal{M}$  iff there is an  $\mathcal{L}$ -formula  $\varphi$  with  $n$  free variables s.t. for  $a_1, \dots, a_n \in M$   $\mathcal{M} \models \varphi(a_1, \dots, a_n) \iff (a_1, \dots, a_n) \in X$

$\{\emptyset\}$  is definable in  $V_\lambda$ ; any hereditary finite set is definable in  $V_\lambda$

**Theorem** (Montague-Levy Reflection Theorem). *Let  $\varphi_1, \dots, \varphi_n$  be any finite list of formulae. Then  $\text{ZFC} \vdash \forall \alpha : \exists \beta > \alpha (\vec{\varphi} \text{ are absolute for } V_\beta)$*

*This is a theorem schema*

*If  $\vec{\varphi}$  are all axioms of ZFC then  $\text{ZFC} \vdash \forall \alpha : \exists \beta > \alpha : (\wedge \vec{\varphi})_{V_\beta}$*

*Proof.* By lengthening the list if necessary assume it is subformula closed. For each  $i \leq n$  s.t.  $\varphi_i = \exists x : \varphi_j$  define

$$F_i : V \rightarrow \text{On}$$

$$F_i(\vec{y}) = \begin{cases} 0 & \text{if } \neg \exists x : \varphi_j(x, \vec{y}) \\ \eta & \text{where } \eta \text{ is least s.t. } \exists x \in V_\eta : \varphi_i(x, \vec{y}) \end{cases}$$

Now define  $G_i : \text{On} \rightarrow \text{On}$  by  $G_i(\gamma) = \sup \{F_i(\vec{y}) \mid \vec{y} \in V_\gamma\}$ .  $G_i$  is well-defined by the Axiom of replacement  $F_i''V_\gamma$  is a set of ordinals so its supremum is an ordinal.

Claim:  $\forall \alpha : \exists \beta > \alpha : \lim(\beta) \wedge \forall \gamma < \beta : \forall i \leq n : G_i(\gamma) \leq \beta$ .

Define by recursion  $\lambda_0 = \alpha$ ,  $\lambda_{k+1} = \max \{\lambda_k + 1, G_1(\lambda_k), \dots, G_n(\lambda_k)\}$ ,  $\beta := \sup \{\lambda_k \mid k \in \omega\}$ . Then  $\lim(\beta)$  and if  $\gamma < \beta$  then  $\gamma < \lambda_k$  for some  $k \in \omega$ . Hence  $G_i(\gamma) \leq G_i(\lambda_k) \leq \lambda_{k+1} < \beta$ . Then apply Lemma 1 to  $V_\beta$ .  $\square$

**Theorem** (Mostowski-Shepherdson Collapsing Lemma). *Let  $W$  be a set and  $R \subseteq W \times W$  be wellfounded s.t.  $(W, R) \models \text{Ax. of Ext. i.e. } u, v \in W, u \neq W \longrightarrow \exists z \in W : zRu \longleftrightarrow \neg zRv$*

*Then there is a unique transitive set  $M$  and unique isomorphism  $\pi : (W, R) \cong (M, \in)$  Additionally, if  $Z \subseteq W$  with  $R \upharpoonright W \times Z = \in \upharpoonright W \times Z$  and  $v \in Z, u \in W$  with  $uRv \longrightarrow u \in Z$  and  $Z$  is transitive then  $\pi \upharpoonright Z = \text{id} \upharpoonright Z$*

*Proof.* Claim 1: If  $\pi$  exists, it is unique.

Suppose we have  $\pi, M$  as above. Let  $u, v \in W$ . If  $uRv$  then  $\pi(u) \in \pi(v)$  as  $\pi$  is an isomorphism. Then  $\{\pi(u) \mid u \in W, uRv\} \subseteq \pi(v)$ .

Further if  $z \in \pi(v)$  then  $z \in M$  by transitivity hence  $z = \pi(x)$  for some  $x \in W$  with  $xRv$  thus  $\pi(v) \subseteq \{\pi(u) \mid u \in W, uRv\}$ . Taken together  $\pi(v) = \{\pi(u) \mid u \in W, uRv\}$ , so if the isomorphism exists it must take this form.

Claim 2: Such a  $\pi$  exists.

Define by recursion on  $R$

$$\pi(v) = \{\pi(u) \mid u \in W, uRv\} \tag{1}$$

Whis is this well defined? A  $R$  is wellfounded there is  $x \in W$  s.t.  $\forall y \in W : \neg yRx$ . For such an  $x$  we have  $\pi(x) = \emptyset$ . Further for  $W \setminus \{x\}$  there is  $x' \in W \setminus \{x\}$  with  $\forall y \in W \setminus \{x\} : \neg yRx'$  so  $\pi(x') = \{\pi(u) \mid u \in W, uRx'\} = \{\pi(u) \mid u = x, uRx'\}$ .

$\pi$  is clearly surjective.

Claim 3:  $\pi$  is injective:

Assume not and take  $t \in$ -minimal s.t. there are  $u \neq v \in W$  with  $t = \pi(u) = \pi(v)$ . As  $u \neq v$  there is some  $x \in W$  with  $xRu \longleftrightarrow \neg xRv$ . Wlog assume  $xRu \wedge \neg xRv$ . Then  $\pi(x) \in \pi(u) = t$ . So we must have that for some  $y$  with  $yRv$   $\pi(y) = \pi(x)$ . But this contradicts the minimality of  $t$ .

Claim 4:  $\pi$  is orderpreserving

“ $\implies$ ” By (1) if  $uRv$  then  $\pi(u) \in \pi(v)$ .

“ $\impliedby$ ” If  $\pi(u) \in \pi(v)$  then for some  $z \in W$  with  $zRv$   $\pi(z) = \pi(u)$ . As  $\pi$  is injective  $z = u$ .

Claim 5 (Also): Suppose for some  $u \in X$   $\pi(u) \neq u$  and take  $v \in$ -minimal such. Then

$$\begin{aligned}\pi(v) &= \{\pi(u) \mid u \in W, uRv\} \\ &= \{\pi(u) \mid u \in W, u \in v\} \\ &= \{u \mid u \in v\} = v\end{aligned}$$

□

**Example.**  $(N, \in) \preceq (V_\kappa, \in)$  with  $\kappa$  inaccessible,  $N$  countable.  $M$  is the collapse of  $(N, \in)$ .  $\pi(\omega_1) = \omega_1^{ck}$  the least non-definable countable ordinal.

$P(\omega) \in N : \pi(\mathcal{P}(\omega)) = \{\pi(y) \mid y \in \mathcal{P}(\omega) \cap N\} = \mathcal{P}(\omega) \cap N$

**Example.** If  $X, R$  is a wellorder then its transitive collapse is  $\text{ot}((X, R))$

**Example.** We can code  $V_\omega$  as a subset of  $\omega$  i.e. there is  $E \subseteq \omega^2$  s.t.  $(\omega, E) \cong (V_\omega, \in)$   
 $M \models x = \mathcal{P}(\omega)$ ,  $x = \mathcal{P}(\omega)^M$ ,  $M \models x$  is uncountable  $\implies \pi(y) = y$ .

Let  $o(m)$  denote  $\text{Ord} \cap M$  then  $o(M) \in \text{Ord}$  is countable.

Here:  $(M, \in)$  countable transitive model of ZFC (ctm). Assume  $M \models \text{CH}$

**Definition** (Poset). A forcing Poset is a triple  $(\mathbb{P}, \leq, \mathbb{1})$  where  $\mathbb{P}$  is a set,  $\leq$  is a pre-order on  $\mathbb{P}$  i.e. transitive and reflexive and  $\mathbb{1} \in \mathbb{P}$  is a largest element:  $\forall p \in \mathbb{P} : p \leq \mathbb{1}$ .

Elements of  $\mathbb{P}$  are called forcing conditions.

$p \leq q$ : “ $p$  extends  $q$ ”

Abuse notation: and use  $\mathbb{P}$  to refer to  $(\mathbb{P}, \leq, \mathbb{1})$ .

**Example.** Normally we use partial orders  $\mathbb{P}$ .  $\mathbb{1}$  ensures the ordering is connected. If it is a partial ordering then  $\mathbb{1}$  is unique.

**Definition.**  $p, q \in \mathbb{P}$  are compatible ( $p \not\perp q$ ) if  $\exists r \in \mathbb{P}$  s.t.  $r \leq p \wedge r \leq q$  i.e. they have a common extension.

$p, q \in \mathbb{P}$  are incompatible ( $p \perp q$ ) iff they are not compatible.

**Example.** In a tree any two elements not along a branch are incompatible. In trees  $(p \not\leq q \wedge q \not\leq p) \implies p \perp q$ . This is not true in general.

**Example.**  $(\mathcal{P}(\omega), \supseteq, \emptyset)$

**Example.** (Infinite subsetsets of  $\omega, \subseteq^*, \omega$ ) with  $p \subseteq^* q : \iff p \setminus q$  is finite.

This is not a partial order e.g.  $\omega \subseteq^* \{42, 43, \dots\}$  and  $\omega \supseteq^* \{42, 43, \dots\}$ .

$p \perp q$  iff  $p \cap q$  is finite

There is no least element.

**Definition.**  $Fn(I, J)$  for sets  $I, J$  is the set of all finite partial functions from  $I$  to  $J$  i.e.  $\{p \subseteq I \times J \mid |p| < \omega, \text{func}(p)\}$

The associated forcing poset has the order  $\supseteq$  and  $\mathbb{1} = \emptyset$ .  $p \leq q \iff p \supseteq q$  i.e.  $p$  extends  $q$  as a function.



$p \not\leq q$  iff they have a common extension iff they are the same on  $\text{dom}(p) \cap \text{dom}(q)$  iff  $p \cup q \in Fn(I, J)$  and if so  $p \cup q \leq p, p \cup q \leq q$ .

$p \in Fn(I, J) \longrightarrow$  a finite approximation of  $f : I \rightarrow J$ ,  $p$  is a condition such an  $f$  must satisfy.

**Definition.** Let  $\mathbb{P}$  be a forcing poset. Then  $D \subseteq \mathbb{P}$  is dense in  $\mathbb{P}$  if  $\forall p \in \mathbb{P} : \exists q \in D : q \leq p$ .

**Example.** Let  $I$  be infinite,  $J \neq \emptyset$  then for  $i \in I$  then  $D_i := \{q \in Fn(I, J) \mid i \in \text{dom}(q)\}$  is dense.

Let  $p \in Fn(I, J), i \in I$ . If  $i \in \text{dom}(p)$  then  $p \in D_i$  otherwise set  $q = p \cup \{(i, j)\}$  where  $j \in J$  is arbitrary. Then  $q \in Fn(I, J)$  and  $q \leq p$  and  $q \in D_i$ .

Also for  $j \in J$  the set  $\{q \in Fn(I, J) \mid j \in \text{ran}(q)\}$  is dense.

not dense:  $L = \{q \in Fn(I, J) \mid (i, j) \in q\}$  if  $|J| \geq 2$ .  $j' \neq j, p = \{(i, j')\} \in Fn(I, J)$  and  $\neg \exists q : q \leq p$  and  $q \in L$ .

**Definition.** Let  $\mathbb{P}$  be a forcing poset. Then  $G \subseteq \mathbb{P}$  is a filter on  $\mathbb{P}$  iff

1.  $1 \in G$
2.  $\forall p, q \in G : \exists r \in G : r \leq p \wedge r \leq q$
3.  $\forall p \in G : \forall q \in \mathbb{P} : p \leq q \implies q \in G$

Note: If  $G \neq \emptyset$  then  $3 \implies 1$ .

**Example.**  $\{1\}$  is a filter if  $\mathbb{P}$  is a partial order

**Example.**  $A \neq \emptyset, \mathbb{P} = \mathcal{P}(A) \setminus \{\emptyset\}, \leq = \subseteq. 1 = A$ .

$F \subseteq A$  is a filter:

1.  $\iff A \in F$
2.  $\iff F$  closed under intersections
3.  $\iff F$  closed under superset

**Example.** For a tree, a filter is a line from the root that stops somewhere

**Example.**  $\mathbb{P} = Fn(I, J), I$  infinite,  $J \neq \emptyset$ . If  $G$  is a filter on  $\mathbb{P}$  then any  $p, q \in G$  agree on  $\text{dom}(p) \cap \text{dom}(q)$  so setting  $f_G := \bigcup G$  is a function with  $\text{dom}(f_G) \subseteq I, \text{ran}(f_G) \subseteq J$ .  $D_i = \{p \in Fn(I, J) \mid i \in \text{dom}(p)\}$  is dense in  $\mathbb{P}$  so if  $\forall i \in I : G \cap D_i \neq \emptyset$  then  $f_G : I \rightarrow J$ .

**Notation.**  $M$  is ctm (of ZFC),  $\mathbb{P} \in M$  we write to mean  $(\mathbb{P}, \leq, 1) \in M$ .

**Example.** If  $I, J \in M, M$  transitive and  $M \models \text{ZFC}$  then  $Fn(I, J) \in M$  as well as the ordering on this poset.

**Definition.** For a forcing poset  $\mathbb{P}, G$  is  $\mathbb{P}$ -generic over  $M$  iff  $G$  is a filter on  $\mathbb{P}$  and  $G \cap D \neq \emptyset$  for all dense  $D \subseteq \mathbb{P}$  s.t.  $D \in M$ .

**Example.**  $Fn(\omega, 2), M$  a ctm for ZFC. Suppose  $G$  is  $\mathbb{P}$ -generic for  $M$ :  $\omega \in M, 2 \in M, Fn(\omega, 2) \in M$ .  $D_i \in M$  for  $i \in \omega$  thus  $f_G : \omega \rightarrow 2$ .

For each  $h \in M$  with  $h : \omega \rightarrow 2$  let  $E_h = \{q \in \mathbb{P} \mid \exists n \in \omega \cap \text{dom}(q) : q(n) \neq h(n)\}$ . Then  $E_h$  is dense: Let  $p \in \mathbb{P}$  with  $h \upharpoonright \text{dom}(p) = p$ .  $\text{dom}(p)$  is finite, so we can pick  $n \notin \text{dom}(p)$  and set  $q = p \cup \{(n, \neg h(n))\}$ . Then  $q \leq p$  and  $q \in E_h$ . Also  $E_h \in M$  so  $G \cap E_h \neq \emptyset$ . Thus  $f_G \neq h$ . Thus  $f_G \notin M$ .

**Lemma** (Generic Filter existence). *Let  $\mathbb{P}$  be any forcing poset, let  $\{D_i \mid i \in \omega\}$  be a countable family of dense subsets of  $\mathbb{P}$  and  $p \in \mathbb{P}$ . Then there is a filter  $G$  on  $\mathbb{P}$  s.t.  $p \in G$  and  $G \cap D_i \neq \emptyset$  for  $i \in \omega$ .*

*Thus for any ctm  $M$  with  $\mathbb{P} \in M$  and any  $p \in \mathbb{P}$  there is a  $\mathbb{P}$ -generic filter  $G$  over  $M$  with  $p \in G$ .*

*Proof.* By recursion and AC for each  $n \in \omega$  choose  $r_{n+1} \in \mathbb{P}$  s.t.  $r_0 = p$  and  $r_{n+1} \leq r_n$  and  $r_{n+1} \in D_n$ . At each step such  $r_{n+1}$  must exist as  $D_n$  is dense.

Set  $G = \{q \in \mathbb{P} \mid \exists n \in \omega r_n \leq q\}$  then  $G$  is a filter: 1,3 are easy. For 2 let  $p, q \in G$  then  $\exists n, m \in \omega : p \geq r_n, q \geq r_m$  so  $p \geq r_{\max\{m, n\}} \in G, q \geq r_{\max\{m, n\}} \in G$ . Also  $p \in G$ , each  $r_n \in G$  so  $r_{n+1} \in G \cap D_n \neq \emptyset$ .  $\square$

**Definition.**  $r \in \mathbb{P}$  is an atom of  $\mathbb{P}$  there are no  $p, q \in r$  s.t.  $p \perp q$ .  $\mathbb{P}$  is atomless iff there are no atoms.

**Example.**  $r \in \mathbb{P}$  with no extension is an atom

$(\mathcal{P}(\omega), \subseteq, \omega)$  every element is an atom

linear order: all elements are atoms

A tree with branches at every node is atomless

$Fn(I, J), J \neq \emptyset$  if  $I$  is finite has atoms. If  $I$  is infinite and  $|J| \geq 2$  then  $Fn(I, J)$  is atomless

**Lemma.** *If  $\mathbb{P} \in M, M$  a ctm of ZFC,  $\mathbb{P}$  atomless and  $G$  is  $\mathbb{P}$ -generic over  $M$  then  $G \notin M$ .*

*Proof.* Let  $D = \mathbb{P} \setminus G$ . Then  $D$  is dense as if  $r \in \mathbb{P}$  there are  $p, q \leq r : p \perp q$ . Then at least one of these must be in  $D$ .

If  $G \in M$  then  $D \in M$  by the axiom of separation in  $M$ . But as  $G$  is  $\mathbb{P}$ -generic over  $M$  we would have  $D \cap G \neq \emptyset$ .  $\square$

**Definition.**  $\tau$  is a  $\mathbb{P}$ -name iff  $\tau$  is a set of ordered pairs s.t.  $\forall (\sigma, p) \in \tau$  we have that  $\sigma$  is a  $\mathbb{P}$ -name and  $p \in \mathbb{P}$  (Definition by recursion).  $V^{\mathbb{P}}$  denotes the class of all  $\mathbb{P}$ -names.

$\emptyset$  is a  $\mathbb{P}$ -name,  $\{(\emptyset, p), (\emptyset, 1)\}$  is a  $\mathbb{P}$ -name ( $p, q \in \mathbb{P}$ )

For any set  $x$  look at the associated graph. Label each node in the tree with some element.

If  $M$  is a transitive model of ZFC with  $\mathbb{P} \in M$  then  $M^{\mathbb{P}} := V^{\mathbb{P}} \cap M = \{\tau \in M \mid (\tau \text{ is a } \mathbb{P}\text{-name})_M\}$

**Definition.** For a  $\mathbb{P}$ -name  $\tau$  and  $G \subseteq \mathbb{P}$  define by recursion

$$val(\tau, G) = \tau_G := \{val(\sigma, G) \mid \exists p \in G : (\sigma, p) \in \tau\}$$

Then  $M[G] := \{\tau_G \mid \tau \in M^{\mathbb{P}}\}$  for a ctm  $M$  with  $\mathbb{P} \in M$ .

**Example.**  $\emptyset_G = \emptyset$

If  $G$  is a filter,  $1 \in G$ , let  $\tau = \{(\sigma, 1), (\theta, 1)\}$  then  $\tau_G = \{val(\sigma, G), val(\theta, G)\} = \{\sigma_G, \theta_G\}$ .

$\tau = \{(\emptyset, p), (\{\emptyset, q\}, r)\}$ ,  $p, r \in G, q \notin G$  then  $\tau_G = \{\emptyset_G, \{\{\emptyset, q\}\}_G\} = \{\emptyset, \emptyset\} = \{\emptyset\}$

$p, r \notin G, q$  does not matter:  $\tau_G = \emptyset$

$p \notin G, r, q \in G$ :  $\tau_G = \{\{\emptyset\}\}$

We want to show that  $M[G]$  is the desired extension of  $M$  which contains  $G$  and everything else needed to ensure  $M[G] \models \text{ZFC}$ . More precisely we will show:

- (1)  $M \subseteq M[G]$
- (2)  $G \in M[G]$
- (3)  $M[G]$  is transitive and countable
- (4)  $M[G] \models \text{ZFC}$
- (5) For any  $N$  with the above properties  $N \supseteq M[G]$

For (i) we find for each  $x \in M$  a name  $\tau \in M^{\mathbb{P}}$ , which for any filter  $G$  on  $\mathbb{P}$  satisfies  $\tau_G = x$

**Definition.** For a forcing poset  $(\mathbb{P}, \leq, 1)$  and any set  $x$  define  $\check{x} := \{(\check{y}, 1) \mid y \in x\}$  (definition by recursion)

Clearly  $\hat{x}$  is a  $\mathbb{P}$ -name

$$\check{2} = \{(\check{0}, 1), (\check{1}, 1)\} = \{(\emptyset, 1), (\{\{\emptyset, 1\}\}, 1)\}$$

**Lemma (1).** *Let  $G$  be a filter on  $1$ . Then*

- (i)  $\forall x \in M : \check{x} \in M^{\mathbb{P}}$  and  $val(\check{x}, G) = x$
- (ii)  $M[G] \supseteq M$

*Proof.* (ii) follows by definition from (i)

For (i)  $\check{x} \in M$  by absoluteness and  $val(\check{x}, G) = x$  by induction as  $val(\check{x}, G) = \{val(\check{y}, G) \mid y \in x\}$   $\square$

**Lemma (3).** *For any filter  $G$  on  $\mathbb{P}$ ,  $M[G]$  is transitive and countable.*

*Proof.* Transitive:  $x \in M[G]$  then  $x = \tau_G$  for some  $\tau \in M^{\mathbb{P}}$  so  $\forall y \in x : y = \sigma_G$  for some  $\sigma \in M^{\mathbb{P}}$  thus  $y \in M^{\mathbb{P}}$

Countable:  $|M[G]| \leq |M^{\mathbb{P}}| \leq |M| \leq |M[G]|$   $\square$

**Definition.** Given  $\mathbb{P}$ , define  $\Gamma := \{(\check{p}, p) \mid p \in \mathbb{P}\}$

$$\Gamma_G = \{\check{p} \mid p \in G\} \stackrel{\text{lem 1}}{=} \{p \mid p \in G\} = G$$

Hence we have

**Lemma (2).**  $\Gamma \in M$  is a  $\mathbb{P}$ -name and  $\Gamma_G = G$  hence  $G \in M[G]$ .

*Proof.*  $\Gamma$  is definable within  $M$  and clearly a  $\mathbb{P}$ -name.  $\square$

**Definition** (Names for unordered and ordered pairs).  $up(\sigma, \tau) := \{(\sigma, \mathbb{1}), (\tau, \mathbb{1})\}$   
 $op(\sigma, \tau) := up(up(\sigma, \sigma), up(\sigma, \tau))$   
Observe:  $\sigma, \tau \in M^{\mathbb{P}} \implies up(\sigma, \tau), op(\sigma, \tau) \in M^{\mathbb{P}}$

**Lemma** (4). *Let  $M, \mathbb{P}$  be as above and  $G$  a filter on  $\mathbb{P}$ . Then  $M[G]$  is a ctm of the axioms of Extensionality, Emptyset, Foundation, Pairing, Infinity and Union.*

*Proof.* We have seen all transitive non-empty sets satisfy Emptyset, Extensionality and Foundation.

$M[G]$  satisfies pairing by observing as  $a, b \in M[G] \implies$  there is  $\sigma, \tau \in M^{\mathbb{P}}$  s.t.  $\tau_G = a, \sigma_G = b$ . Then by above  $up(\sigma, \tau) \in M^{\mathbb{P}}$  and  $(up(\sigma, \tau))_G = \{\sigma_G, \tau_G\} = \{a, b\}$ .

$M[G]$  satisfies the Axiom of Infinity as  $\omega \in M[G]$  and absoluteness.

Union: Let  $a \in M[G]$ . Fix  $\tau \in M^{\mathbb{P}}$  s.t.  $a = \tau_G$ .

set  $\pi = \{(\theta, p) \mid \exists(\sigma, q) \in \tau, \exists r \in \mathbb{P} : ((\theta, r) \in \sigma \wedge p \leq r \wedge p \leq q)\}$

Claim.  $\bigcup a = b := \pi_G$

$\subseteq$ : Let  $d \in \bigcup a$  then  $\exists q \in G : \exists \sigma : (\sigma, q) \in \tau$  and  $d \in \sigma_G$ . Thus there is also  $r \in G, \theta$  s.t.  $(\theta, r) \in \sigma$  and  $\theta_G = d \in \pi_G$

$\supseteq$ : Let  $d \in \pi_G$ . Then  $\exists p \in G, \exists \theta$  s.t.  $(\theta, p) \in \pi$  and  $\theta_G = d, p \leq r, p \leq q \implies \theta_G \in \sigma_G$  and  $\sigma_G \in \tau_G = a \implies \theta_G \in \bigcup a$   $\square$

**Example.**  $\mathbb{P} = Fn(I, J), I, J \in M, G$  a filter then  $\bigcup G$  is a function. We have a name  $\Gamma$  for  $G$  so we can write a name for  $\bigcup G$  as above. But there is a much more natural name for  $\bigcup G$ .

$$\check{f} = \left\{ (op(\check{i}, \check{j}), p) \mid p \in \mathbb{P} \wedge (i, j) \in p \right\}$$

We have seen  $P, G \in M[G]$ . What about  $\mathbb{P} \setminus G =: C$ ?

Candidate name  $\check{C} = \{(\check{p}, q) \mid p, q \in \mathbb{P}, p \perp q\}$

$\check{C}_G = \{p \in \mathbb{P} \mid \exists q \in G : p \perp q\}$ . As all elements of a filter are compatible:  $\check{C}_G \cap G = \emptyset$

Do we have  $G \cup \check{C}_G = \mathbb{P}$ ?

Let  $p \in \mathbb{P}$  set  $D_p = \{q \in \mathbb{P} \mid p \perp q \vee q \leq p\}$ .  $D_p \in M$  and  $D_p$  is dense ( $r \in \mathbb{P}, r \perp p \implies r \in D_p$  otherwise  $\exists q \in \mathbb{P} : q \leq r, q \leq p \implies q \in D_p$ )

If  $G$  is generic we find  $q \in G \cap D_p \implies q \in D_p$

If  $q \perp p$  then  $p \in \check{C}_G$ . If  $q \leq p$  then  $p \in G \implies$  is complement.

**Lemma** (5). *Let  $N$  be a transitive model of ZFC with  $M \subseteq N$  and  $G \in N$ . Then  $M[G] \subseteq N$ .*

*Proof.* The definition of  $val(\tau, G) = \tau_G$  is absolute between transitive models of ZFC so if  $M^{\mathbb{P}} \subseteq M \subseteq N$  and  $G \in N$  then  $val(\tau, G) \in N$  for any  $\tau \in M^{\mathbb{P}}$ . Thus  $M[G] \subseteq N$ .  $\square$

**Lemma** (6). *For all  $\tau \in M^{\mathbb{P}}$*

$$(1) \ rk(\tau_G) \leq rk(\tau)$$

$$(2) \ o(M[G]) = o(M) \text{ i.e. } On \cap M[G] = On \cap M$$

*Proof.* (1) By  $\in$ -induction on  $\tau$ :

$$\begin{aligned}
rk(\tau_G) &= \sup \{rk(x) + 1 \mid x \in \tau_G\} \\
&= \sup \{rk(\sigma_G) + 1 \mid (\sigma, p) \in \tau \wedge p \in G\} \\
&\leq \sup \{rk(\sigma_G) + 1 \mid \sigma \in \text{dom}(\tau)\} \\
&\stackrel{IH}{\leq} \sup \{rk(\sigma) + 1 \mid \sigma \in \text{dom}(\tau)\} \\
&\leq rk(\tau)
\end{aligned}$$

(2)  $\alpha \in M[G] \cap \text{On} \implies \alpha = \tau_G$  for some  $\tau \in M^{\mathbb{P}}$  but  $\alpha = rk(\alpha) = rk(\tau_G) \leq rk(\tau) \in M \cap \text{On}$  as  $\tau \in M \models \text{ZFC}$  so  $\alpha \in M$ . Thus  $o(M[G]) \subseteq o(M)$  but as  $M \subseteq M[G]$  we also have  $o(M) \subseteq o(M[G])$ .  $\square$

We need to show  $M[G] \models \text{Axiom of Separation}$  i.e. for  $\varphi$  a formula of  $\mathcal{L}_{\in}$ ,  $\vec{a}, b \in M[G]$  we have  $\exists s \in M[G]$  s.t.  $s = \{z \mid z \in b \wedge \varphi(z, \text{veca})_{M[G]}\}$

Let's look at a simpler case  $b = \omega, \varphi = z \in a, a = \tau_G$  so we need to find a name  $\sigma \in M^{\mathbb{P}}$  s.t.  $\sigma_G = \tau_G \cap \omega$ . Why not take  $\tau_G \restriction \omega$ ? If  $\tau_G \notin M$  then we can't assume  $\tau_G \cap \omega \in M$ . If  $\tau = \{(\check{x}_i, p_i) \mid i \in I\}$  then  $\sigma = \{(\check{x}_i, p_i) \mid i \in I \wedge x_i \in \omega\}$ . This works but in general we cannot tell for  $\pi \in \text{dom}(\tau)$  if  $\pi_G \in \omega$  or not without knowing what  $G$  is. For example, suppose  $(\pi, p) \in \tau, \pi = \{(\check{n}, p_n) \mid n \in \omega\}$ . Then  $\pi_G \in \omega$  iff there is some  $n \in \omega$  s.t.  $\forall m \leq n : p_m \in G$  and  $\forall m > n : p_m \notin G$ . Thus if  $q \in \mathbb{P}$ , new  $p \geq q, \forall m \leq n : p_m \geq q$  and  $\forall m > n : p_m \perp q$  then  $q \in G \implies \pi_G \in \tau_G \cap \omega$ . So we could put  $(\pi, q)$  in  $\sigma$ .

**Definition.**  $\mathcal{FL}_{M, \mathbb{P}}$  is the collection of all formulae in the language  $\mathcal{L}_{\in}$  extended with constant symbols for each name in  $M^{\mathbb{P}}$ .

**Example.** If  $\varphi$  is a formula of  $\mathcal{L}_{\in}$  with two free variables then  $\varphi(\check{n}, \tau)$  is a sentence in  $\mathcal{FL}_{M, \mathbb{P}}$

**Definition.** Let  $\psi$  be a sentence of  $\mathcal{FL}_{M, \mathbb{P}}$ . Then  $M[G] \models \psi$  has the usual model theoretic meaning with  $\in$  interpreted as membership and any constant symbol or  $P$ -name  $\tau$  is interpreted as  $\tau_G$ .

**Definition.** For a sentence  $\psi$  of  $\mathcal{FL}_{M, \mathbb{P}}$  we say  $p \Vdash_{\mathbb{P}, M} \psi$  if for all  $\mathbb{P}$ -generic filter  $G$  over  $M$  with  $p \in G$  we have  $M[G] \models \psi$ .

We omit subscripts  $\mathbb{P}, M$  when this is clear from the context.

**Example.** (example from above):  $q \Vdash \pi \in \tau \cap \check{\omega}$

**Example.** Let  $\Gamma : \{(\check{p}, p) \mid p \in \mathbb{P}\}$  be the name for the generic.

If  $p \leq q$  then  $p \Vdash \check{q} \in \Gamma$  i.e.  $\forall G$  s.t.  $p \in G$  we have  $M[G] \models \check{q} \in \Gamma$  i.e.  $M[G] \models \check{q}_G \in \Gamma_G$  i.e.  $M[G] \models q \in G$  i.e.  $q \in G$ .

$\mathbb{1} \Vdash \psi$  iff  $M[G] \models \psi$  for all  $\mathbb{P}$ -generic filters  $G$  over  $M$  e.g.  $\mathbb{1} \Vdash \text{Axiom of Union}$  by Lemma 4.

If  $\mathbb{P}$  is atomless and  $p \perp q \in \mathbb{P}$  then  $p \Vdash \check{q} \notin \Gamma$ .

$\mathbb{1} \Vdash \check{p} \in \Gamma$  and  $\mathbb{1} \nVdash \check{p} \notin \Gamma$

We may have that  $p \nVdash \psi$  and  $p \nVdash \neg\psi$

$\mathbb{1} \Vdash \check{\omega} \in \omega \dot{+} 1$

**Lemma.** *If  $p \Vdash \varphi$  and  $q \leq p$  then  $q \Vdash \varphi$*

*Proof.* If  $q \leq p$  and  $q \in G$  for a filter  $G$  then  $p \in G$  by upwards closure.  $\square$

Back to the example: Our proposal is  $\sigma = \{(\check{n}, p) \mid n \in \omega \wedge p \in \mathbb{P} \wedge p \Vdash \check{n} \in \tau\}$ .  $\sigma$  is a name but is it definable in  $M$  i.e. is  $\sigma \in M^{\mathbb{P}}$ ?

**Problem 1:** We defined  $\Vdash$  using generics for  $M$  and these are generally not in  $M$ , so there is no obvious reason to think that  $\sigma \in M$ .

**Problem 2:** How can we be sure  $\sigma_G = \omega \cap \tau_G$ .  $\sigma_G \subseteq \omega \cap \tau_G$  as if  $G$  is a generic with  $p \in G$  then  $n \in \sigma \iff p \Vdash \check{n} \in \tau \implies M[G] \models \check{n} \in \tau \implies n \in \tau_G \cap \omega$ .

But how do we know for every  $n \in \omega \cap \tau_G$  there is some  $p \in \mathbb{P}$  s.t.  $p \Vdash \check{n} \in \tau$ ?

**Lemma** (Truth Lemma). *Let  $\psi$  be a sentence in  $\mathcal{FL}_{M, \mathbb{P}, G}$   $\mathbb{P}$ -generic over  $M$  then  $M[G] \models \psi$  iff there is a  $p \in G$  s.t.  $p \Vdash \psi$ .*

Note:  $\Leftarrow$  is obvious from the definition of  $\Vdash$ .

**Lemma** (Definability Lemma). *Let  $\varphi$  be a formula of  $\mathcal{L}_\in$  with  $FV(\varphi) = \{x_1, \dots, x_n\}$ . Then  $\{(p, \mathbb{P}, \leq, \mathbb{1}, \theta_1, \dots, \theta_n) \mid (\mathbb{P}, \leq, \mathbb{1}) \text{ is a forcing poset in } M, p \in \mathbb{P}, \theta_1, \dots, \theta_n \in M^{\mathbb{P}}, p \Vdash \varphi(\theta_1, \dots, \theta_n)\}$  is definable (without parameters) in  $M$ .*

Using these, we see  $\sigma$  works:

Definability Lemma for  $\varphi = x \in y \implies \sigma \in M^{\mathbb{P}}$

Truth Lemma:  $(n \in \tau_G \cap \omega)_{M[G]}$  i.e.  $M[G] \models \check{n} \in \tau \cap \check{\omega} \implies \exists p \in G : p \Vdash \check{n} \in \tau \cap \check{\omega}$  So  $p \Vdash \check{n} \in \tau \implies (\check{n}, p) \in \sigma_G$

**Lemma** (1). *For any  $\mathbb{P} \in M$  and sentences  $\varphi, \psi \in \mathcal{FL}_{M, \mathbb{P}}$*

1. *No  $p$  forces both  $\varphi$  and  $\neg\varphi$*
2. *If  $\varphi$  and  $\psi$  are logically equivalent then  $p \Vdash \varphi$  iff  $p \Vdash \psi$*
3.  *$p \Vdash \varphi$  and  $q \leq p$  then  $q \Vdash \varphi$*
4.  *$p \Vdash \varphi \wedge \psi$  iff  $p \Vdash \varphi \wedge p \Vdash \psi$*

**Lemma.**  $M[G] \models$  *Axiom of Separation*

*Proof.* Fix a formula  $\varphi$  of  $\mathcal{L}_\in$  with  $y \notin FV(\varphi)$ . We need to show that  $\forall z : \forall v_1, \dots, v_n \in M[G] : \exists y \in M[G] : \forall x \in M[G] : (x \in y \leftrightarrow (x \in z \wedge \varphi(x, z, \vec{v}))_{M[G]})$

Fix  $\pi, \sigma^1, \dots, \sigma^n \in M^{\mathbb{P}}$  i.e.  $\pi_G, \sigma_G^1, \dots, \sigma_G^n \in M[G]$ .

Set  $S := \{x \in \pi_G \mid \varphi(x, \pi_G, \vec{\sigma}_G)_{M[G]}\}$ . We show  $S \in M[G]$  so we need  $\tau \in M^{\mathbb{P}}$  s.t.  $\tau_G = S$ .

Let  $\vec{\varphi}(x)$  denote  $\varphi(x, \pi, \vec{\sigma}) \in \mathcal{FL}_{M, \mathbb{P}}$ .

Now  $S = \{\theta_G \mid \theta \in \text{dom}(\pi) \wedge M[G] \models (\theta \in \pi \wedge \vec{\varphi}(\theta))\}$ . Define  $\tau := \{(\theta, p) \mid \theta \in \text{dom}(\pi) \wedge p \in \mathbb{P} \wedge p \Vdash \vec{\varphi}(\theta)\}$ .

Then  $\tau \in M^{\mathbb{P}}$  by the Definability Lemma and  $\tau_G = \{\theta_G \mid \theta \in \text{dom}(\pi) \wedge \exists p \in G : p \Vdash (\theta \in \pi \wedge \vec{\varphi}(\theta))\}$ . By definition of forcing  $\tau_G \subseteq S$ . To prove  $S \subseteq \tau_G$  fix  $\theta_G \in S$  with  $\theta \in \text{dom}(\pi)$  then  $M[G] \models \pi \wedge \vec{\varphi}(\theta)$  so by the Truth Lemma we can find  $p \in G$  s.t.  $p \Vdash \theta \in \pi \wedge \vec{\varphi}(\theta)$ . Thus  $(\theta, p) \in \tau$  so  $\theta_G \in \tau_G$  thus  $S \in M[G]$ .  $\square$

**Theorem.**  $M[G] \models \text{ZFC}$

*Proof.* Powerset: It is enough to prove that if  $a \in M[G]$  then  $\exists b \in M[G]$  s.t.  $\mathcal{P}(a) \cap M[G] \subseteq b$ . Then by separation  $\mathcal{P}(a) \cap M[G] \in M[G]$ .

Fix  $\tau \in M^{\mathbb{P}}$  s.t.  $\tau_G = a$ . Let  $Q = (\mathcal{P}(\text{dom}(\tau) \times \mathbb{P}))_M$  i.e. all names  $\theta \in m^{\mathbb{P}}$  s.t.  $\theta \subseteq \text{dom}(\tau)$ .

Let  $\pi = Q \times \{1\}$  and  $b = \pi_G = \{\theta_G \mid \theta \in Q\}$ . Now let  $c \in \mathcal{P}(a) \cap M[G]$ , we show  $c \in b$ . Fix  $\gamma \in M^{\mathbb{P}}$  s.t.  $\gamma_G = c$  and let  $\theta = \{(\sigma, p) \mid \sigma \in \text{dom} \tau \wedge p \Vdash \sigma \in \gamma\}$

$\theta \in M$  by the Definability Lemma. Since  $\theta \in Q$ , we're done if we can show  $\theta_G = c$ .  $\theta_G \subseteq c$  as  $\theta_G = \{\sigma_G \mid \sigma \in \text{dom}(\tau), \exists p \in G : p \Vdash \sigma \in \gamma\}$  and by definition of  $\Vdash$  such  $\sigma_G \in \gamma_G = c$ . Since  $c \subseteq a = \tau_G$  each element of  $c$  is some  $\sigma_G$  for some  $\sigma \in \text{dom}(\tau)$ . Since  $\sigma_G \in c = \gamma_G$  by the Truth Lemma there is  $p \in G$  s.t.  $p \Vdash (\sigma, p) \in \theta$  and  $\sigma_G \in \theta_G$ , so  $c \supseteq \theta_G$ .

Replacement: Let  $a \in M[G]$   $(\forall x \in a : \exists y : \vec{\varphi}(x, y))_{M[G]}$  and find  $b \in M[G]$  s.t.  $(\forall x \in a : \exists y \in b : \vec{\varphi}(x, y))_{M[G]}$  so  $\vec{\varphi} \in \mathcal{FL}_{M, \mathbb{P}}$  is an abbreviation for  $\varphi$  with names/constants as parameters as in the proof of the Axiom of Separation.

Say  $a = \tau_G$  so  $M[G] \models \forall x \in \tau : \exists y : \vec{\varphi}(x, y)$ . Working in  $M$  using the Definability Lemma let  $Q$  be a set of names s.t.  $\forall p \in \mathbb{P} : \forall \sigma \in \text{dom} \tau$  if there is  $\theta$  s.t.  $p \Vdash \sigma, \vec{p}$  then at least one such  $\theta \in Q$ .

How do we know such a set (not a proper class)  $Q$  exists? By the Montague-Levy Reflection Theorem in  $M : \forall p : \forall \sigma : \exists \theta$  s.t.  $p \Vdash \varphi(\sigma, \theta)$  holds in  $M$  then it holds restricted to some  $(V_\sigma)_M$  s.t.  $\mathbb{P}, \tau \in (V_\sigma)_M$  so we could take  $Q = M^{\mathbb{P}} \cap (V_\sigma)_M$ .

Let  $\pi = Q \times \{1\}$  and  $b = \pi_G = \{\theta_G \mid \theta \in Q\}$ . Fix  $x \in a$  so  $x = \sigma_G$  for some  $\sigma \in \text{dom}(\tau)$ . Then  $M[G] \models \vec{\varphi}(\sigma, \theta)$  and then by the Truth Lemma there is  $p \in G$  s.t.  $p \Vdash \vec{\varphi}(\sigma, \theta)$ . Then there is a  $\theta \in G$  s.t.  $p \Vdash \vec{\varphi}(\sigma, \theta)$  if  $y = \theta_G$  then  $y \in b$  and  $\vec{\varphi}(x, y)_{M[G]}$ .  $\square$

**Lemma** (1 (cont'd)). 5.  $p \Vdash \neg \varphi$  iff  $\neg \exists q \leq p : q \Vdash \varphi$  and  $p \Vdash \varphi$  iff  $\neg \exists q \leq p : q \Vdash \neg \varphi$

6.  $p \Vdash \varphi \longrightarrow \psi$  iff  $\neg \exists q \leq p : (q \Vdash \varphi \wedge q \Vdash \neg \psi)$

7.  $p \Vdash \varphi \wedge \psi$  iff  $\{q \leq p \mid q \Vdash \varphi \text{ or } q \Vdash \psi\}$  is dense below  $p$ .

Using the Truth Lemma. 5.  $(\longrightarrow)$  is clear.

$(\longleftarrow)$  Assume  $p \nVdash \neg \varphi$  then we can find some generic  $G$  with  $p \in G$  and  $M[G] \models \varphi$ .

By the Truth Lemma there is  $r \in G$  s.t.  $r \Vdash \varphi$ . Since  $G$  is a filter we can find  $q \in G$  s.t.  $q \leq r$  and  $q \leq p$  then  $q \Vdash \varphi$  by (3). So  $p \Vdash \neg \varphi$ .

6. follows from (5) and (4).

7.

$$\begin{aligned} p \nVdash \neg \varphi \longrightarrow \psi &\stackrel{(6)}{\iff} \exists r \leq p : (r \Vdash \neg \varphi \wedge r \Vdash \neg \psi) \\ &\stackrel{(5)}{\iff} \exists r \leq p : \forall q \leq r : q \Vdash \varphi \wedge q \nVdash \psi \end{aligned}$$

so  $p \Vdash \varphi \vee \psi \iff \forall r \leq p : \exists q \leq r : (q \Vdash \varphi \vee q \Vdash \psi)$

Observe  $M[G] \Vdash \varphi$  and  $p \in G$  then there is a  $q \in G$  with  $q \leq p$  and  $q \Vdash \varphi$ .

□

**Lemma.** *Let  $\varphi(x) \in \mathcal{FL}_{M, \mathbb{P}}$  with only one free variable  $x$ .*

1.  $p \Vdash \forall x : \varphi(x)$  iff  $p \Vdash \varphi(\tau)$  for all  $\tau \in M^{\mathbb{P}}$
2.  $p \Vdash \exists x : \varphi(x)$  iff  $\{q \leq p \mid \exists \tau \in M^{\mathbb{P}} : q \Vdash \varphi(\tau)\}$  is dense below  $p$ .

*Proof.* 1.  $M[G] \models \forall x : \varphi(x)$  iff  $M[G] \models \varphi(\tau)$  for all  $\tau \in M^{\mathbb{P}}$  so  $p \Vdash \forall x : \varphi(x)$  iff for all  $\tau \in M^{\mathbb{P}}$  and all generics  $G$  with  $p \in G$   $M[G] \models \varphi(\tau)$  iff  $\forall \tau \in M^{\mathbb{P}} : p \Vdash \varphi(\tau)$ .

2. By Lemma 1(5) and Lemma 2(1)  $p \Vdash \neg \forall x : \neg \varphi(x) \iff$  for all  $r \leq p : r \nVdash \neg \varphi(x)$  iff by (1)  $\forall r \leq p : \exists \tau \in M^{\mathbb{P}} : r \nVdash \neg \varphi(\tau)$  iff by (5)  $\forall r \leq p : \exists \tau \in M^{\mathbb{P}} : \exists q \leq r : q \Vdash \varphi(\tau)$

□