This project is purposefully open ended, giving you some freedom in how you decide to understand the dynamics. You are free to discuss the project with your classmates, and this is in fact encouraged. However, each of you must submit your *own* version of the analysis, which is **not** allowed to be copied verbatim from your classmates: make the work your own!

Programming exercise 1

Respiratory oscillations in fungi and bacteria are a well studied phenomenon in nature. Here we will study a simplified model of respiration constructed using the metabolism of *Klebsiella aerogenes*. In the system of reactions shown in Equation 1,

$$\begin{aligned} A &\longrightarrow Y \\ B &\longleftrightarrow X \\ X + Y &\longrightarrow P, \end{aligned} \tag{1}$$

X and Y denote the concentration of oxygen and some nutrient, and A and B are metabolite sources, modeled as external parameters whose concentrations can be seen as constant. P is the final product of respiration, carbon dioxide. These reactions can be written as a system of nonlinear differential equations using kinetic expressions for the reaction rates, as shown in Equation 2,

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t} &= k_1 B - k_2 X - \frac{k_3 XY}{(1+k_4 X)^2} \\ \frac{\mathrm{d}Y}{\mathrm{d}t} &= k_5 A - \frac{k_3 XY}{(1+k_4 X)^2}, \end{aligned} \tag{2}$$

where k_1, k_2, k_3, k_4, and k_5 are kinetic parameters.

1. Read "Time-periodic oscillations in a model for the respiratory process of a bacterial culture" by Fairén and Velarde (1979) in the Journal of Mathematical Biology, and see if you agree with them that Equation 2 can be non-dimensionalized to,

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= b - x - \frac{xy}{1+qx^2} \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= a - \frac{xy}{1+qx^2}. \end{aligned} \tag{3}$$

   Supply a meaningful biological interpretation of a, b, and q.

2. Perform a full stability analysis on the system with all tools developed in the lecture. This includes determining the stationary states, the Jacobian, and evaluating the Jacobian at the steady state. Find out how trace T and determinant Δ of the Jacobian depend on the parameters, a, b and q. Try to derive equations for which $T = 0$ and $T^2 = 4\Delta$. Plot these lines in the (a, b)-parameter space for a fixed value of q (e.g. $q = 1$).
   Hints:

   1. You should find that the steady state for x is $\bar{x} = b - a$. Therefore, only parameter combinations, in which $b > a$ should be considered.
   2. The steady state of y is
   $$\bar{y} = \frac{a}{\bar{x}}\left(1 + q\bar{x}^2\right).$$
   3. The Jacobian at the steady state is
   $$\mathbf{A} = \begin{pmatrix} -1 - \frac{a}{b-a}\frac{1-q(b-a)^2}{1+q(b-a)^2} & -\frac{b-a}{1+q(b-a)^2} \\ -\frac{a}{b-a}\frac{1-q(b-a)^2}{1+q(b-a)^2} & -\frac{b-a}{1+q(b-a)^2} \end{pmatrix}. \tag{4}$$
   (Check also the reference given above)

3. Numerically integrate the system for different parameter values and provide time and phase plots for stable/unstable nodes/foci. Choose suitable initial conditions. Confirm your analytical results with numerical integration or, if you had problems deriving the analytical results, investigate the system using numerical integration only.

Report all your results in a Jupyter Notebook and upload the notebook by **Wednesday, May 24, 2023**.