Questions

1. Analytically solve the following initial value problems using the method of the *variation of the constant*:
   (a) $\dot{x} + x \cdot t = t, \quad x(0) = 0$
   (b) $y' + \frac{y}{x} = \cos x, \quad y(\pi/2) = 2$

2. According to *Torricelli's law*, the velocity of a liquid flowing from a hole of a container is dependent on the height of fluid above the opening by $v = \sqrt{2gh}$, where h is the height of the fluid and $g = 9.81\,\mathrm{m\,s^{-2}}$ is the acceleration due to gravity. If the hole has an area F, the rate of volume leaving the container is given by $F \cdot v$, and therefore the volume in the container changes according to

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\gamma\sqrt{h} \quad \text{with} \quad \gamma = \sqrt{2g} \cdot F.$$

   (a) Consider a cylindrical container with (constant) cross-section A. Set up the differential equation for the height of fluid $h(t)$ as function of time. Solve this equation and sketch the solution.
   (b) When is the container empty when the initial height was $h(0) = h_0$?
   (c) Verify your result experimentally by using an empty drinks can of your choice.
   (d) Now consider a conical container, for which the cross-section A is proportional to h^2, with a hole in the bottom. Set up the corresponding differential equation for $h(t)$ and solve.
   (e) Can you design a container for which $\frac{\mathrm{d}h}{\mathrm{d}t} = \text{const.}$?

3. Assume that the population of rhinos in a nature reserve can be modelled by the logistic growth equation, i.e. that, without other factors, $\dot{x} = rx\left(1 - \frac{x}{K}\right)$, where x denotes the rhino population, r the relative growth rate at small populations, and K the carrying capacity of the nature reserve. Now assume that by illegal hunting, every year a certain fraction of rhinos is killed. The differential equation now changes to

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{K}\right) - h \cdot x,$$

   where the rate constant h quantifies the illegal hunting activities.
   (a) Calculate the steady states of the system as function of the constant h.
   (b) Sketch a bifurcation diagram using h as the bifurcation parameter.
   (c) Can you identify a critical parameter value?
   (d) What happens at this value to the stability of the steady states?

Submit your solutions by **Wednesday, April 19, 2023**.