QBio 202 - SS 2023 - Exercises 0

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1a) $\frac{d}{dx}\left(\ln(\sin(x))\right)$

$= \cos(x) \cdot \frac{1}{\sin(x)} = \cot(x)$

1b) $\frac{d}{dx}\left(\sin(\ln(x))\right)$

$= \frac{1}{x} \cdot \cos(\ln(x))$

1c) $\frac{d}{dx}\left(e^{1+x^2}\right)$

$= 2x \cdot e^{1+x^2}$

1d) $\frac{d}{dx}\left(\cos(x) \cdot e^{1-2x^2}\right)$

$= -\sin(x) \cdot e^{1-2x^2} + \cos(x) \cdot (-4x) \cdot e^{1-2x^2}$

$= -e^{1-2x^2}\left(4x\cos(x) + \sin(x)\right)$

2a) $\int_0^1 (x+5)^4 \, dx$

$\overset{1)}{=} \int_5^6 u^4 \, du$

$= \left[\frac{1}{5} u^5 \right]_5^6 = \frac{1}{5} (6^5 - 5^5) = \frac{4651}{5} = \underline{\underline{930.2}}$

1) Substitution
$u = x + 5$
$\Rightarrow \frac{du}{dx} = 1$
$\Rightarrow dx = du$

2b) $\int \sin(7x-3) \, dx$

$\overset{1)}{=} \frac{1}{7} \cdot \int \sin(u) \, du$

$= -\frac{1}{7} \cos(u) + C$

$\overset{2)}{=} -\frac{1}{7} \cos(7x-3) + C$

where C is the constant of integration

1) Substitution
$u = 7x - 3$
$\Rightarrow$
$\frac{du}{dx} = 7$
$\Rightarrow dx = \frac{1}{7} du$

2) Undo substitution
$u = 7x - 3$

2c) $\int x \cdot \sin(2x^2) \, dx$

$\overset{1)}{=} \frac{1}{4} \int \sin(u) \, du$

$= -\frac{1}{4} \cos(u) + C$

$\overset{2)}{=} -\frac{1}{4} \cos(2x^2) + C$

where C is the constant of integration

1) Substitution
$u = 2x^2$
$\Rightarrow \frac{du}{dx} = 4x$
$\Rightarrow dx = \frac{du}{4x}$

2) Undo substitution
$u = 2x^2$

2d) $\int e^{\cos(x)} \cdot \sin(x)\, dx$

$= - \int e^{u}\, du$

$= - e^{u} + C$

$= - e^{\cos(x)} + C$

where C is the constant of integration

1) Substitution

$u = \cos(x)$

$\Rightarrow \frac{du}{dx} = -\sin(x)$

$\Rightarrow dx = -\frac{du}{\sin(x)}$

2) Undo substitution

$u = \cos(x)$

3a) The eigenvalues λ of $A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$ are determined by the following equation:

$\det(A - \lambda I) = 0$

$\Leftrightarrow$

$\det \begin{pmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix} = 0$

$\Leftrightarrow$

$(2-\lambda)\cdot(3-\lambda) - 2 = 0$

$\Leftrightarrow$

$6 - 2\lambda - 3\lambda + \lambda^2 - 2 = 0$

$\Leftrightarrow$

$\lambda^2 - 5\lambda + 4 = 0$

$\Leftrightarrow$

$\lambda = \frac{5}{2} \mp \sqrt{\left(\frac{5}{2}\right)^2 - 4} = \frac{5}{2} \mp \sqrt{\frac{25-16}{4}}$

$= \frac{1}{2}(5 \mp 3)$

3a) Continued...

Therefore, the matrix A has the two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 4$.

Cross-checks with λ_i being the eigenvalues of A:

i) $tr(A) = 2 + 3 = \sum_i \lambda_i$ ✓

ii) $det(A) = 6 - 2 = \prod_i \lambda_i$ ✓

The eigenvectors $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ corresponding to the eigenvalue λ are determined by the following equation:

$$(A - \lambda I)\vec{v} = \vec{0}$$

$\Leftrightarrow$

$$\begin{pmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\underline{\lambda = 1}$:

$$\begin{pmatrix} 2-1 & 1 \\ 2 & 3-1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Leftrightarrow \quad v_1 + v_2 = 0$

$\Leftrightarrow \quad v_2 = -v_1$

$\Leftrightarrow$

$$\vec{v} \in \left\{ \begin{pmatrix} z \\ -z \end{pmatrix} ; z \in \mathbb{C} \right\}$$

Cross-check:

$$\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} ✓$$

Therefore, e.g. $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 1$.

3a) Continued ...

$\underline{\lambda = 4:}$

$$\begin{pmatrix} 2-4 & 1 \\ 2 & 3-4 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Leftrightarrow$

$$-2v_1 + v_2 = 0$$

$\Leftrightarrow$ $v_2 = 2v_1$

$\Leftrightarrow$

$$\vec{v} \in \left\{ \begin{pmatrix} z \\ 2z \end{pmatrix} ; z \in \mathbb{C} \right\}$$

Therefore, e.g. $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = 4$.

Cross-check:

$$\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} = 4 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \checkmark$$

3b) The eigenvalues λ of $A = \begin{pmatrix} -1 & 2 \\ -4 & 3 \end{pmatrix}$

are determined by the following equation:

$$\det(A - \lambda I) = 0$$

$\Leftrightarrow$

$$\det \begin{pmatrix} -1-\lambda & 2 \\ -4 & 3-\lambda \end{pmatrix} = 0$$

$\Leftrightarrow$

$$(-1-\lambda)(3-\lambda) + 8 = 0$$

$\Leftrightarrow$

$$-3 + \lambda - 3\lambda + \lambda^2 + 8 = 0$$

$\Leftrightarrow$

$$\lambda^2 - 2\lambda + 5 = 0$$

$\Leftrightarrow$

$$\lambda = 1 \mp \sqrt{1-5} = 1 \mp 2i$$

Therefore, A has the two conjugate complex eigenvalues $\lambda_1 = 1 - 2i$ and $\lambda_2 = 1 + 2i$.

Cross-check with λ_i being the eigenvalues of A:

i) $\operatorname{tr}(A) = -1 + 3 = \sum_i \lambda_i$ ✓

ii) $\det(A) = -3 + 8 = \prod_i \lambda_i$ ✓

3b) continued...

The eigenvector $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ corresponding to the eigenvalue λ are determined by the following equation:

$$(A - \lambda I) \cdot \vec{v} = \vec{0}$$

$\Leftrightarrow$

$$\begin{pmatrix} -1-\lambda & 2 \\ -4 & 3-\lambda \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\underline{\lambda = 1 - 2i:}$

$$\begin{pmatrix} -1-(1-2i) & 2 \\ -4 & 3-(1-2i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Leftrightarrow$

$$(-2+2i)v_1 + 2v_2 = 0$$
$$\wedge \; -4v_1 + (2+2i)v_2 = 0$$

$\Leftrightarrow$

$$v_2 = (1-i)v_1$$

$\Leftrightarrow$

$$\vec{v} \in \left\{ \begin{pmatrix} z \\ (1-i)z \end{pmatrix} ; z \in \mathbb{C} \right\}$$

Cross-check:

$$\begin{pmatrix} -1 & 2 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} -1+2-2i \\ -4+3-3i \end{pmatrix}$$
$$= \begin{pmatrix} 1-2i \\ -1-3i \end{pmatrix} = (1-2i) \begin{pmatrix} 1 \\ 1-i \end{pmatrix} \checkmark$$

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3b) Continued...

$\lambda = 1+2i$:

If a real matrix has a non-real (complex) eigenvalue, both that eigenvalue and the corresponding eigenvectors come in complex pairs.

Therefore, e.g. $\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = 1+2i$.

Cross-check:

$$\begin{pmatrix} -1 & 2 \\ -4 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1+i \end{pmatrix} = \begin{pmatrix} -1+2+2i \\ -4+3+3i \end{pmatrix}$$

$$= \begin{pmatrix} 1+2i \\ -1+3i \end{pmatrix}$$

$$= (1+2i) \begin{pmatrix} 1 \\ 1+i \end{pmatrix} \checkmark$$

3c) The eigenvalues λ of

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 2 & 1 \\ -4 & 1 & 2 \end{pmatrix}$$

are determined by the following equation:

$$\det(A - \lambda I) = 0$$

$\Leftrightarrow$

$$\det \begin{pmatrix} 2-\lambda & 0 & 0 \\ -1 & 2-\lambda & 1 \\ -4 & 1 & 2-\lambda \end{pmatrix} = 0$$

$\Leftrightarrow$

$$(2-\lambda)\left((2-\lambda)^2 - 1\right) = 0$$

$\Leftrightarrow$

$$(2-\lambda)(\lambda^2 - 4\lambda + 3) = 0$$

Auxiliary computation:

$$\lambda^2 - 4\lambda + 3 = 0 \Leftrightarrow \lambda = 2 \mp \sqrt{4-3} = 2 \mp 1$$

$\Leftrightarrow$

$$(2-\lambda)(1-\lambda)(3-\lambda) = 0$$

Therefore, the matrix A has the three eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$.

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3c) Continued ...

Cross-checks with λ_i being the eigenvalues of A:

i) $tr(A) = 2+2+2 = \sum_i \lambda_i$ ✓

ii) $det(A) = 2 \cdot (4-1) = \prod_i \lambda_i$ ✓

The eigenvector

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

corresponding to the eigenvalue λ are determined by the following equation:

$$(A - \lambda I) \cdot \vec{v} = \vec{0}$$

$\Leftrightarrow$

$$\begin{pmatrix} 2-\lambda & 0 & 0 \\ -1 & 2-\lambda & 1 \\ -4 & 1 & 2-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

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3c) continued...

$\underline{\lambda_1 = 1:}$

$$\begin{pmatrix} 2-1 & 0 & 0 \\ -1 & 2-1 & 1 \\ -4 & 1 & 2-1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Leftrightarrow$

$$v_1 = 0$$
$$\wedge\ v_2 + v_3 = 0$$

$\Leftrightarrow$

$$\vec{v} \in \left\{ \begin{pmatrix} 0 \\ z \\ -z \end{pmatrix} ; z \in \mathbb{C} \right\}$$

Therefore, e.g. $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 1$.

Cross-check:

$$\begin{pmatrix} 2 & 0 & 0 \\ -1 & 2 & 1 \\ -4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \checkmark$$

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3c) Continued ...

$\underline{\lambda = 2:}$

$$\begin{pmatrix} 2-2 & 0 & 0 \\ -1 & 2-2 & 1 \\ -4 & 1 & 2-2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Leftrightarrow$

$v_3 = v_1$

$\wedge \; v_2 = 4 v_1$

$\Leftrightarrow$

$$\vec{v} \in \left\{ \begin{pmatrix} z \\ 4z \\ z \end{pmatrix} ; z \in \mathbb{C} \right\}$$

Therefore, e.g. $\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = 2$.

Cross-check:

$$\begin{pmatrix} 2 & 0 & 0 \\ -1 & 2 & 1 \\ -4 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} \checkmark$$

3c) continued...

$\underline{\lambda = 3:}$

$$\begin{pmatrix} 2-3 & 0 & 0 \\ -1 & 2-3 & 1 \\ -4 & 1 & 2-3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Leftrightarrow$

$$v_1 = 0$$
$$\wedge \; v_3 = v_2$$

$\Leftrightarrow$

$$\vec{v} \in \left\{ \begin{pmatrix} 0 \\ z \\ z \end{pmatrix} ; z \in \mathbb{C} \right\}.$$

Therefore, e.g. $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is an eigenv~~ector of A~~ corresponding to the eigenvalue $\lambda_3 = 3$.

Cross-check:

$$\begin{pmatrix} 2 & 0 & 0 \\ -1 & 2 & 1 \\ -4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} = 3 \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \checkmark$$

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