Questions

1. *Harmonic oscillator*
An important system in physics and biology is the harmonic oscillator. The second-order linear differential equation is given by
$$\ddot{x} + \omega^2 x = 0, \quad \text{where} \quad \ddot{x} = \frac{d^2 x}{dt^2}.$$
By introducing the variable $y = \dot{x}$, this second-order equation can be rewritten as a system of two first-order equations:
$$\begin{array}{rcl} \dot{x} & = & y \\ \dot{y} & = & -\omega^2 x \end{array} \quad \text{or} \quad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$
   (a) Determine the general solution $x(t) = c_1 u_1 e^{\lambda_1 t} + c_2 u_2 e^{\lambda_2 t}$ by finding the eigenvalues $\lambda_{1/2}$ and eigenvectors $u_{1/2}$ of the matrix $A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$
   (b) By writing the integration constants as the complex conjugate pair $c_1 = \frac{A}{2} e^{i\varphi}$ and $c_2 = \frac{A}{2} e^{-i\varphi}$, show that the general solution can be written as $x(t) = A\cos(\omega t + \varphi)$.
   (c) How do amplitude A and phase shift φ depend on the initial conditions $x_0 = x(0)$ and $y_0 = y(0) = \dot{x}(0)$? (Hint: you will need that $\tan\varphi = \sin\varphi / \cos\varphi$ and $\cos^2\varphi + \sin^2\varphi = 1$)
   (d) Show that the quantity $E = \frac{\dot{x}^2}{2} + \frac{\omega^2}{2} x^2$ is a conserved quantity. For this, show that $\frac{dE}{dt} = 0$. Hint: you *don't* need the solutions to the differential equations. The differential equation $\ddot{x} + \omega^2 x = 0$ is enough.

2. *Damped oscillator*
The harmonic oscillator is an idealisation. In reality, all systems lose energy due to friction. This is reflected by a term proportional to the velocity. Consider
$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0$$
with some positive γ.
   (a) Rewrite the equation as a system of two linear equations.
   (b) Show that the characteristic polynomial is $\lambda^2 + 2\gamma\lambda + \omega^2$ and that for all $\gamma > 0$ the steady state is stable.
   (c) For $\gamma = \omega$ the system is degenerate and has two identical eigenvalues. Show that the function $x(t) = t \cdot e^{-\omega t}$ solves the equation $\ddot{x} + 2\omega\dot{x} + \omega^2 x = 0$. (This is called the *critically damped case*)

3. *Forced oscillator*
We now consider a harmonic oscillator with a periodic external force. The equation reads
$$\ddot{x} + 2\gamma\dot{x} + \omega^2 = f_0 \cos\Omega t.$$
This equation is much easier to solve in the complex number space.
   (a) Make the assumption that the particular solution also describes an oscillation with frequency Ω and show that the function $x(t) = Ae^{i(\Omega t - \varphi)}$ solves
$$\ddot{x} + 2\gamma\dot{x} + \omega^2 = f_0 e^{i\Omega t}$$
when
$$A\left(-\Omega^2 + 2i\gamma\Omega + \omega^2\right) = f_0 e^{i\varphi t}.$$
   (b) Derive expressions how the amplitude A and the phase shift φ depend on the parameters and sketch these values as functions of the forcing period Ω for different values of γ.
   (c) What happens in the case with no friction $\gamma = 0$ when the forcing frequency approaches the *eigenfrequency* ω?

Submit your solutions by **Wednesday, April 26, 2023**.