Ecuaciones Lineales. Método LU.

$Ax = b$ se factoriza A de la forma $A = LU$, donde L y U son dos matrices triangulares.

$$Ax = b \Rightarrow LUx = b \Rightarrow \left\{ \begin{array}{l} Lz = b \\ Ux = z \end{array} \right.$$

Para mayor claridad describimos el método con un ejemplo genérico de un sistema con 4 ecuaciones y 4 incógnitas.

Obención de L y U.

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & L_{32} & 1 & 0 \\ L_{41} & L_{42} & L_{43} & 1 \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{pmatrix}$$

Fila L	$*$	Columna U		
1	$*$	1	$\Rightarrow$	$U_{11} = A_{11}$
1	$*$	2	$\Rightarrow$	$U_{12} = A_{12}$
1	$*$	3	$\Rightarrow$	$U_{13} = A_{13}$
1	$*$	4	$\Rightarrow$	$U_{14} = A_{14}$
				$\Rightarrow U_{1j} = A_{1j}$
2	$*$	1	$\Rightarrow$	$L_{21} = \frac{1}{U_{11}} A_{21}$
2	$*$	2	$\Rightarrow$	$U_{22} = A_{22} - L_{21}U_{12}$
2	$*$	3	$\Rightarrow$	$U_{23} = A_{23} - L_{21}U_{13}$
2	$*$	4	$\Rightarrow$	$U_{24} = A_{24} - L_{21}U_{14}$
				$\Rightarrow U_{2j} = A_{2j} - L_{21}U_{1j}$
3	$*$	1	$\Rightarrow$	$L_{31} = \frac{1}{U_{11}} A_{31}$
3	$*$	2	$\Rightarrow$	$L_{32} = \frac{1}{U_{22}} (A_{32} - L_{31}U_{12})$
3	$*$	3	$\Rightarrow$	$U_{33} = A_{33} - (L_{31}U_{13} + L_{32}U_{23})$
3	$*$	4	$\Rightarrow$	$U_{34} = A_{34} - (L_{31}U_{14} + L_{32}U_{24})$
				$\Rightarrow U_{3j} = A_{3j} - \sum_{k=1}^{3-1} L_{3k}U_{kj}$
4	$*$	1	$\Rightarrow$	$L_{41} = \frac{1}{U_{11}} A_{41}$
4	$*$	2	$\Rightarrow$	$L_{42} = \frac{1}{U_{22}} (A_{42} - L_{41}U_{12})$
4	$*$	3	$\Rightarrow$	$L_{43} = \frac{1}{U_{33}} (A_{43} - L_{41}U_{13} - L_{42}U_{23})$
4	$*$	4	$\Rightarrow$	$U_{44} = A_{44} - (L_{41}U_{14} + L_{42}U_{24} + L_{43}U_{34})$

De aquí inducimos para el caso general:

$$U_{ij} = A_{ij} - \sum_{k=1}^{i-1} L_{ik}U_{kj} \quad i \leq j \quad ; \quad U_{ij} = 0 \quad i > j$$

$$L_{ij} = \frac{1}{U_{jj}} (A_{ij} - \sum_{k=1}^{j-1} L_{ik}U_{kj}) \quad i > j \quad ; \quad L_{ij} = 0 \quad i < j \quad ; \quad L_{ii} = 1$$

$$Ax = b \Rightarrow LUx = b \Rightarrow \left\{ \begin{array}{l} Lz = b \\ Ux = z \end{array} \right.$$

$$Lz = b \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & L_{32} & 1 & 0 \\ L_{41} & L_{42} & L_{43} & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

$$\begin{array}{lll} z_1 = b_1 & & \\ L_{21}z_1 + z_2 = b_2 & \Rightarrow & z_2 = b_2 - L_{21}z_1 \\ L_{31}z_1 + L_{32}z_2 + z_3 = b_3 & \Rightarrow & z_3 = b_3 - (L_{31}z_1 + L_{32}z_2) \\ L_{41}z_1 + L_{42}z_2 + L_{43}z_3 + z_4 = b_4 & \Rightarrow & z_4 = b_4 - (L_{41}z_1 + L_{42}z_2 + L_{43}z_3) \end{array}$$

De aquí inducimos para el caso general de n ecuaciones con n incógnitas:

$$z_1 = b_1 \quad ; \quad z_i = b_i - \sum_{j=1}^{i-1} L_{ij} z_j$$

$$Ux = z \Rightarrow \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$$

$$\begin{array}{lll} U_{44}x_4 = z_4 & \Rightarrow & x_4 = \frac{1}{U_{44}} z_4 \\ U_{33}x_3 + U_{34}x_4 = z_3 & \Rightarrow & x_3 = \frac{1}{U_{33}}(z_3 - U_{34}x_4) \\ U_{22}x_2 + U_{23}x_3 + U_{24}x_4 = z_2 & \Rightarrow & x_2 = \frac{1}{U_{22}}(z_2 - U_{23}x_3 - U_{24}x_4) \\ U_{11}x_1 + U_{12}x_2 + U_{13}x_3 + U_{14}x_4 = z_1 & \Rightarrow & \\ x_1 = \frac{1}{U_{11}}(z_1 - U_{12}x_2 - U_{13}x_3 - U_{14}x_4) & & \end{array}$$

De aquí inducimos para el caso general de n ecuaciones con n incógnitas:

$$x_n = \frac{1}{U_{nn}} z_n \quad ; \quad x_i = \frac{1}{U_{ii}}(z_i - \sum_{j=i+1}^{n} U_{ij} x_j)$$

Ejemplo.

$$Ax = b \quad ; \quad A = \begin{pmatrix} 9 & 1 & 1 \\ 2 & 10 & 3 \\ 3 & 4 & 11 \end{pmatrix} \qquad b = \begin{pmatrix} 10 \\ 19 \\ 0 \end{pmatrix}$$

1. Factorización $A = LU$.

$$\begin{pmatrix} 9 & 1 & 1 \\ 2 & 10 & 3 \\ 3 & 4 & 11 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$$

Igualando términos obtenemos:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2/9 & 1 & 0 \\ 1/3 & 99/264 & 1 \end{pmatrix} \quad ; \quad U = \begin{pmatrix} 9 & 1 & 1 \\ 0 & 88/9 & 25/9 \\ 0 & 0 & 9.625 \end{pmatrix}$$

2. Resolución de $Lz = b$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 2/9 & 1 & 0 \\ 1/3 & 99/264 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 19 \\ 0 \end{pmatrix} \Longrightarrow z = \begin{pmatrix} 10 \\ 151/9 \\ -9.625 \end{pmatrix}$$

3. Resolución de $Ux = z$.

$$\begin{pmatrix} 9 & 1 & 1 \\ 0 & 88/9 & 25/9 \\ 0 & 0 & 9.625 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 151/9 \\ -9.625 \end{pmatrix} \Longrightarrow x = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Ejercicio.

$$Ax = b \quad ; \quad A = \begin{pmatrix} 1 & 2 & -1 \\ -2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Resumen de fórmulas.

$Ax = b$ se factoriza A de la forma $A = LU$, donde L y U son dos matrices triangulares.

$$Ax = b \Rightarrow LUx = b \Rightarrow \begin{cases} Lz = b \\ Ux = z \end{cases}$$

Los indices $i, j, k = 0, \ldots, n-1$ cuando se programe en C++.

Si $i < j$

$$L_{ij} = 0 \ \ ; \ \ U_{ij} = A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj}$$

Si $i = j$

$$L_{ii} = 1 \ \ ; \ \ U_{ij} = A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj}$$

Si $i > j$

$$U_{ij} = 0 \ \ ; \ \ L_{ij} = \frac{1}{U_{jj}} (A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}) \ \ ;$$

$$z_0 = b_0 \ \ ; \ \ z_i = b_i - \sum_{j=0}^{i-1} L_{ij} z_j \ \ ; \ \ x_{n-1} = \frac{1}{U_{n-1n-1}} z_{n-1} \ \ ; \ \ x_i = \frac{1}{U_{ii}} (z_i - \sum_{j=i+1}^{n-1} U_{ij} x_j)$$