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INFORMATION
TECHNOLOGY

FIT2004
Algorithms and Data Structures

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Referencing materials by
Nathan Companez, Aamir Cheema, Arun Konagurthu and Lloyd Allison

GROUP OF EIGHT AUSTRALIA

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Ready?

Agenda

- Proof of Correctness
  - Loop invariants
  - Termination

Agenda

- Proof of Correctness
  - Loop invariants
  - Termination

Covered in Lecture 02 using sorting algorithms for case study

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Let us begin…

Proof of Correctness
for Algorithms

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- Why?

Proof of Correctness
for Algorithms

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- Why?
- Why not just program it out and run it?

Proof of Correctness
for Algorithms

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- Why?
- Why not just program it out and run it?
  - Development cost can be costly
  - Resources can be limited

Proof of Correctness
for Algorithms

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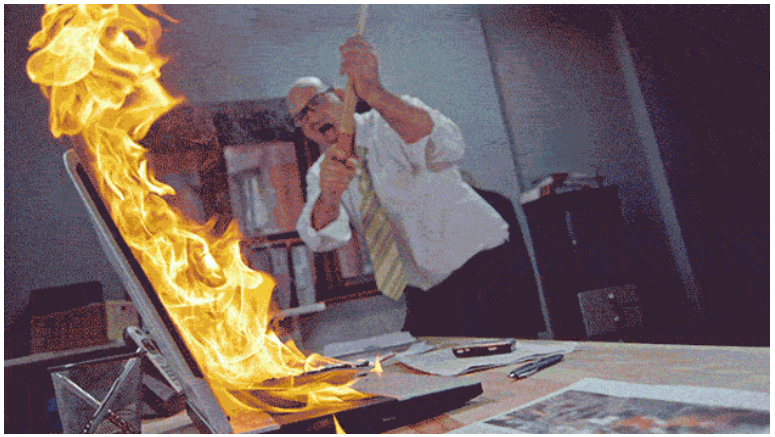
- Why?
- Why not just program it out and run it?
  - Development cost can be costly
  - Resources can be limited
  - Damage can happen!



Consequences of errors

Explosion of unmanned Ariane 5 rocket in 1996

- Exploded within 40 seconds after launch
- Horizontal velocity incorrectly computed
- Loss ~$7 Billion dollars

Consequences of errors

American Patriot Missile battery in Saudi Arabia failed to intercept an incoming Iraqi Scud Missile

- Killed 28 US soldiers
- Incorrect computation of the time since boot

Consequences of errors

Incorrect maps almost started a war between Costa Rica and Nicaragua

Proof of Correctness
for Algorithms

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APPS & SOFTWARE

People are still driving into lakes because their GPS tells them to

Proof of Correctness
for Algorithms

Proof of Correctness
for Algorithms

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- Why?
- Why not just program it out and run it?
  - Development cost can be costly
  - Resources can be limited
  - Things can go really wrong

- But you can’t argue against testing right?

Proof of Correctness
for Algorithms

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- Why?
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  - How often you think your solution work…

Proof of Correctness
for Algorithms

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- But you can't argue against testing right?
  - How often you think your solution work…
    THEN SCUMBAG IAN MINUS YOUR MARKS???

Proof of Correctness
for Algorithms

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- Why?
- Why not just program it out and run it?
  - Development cost can be costly
  - Resources can be limited
  - Things can go really wrong

- But you can't argue against testing right?
  - How often you think your solution work…
  - You can't test everything…
  - But you can apply it when you design algorithms!

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Questions?

Proof of Correctness
for Algorithms

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- So how do we prove?
  - Termination
  - Loop invariant

Proof of Correctness
for Algorithms

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- Termination
  - Program needs to end to return the result
  - If it doesn't end, then you don't have your result

Proof of Correctness
for Algorithms

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- Termination
  - Program needs to end to return the result
  - If it doesn't end, then you don't have your result

- Loop invariant = constants in a loop
  - What keeps happening over and over
  - Will lead to the solution/ rightness/ result

Proof of Correctness
for Algorithms

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- Let us have a relatable example
- Getting a Degree from Monash

Proof of Correctness
for Algorithms

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- Let us have a relatable example

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  - You have a loop go through semester after semester…

Proof of Correctness
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Proof of Correctness
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  - What must not change for you to get your degree in every semester?

Proof of Correctness
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- Let us have a relatable example

- Getting a Degree from Monash
  - You have a loop go through semester after semester…
    - Only when this loop terminate, you get your degree
  - What must not change for you to get your degree in every semester?
    - Number of sleep per night?
    - Number of anime you watch?
    - Number of games you play?

Proof of Correctness
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    - Passing at least 1 unit per semester!

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- Let us have a relatable example

- Getting a Degree from Monash
  - You have a loop go through semester after semester…
    - Only when this loop terminate, you get your degree
  - What must not change for you to get your degree in every semester?
    - Number of sleep per night?
    - Number of anime you watch?
    - Number of games you play?
    - Passing at least 1 unit per semester!
      - Eventually, you run out of units…

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Questions?

Proof of Correctness
for Algorithms

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- Now let us look at the actual algorithm examples

Proof of Correctness
for Finding Minimum

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- We'll use our code one (sent via Slack)

```
1   def find_min(array):
2       """
3       Find the minimum...
4       Does this work?
5       """
6       my_min = array[0]
7       index = 1
8       while index < len(array):
9           if array[index] < my_min:
10              my_min = array[index]
11          index = index + 1
12      return my_min
```

Proof of Correctness
for Finding Minimum

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- Does it terminate?
- What is the loop invariant?

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Proof of Correctness
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Proof of Correctness
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- Does it terminate?
  - Yes
  - But how do you explain it?
    - array is finite
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    - my_min initialized to the first element

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  - How would you write it?
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    - my_min has the minimum value in array[0...index]

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- Combination of the above explains why the algorithm works!

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10              my_min = array[index]
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```

connect to the termination

Index eventually reach the end of array; ie we have the minimum value of the entire array

Proof of Correctness
for Finding Minimum

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    - array is finite
    - index starts from 1
    - At the end of each loop, increment by 1 and we will eventually reach the end of the array to terminate
- What is the loop invariant?
  - How would you write it?
    - my_min initialized to the first element } Initialization
    - my_min has the minimum value in array[0...index]
- Combination of the above explains why the algorithm works!

Proof of Correctness
for Finding Minimum

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- Does it terminate?
  - But how do you explain it?
    - array is finite
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9           if array[index] < my_min:
10              my_min = array[index]
11          index = index + 1
12      return my_min
```

- What is the loop invariant?
  - How would you write it?
    - my_min initialized to the first element — Initialization
    - my_min has the minimum value in array[0...index] — Maintenance
- Combination of the above explains why the algorithm works!

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Questions?

Proof of Correctness
for Algorithms

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- This is commonly asked in the exam

Proof of Correctness
for Algorithms

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- This is commonly asked in the exam
  - We will show you an algorithm (not code usually)
  - Then ask you to explain why it is correct
  - <past year.jpg>

Proof of Correctness
for Algorithms

(b) (2 marks) Write a useful loop invariant for the following function which computes the sum of all even numbers in a list. You must write the loop invariant that holds at the end of each iteration of the for loop (write next to #INVARIANT). Using the loop invariant, prove that the function correctly computes the sum of all even numbers in the list.

```
def sumOfEvens(aList):
    total = 0
    n = len(aList)
    for i in range(n):
        if aList[i]%2 == 0:
            total += aList[i]

        #INVARIANT:

    return total
```

#INVARIANT: total is the sum of all even numbers in aList[:i+1]
At the end of the last iteration, i = n -1. Thus, total is the sum of all even numbers in aList[:n] when the for loop terminates, i.e., total is the sum of all even numbers in the list.

len -1

Proof of Correctness
for Algorithms

(b) (1 mark) Write a loop invariant for the Floyd-Warshall algorithm that can be used to show that the algorithm correctly computes all-pairs shortest distances.

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Questions?

Proof of Correctness
for Binary Search

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- Now let us try to binary search
  - Something we are all familiar with…
  - OR DO WE?

Proof of Correctness
for Binary Search

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- Now let us try to binary search
  - Something we are all familiar with…
  - OR DO WE?

```
1   def binary_search(array, key):
2       """
3       Binary search for key
4       Does this work?
5       Note: We don't terminate it earlier when we find the key because we use lo for the index of key
6       """
7       lo = 0
8       hi = len(array)
9       while lo < hi:
10          mid = (lo+hi) // 2
11          if key >= array[mid]:
12              lo = mid
13          else:
14              hi = mid
15      if len(array) > 0 and array[lo] == key:
16          print("key found at index " + str(lo))
17      else:
18          print("key not found")
```

Proof of Correctness
for Binary Search

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- Same questions
  - Do it terminate?
  - What is the loop invariant?

```
def binary_search(array, key):
    """
    Binary search for key
    Does this work?
    Note: We don't terminate it earlier when we find the key because we use lo for the index of key
    """
    lo = 0
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Proof of Correctness
for Binary Search

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- It doesn't terminate
  - Terminates when lo >= hi
  - In the loop, lo < mid < hi and each iteration move lo or hi to mid
  - So eventually lo and hi will meet it right?

```
1  def binary_search(array, key):
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Proof of Correctness
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- It doesn't terminate
  - Terminates when lo >= hi
  - In the loop, lo < mid < hi and each iteration move lo or hi to mid
  - So eventually lo and hi will meet it right?
  - No, we have a situation…
    - Lo = 5
    - Hi = 6
    - So mid = 5
    - Then what is lo = mid?

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    - Then what is lo = mid?
    - How would you fix this?

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    - Change to while lo < hi – 1?

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Proof of Correctness
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- It doesn't terminate
  - Terminates when lo >= hi
  - In the loop, lo < mid < hi and each iteration move lo or hi to mid
  - So eventually lo and hi will meet it right?
  - No, we have a situation…
    - Lo = 5
    - Hi = 6 6 is exclusive, since only index 0 - 5
    - So mid = 5
    - Then what is lo = mid? = 5, hi -1 =5
    - How would you fix this?
    - Change to while lo < hi – 1?
    - Yes because hi is exclusive! Initialized to len(array)!
    - So search space shrink till size of 1

```
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    Binary search for key
    Does this work?
    Note: We don't terminate it earlier when we find
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```

Proof of Correctness
for Binary Search

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- It doesn't terminate
- So please be careful, certain test cases can cause your program to run forever…
  - In your assignment, I set a time limit before I kill off your processing thread

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Questions?

Proof of Correctness
for Binary Search

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- What is the loop invariant?

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Proof of Correctness
for Binary Search

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array[lo....hi-1] inclusive
array[lo.....hi] exclusive

- What is the loop invariant?
  - If the key exist in array[0…N] then the key exist in array[lo…hi]

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Proof of Correctness
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  - If the key exist in array[0…N] then the key exist in array[lo…hi]
  - Kinda make sense from the if-else

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Proof of Correctness
for Binary Search

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- What is the loop invariant?
  - If the key exist in array[0…N] then the key exist in array[lo…hi]
  - Kinda make sense from the if-else
  - And there isn't a need to be so complex…

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        print("key found at index " + str(lo))
    else:
        print("key not found")
```

Proof of Correctness
for Binary Search

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- What is the loop invariant?
  - If the key exist in array[0…N] then the key exist in array[lo…hi]
  - Kinda make sense from the if-else
  - And there isn't a need to be so complex…
- But you can do the following:
  1. Define the invariant
  2. Code based on the invariant

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Questions?

Proof of Correctness

TL;DR

- Termination
- Loop invariant

Proof of Correctness

TL;DR

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- Termination
  - What updates/ step to ensure the loop will be exited
- Loop invariant

Proof of Correctness

TL;DR

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- Termination
  - What updates/ step to ensure the loop will be exited
  - Or function will reach base case?
- Loop invariant

Proof of Correctness

TL;DR

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- Termination
  - What updates/ step to ensure the loop will be exited for iteration
  - Or function will reach base case? for recursion
- Loop invariant
  - What doesn't change?
  - But what doesn't change but help you reach the output? Or closer towards the answer.

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Questions?

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Thank you