# Inverse Matrices

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## Contents



## 1 Inverse Matrices

The **inverse matrix** $A^{-1}$ of a matrix $A$ is a matrix such that $A^{-1}A = I$, where $I$ is the identity matrix.

Inverse matrices are *commutative*, i.e. $A^{-1}A = AA^{-1} = I$. This is to say, if we have an inverse matrix $A^{-1}$ such that $A^{-1}A = I$, and another matrix $B$ such that $AB = I$, then $A^{-1}$ must equal $B$:

$$\begin{aligned} AB &= I \\ A^{-1}AB &= A^{-1}I \\ B &= A^-1 \end{aligned}$$

Inverse matrices are also *unique*. In other words, if $AB = AC = I$, then $B = C$:

$$\begin{aligned} AB = AC = I \\ A^{-1}AB = A^{-1}AC = A^{-1} \\ B = C = A^{-1} \end{aligned}$$

Not all matrices have inverses. Matrices that do have inverses are called **invertible** (or **nonsingular**). This is useful as when $A\mathbf{x} = \mathbf{b}$ has a unique solution, and $A$ is square and invertible, then $\mathbf{x} = A^{-1}\mathbf{b}$.

### 1.1 Properties of Inverse Matrices

The inverse of an inverse matrix is the original matrix:

$$(A^{-1})^{-1} = A \tag{1}$$

The inverse of two matrices multiplied by each other is:

$$(AB)^{-1} = B^{-1}A^{-1} \tag{2}$$

## 2 Finding Inverse Matrices

### 2.1 $2 \times 2$

Given a $2 \times 2$ matrix $A$ such that:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Call $ad - bc$ the **determinant** of $A$. If $ad - bc \neq 0$, then:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{3}$$

Note that if $ad = bc$, then the matrix is not invertible.

### 2.2 Larger Than $3 \times 3$

Consider the **elementary matrix** $E$: a matrix obtained from $I$ by performing a single row operation. Notice that applying that single row operation to another matrix $A$ is equivalent to computing $EA$.

If $A$ is invertible, then its reduced row echelon form is $I$. This is because if $A$ is invertible, then there is a unique solution to $A\mathbf{x} = \mathbf{b}$ (which is $A^{-1}\mathbf{b}$), which means that there must be a pivot in every column of the RREF of $A$. If there is a pivot in every column of $A$, then by the definition of RREF those pivots must be 1, and every other number is 0, which becomes the identity matrix.

Therefore, if $A$ is invertible, then we can row reduce $A$ to $I$ using row operations. We can encode these row operations using elementary matrices like so:

$$E_k E_{k-1} \cdots E_1 A = I$$

Thus, it can be seen that in general, the inverse of $A$ is:

$$A^{-1} = E_k E_{k-1} \cdots E_1 \tag{4}$$

To find this series of elementary matrices, we can create the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$. When we row reduce this augmented matrix such that the $A$ part becomes $I$, the $I$ part of the augmented matrix will become $A^{-1}$ (the series of row operations encoded as the product of elementary matrices).