Matrix Operations

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Contents

1 Matrix Addition

If we think of two matrices as linear transformations, **matrix addition** will only be defined if their codomain and domain are the same; in other words, $A + B$ is only defined if A and B are the same size.

If $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ and $B = \begin{bmatrix} b_{ij} \end{bmatrix}$,

$$\boxed{A + B = \begin{bmatrix} a_{ij} + b_{ij} \end{bmatrix}} \tag{1}$$

2 Matrix Multiplication

If A and B are matrices, then we can consider their associated linear transformations T and S. We define **matrix multiplication** to be analogous to the composition of two transformations, i.e. $T \circ S$.

This requires the codomain of S to be the domain of T. In other words, if A is an $m \times n$ matrix, B must be $n \times p$.

We know that multiplying a matrix by a column vector is done by forming a new matrix where each entry in each row is the dot product between the corresponding row in the matrix and the column vector. For matrix-matrix multiplication, we extend that by considering B to be a matrix of columns $b_1, b_2, \ldots, b_p$. Then,

$$\boxed{AB = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix}} \tag{2}$$

2.1 Multiplication Properties

Matrix multiplication is **associative**:

$$A(BC) = (AB)C \tag{3}$$

However, matrix multiplication is *not commutative*:

$$AB \neq BA \tag{4}$$

We can understand this through the analogue of the composition of two transformation functions. It can be seen that given two functions f and g, $f \circ g \neq g \circ f$.

2.2 Identity

The $n \times n$ **identity matrix** I_n is the matrix that implements the identity transformation:

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & & \\ 0 & & & 1 \end{bmatrix}$$

The identity matrix has 1s on the diagonal and 0s everywhere else. It satisfies the following equality for any $n \times n$ matrix A:

$$A \cdot I_n = I_n \cdot A = A \tag{5}$$

3 Matrix Transposition

The **transpose** A^T of a matrix A is A with the rows and columns switched. Thus, if A was a $m \times n$ matrix, then A^T would be a $n \times m$ matrix:

$$\boxed{A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad A^T = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ a_{12} & \cdots & a_{m2} \\ \vdots & & \\ a_{1n} & \cdots & a_{mn} \end{bmatrix}} \tag{6}$$

3.1 Transposition Properties

Two useful properties of transpose:

$$(A + B)^T = A^T + B^T \tag{7}$$

$$(AB)^T = B^T A^T \tag{8}$$