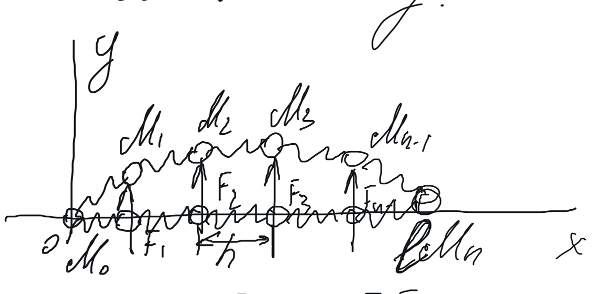
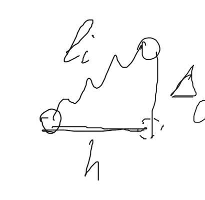
Основы вариационного исчисления

1) Переход от конечного числа степеней свободы к бесконечному.



P – сила натяжения пружин



$l_i = \sqrt{h^2 + \Delta y_i^2}$

$U = U_{int} + U_{ext}$ – полная потенциальная энергия системы

$U_{int} = \sum_{i=1}^{n-1} P \Delta l_i$, $\Delta l_i = l_i - \underbrace{l_i^{(0)}}_{h} = \sqrt{h^2 + \Delta y_i^2} - h$

$\Delta y_i \ll h$ $\Delta l_i = h\left(\sqrt{1 + \frac{\Delta y_i^2}{h^2}} - 1\right) \approx h\left(1 + \frac{\Delta y_i^2}{2h^2} - 1\right) =$

$= \frac{\Delta y_i^2}{2h}$

$U_{int} = \frac{P}{2h} \sum_{i=1}^{n-1} \Delta y_i^2 = \frac{P}{2h} \sum_{i=1}^{n-1} (y_i - y_{i-1})^2$

$U_{ext} = -\sum_{i=1}^{n-1} F_i y_i$

$U(y_1, y_2, \dots, y_{n-1}) = \frac{P}{2h} \sum_{i=1}^{n-1} (y_i - y_{i-1})^2 - \sum_{i=1}^{n-1} F_i y_i$

$\frac{\partial U}{\partial y_k} = 0$, $k = 1, 2, \dots, n-1$

$\frac{\partial}{\partial y_k}\left(\frac{P}{2h}(y_k - y_{k-1})^2 + \frac{P}{2h}(y_{k+1} - y_k)^2 - F_k y_k\right) = 0$

$\frac{P}{h}(y_k - y_{k-1}) - \frac{P}{h}(y_{k+1} - y_k) - F_k = 0$

$\frac{P}{h}(2y_k - y_{k-1} - y_{k+1}) = F_k$

$$\boxed{\begin{cases} \frac{P}{h}(y_{k+1} - 2y_k + y_{k-1}) = -F_k \\ y_0 = y_n = 0 \end{cases}}$$

Задача: $F_i = F = \text{Const}$

$\frac{P}{h}(y_{i+1} - 2y_i + y_{i-1}) = -F$

$y_{i+1} - 2y_i + y_{i-1} = -\frac{hF}{P}$

$y_0 = y_n = 0$

$y_2 - 2y_1 = -\frac{Fh}{P}$ $y_2 - y_1 = -\frac{Fh}{P} + y_1$

$y_3 - 2y_2 + y_1 = -\frac{Fh}{P}$, $y_3 - y_2 = -\frac{Fh}{P} + y_2 - y_1 = -\frac{2Fh}{P} + y_1$

$y_4 - 2y_3 + y_2 = -\frac{Fh}{P}$, $y_4 - y_3 = -\frac{Fh}{P} + y_3 - y_2 = -\frac{3Fh}{P} + y_1$

$y_2 = -\frac{Fh}{P} + 2y_1$ 1, 3, 6,

$y_3 = -\frac{3Fh}{P} + 3y_1$

$y_4 = -\frac{6Fh}{P} + 4y_1$

. . . .

$y_k = ky_1 - \frac{k(k-1)}{2}\frac{Fh}{P}$

$y_n = 0 = ny_1 - \frac{n(n-1)}{2}\frac{Fh}{P} \Rightarrow y_1 = \frac{n-1}{2}\frac{Fh}{P}$

$y_k = \frac{k(n-1)}{2}\frac{Fh}{P} - \frac{k(k-1)}{2}\frac{Fh}{P} = \frac{Fh}{P}\frac{k}{2}(n-1-k+1)$

$$\boxed{y_k = \frac{Fh}{2P}k(n-k) = \frac{Fh}{2P}k(n-k)}$$

$n \to \infty$ $h \to 0$, $nh \to l$

U $\sum_{k=0}^{n}\left(\frac{P}{2}\left(\frac{y_{k+1} - y_k}{h}\right)^2 - f_k y_k\right)h$

$f_k = \frac{F_k}{h}$ – сила, приходящаяся на единицу длины.

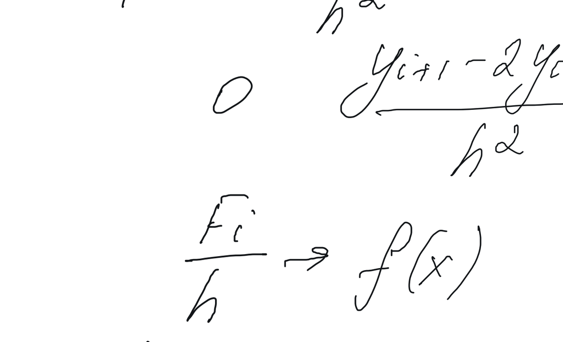
$y_k \to y(x)$, $\frac{y_{k+1} - y_k}{h} \to y'(x)$ $f_i \Rightarrow f(x)$

$$\boxed{U = \int_0^l dx\left(\frac{P}{2}y'^2(x) - f(x)y(x)\right)}$$

– потенциальная энергия натянутой струны

$y(0) = y(l) = 0$

$y(x) = ?$



$\frac{P}{h}(y_{i+1} - 2y_i + y_{i-1}) = -F_i$

$P\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = -\frac{F_i}{h}$

0 $\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \to y''(x)$

$\frac{F_i}{h} \to f(x)$

$$\begin{cases} Py''(x) = -f(x) \\ y(0) = y(l) = 0 \end{cases}$$

$f(x) = f_0$

$y'' = -\frac{f_0}{P}$ $y'(x) = -\frac{f_0 x}{P} + C_1$

$y(x) = -\frac{f_0 x^2}{2P} + C_1 x + C_2$

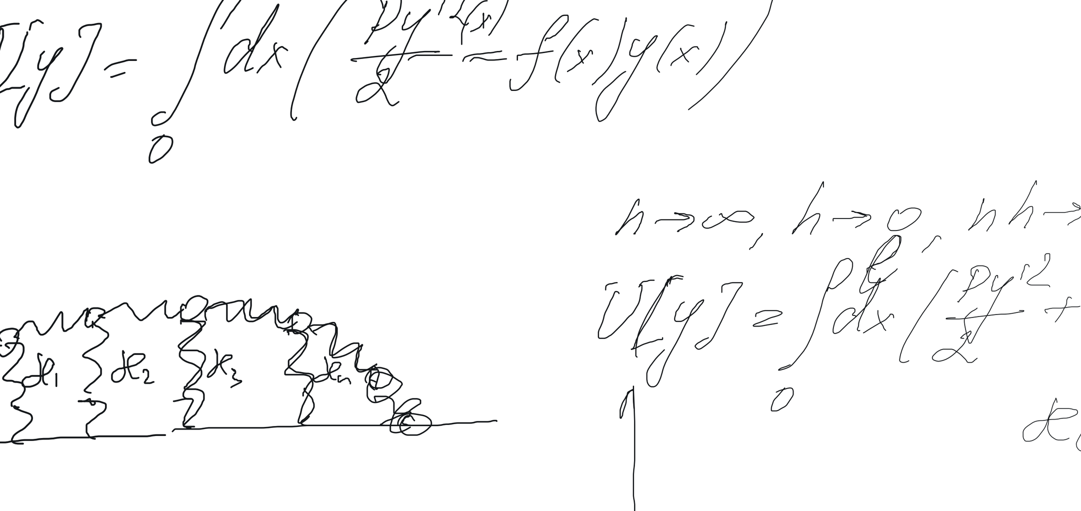
$y(0) = C_2 = 0$, $y(l) = -\frac{f_0 l^2}{2P} + C_1 l = 0$

$C_1 = \frac{f_0 l}{2P}$

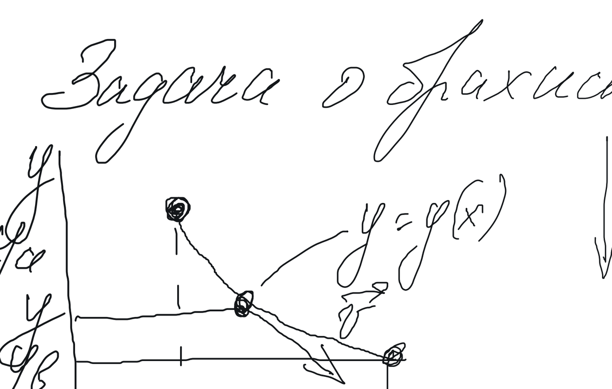
$y(x) = \frac{f_0 l x}{2P} - \frac{f_0 x^2}{2P} = \frac{f_0 x(l - x)}{2P}$

$U(y_1, y_2, \dots, y_{n-1}) \xrightarrow[n \to \infty]{} U[y]$

Опр. Функционалом $S[y]$ называется отображение некоторого класса функций $y = y(x)$, $x \in [a, b]$ на множество вещественных (комплексных) чисел.



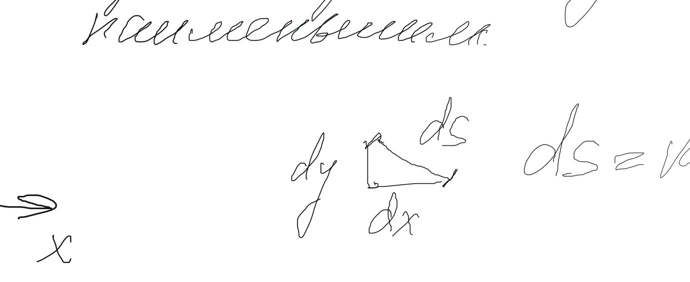
$U[y] = \int_0^l dx\left(\frac{Py'^2(x)}{2} - f(x)y(x)\right)$



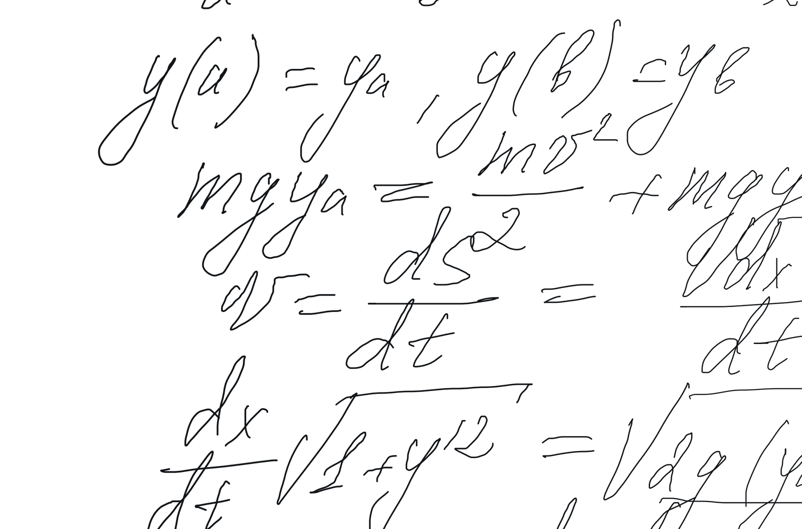
$n \to \infty$, $h \to 0$, $nh \to l$ $\frac{\varkappa_i}{h} \to \varkappa(x)$

$U[y] = \int_0^l dx\left(\frac{Py'^2}{2} + \frac{\varkappa(x)y^2(x)}{2} - fy\right)$

$\varkappa(x) = \varkappa = \text{Const}$



Задача о брахистохроне



Найти профиль горки, при котором время спуска будет наименьшим.

$ds = \sqrt{dx^2 + dy^2}$

$y(a) = y_a$, $y(b) = y_b$

$mgy_a = \frac{mv^2}{2} + mgy \Rightarrow v = \sqrt{2g(y_a - y)}$

$v = \frac{ds}{dt} = \frac{\sqrt{dx^2 + dy^2}}{dt} = \frac{dx}{dt}\sqrt{1 + y'^2}$, $y' = \frac{dy}{dx}$

$\frac{dx}{dt}\sqrt{1 + y'^2} = \sqrt{2g(y_a - y)}$

$dt = \frac{dx\sqrt{1 + y'^2}}{\sqrt{2g(y_a - y)}} \Rightarrow \int_0^T dt = \int_a^b \frac{\sqrt{1 + y'^2}\,dx}{\sqrt{2g(y_a - y)}}$

$$\boxed{T[y] = \int_a^b \frac{\sqrt{1 + y'^2}\,dx}{\sqrt{2g(y_a - y)}}}$$

$y = ? \Rightarrow T[y] = \min$