Introduction to Linear Regression

GR 5205 / GU 4205
Section 2/ Section 3

Columbia University
Xiaofei Shi

Course setup

Basic administrative details:

Course website https://innerpeas.github.io/LRM_index.html

- Instructor: Xiaofei Shi xs2427@columbia.edu OH: Tuesday 1:30 - 2:30
- TA: Navid Ardeshir na2844@columbia.edu OH: Monday 11:00 - 12:00

  Daran Xu dx2207@columbia.edu OH: Monday 3:00 - 4:00

  Office hours are on Zoom: 969 8687 3013
- Textbook: Kutner, Nachtsheim, Neter: *Applied Linear Regression Models*
- We will use Courseworks to post lecture notes and homework
- We will use Piazza for announcements and discussions:

  GU 5205/GU 4205 Linear Regression Models

Tentative Evaluation Plan

20% Final + 30% Midterm + 50% max{homework average, exam average}

- Homework: 5 homework in total；
- Midterm: October 27th & November 29th, in class;
- Final: See school schedule;
- Participation: Piazza, recitations, class survey.

Class participation

- In person class only, but subject to changes
- According to the current return to school procedure, all class will be in person. In order to protect everyone in the classroom while provide the education we promised you,

  - Please wear a facial mask that covers your nose and mouth

  - Please keep 6-feet social distance

  - Follow the gateway testing

Prerequisites

Assume working knowledge of/proficiency with:

- Probability
- Linear Algebra
- Statistics
- Programming (Python, R or Matlab)
- Formal mathematical thinking

If you fall short on any one of these things, it’s certainly possible to catch up; but don’t hesitate to talk to us.

Goal of this course

- Data analysis
- R, Python, Matlab, Mathematica output of linear regression models
- Aggregated with probability and (statistical) inference

Probability Review:
A (scalar) random variable X

- Distribution: Discrete type v.s. Continuous type

More details will be covered in Probability class.

Probability Review:
A (scalar) random variable X

In this course, we mainly focus on random variables with probability density functions (pdf)

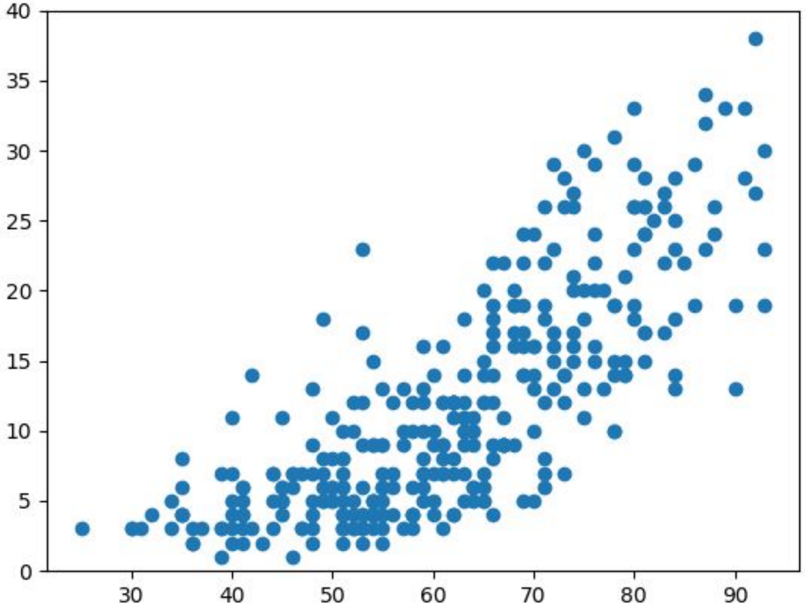
- Distribution $\mathbb{P}[X \leq x] = \int_{-\infty}^{x} p(y)dy$
- Expectation $$\mathbb{E}[X] = \int_{-\infty}^{\infty} xp(x)dx$$
- Variance $Var[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \int_{-\infty}^{\infty} (x - \mathbb{E}[X])^2 p(x)dx$

Probability Review: Random vector (X,Y)

- Joint distribution $\mathbb{P}[X \leq x, Y \leq y] = \int_{-\infty}^{x} \int_{-\infty}^{y} p_{X,Y}(x,y)dydx$
- Marginal distribution $p_X(x) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)dy$
- Conditional probability $p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)}$
- Conditional expectation $\mathbb{E}[Y|X=x] = \int_{-\infty}^{\infty} y\, p_{Y|X}(y|x)dy$
- Conditional variance $Var[Y|X=x] = \int_{-\infty}^{\infty} (y - \mathbb{E}[Y|X=x])^2\, p_{Y|X}(y|x)dy$

Notice that the conditional expectation and variance are both depend on the value of x. It is generally reasonable to assume that the conditional mean and variance functions are continuous.

Regression analysis
What are we interested in?



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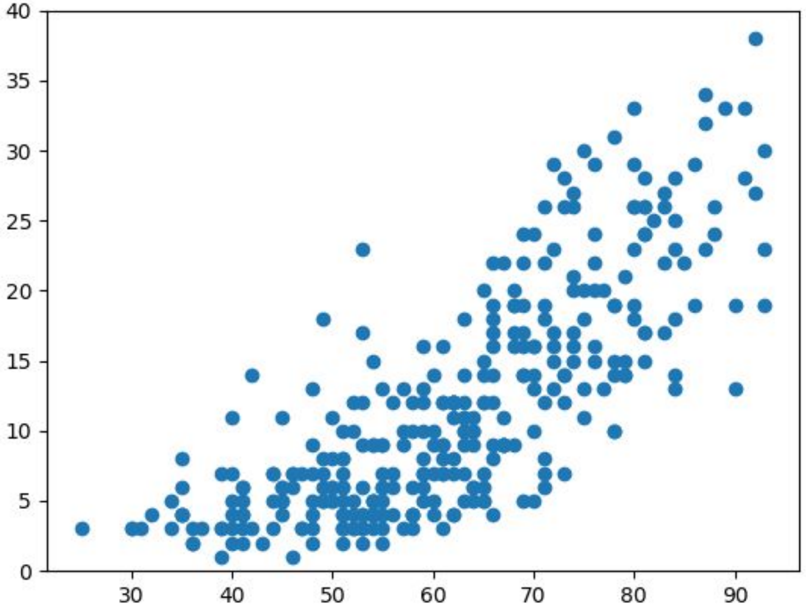


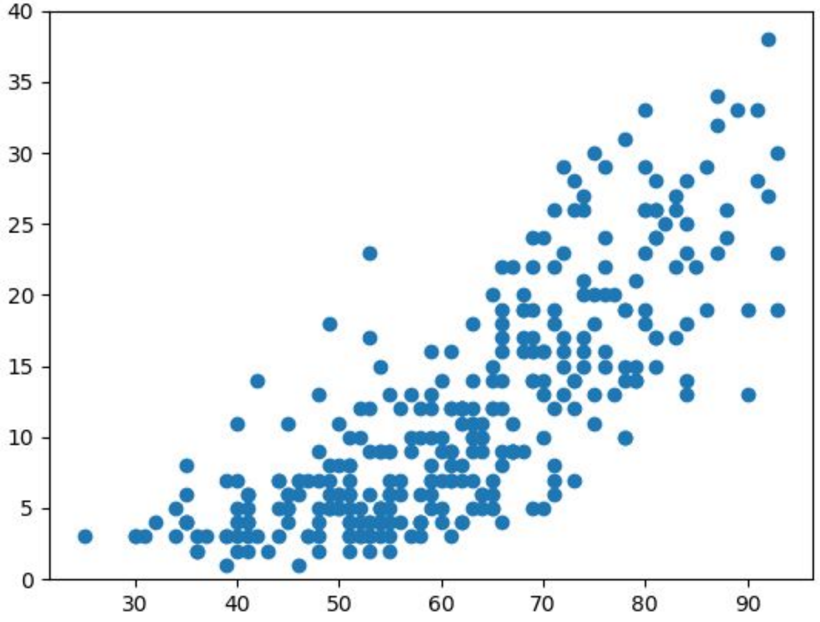
- We are most interested in the conditional distribution of Y given X.

Regression analysis
What are we interested in?

- Typical question: what is the distribution of Y|X=50 compare to Y|X=70?
- Life gets easier if we can justify further assuming the linear relationship

$$E[Y|X=x] = \beta_0 + \beta_1 x$$

$$Var[Y|X=x] = \text{constant}$$



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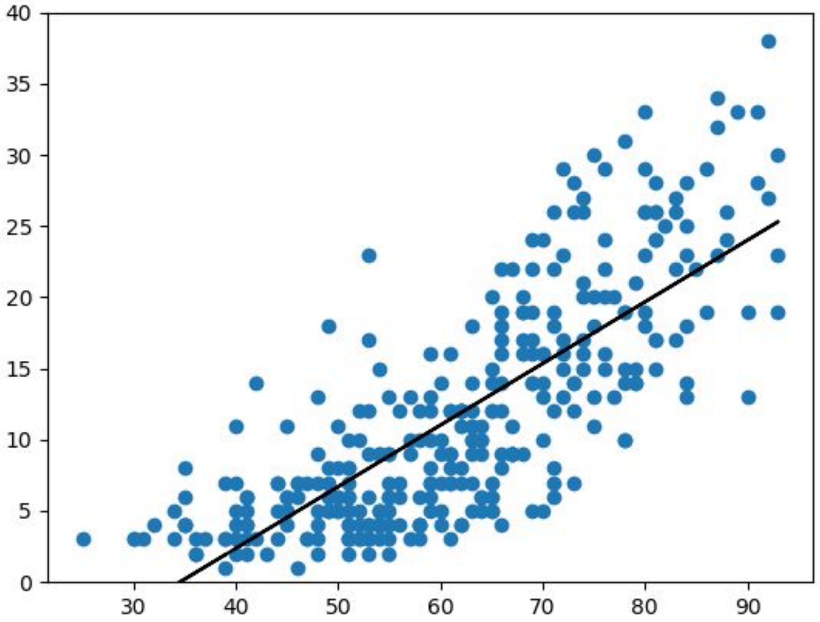
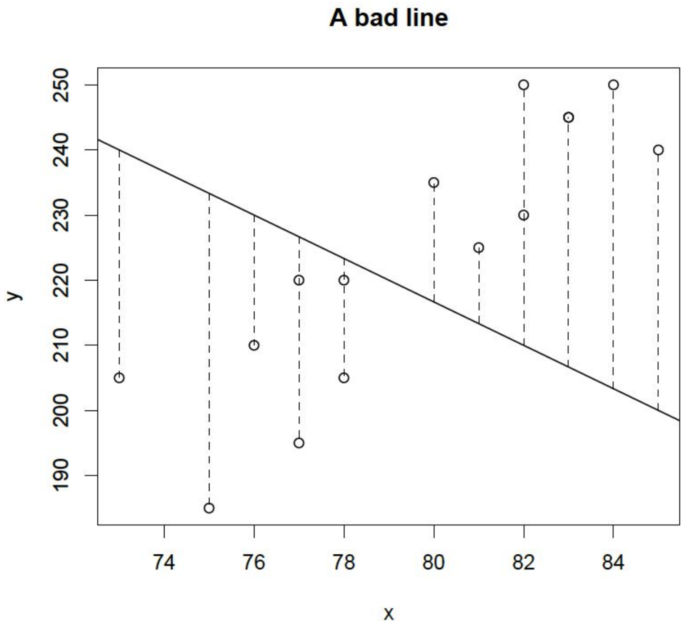
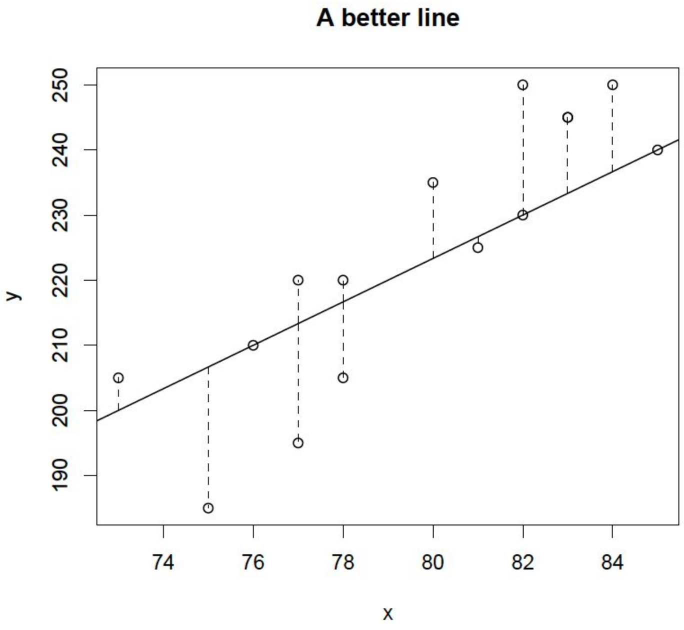


In other words...
The Simple Linear Regression Model

- Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be samples from (X, Y)
- If the SLR model holds, we write $Y_i = \beta_0 + X_i \beta_1 + \epsilon_i$,
- Here, ϵ_i satisfies $\mathbb{E}[\epsilon_i] = 0$ and $\mathbb{E}[\epsilon_i \epsilon_j] = \sigma^2 \delta_{ij}$
- Model parameters: $\beta_0, \beta_1, \sigma^2$

How to fit the model?





Mean Squared Error (MSE)

- How to fit the model? Mean Squared Error minimizer!

$$\min_{\beta_0,\beta_1} \mathbb{E}\left[(Y - \beta_0 - \beta_1 X)^2\right]$$

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- Matrix form:

$$\min_{\beta_0,\beta_1} Q := \|y - \beta_0 - \beta_1 x\|^2$$

Can also be found in...

- Ridge Regression

$$\min_{\beta_0,\beta_1} \|y - \beta_0 - \beta_1 x\|^2 + \lambda|\beta_1|^2$$

- Lasso

$$\min_{\beta_0,\beta_1} \|y - \beta_0 - \beta_1 x\|^2 + \lambda|\beta_1|$$

- Autoregression(AR) model

$$\min_{\beta_0,\beta_1} \sum_{i=1}^{n-1} (x_{i+1} - \beta_0 - \beta_1 x_i)^2$$

- And a lot of other topics in convex optimization... e.g. fused lasso/trend filtering

References and further reading

- Kutner, Nachtsheim, Neter: *Applied Linear Regression Models* Chapter 1 & Appendix 1
- Agresti: *Foundations of Linear and Generalized Linear Models* Chapter 1