Managing Transaction Costs in Dynamic Trading

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Abstract

We explicitly solve the lifetime investment-consumption problem of investors trading in an incomplete market where asset returns are partially predictable but trading is costly. The solution is expressed in terms of the unique, global solution of a risk-sensitive Riccati system. We show that the optimal trading strategy targets a portfolio that is optimal for a frictionless version of the model where asset returns have been adjusted for costs, in that they are expressed on a net rather than gross basis. The legacy portfolio (the inherited undesirable positions) are then traded away in line with a backward-looking optimal execution problem. Thus, the investment process is separated into an investment stage where a desired or target portfolio is designed using a model of time-varying predictable net returns, and a execution stage that disposes of any unwanted or legacy assets as efficiently as possible assuming there are no-excess returns to any of these assets.

JEL Classification: G11, G12.

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1 Introduction

Modern quantitative investment processes are often structured with an investment team producing stock order lists and an independent trading desk executing these orders. Investors today care as much about transaction costs as they do about expected returns and risk (because the funds they manage are very large or because they trade at high-frequency). In this paper, we provide conditions under which this separation of tasks is optimal.

More precisely, we characterize the optimal dynamic portfolio in a continuous-time setup assuming investors maximize the welfare derived from a consumption stream in an incomplete market where returns to the securities are partially predictable but costly to trade.[1] We derive an explicit solution to this problem in terms of the solution to a risk-sensitive Riccati equation. We show that the optimal trading strategy is to target a portfolio that is the optimal solution to a frictionless (or "no-cost") dynamic portfolio problem. In this problem the asset returns have been pre-adjusted

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[1]Specifically we assume a CARA utility function, an incomplete market where the evolution of the economic states are described by a linear time series model, costs are quadratic (linear price impact) and the investment horizon is infinite. Most of these assumption can be relaxed but we keep the model as simple as possible.

for costs; that is they have been expressed on a net rather than gross basis. The legacy portfolio (the inherited undesirable positions) are then traded away in line with a backward-looking optimal execution problem. Thus the investment process is separated into an investment stage where a desired or target portfolio is designed using a model of time-varying predictable net returns, and a execution stage that disposes of any unwanted or legacy assets as efficiently as possible.

Critical in this separation result is the optimal adjustment to returns for trading costs. To describe this adjustment process, assume there is a shock to the economy that results in a temporary investment opportunity – some securities now have positive conditional expected returns. In order to decide whether to trade into these securities, our investor must know, of course, the costs of trading each security. But the investor also needs to know how long each of these investment opportunities are likely to persist given it is costly to take a position. The more persistent the opportunity, the smaller the cost per unit time of taking an exposure. However our investors are risk averse too, and so they also want to know the likelihood of further shocks arriving that change these opportunities; this way they can try to limit future possible trading costs as new information arrives.

To effect the adjustment, the investor estimates the expected risk-adjusted round-trip cost of trading into and out of a position so as to capture as much of the conditional expected return as possible. The adjustment for costs is the instantaneous change in the round trip costs. The adjustment is done both as reduction to the expected return to a security (a mean cost adjustment) and as an increase in the volatility of the innovation to the securities (a execution risk adjustment). The adjustment thus creates a wedge between the gross and net return process which is the greater the greater the cost of trading, the faster the expected return decay rate and the more volatile the forecast. Having adjusted gross returns for costs, the investor then designs the optimal target portfolio as if markets were frictionless.

In our setup, we assume investors maximize their utility from a consumption stream. In a dynamic framework, this has two immediate advantages (over extended mean-variance performance criteria) that we exploit. Firstly, our investors have an incentive to intertemporally smooth their portfolio returns. Thus our optimal portfolio exhibits the hedging demands of Merton (1971). These portfolios hedge the investor against adverse changes in the opportunity set. Thus our investors benefit from the predictability of returns, even in the absence of costs. In two recent papers, Gârleanu and Pedersen [3, 4] extended the mean-variance performance criterion to multi-period and continuous-time settings. Though their optimal dynamic portfolio shares many of the characteristics of ours, it abstracts away from these hedging motives. Thus in their framework, the investor's optimal portfolio tends towards the myopic mean-variance portfolio as costs become small.

The remainder of this paper is organized as follows. Section 2 introduces the model, and Section 3 formally derives candidates for the optimal controls and value function using heuristic stochastic control arguments. Section 4 in turn presents a rigorous verification theorem, which shows that the heuristics indeed always lead to the well-defined solution of our problem. Section 5 then discusses the qualitative properties of the solution, in particular, the separation into an optimization and execution problem. The quantitative properties of the results are explored in a concrete example in Section 6. All proofs are delegated to the appendices for better readability.

2 Model

Throughout, we fix a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ supporting a k-dimensional standard Brownian motion $\{B(t)\}_{t\geq 0}$.

2.1 Financial Market

We consider a financial market with $1+m$ assets. The first one is safe and earns a constant interest rate $r > 0$. The other m assets are risky, with dynamics

$$dP(t) = (\bar{\mu} + C_x X(t) + rP(t))\, dt + \sigma_p dB(t), \qquad P(0) = p_0 \in \mathbb{R}^m. \tag{2.1}$$

Here, the volatility matrix $\sigma_p \in \mathbb{R}^{m\times k}$ is deterministic and the corresponding covariance matrix $\sigma_p \sigma_p^\top$ is invertible. The expected excess returns are composed of a constant $\bar{\mu} \in \mathbb{R}^m$ and a mean-reverting component, which is the product of a scaling matrix $C_x \in \mathbb{R}^{m\times d}$ and an $\mathbb{R}^d$-valued Ornstein-Uhlenbeck process:

$$dX(t) = A_x X(t)dt + \sigma_x dB(t), \qquad X(0) = x_0 \in \mathbb{R}^d. \tag{2.2}$$

Here, $\sigma_x \in \mathbb{R}^{d\times k}$ and the corresponding covariance matrix $\sigma_x \sigma_x^\top$ is invertible; the matrix A_x is stable (i.e., all real parts of its eigenvalues are negative), so that $X(t)$ is indeed mean reverting around zero and the expected excess returns of the risky assets in turn fluctuate around $\bar{\mu}$.

2.2 Transaction Costs and Goal Functionals

As in [3, 4], trades in the risky assets incur quadratic costs $\frac{1}{2}\dot{\theta}(t)^\top \Lambda \dot{\theta}(t)$ for a symmetric, positive definite matrix $\Lambda \in \mathbb{R}^{m\times m}$, levied on the rates with which the risky positions are adjusted:

$$d\theta(t) = \dot{\theta}(t)dt, \qquad \theta(0) = \theta_0 \in \mathbb{R}^m. \tag{2.3}$$

The wealth process of an agent who trades at rate $\dot{\theta}(t)$, consumes at rate $c(t)$ and receives a deterministic endowment at rate $y(t)$, in turn has dynamics

$$dW^{c,\dot{\theta}}(t) = \Big(rW^{c,\dot{\theta}}(t) + y(t) - c(t) + \theta(t)^\top(\bar{\mu} + C_x X(t)) - \frac{1}{2}\dot{\theta}(t)^\top \Lambda \dot{\theta}(t)\Big)dt + \theta(t)^\top \sigma_p dB(t), \tag{2.4}$$
$$W^{c,\dot{\theta}}(0) = w_0 \in \mathbb{R}.$$

For agents with constant absolute risk aversion $\beta > 0$ and time-discount rate $\delta > 0$, we consider the problem of maximizing lifetime utility from consumption:

$$\mathbb{E}\left[\int_0^\infty -e^{-\delta t - \beta c(t)} dt\right] \to \max! \tag{2.5}$$

Here, in order to rule out doubling strategies and excessive borrowing, the risky positions $\theta^{c,\dot{\theta}}(t)$ and the wealth $W^{c,\dot{\theta}}(t)$ corresponding to the consumption and trading rates $c(t)$, $\dot{\theta}(t)$ need to satisfy suitable integrability and transversality conditions, see Section 4.

3 Control Heuristics

We now identify candidates for the value function and optimal controls for (2.5) using formal stochastic control arguments. A rigorous verification theorem is in turn presented in Section 4.

The value function $J(t, x, w, \theta)$ for the control problem (2.5) depends on time t (through the time-dependent endowment rate), investment opportunities (through the mean-reverting component x of the expected returns), wealth w, and on the positions θ in the risky assets. Standard arguments suggest that it solves the HJB equation

$$
\begin{aligned}
0 = {} & \frac{\partial J}{\partial t} + \theta^\top \sigma_p \sigma_x^\top \frac{\partial^2 J}{\partial w \partial x} + \frac{1}{2} \mathrm{tr} \left(\sigma_x \sigma_x^\top \frac{\partial^2 J}{\partial x^2} \right) + x^\top A_x^\top \frac{\partial J}{\partial x} + \frac{1}{2} \theta^\top \sigma_p \sigma_p^\top \theta \frac{\partial^2 J}{\partial w^2} \\
& + \sup_{c, \dot{\varphi}} \left\{ \dot{\theta}^\top \frac{\partial J}{\partial \theta} + \left(rw + y - c + \theta^\top (\bar{\mu} + C_x x) - \frac{1}{2} \dot{\theta}^\top \Lambda \dot{\theta} \right) \frac{\partial J}{\partial w} - e^{-\beta c - \delta t} \right\}.
\end{aligned} \tag{3.1}
$$

Pointwise optimization yields the optimal controls in feedback form,

$$
c = -\frac{1}{\beta} \left(\ln \frac{\partial J}{\partial w} - \ln \beta + \delta t \right), \qquad \dot{\theta} = \Lambda^{-1} \frac{\partial J / \partial \theta}{\partial J / \partial w}.
$$

To proceed, we set $\gamma = r\beta$ (this is the risk aversion of the indirect utility function J), and make the exponential-quadratic ansatz

$$
\begin{aligned}
J(t, x, w, \theta) = {} & -J_0 \exp \left(-\gamma w + \gamma \int_0^t e^{r(t-u)} y(u) du - \delta t \right) \\
& \times \exp \left(\gamma \left(-\frac{1}{2} x^\top \Pi_{xx} x + \Pi_x^\top x + \frac{1}{2} \theta^\top \Pi_{\theta\theta} \theta + \Pi_\theta^\top \theta + \theta^\top \Pi_{\theta x} x \right) \right),
\end{aligned} \tag{3.2}
$$

with coefficients $\Pi_{xx} \in \mathbb{R}^{d \times d}$, $\Pi_{x\theta} \in \mathbb{R}^{m \times d}$, $\Pi_{\theta\theta} \in \mathbb{R}^{m \times m}$, $\Pi_x \in \mathbb{R}^d$, $\Pi_\theta \in \mathbb{R}^m$, $J_0 \in \mathbb{R}$ to be determined. Then, the candidate optimal controls simplify to

$$
e^{-\beta c - \delta t} = \frac{1}{\beta} \frac{\partial J}{\partial w} = -rJ,
$$

$$
c = r \left(w - \int_0^t e^{r(t-u)} y(u) du + \frac{1}{2} x^\top \Pi_{xx} x - \Pi_x^\top x - \frac{1}{2} \theta^\top \Pi_{\theta\theta} \theta - \Pi_\theta^\top \theta - \theta^\top \Pi_{\theta x} x - \frac{\ln r J_0}{\gamma} \right), \tag{3.3}
$$

as well as

$$
\dot{\theta} = -\Lambda^{-1} \left(\Pi_{\theta\theta} \theta + \Pi_\theta + \Pi_{\theta x} x \right). \tag{3.4}
$$

After plugging these expressions back into the HJB equation and cancelling the common term J, it follows that

$$
\begin{aligned}
0 = {} & r - \delta - r \ln r J_0 - \frac{\gamma}{2} \mathrm{tr} \left(\sigma_x^\top \Pi_{xx} \sigma_x \right) - \frac{\gamma}{2} \left(\Pi_{\theta\theta} \theta + \Pi_\theta + \Pi_{\theta x} x \right)^\top \Lambda^{-1} \left(\Pi_{\theta\theta} \theta + \Pi_\theta + \Pi_{\theta x} x \right) \\
& - \gamma \theta^\top \left(\bar{\mu} + C_x x \right) - \gamma x^\top A_x^\top \left(\Pi_{xx} x - \Pi_x - \Pi_{\theta x}^\top \theta \right) + \frac{1}{2} \gamma^2 \theta^\top \sigma_p \sigma_p^\top \theta \\
& + \gamma^2 \theta^\top \sigma_p \sigma_x^\top \left(\Pi_{xx} x - \Pi_x - \Pi_{\theta x}^\top \theta \right) - r\gamma \left(-\frac{1}{2} x^\top \Pi_{xx} x + \Pi_x^\top x + \frac{1}{2} \theta^\top \Pi_{\theta\theta} \theta + \Pi_\theta^\top \theta + \theta^\top \Pi_{\theta x} x \right) \\
& + \frac{\gamma^2}{2} \left(\Pi_{xx} x - \Pi_x - \Pi_{\theta x}^\top \theta \right)^\top \sigma_x \sigma_x^\top \left(\Pi_{xx} x - \Pi_x - \Pi_{\theta x}^\top \theta \right).
\end{aligned}
$$

Comparison of coefficients for the terms quadratic in θ in turn gives

$$
\Pi_{\theta\theta} \Lambda^{-1} \Pi_{\theta\theta} + r \Pi_{\theta\theta} - \gamma \left(\Pi_{\theta x} \sigma_x - \sigma_p \right) \left(\Pi_{\theta x} \sigma_x - \sigma_p \right)^\top = 0. \tag{3.5}
$$

For the terms quadratic in x we obtain

$$
\gamma \Pi_{xx} \sigma_x \sigma_x^\top \Pi_{xx} + r \Pi_{xx} - A_x^\top \Pi_{xx} - \Pi_{xx} A_x - \Pi_{\theta x}^\top \Lambda^{-1} \Pi_{\theta x} = 0. \tag{3.6}
$$

Finally, comparison of coefficients for the cross terms that depend both on θ and x yields

$$\gamma\sigma_p\sigma_x^\top\Pi_{xx} - C_x - \left(\Pi_{\theta\theta}\Lambda^{-1} + \frac{r}{2}I_m\right)\Pi_{\theta x} - \Pi_{\theta x}\left(\frac{r}{2}I_d - A_x + \gamma\sigma_x\sigma_x^\top\Pi_{xx}\right) = 0. \tag{3.7}$$

Once an appropriate solution $(\Pi_{\theta\theta}, \Pi_{xx}, \Pi_{\theta x})$ of the coupled matrix Riccati equations (3.5),(3.6), (3.7) is identified, the remaining coefficients of the candidate value function can be computed directly. To wit, comparison of coefficients for the terms that only depend on θ or x, respectively, leads to the following linear equations for Π_θ and Π_x:

$$0 = \gamma\left(\Pi_{\theta x}\sigma_x - \sigma_p\right)\sigma_x^\top\Pi_x - \left(\Pi_{\theta\theta}\Lambda^{-1} + rI_m\right)\Pi_\theta - \bar{\mu}, \tag{3.8}$$

$$0 = \left(rI_d - A_x + \gamma\sigma_x\sigma_x^\top\Pi_{xx}\right)^\top\Pi_x + \Pi_{\theta x}^\top\Lambda^{-1}\Pi_\theta. \tag{3.9}$$

Finally, comparison of coefficients for the terms independent of all state variables gives

$$J_0 = \frac{1}{r}\exp\left(\frac{1}{r}\left(r - \delta - \frac{\gamma}{2}\Pi_\theta^\top\Lambda^{-1}\Pi_\theta + \frac{\gamma^2}{2}\Pi_x^\top\sigma_x\sigma_x^\top\Pi_x - \frac{\gamma}{2}\mathrm{tr}\left(\sigma_x^\top\Pi_{xx}\sigma_x\right)\right)\right). \tag{3.10}$$

4 Verification Theorem

Our first main result states that the coupled matrix Riccati equations (3.5),(3.6), (3.7) indeed have a global solution. For better readability, the lengthy proof is delegated to Appendix A.

Theorem 4.1. *There exists a unique solution $\Pi_{\theta\theta}, \Pi_{xx}, \Pi_{\theta x}$ of the matrix Riccati system* (3.5),(3.6), (3.7) *for which $\Pi_{\theta\theta}, \Pi_{xx}$ are positive semidefinite.*

The remaining coefficients of the candidate value function (3.2) are in turn also well defined by (3.8), (3.9), and (3.10). Again, the proof is delegated to Appendix A.

Corollary 4.2. *There exists a unique solution Π_θ, Π_x of the linear system* (3.8), (3.9).

With the candidate (3.2) at hand, we now present a verification theorem, which shows it indeed is the value function for the lifetime consumption problem (2.5) and the corresponding feedback controls (3.3),(3.4) are optimal. In order to rule out doubling strategies and excessive borrowing ("Ponzi schemes"), we focus on *admissible* policies c, $\dot{\theta}$ for which (i) the local martingale

$$\gamma\int_0^\cdot J\left(t, X(t), W^{c,\dot{\theta}}(t), \theta^{c,\dot{\theta}}(t)\right)\left(\left(\Pi_{\theta x}\sigma_x - \sigma_p\right)^\top\theta(t) - \sigma_x^\top\left(\Pi_{xx}X(t) - \Pi_x\right)\right)^\top dB(t) \tag{4.1}$$

is a true martingale and (ii) the following transversality condition holds:

$$\lim_{\tau\to\infty}\mathbb{E}\left[J(\tau, X(\tau), W^{c,\dot{\theta}}(\tau), \theta^{c,\dot{\theta}}(\tau))\right] \to 0. \tag{4.2}$$

Both of these conditions are satisfied if the risky positions and the negative part of the agent's wealth are integrable enough (which rules out doubling strategies and Ponzi schemes). In particular, both (4.1) and (4.2) hold for controls for which the corresponding risky positions are uniformly bounded and wealth is bounded from below. With substantially more sophisticated estimates, we can also show that our candidate policy (3.3),(3.4) also satisfies these requirements, see Appendix C.

Theorem 4.3. *For any admissible investment/consumption policy $(\dot{\theta}, c)$, we have*

$$J\left(0, x_0, w_0, \theta_0\right) \geq \mathbb{E}\left[\int_0^\infty -e^{-\delta t - \beta c(t)}dt\right].$$

This upper bound is attained by the feedback policy (3.3),(3.4)*. Therefore, this policy is optimal for the lifetime consumption problem* (2.5) *and J is the corresponding value function.*

5 Characterisation of the optimal portfolio holdings

In this section we offer two characterisations of the optimal portfolio.

Our first characterisation is close in spirit to Garleanu and Pedersen [3, 4], however there two significant differences. As in [3, 4], our investor forecasts the evolution of the states over time, and at each point over this path solves for the desired instantaneous mean-variance optimal portfolio. The target portfolio is a weighted average of these instantaneous portfolios where the weights depend on trading costs. Thus the investor charts her way through a path of desired portfolios as close as possible given the costs of trading. However there are two distinct differences relative to [3, 4]: firstly, our approach takes into account Merton's intertemporal heging motives; investors hedge themselves agains adverse changes to the investment opportunity set. Secondly our approach incorporates execution costs into the construction of the instantaneous mean-variance efficient portfolio. The mean-variance efficient portfolio is calculated with respect to expected returns on a *net* basis (i.e., after costs) not on a gross basis.

The second characterisation builds on this to show that the optimal dynamic problem can be separated into an frictionless dynamic portfolio problem (i.e., one without trading costs) but with the assets returns adjusted to be on a *net* basis (i.e., after costs) and an optimal execution problem. In this characterisation the portfolio, $\theta(t)$, can be decomposed at any time t into a desired or target portfolio, $\widehat{\theta}^*(t)$ and a legacy portfolio, $\breve{\theta}(t) = \theta(t) - \widehat{\theta}^*(t)$; the target portfolio being the solution to the frictionless problem and the legacy portfolio being the unwanted holdings which are traded away as efficiently as possible.

We start by defining the optimal target portfolio.

Definition 5.1. *The target holdings $\widehat{\theta}^*(t)$ maximize the opportunity set*

$$G(w(t), \theta(t), x(t)) = w + \frac{1}{2}x^\top \Pi_{xx} x - \Pi_x^\top x - \frac{1}{2}\theta^\top \Pi_{\theta\theta}\theta - \Pi_\theta^\top \theta - \theta^\top \Pi_{\theta x} x$$

given the state x. This maximum is achieved by the target portfolio in (3.4)*,*

$$\widehat{\theta}^*(t) = -\Pi_{\theta\theta}^{-1}\left(\Pi_{\theta x} x + \Pi_\theta\right).$$

The states are exogenous to the investor's portfolio decision. Therefore given the current states, the investor would ideally hold the target portfolio to maximize her opportunity set and hence welfare. If this were the case, either through luck or design, there is no motive trade. To make this maximisation explicit we can rewrite the quadratic term in the expression for the opportunity set (by completing the square) as

$$\begin{aligned} G(w,\theta,x) = w &+ \begin{bmatrix} x \\ \theta - \widehat{\theta}^* \end{bmatrix}^T \begin{bmatrix} \Pi_{xx} + \Pi_{\theta x}^T \Pi_{\theta\theta}^{-1}\Pi_{\theta x} & 0 \\ 0 & -\Pi_{\theta\theta} \end{bmatrix} \begin{bmatrix} x \\ \theta - \widehat{\theta}^* \end{bmatrix} \\ &+ \left(\Pi_x + \Pi_\theta^T \Pi_{\theta\theta}^{-1}\Pi_{\theta x}\right) x + \frac{1}{2}\Pi_\theta^T \Pi_{\theta\theta}^{-1}\Pi_\theta. \end{aligned} \tag{5.1}$$

As Π_{xx} and $\Pi_{\theta\theta}$ are positive semidefinite, the first element on the principal diagonal of the quadratic term is positive semidefinite and the second element is negative semidefinite. The maximum of the opportunity set therefore occurs when $\theta = \widehat{\theta}^*$. This transformation effectively decomposes the opportunity set into the maximal expected positive contribution to welfare from the extra return premium, i.e. $x^\top \left(\Pi_{xx} + \Pi_{\theta x}^T \Pi_{\theta\theta}^{-1}\Pi_{\theta x}\right) x$, and a negative component to account for the costs of trading away any legacy portfolio.

The following lemma clarifies this interpretation. It shows that if actual holdings differ from the target portfolio, the optimal trading rule can be understood as trading away the legacy portfolio as efficiently as possible.

Lemma 5.2. *The optimal policy trade policy can be re-written as*

$$\dot{\theta}^* = -\Lambda^{-1}\Pi_{\theta\theta}\left(\theta - \widehat{\theta}^*\right) \tag{5.2}$$

and can be understood as trading away the legacy portfolio with speed $\Lambda^{-1}\Pi_{nn}$ (where $-\Lambda^{-1}\Pi_{nn}$ is a stable matrix).

Equation (5.2) describes the optimal trading rule: the investor trades optimally towards a desired target portfolio where the speed of trading $\Lambda^{-1}\Pi_{nn}$ depends on transactions costs.

5.1 Instantaneous mean-variance problem

We define the instantaneous mean-variance portfolio as the portfolio that maximises the instantaneous risk adjusted expected excess return. The excess return to the portfolio is $\theta(t)^\top(\bar{\mu}+C_xX(t))$ from equation (2.4). The instantaneous risk is the innovation to the opportunity set (not just wealth as in the myopic mean-variance problem). An application of Itô's formula to (5.1)) yields

$$dG(w(t),x(t)) = \mu_G dt + \left(\theta(t)^T\left(\Pi_{\theta x}\sigma_x - \sigma_p\right) - \left(X(t)^T\Pi_{xx} - \Pi_x^T\right)\sigma_x\right)dB(t) \tag{5.3}$$

where he drift term, μ_G, in equation (5.3) generally depends on the choice of consumption and trading intensity. However, along the optimal path, we shall show that this drift term is a function of the risk free rate. We denote the innovation to the opportunity set by $\eta_t^\top dB_t$, where

$$\eta_t = \left(\Pi_{\theta x}\sigma_x - \sigma_p\right)^\top\theta(t) - \sigma_x^\top\left(\Pi_{xx}X(t) - \Pi_x\right). \tag{5.4}$$

The next definition defines the instantaneous mean-variance portfolio in terms of these measures of risk and return.

Definition 5.3. *The instantaneous mean variance portfolio $\theta^{iMV}(t)$ are the positions that maximise the following utility*

$$-\frac{1}{\gamma}E\left[e^{-\gamma\left(\theta(t)^\top(\bar{\mu}+C_x\bar{X}(t))dt+\eta_t^\top dB_t\right)}\right]$$

The maximum is achieved by the portfolio

$$\theta^{iMV}(t) = \frac{1}{\gamma}\left(\begin{bmatrix} -\Pi_{\theta x}^T \\ I\end{bmatrix}^\top\mathbf{\Sigma}\begin{bmatrix} -\Pi_{\theta x}^T \\ I\end{bmatrix}\right)^{-1}\left(C_xX(t)+\bar{\mu} - \begin{bmatrix} -\Pi_{\theta x}^T \\ I\end{bmatrix}^\top\mathbf{\Sigma}\begin{bmatrix}\Pi_{xx}X(t)-\Pi_x \\ 0\end{bmatrix}\right) \tag{5.5}$$

where $\mathbf{\Sigma} = \begin{bmatrix}\sigma_x \\ \sigma_p\end{bmatrix}\begin{bmatrix}\sigma_x \\ \sigma_p\end{bmatrix}^\top$.

The next Lemma relates the instantaneous mean-variance portfolio to the target portfolio and thus gives an interpretation of the adjustment to expected returns for costs.

Lemma 5.4. *The target portfolio is a weighted average of current and future expected instantaneous mean-variance portfolios. To wit, with $H_t = e^{-\Lambda^{-1}\Pi_{\theta\theta}t}$, we have*

$$\widehat{\theta}^*(t) = E_t\left[\int_t^\infty e^{-r(v-t)}H_{v-t}\theta_v^{iMV}dv\right] / \left[\int_t^\infty e^{-r(v-t)}H_{v-t}dv\right]$$

Here, the weight or discount rate H_t is inversely related to the the trading speed $-\Lambda^{-1}\Pi_{\theta\theta}$: with lower costs and therefore faster trading speeds – the effective discount rate is larger and so expectation horizon is shorter – and hence the target portfolio is closer to the current instantaneous mean-variance portfolio, .

Proof. Given in the Appendix. □

The target portfolio is a weighted average of expected instantaneous mean-variance portfolios along the expected optimal path. This characterisation applies also within the certainty equivalent framework of [3] and [8]. In this linear-quadratic framework the instantaneous mean-variance portfolio is the unadjusted myopic mean-variance portfolio, $\frac{1}{\gamma}\left(\sigma_p\sigma_p^T\right)^{-1}(C_xX(t)+\bar{\mu})$. The target portfolio is then weighted average of these mean-variance portfolios. In our dynamic risk-sensitive utility framework, the instantaneous mean-variance includes an adjustment for both intertemporal hedging and execution risk (which disappear in the certainty equivalent framework); with the target portfolio then being a weighted average of these portfolios over the projected future. So the adjustments for the intertemporal nature of the problem are partially done within the instantaneous mean-variance problem and partially in the intertemporal weighting procedure.

5.2 Optimal Frictionless Dynamic Portfolio Problem

In this section we offer an alternative characterisation of the optimal target portfolio. Asset returns are adjusted to be net of "endogenous" trading costs. These trading costs take into account the estimated cost of trading in and out of the asset and spread these costs over the expected holding period of the asset. Thus if the conditional excess return is expected to persist, the trade costs per unit time are lower than if the excess return is expected to decay more rapidly. Once the returns are adjusted, the investor chooses her portfolio as if there were no costs. Thus the target portfolio is the optimal portfolio to a frictionless problem but one where the returns are measured net of costs.

Corollary 5.5. *The target portfolio, $\widehat{\theta}^*(t)=-\Pi_{\theta\theta}^{-1}\left(\Pi_{\theta x}X(t)+\Pi_\theta\right)$, can be rewritten as*

$$\widehat{\theta}^*(t)=\frac{1}{\gamma}\left(\begin{bmatrix}-\Pi_{\theta x}^T\\ I\end{bmatrix}^T\mathbf{\Sigma}\begin{bmatrix}-\Pi_{\theta x}^T\\ I\end{bmatrix}\right)^{-1}\left(\left(C_x-\Pi_{\theta x}A_x\right)X(t)+\bar{\mu}-\begin{bmatrix}-\Pi_{\theta x}^T\\ I\end{bmatrix}^\top\mathbf{\Sigma}\begin{bmatrix}\Pi_{xx}X(t)-\Pi_x\\ 0\end{bmatrix}\right) \tag{5.6}$$

Proof. This follows directly from noting that $\widehat{\theta}^*(t)=-\Pi_{\theta\theta}^{-1}\left(\Pi_{\theta x}X(t)+\Pi_\theta\right)$ and an arrangement of equations (3.5), (3.8) and (3.9). □

This expression for the target portfolio looks similar to the instantaneous mean-variance in equation (5.5) except for the additional term $-\Pi_{\theta x}A_xX(t)$. Informed by this, we define the trade premium $z(t)$ as the m-vector of the expected present value of remaining trade costs to be paid on each asset. The trade premium on the m risky assets is $z(t)=-\Pi_{\theta x}X(t)$. For the moment, we take this result as given. If $z(t)$ is the total remaining trade charge to be levied on the assets, then its rate of change, $dz(t)$, is the instantaneous charge paid on each asset. Using equation (2.2), this instantaneous trade charge can be expressed as $dz=-\Pi_{\theta x}dX=-\Pi_{\theta x}\left(A_sX+\sigma_xdB\right)$. The trade premium is thus a stochastic process because innovations to the opportunity set change expected excess returns and therefore expected trade premiums. Net returns are defined as gross returns minus this instantaneous trade charge and we refer to the supporting prices as trade prices, $\hat{p}(t)$, where

$$d\hat{p}(t)=\left(r\hat{p}(t)+\bar{\mu}+\left(C_x-\Pi_{\theta x}A_x\right)X(t)\right)dt+\begin{bmatrix}-\Pi_{\theta x}^T\\ I\end{bmatrix}^T\begin{bmatrix}\sigma_x\\ \sigma_p\end{bmatrix}dB. \tag{5.7}$$

Trading costs alter the drift of trade prices by the expected change in the trade premium. We denote this conditional expected net return as $\hat{C}_xX(t)$ where $\hat{C}_x=C_x-\Pi_{\theta x}A_x$. However trade costs also impact the variance of net returns, as innovations to the opportunity set change the trade premium. We shall call this additional risk - the execution risk. Thus the instantaneous variance of net returns, denoted as $\hat{\mathbf{\Sigma}}_{pp}$, is

$$\hat{\mathbf{\Sigma}}_{pp} = \begin{bmatrix} -\Pi_{\theta x}^{\top} \\ I \end{bmatrix}^T \underbrace{\begin{bmatrix} \sigma_x \\ \sigma_p \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_p \end{bmatrix}^{\top}}_{\mathbf{\Sigma}} \begin{bmatrix} -\Pi_{\theta x}^{\top} \\ I \end{bmatrix} \tag{5.8}$$

This can be contrasted to the variance of gross returns $\mathbf{\Sigma}_{pp} = \left(\sigma_p \sigma_p^{\top}\right)$.

The target portfolio appears to have the structure of an optimal portfolio to a frictionless dynamic portfolio problem. It is a myopic mean variance portfolio plus a hedging portfolio. In this case, it is the myopic mean variance portfolio associated with the net return process, $\frac{1}{\gamma}\hat{\mathbf{\Sigma}}_{pp}^{-1}\hat{C}_s s(t)$. In the rest of this section, we show that the second term in (5.6) is indeed a hedging portfolio.

We now state this problem precisely. The investor chooses her consumption, $\hat{c}(t)$, so as to maximise the same welfare function as before,

$$\mathbb{E}\left[\int_0^{\infty} -e^{-\delta t - \gamma \hat{c}(t)} dt\right] \tag{5.9}$$

The budget equation must be modified too. Firstly because there are no costs; and secondly the investor now receives an income stream, $\hat{y}(t) = \frac{1}{2} r X(t)^T \Pi_{\theta x}^T \Pi_{\theta\theta}^{-1} \Pi_{\theta x} X(t)$ as a compensation for the loss in wealth resulting from buying the assets at trade prices; $\hat{y}(t)$ is exogenous being a function of economic states only. The evolution of wealth therefore follows the process

$$d\hat{w}(t) = \Big(r\hat{w}(t) + \hat{y}(t) - \hat{c}(t) + \widehat{\theta}(t)^T \left(C_x - \Pi_{\theta x} A_x\right) s(t)\Big) dt + \widehat{\theta}(t)^T \begin{bmatrix} -\Pi_{\theta x}^{T} \\ I \end{bmatrix}^T \begin{bmatrix} \sigma_x \\ \sigma_p \end{bmatrix} dB \tag{5.10}$$

This describes a dynamic portfolio problem where the investor chooses her portfolio costlessly so as to maximise the utility from her consumption in the face of predictable changes in returns. We summarise this frictionless dynamic portfolio problem.

Problem FDP (Frictionless Dynamic Portfolio Problem): Given the economic states $X(t)$, and wealth, $\hat{w}(t)$, whose evolution is described by the transition equations (2.2) and (5.10) respectively, the investor chooses portfolio holdings $\widehat{\theta}(t)$ and the consumption $\hat{c}$ to maximise the welfare function (5.9) subject to the transversality condition $\lim_{t\to\infty} E\left(e^{-\delta t - r\beta\hat{w}(t)}\right) = 0$.

Similar problems have been studied in the literature; here, we offer a new, compact and explicit solution to this problem using a very similar approach to the one developed earlier for the Dynamic Portfolio Control (DPC) problem with Frictions. This solution also describes the optimal consumption and portfolio in terms of the solution to a risk-sensitive Riccati equation. Though the treatment is similar to the DPC problem there are a few differences

1. The only states space in the FDP problem are the economic states $\dot{X(t)}$. We denote this matrix as $\hat{\Pi}$.

2. The stabilising solution to the Riccati equation, $\hat{\Pi}$, is guaranteed to be positive definite.

For better readability, we relegate all the details to the Appendix. We simply state the solution to the FDP problem in the next corollary.

Corollary 5.6. *The optimal portfolio $\widehat{\theta}^*(t)$ to the frictionless dynamic portfolio problem is the target portfolio of the full dynamic portfolio problem as given in Corollary 5.5. The corresponding value function is*

$$\hat{J}_0(\hat{w}, X, t) = -ke^{-\delta t - \gamma\left[\hat{w} + \frac{1}{2}X^T\Pi_{xx}X - \Pi_x X\right]}$$

where $k = \frac{1}{r}e^{-\frac{\delta - r}{r} - \frac{\beta}{2}\mathrm{tr}\left(\sigma_x^\top \Pi_{xx}\sigma_x\right)}$ *and optimal consumption choice is*

$$\hat{c}^*(t) = r\left(\hat{w}(t) + \frac{1}{2}X(t)^T\Pi_{xx}X(t)\right) + \frac{\delta - r}{\gamma} - \frac{1}{2}\mathrm{tr}\left(\sigma_x^\top \Pi_{xx}\sigma_x\right)$$

Thus solving for the optimal target portfolio in the presence of transaction costs is equivalent to solving for the optimal portfolio assuming no explicit trading costs but adjusting the return process for the estimated cost of the trade. solution to an optimal execution problem.

6 Example

In this section, we use a concrete example to illustrate the quantitative and qualitative implications of our main result. To wit, we study how the presence of trading costs affects a standard example from the portfolio choice literature [1], namely the use of price-dividend ratios as a trading signal.

6.1 Model Parameters

For the parameters of the frictionless model, we follow [11] and use the estimates obtained by Barberis [1] from a long time series of US equity index data. Adapted to our notation,[2] these (monthly) parameter estimates are summarized in Table 1.

Parameter	Value
Interest rate r	0.0014
Price volatility σ_p	0.0435
Constant return $\bar{\mu}$	0.0034
State volatility σ_x	0.0189
State mean reversion $-A_x$	0.0226
State weight C_x	0.0435
Time discount rate δ	0.0052
Risk Aversion $\gamma = r\beta$	1.79×10^{-9}
Trading Cost Λ	0.80×10^{-10}

Table 1: Model Parameters

In the present context, the relative magnitudes of risk aversion and trading costs play a crucial role. To wit, risk aversion pins down the average size of the optimal investment in the absence of trading costs. This in turn determines the extent to which the optimal trading decisions are

[2]Without trading costs [7, 11]., the asset price dynamics only enter the solution through the Sharpe ratio, because all models with the same Sharpe ratio span the same set of payoffs in this case. However, since the portfolios generating these payoff streams are different, this equivalence breaks down with trading costs. In this paper, like in other settings with quadratic trading costs [3, 4], we therefore focus on the frictionless specification that is most tractable in our framework: an arithmetic model with constant volatility and Ornstein-Uhlenbeck returns, and preferences with constant absolute risk aversion. In the frictionless version of the model, this leads to the same solution as for a geometric model and preferences with constant relative risk aversion, up to reparametrizing the optimal strategy in terms of numbers of shares rather than risky fractions [7].

affected by the quadratic price impact costs. To illustrate the effects of trading costs on a typical hedge fund portfolio, we choose $\gamma = 1.29 \times 10^{-6} \times 0.0014 = 1.79 \times 10^{-9}$, which leads to an average fund size of one billion for the frictionless myopic portfolio $\frac{\bar{\mu}}{\gamma\sigma_p^2}$.

For the trading cost, we assume that trading 25% of this average fund size over a month reduces the corresponding expected return by 1% of the fund size. This leads to $\Lambda = 0.80 \times 10^{-10}$. These choices of the preference and trading costs parameters are also summarized in Table 1. For comparison, we also report the corresponding results for a larger transaction cost that reduces expected returns by 4% of the fund size per month.

6.2 Results

For these parameter values, we now compare our optimal portfolio to its counterpart for mean-variance preferences over wealth changes [3, 4]. In the frictionless version of their model, portfolio choice is myopic, in that the optimal frictionless portfolio $\frac{\bar{\mu}+C_x X_t}{\gamma\sigma_p^2}$ only depends on the risk-return tradeoff, but does not hedge against future changes in the investment opportunity set. With transaction costs, [4, Equations (10) and (12)] show that it is optimal to track a weighted average of all future values of this frictionless portfolio with a constant trading speed. For the Ornstein-Uhlenbeck dynamics (1.1), (1.2), this leads to

$$\dot{\theta}_{GP} = M_{rate}\left(\frac{\bar{\mu}}{\gamma\sigma_p^2} + \frac{b}{b - A_x}\frac{C_x X_t}{\gamma\sigma_p^2}\right),$$

where

$$M_{rate} = \sqrt{\frac{\gamma\sigma_p^2}{\Lambda} + \frac{\delta^2}{4}} - \frac{\delta}{2} \quad \text{and} \quad b = \sqrt{\frac{\gamma\sigma_p^2}{\Lambda} + \frac{\delta^2}{4}} + \frac{\delta}{2}.$$

By Theorem 3.3, the optimal portfolio in our model with exponential preferences tracks the target portfolio $-\Pi_{\theta\theta}^{-1}(\Pi_\theta + \Pi_{\theta x}X_t)$ with the constant trading speed $-\Pi_{\theta\theta}/\Lambda$. In this model, both the target portfolio and the trading speed now depend on the correlation parameter ρ unlike for the mean-variance model, where this parameter does not influence the optimal trading strategy either without or with transaction costs.

Correlation ρ	Portfolio Constant $-\Pi_{\theta\theta}^{-1}\Pi_\theta$	Portfolio Gain $-\Pi_{\theta\theta}^{-1}\Pi_{\theta x}$
0	0.98	12.18
-0.1	1.05	12.72
-0.3	1.25	14.07
-0.5	1.58	16.07
-0.7	2.28	19.58
-0.9	5.09	29.28

Table 2: Target portfolios in the exponential model for transaction costs $\Lambda = 0.8 \times 10^{-10}$. For the mean-variance model of [4], the corresponding portfolio constant 1 and portfolio gain 12.32 do not depend on ρ. (All values are multiplied by the fund size 10^{-9}.)

The numerical values of the trading targets are collected in Tables 2 and 3, which report the constant and linear parts of the target portfolios for different values of ρ ranging from $\rho = 0$ to $\rho = -0.9$ (which is close to the value $\rho = -0.9135$ estimated by [1]). We see that if the correlation parameter is close to zero, then both the constant and the linear parts of the [4] target portfolio are larger than in the exponential model. As ρ becomes more negative, the sign of this effect is

Correlation ρ	Portfolio Constant $-\Pi_{\theta\theta}^{-1}\Pi_{\theta}$	Portfolio Gain $-\Pi_{\theta\theta}^{-1}\Pi_{\theta x}$
0	0.91	10.05
-0.1	0.98	10.57
-0.3	1.18	11.89
-0.5	1.52	13.87
-0.7	2.23	17.36
-0.9	5.10	27.15

Table 3: Target portfolios in the exponential model for larger transaction costs $\Lambda = 3.2 \times 10^{-10}$. For the mean-variance model of [4], the corresponding portfolio constant 1 and portfolio gain 10.68 do not depend on ρ. (All values are multiplied by the fund size 10^{-9}.)

quickly reversed, and the target positions become much larger in the exponential model than in its mean-variance counterpart. This second effect is due to intertemporal hedging against changing investment opportunities, which is well known to drives a substantial wedge between the frictionless optimal portfolios with exponential and quadratic preferences. To wit, negative price shocks then tend to be offset by better future investment opportunities, and this extra diversification – which is ignored by the quadratic model – scales up the optimal risky investments by a substantial amount. Unless the correlation is very weak, this effect also dominates with transaction costs.

With larger transaction costs, the size of the target portfolio is scaled back somewhat but, for strong correlation, in particular the constant part of the target does not change much.

Correlation ρ	Trading Speed $-\Pi_{\theta\theta}/\Lambda$	Trading Speed $-\Pi_{\theta\theta}/\Lambda$ for large costs
0	0.65	0.10
-0.1	0.65	0.10
-0.3	0.65	0.10
-0.5	0.64	0.09
-0.7	0.63	0.09
-0.9	0.58	0.08

Table 4: Trading speeds in the exponential model for transaction costs $\Lambda = 0.8 \times 10^{-10}$ and for larger transaction costs $\Lambda = 3.2 \times 10^{-10}$. For the mean-variance model of [4], the corresponding trading speeds 0.65 and 0.10 do not depend on ρ.

Table 4 compares the corresponding trading speeds to their counterparts in the mean-variance model of [4]. Clearly, the dependence on the correlation parameter ρ is weaker here in line with asymptotic results for small transaction costs [10]. However, for substantially negative correlation as in [1], the larger portfolio size due to intertemporal hedging does slow down trading. The magnitude of this effect becomes considerably bigger for less persistent trading signals. This is illustrated in Table 5, which reports the optimal trading speeds when the mean-reversion speed $-A_x$ is halved.

In this case, the optimal trading speeds in the mean-variance model remain unchanged, but the optimal trading speeds in the exponential model drop substantially for stronger correlations.

Correlation ρ	Trading Speed $-\Pi_{\theta\theta}/\Lambda$	Trading Speed $-\Pi_{\theta\theta}/\Lambda$ for large costs
0	0.65	0.10
-0.1	0.65	0.10
-0.3	0.65	0.10
-0.5	0.64	0.09
-0.7	0.63	0.08
-0.9	0.58	0.06

Table 5: Trading speeds in the exponential model with slower state mean reversion $-A_x = 0.0226/2$ for transaction costs $\Lambda = 0.8 \times 10^{-10}$ and for larger transaction costs $\Lambda = 3.2 \times 10^{-10}$. For the mean-variance model of [4], the corresponding trading speeds 0.65 and 0.10 do not depend on ρ.

A Proof of Theorem 4.1 and Corollary 4.2

In this appendix, we prove global existence for the system of Riccati equations that defines our candidate value function (3.2) and in turn the corresponding feedback controls (3.3), (3.4).

A.1 Notation

We first introduce some notation and concepts that are used throughout the proof without further mention. The *Kronecker product* of matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m'\times n'}$ is denoted by

$$A \otimes B := \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix} \in \mathbb{R}^{mm'\times nn'}.$$

The *vectorization* $\mathrm{vec}(A)$ of a matrix $A \in \mathbb{R}^{m\times n}$ is the $mn \times 1$ column vector obtained by stacking the columns of the matrix A on top of one another:

$$\mathrm{vec}(A) = [a_{1,1}, \ldots, a_{m,1}, a_{1,2}, \ldots, a_{m,2}, \ldots, a_{1,n}, \ldots, a_{m,n}]^\top.$$

The *operator norm* of a matrix or vector is denoted by $\|\cdot\|$, whereas $|\cdot|$ refers to the determinant of a square matrix. A symmetric matrix $A \in \mathbb{R}^{m\times m}$ is *positive semidefinite* if $z^\top A z \geq 0$ for all $z \in \mathbb{R}^m$ and *positive definite* if this inequality is strict for all $z \in \mathbb{R}^m \setminus \{0\}$. A matrix $A \in \mathbb{R}^{m\times m}$ is *stable* is all the eigenvalues of A have strictly negative real parts.

A.2 Aggregation of the Riccati Equations

We start the proof of Theorem 4.1 by aggregating the three coupled matrix Riccati equations (3.5),(3.6), (3.7) into a higher-dimensional system. To this end, write

$$\Pi = \begin{bmatrix} \Pi_{xx} & -\Pi_{\theta x}^\top \\ -\Pi_{\theta x} & -\Pi_{\theta\theta} \end{bmatrix} \tag{A.1}$$

and define the function $f : \mathbb{R}^{(d+m)\times(d+m)} \times \mathbb{R} \to \mathbb{R}^{(d+m)\times(d+m)}$ by

$$f(\Pi; \epsilon) = \gamma \Pi B_1 \sigma_x \sigma_x^\top B_1^\top \Pi - \epsilon \Pi B_2 \Lambda^{-1} B_2^\top \Pi + \Pi A + A^\top \Pi + Q, \tag{A.2}$$

where

$$B_1 = \begin{bmatrix} I_d \\ 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ I_m \end{bmatrix}, \qquad A = \begin{bmatrix} \frac{r}{2} I_d - A_x & \gamma\sigma_x\sigma_p^\top \\ 0 & \frac{r}{2} I_m \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & -C_x \\ -C_x & \gamma\sigma_p\sigma_p^\top \end{bmatrix}.$$

For $\epsilon = 1$, the off-diagonal terms of $f(\Pi;\epsilon)$ then coincide with the left-hand side of equation (3.7) and the diagonal terms match the left-hand sides of equations (3.5) and (3.6), respectively. Therefore, solving equations (3.5) - (3.7) is equivalent to solving $f(\Pi;1) = 0$.

A.3 Evolution as a Function of the Cost Size

The scalar parameter ϵ is introduced in (A.2) to modulate the size of the transaction costs in equations (3.5)–(3.7):

$$0 = \epsilon\Pi_{\theta\theta}\Lambda^{-1}\Pi_{\theta\theta} + r\Pi_{\theta\theta} - \gamma\left(\Pi_{\theta x}\sigma_x - \sigma_p\right)\left(\Pi_{\theta x}\sigma_x - \sigma_p\right)^\top, \tag{A.3}$$

$$0 = \gamma\Pi_{xx}\sigma_x\sigma_x^\top\Pi_{xx} + r\Pi_{xx} - A_x^\top\Pi_{xx} - \Pi_{xx}A_x - \epsilon\Pi_{\theta x}^\top\Lambda^{-1}\Pi_{\theta x}, \tag{A.4}$$

$$0 = \gamma\sigma_p\sigma_x^\top\Pi_{xx} - C_x - \left(\epsilon\Lambda^{-1}\Pi_{\theta\theta} + \frac{r}{2}I_m\right)^\top\Pi_{\theta x} - \Pi_{\theta x}\left(\frac{r}{2}I_d - A_x + \gamma\sigma_x\sigma_x^\top\Pi_{xx}\right). \tag{A.5}$$

As ϵ tends to zero the costs become prohibitively large and we obtain an explicit solution, that corresponds to the no-trade solution for the lifetime consumption problem (2.5). To wit, for $\epsilon = 0$, (A.3), (A.4), (A.5) simplify to

$$0 = r\Pi_{\theta\theta} - \gamma\left(\Pi_{\theta x}\sigma_x - \sigma_p\right)\left(\Pi_{\theta x}\sigma_x - \sigma_p\right)^\top, \tag{A.6}$$

$$0 = \gamma\Pi_{xx}\sigma_x\sigma_x^\top\Pi_{xx} + r\Pi_{xx} - A_x^\top\Pi_{xx} - \Pi_{xx}A_x, \tag{A.7}$$

$$0 = \Pi_{\theta x}\left(rI_d - A_x + \gamma\sigma_x\sigma_x^\top\Pi_{xx}\right) + C_x - \gamma\sigma_p\sigma_x^\top\Pi_{xx}. \tag{A.8}$$

(A.7) is evidently solved by $\Pi_{xx} = 0$, and this is the unique positive semidefinite solution by [12, Theorem 13.7]. With $\Pi_{xx} = 0$, the other two equations (A.6) and (A.8) in turn have the unique explicit solutions

$$\Pi_{\theta x} = -C_x\left(rI_d - A_x\right)^{-1},$$
$$\Pi_{\theta\theta} = \gamma\left(\sigma_p + C_x\left(rI_d - A_x\right)^{-1}\sigma_x\right)\left(\sigma_p + C_x\left(rI_d - A_x\right)^{-1}\sigma_x\right)^\top.$$

Here, $\Pi_{\theta\theta}$ is positive definite. In summary, the aggregate system $f(\Pi;0) = 0$ has the solution

$$\Pi_0 = \begin{bmatrix} 0 & C_x\left(rI_d - A_x\right)^{-1} \\ C_x\left(rI_d - A_x\right)^{-1} & -\gamma\left(\sigma_p + C_x\left(rI_d - A_x\right)^{-1}\sigma_x\right)\left(\sigma_p + C_x\left(rI_d - A_x\right)^{-1}\sigma_x\right)^\top \end{bmatrix}. \tag{A.9}$$

Starting from this fully-explicit solution of $f(\Pi;\epsilon)$ for $\epsilon = 0$ (infinitely large transaction costs), our goal now is to show that solutions also exist for larger ϵ, corresponding to finite levels of transaction costs. To this end, notice that *if* solutions

$$\Pi(\epsilon) = \begin{bmatrix} \Pi_{xx}(\epsilon) & -\Pi_{\theta x}(\epsilon)^\top \\ -\Pi_{\theta x}(\epsilon) & -\Pi_{\theta\theta}(\epsilon) \end{bmatrix}$$

of $f(\Pi;\epsilon) = 0$ exist and depend on ϵ in a smooth manner, then we can differentiate the equation with respect to ϵ and obtain a matrix-valued initial value problem that describes the evolution of the solution starting from the explicit large-cost limit:

$$\Pi'(\epsilon)\mathcal{A}(\Pi(\epsilon);\epsilon) + \mathcal{A}(\Pi(\epsilon);\epsilon)^\top\Pi'(\epsilon) - \Pi(\epsilon)B_2\Lambda^{-1}B_2^\top\Pi(\epsilon) = 0, \quad \Pi(0) = \Pi_0, \tag{A.10}$$

where

$$\begin{aligned}\mathcal{A}(\Pi;\epsilon) &= \gamma B_1\sigma_x\sigma_x^\top B_1^\top\Pi - \epsilon B_2\Lambda^{-1}B_2^\top\Pi + A \qquad \text{(A.11)} \\ &= \begin{bmatrix} \gamma\sigma_x\sigma_x^\top\Pi_{xx} + \frac{r}{2}I_d - A_x & \gamma\sigma_x\left(\sigma_p - \Pi_{\theta x}\sigma_x\right)^\top \\ \epsilon\Lambda^{-1}\Pi_{\theta x} & \epsilon\Lambda^{-1}\Pi_{\theta\theta} + \frac{r}{2}I_m \end{bmatrix}.\end{aligned}$$

A.4 Wellposedness of the Evolution Equation

Using the properties of the matrix-valued function $\mathcal{A}(\cdot;\epsilon)$ established in Appendix B, we now prove the *global* existence and uniqueness of the matrix ODE (A.10) for arbitrary values of ϵ, without *any* constraints on the model parameters. In particular, this establishes the existence result for our original Riccati system (A.3), (A.4), (A.5) stated in Theorem 4.1.

Theorem A.1. *(i) There exists a unique C^1 solution $\epsilon \mapsto \Pi(\epsilon)$ of* (A.10) *on $[0,\infty)$.*

(ii) For every $v \in \mathbb{R}^{(d+m)\times 1}$, $\epsilon \mapsto v^\top \Pi(\epsilon) v$ is increasing on $[0,\infty)$.

(iii) For every $\epsilon \geq 0$, $f(\Pi(\epsilon);\epsilon) = 0$, and $B_1^\top \Pi(\epsilon) B_1 \in \mathbb{R}^{d\times d}$ and $-B_2^\top \Pi(\epsilon) B_2 \in \mathbb{R}^{m\times m}$ are both symmetric positive-semidefinite matrices. Moreover, $B_1^\top \Pi(\epsilon) B_1$, $-B_1^\top \Pi(\epsilon) B_2$ and $B_2^\top \Pi(\epsilon) B_2$ solve the matrix Riccati equations (A.3), (A.4), (A.5).

Proof. We prove the assertions of the theorem as follows. First, we establish *local* existence for (A.10) on a maximum interval of existence $[0,\epsilon_+)$. Then, we show that properties (ii) and (iii) hold on $[0,\epsilon_+)$. Finally, we prove that the local solutions $\Pi(\epsilon)$ remains uniformly bounded in ϵ. This implies that $\epsilon_+ = \infty$, so that the solutions are in fact global and properties (ii) and (iii) in turn hold for arbitrary values of ϵ.

Step 1: Establish local existence and uniqueness for (A.10). Using the Kronecker product and the vectorization operator, the matrix ODE (A.10) can be rewritten as

$$\begin{cases} \operatorname{vec}(\Pi(\epsilon))' = \left(I_{d+m} \otimes \mathcal{A}(\Pi(\epsilon);\epsilon)^\top + \mathcal{A}(\Pi(\epsilon);\epsilon)^\top \otimes I_{d+m}\right)^{-1} \operatorname{vec}\left(\Pi(\epsilon) B_2 \Lambda^{-1} B_2^\top \Pi(\epsilon)\right), \\ \operatorname{vec}(\Pi(0)) = \operatorname{vec}(\Pi_0). \end{cases} \tag{A.12}$$

Here $\left(I_{d+m} \otimes \mathcal{A}(\Pi;\epsilon)^\top + \mathcal{A}(\Pi;\epsilon)^\top \otimes I_{d+m}\right)$ is invertible if and only if the determinant of $\mathcal{A}$ is not 0. At $\epsilon = 0$,

$$|\mathcal{A}(\Pi(0);0)| = \left|\begin{bmatrix} \frac{r}{2} I_d - A_x & \gamma\sigma_x\left(\sigma_p - \Pi_{\theta x}(0)\sigma_x\right)^\top \\ 0 & \frac{r}{2} I_m \end{bmatrix}\right| = \frac{r}{2}\left|\frac{r}{2} I_d - A_x\right| > 0.$$

Hence there exists a neighborhood of $(\Pi(0),0)$ such that $\mathcal{A}(\Pi;\epsilon)$ is invertible. By Cramer's rule, each entry of the matrix $(I_{d+m} \otimes \mathcal{A}(\Pi;\epsilon)^\top + \mathcal{A}(\Pi;\epsilon)^\top \otimes I_{d+m})^{-1}\operatorname{vec}(\Pi B_2 \Lambda^{-1} B_2^\top \Pi)$ is the ratio of determinants of two matrices, and therefore the ratio of two polynomials. Hence, the entries are jointly continuous in (Π,ϵ) and locally Lipschitz in Π in a neighborhood of $(\Pi(0),0)$. By [2, Theorem 1.261], the initial-value problem (A.12) therefore has a unique solution on a maximum interval of existence $[0,\epsilon_+)$.

Step 2: Link (A.10) *to the original Riccati equations* (A.3) - (A.5) *on* $[0,\epsilon_+)$. As $f(\Pi(0);0) = f(\Pi_0;0) = 0$, integrating both sides of the ODE (A.10) with respect to ϵ yields

$$\begin{aligned} f(\Pi(\epsilon);\epsilon) &= f(\Pi(\epsilon);\epsilon) - f(\Pi(0);0) \\ &= \int_0^\epsilon \left(\Pi'(\epsilon')\mathcal{A}(\Pi(\epsilon');\epsilon') + \mathcal{A}(\Pi(\epsilon');\epsilon')^\top \Pi'(\epsilon') - \Pi(\epsilon') B_2 \Lambda^{-1} B_2^\top \Pi(\epsilon')\right) d\epsilon' = 0. \end{aligned}$$

The solution of the matrix ODE (A.10) therefore indeed yields a solution of $f(\Pi(\epsilon);\epsilon) = 0$, and in turn a solution of the coupled system of algebraic Riccati equations (A.3) - (A.5) as asserted in (iii).

Step 3: Establish properties of the local solution. Next, we show that the matrices $\Pi_{xx}(\epsilon)$ and $\Pi_{\theta\theta}(\epsilon)$ are positive semidefinite for $\epsilon \in [0,\epsilon_+)$. We start with $\Pi_{\theta\theta}(\epsilon) = -B_2^\top \Pi(\epsilon) B_2$. Fixing

$\epsilon \in [0, \epsilon_+)$ and treating $\Pi_{\theta x}(\epsilon) = -B_1^\top \Pi(\epsilon) B_2$ as given, $\Pi_{\theta\theta}(\epsilon)$ is the solution (A.3). By [12, Theorem 13.7], positive semidefiniteness of $\Pi_{\theta\theta}(\epsilon)$ is equivalent to stability of $-\epsilon\Lambda^{-1}\Pi_{\theta\theta}(\epsilon) - \frac{r}{2}I_m$. This is in turn equivalent to establishing that the eigenvalues of $\epsilon\Lambda^{-1/2}\Pi_{\theta\theta}(\epsilon)\Lambda^{-1/2}$ are all greater than $-r/2$ for every $\epsilon \in [0, \epsilon_+)$. To this end, we assume by contradiction that there exists $\epsilon_0 \in [0, \epsilon_+)$ such that the smallest eigenvalue of $\epsilon_0\Lambda^{-1/2}\Pi_{\theta\theta}(\epsilon_0)\Lambda^{-1/2}$ is less than or equal to $-r/2$. Define

$$f_\theta(\epsilon) = \min\left\{\epsilon\theta^\top\Lambda^{-1/2}\Pi_{\theta\theta}(\epsilon)\Lambda^{-1/2}\theta : \theta \in \mathbb{R}^m, \|\theta\| = 1\right\}, \tag{A.13}$$

which is continuous in ϵ by the maximum theorem. As $f_\theta(0) = 0$ and by the variational characterization of the smallest eigenvalue, it follows that there exists $\epsilon_\theta \in (0, \epsilon_0]$ such that $f_\theta(\epsilon_\theta) = -r/2$. Again by the variational characterization of the smallest eigenvalue $-r/2$, there exists $\theta(\epsilon_\theta) \in \mathbb{R}^m$ with $\|\theta(\epsilon_\theta)\| = 1$ such that

$$\epsilon_\theta\Lambda^{-1/2}\Pi_{\theta\theta}(\epsilon_\theta)\Lambda^{-1/2}\theta(\epsilon_\theta) = -\frac{r}{2}\theta(\epsilon_\theta).$$

Since $\epsilon_\theta > 0$, it follows that

$$\theta(\epsilon_\theta)^\top\Lambda^{-1/2}\Pi_{\theta\theta}(\epsilon_\theta)\Lambda^{-1/2}\theta(\epsilon_\theta) = -\frac{1}{\epsilon_\theta}\frac{r}{2}\theta(\epsilon_\theta)^\top\theta(\epsilon_\theta) = -\frac{r}{2\epsilon_\theta}.$$

After multiplying the Riccati equation (A.3) by $\left(\Lambda^{-1/2}\theta(\epsilon_\theta)\right)^\top$ and $\Lambda^{-1/2}\theta(\epsilon_\theta)$ from the left and right, respectively, and plugging in these two identities, we arrive at the desired contradiction:

$$\begin{aligned}
0 &\leq \gamma\left(\Lambda^{-1/2}\theta(\epsilon_\theta)\right)^\top\left(\Pi_{\theta x}(\epsilon_\theta)\sigma_x - \sigma_p\right)\left(\Pi_{\theta x}(\epsilon_\theta)\sigma_x - \sigma_p\right)^\top\Lambda^{-1/2}\theta(\epsilon_\theta) \\
&= \epsilon_\theta\left(\Lambda^{-1/2}\theta(\epsilon_\theta)\right)^\top\Pi_{\theta\theta}(\epsilon_\theta)^\top\Lambda^{-1}\Pi_{\theta\theta}(\epsilon_\theta)\Lambda^{-1/2}\theta(\epsilon_\theta) + r\left(\Lambda^{-1/2}\theta(\epsilon_\theta)\right)^\top\Pi_{\theta\theta}(\epsilon_\theta)\Lambda^{-1/2}\theta(\epsilon_\theta) \\
&= \frac{1}{\epsilon_\theta}\left\|\epsilon_\theta\Lambda^{-1/2}\Pi_{\theta\theta}(\epsilon_\theta)\Lambda^{-1/2}\theta(\epsilon_\theta)\right\|^2 + r\theta(\epsilon_\theta)^\top\Lambda^{-1/2}\Pi_{\theta\theta}(\epsilon_\theta)\Lambda^{-1/2}\theta(\epsilon_\theta) \\
&= \frac{r^2}{4\epsilon_\theta} - \frac{r^2}{2\epsilon_\theta} = -\frac{r^2}{4\epsilon_\theta} < 0.
\end{aligned}$$

Therefore, for arbitrary $\epsilon \in [0, \epsilon_+)$, we can conclude that $-\epsilon\Lambda^{-1}\Pi_{\theta\theta}(\epsilon) - \frac{r}{2}I_m$ is stable. Hence, the positive semidefiniteness of $\Pi_{\theta\theta}$ follows from [12, Theorem 13.7].

To establish the positive semidefiniteness of $\Pi_{xx}(\epsilon) = B_1^\top\Pi(\epsilon)B_1$, we first show the positive defniteness of $\Pi_{xx}(\epsilon) + \widetilde{A}_x$, where $\widetilde{A}_x$ is the unique positive definite solution of

$$\gamma\widetilde{A}_x\sigma_x\sigma_x^\top\widetilde{A}_x + A_x^\top\widetilde{A}_x + \widetilde{A}_xA_x = 0.$$

(Existence and uniqueness once again follow from [12, Theorem 13.7].) Suppose by contradiction that for the smallest eigenvalue of $\Pi_{xx}(\epsilon) + \widetilde{A}_x$ is less than or equal to 0 for some $\epsilon \in [0, \epsilon_+)$. As above, the continuity of the smallest eigenvalues and the existence of the corresponding eigenvectors show that there exists $\epsilon_x \in [0, \epsilon_+)$ and $x(\epsilon_x) \in \mathbb{R}^d$ with $\|x(\epsilon_x)\| = 1$ such that

$$\Pi_{xx}(\epsilon_x)x(\epsilon_x) = -\widetilde{A}_xx(\epsilon_x).$$

Multiplying the Riccati equation (A.4) by $x(\epsilon_x)^\top$ and $x(\epsilon_x)$ from the left and right, respectively, plugging in this identity and rearranging terms, we obtain

$$
\begin{aligned}
0 &\le \epsilon_x x(\epsilon_x)^\top \Pi_{\theta x}(\epsilon_x)^\top \Lambda \Pi_{\theta x}(\epsilon_x) x(\epsilon_x) \\
&= x(\epsilon_x)^\top \left(\gamma \Pi_{xx}(\epsilon_x)\sigma_x\sigma_x^\top \Pi_{xx}(\epsilon_x) + r\Pi_{xx}(\epsilon_x) - A_x^\top \Pi_{xx}(\epsilon_x) - \Pi_{xx}(\epsilon_x) A_x \right) x(\epsilon_x) \\
&= \gamma x(\epsilon_x)^\top \widetilde{A}_x \sigma_x\sigma_x^\top \widetilde{A}_x x(\epsilon_x) - r x(\epsilon_x)^\top \widetilde{A}_x x(\epsilon_x) + x(\epsilon_x)^\top A_x^\top \widetilde{A}_x x(\epsilon_x) + x(\epsilon_x)^\top \widetilde{A}_x A_x x(\epsilon_x) \\
&= -r x(\epsilon_x)^\top \widetilde{A}_x x(\epsilon_x) + x(\epsilon_x)^\top \left(\gamma \widetilde{A}_x \sigma_x\sigma_x^\top \widetilde{A}_x + A_x^\top \widetilde{A}_x + \widetilde{A}_x A_x \right) x(\epsilon_x) \\
&= -r x(\epsilon_x)^\top \widetilde{A}_x x(\epsilon_x) < 0.
\end{aligned}
$$

This contradicts the positive definiteness of $\widetilde{A}_x$, so $\Pi_{xx}(\epsilon)+\widetilde{A}_x$ is indeed positive definite as asserted above. Next, notice that

$$
\begin{aligned}
&\left(\Pi_{xx}(\epsilon) + \widetilde{A}_x\right)\left(\gamma\sigma_x\sigma_x^\top \Pi_{xx}(\epsilon) + \frac{r}{2} I_d - A_x\right) + \left(\gamma\sigma_x\sigma_x^\top \Pi_{xx}(\epsilon) + \frac{r}{2} I_d - A_x\right)^\top \left(\Pi_{xx}(\epsilon) + \widetilde{A}_x\right) \\
&= r\widetilde{A}_x + \epsilon \Pi_{\theta x}(\epsilon)^\top \Lambda^{-1} \Pi_{\theta x}(\epsilon) + \Pi_{xx}(\epsilon)\sigma_x\sigma_x^\top \Pi_{xx}(\epsilon) + \Pi_{xx}(\epsilon)\sigma_x\sigma_x \widetilde{A}_x + \widetilde{A}_x \sigma_x\sigma_x^\top \Pi_{xx}(\epsilon) \\
&\quad - A_x^\top \widetilde{A}_x - \widetilde{A}_x A_x \\
&= r\widetilde{A}_x + \epsilon \Pi_{\theta x}(\epsilon)^\top \Lambda^{-1} \Pi_{\theta x}(\epsilon) + \gamma \Pi_{xx}(\epsilon)\sigma_x\sigma_x^\top \Pi_{xx}(\epsilon) + \gamma \Pi_{xx}(\epsilon)\sigma_x\sigma_x \widetilde{A}_x + \gamma \widetilde{A}_x \sigma_x\sigma_x^\top \Pi_{xx}(\epsilon) \\
&\quad + \gamma \widetilde{A}_x \sigma_x\sigma_x^\top \widetilde{A}_x \\
&= r\widetilde{A}_x + \epsilon \Pi_{\theta x}(\epsilon)^\top \Lambda^{-1} \Pi_{\theta x}(\epsilon) + \gamma \left(\Pi_{xx}(\epsilon) + \widetilde{A}_x\right) \sigma_x\sigma_x^\top \left(\Pi_{xx}(\epsilon) + \widetilde{A}_x\right)
\end{aligned}
$$

is also positive definite since it is the sum of a positive definite matrix and two positive semidefinite matrices. As a consequence, the Lyapunov Lemma as in [12, Lemma 3.18] yields the stability of $-\left(\gamma\sigma_x\sigma_x^\top \Pi_{xx}(\epsilon) + \frac{r}{2} I_d - A_x\right)$. As the matrix $\Pi_{xx}(\epsilon)$ solves the Riccati equation (A.4) (for fixed $\epsilon \in [0, \epsilon_+)$ and $\Pi_{\theta x}(\epsilon)$), it in turn follows from [12, Theorem 13.7] that $\Pi_{xx}(\epsilon)$ is indeed positive semidefinite for all $\epsilon \in [0, \epsilon_+)$.

Step 4: Establish the monotonicity of the local solution. From Steps 2 and 3, we know that $-B_2^\top \Pi(\epsilon) B_2$ and $B_1^\top \Pi(\epsilon) B_1$ are the positive semidefinite solutions of the matrix Riccati equations (A.3) and (A.4) respectively. Hence, by Lemma B.1, we know the matrix $-\mathcal{A}(\Pi(\epsilon);\epsilon)$ from (A.11) is stable. As a consequence,[3] we can therefore rewrite (A.10) as

$$
\Pi'(\epsilon) = \int_0^\infty e^{-\mathcal{A}(\Pi(\epsilon);\epsilon)^\top \tau} \Pi(\epsilon) B_2 \Lambda^{-1} B_2^\top \Pi(\epsilon) e^{-\mathcal{A}(\Pi(\epsilon);\epsilon)\tau} d\tau.
$$

This shows that $\Pi'(\epsilon)$ is positive semidefinite for every $\epsilon \in [0, \epsilon_+)$ by [12, Lemma 3.18], and in turn yields the asserted monotonicity:

$$
\left(v^\top \Pi(\epsilon) v\right)' = v^\top \Pi'(\epsilon) v \ge 0, \quad \text{for } v \in \mathbb{R}^{d+m}.
$$

Step 5: Show that the local solution is global. To ease notation, define

$$
y(\epsilon) := \Lambda^{-1/2}\left(\Pi_{\theta x}(\epsilon)\sigma_x - \sigma_p\right) \in \mathbb{R}^{m\times k}, \tag{A.14}
$$

and notice that $\epsilon y(\epsilon) y(\epsilon)^\top + \frac{r^2}{4} I_m$ is positive definite for every $\epsilon \in (0, \epsilon_+)$. With this notation, (A.5) can be rewritten as

$$
\begin{aligned}
\Lambda^{-1/2}\Pi_{\theta x}(\epsilon) = -&\left(\epsilon y(\epsilon) y(\epsilon)^\top + \frac{r^2}{4} I_m\right)^{-1/2} \\
&\times \left(\gamma y(\epsilon)\sigma_x^\top \Pi_{xx}(\epsilon) + \Lambda^{-1/2}\left(\Pi_{\theta x}(\epsilon)\left(\frac{r}{2} I_d - A_x\right) + C_x\right)\right).
\end{aligned} \tag{A.15}
$$

[3]For a strictly negative number $a < 0$ any real b can be written as $b = \int_0^\infty e^{a\tau} ab \, d\tau$. Likewise, for a stable matrix $A \in \mathbb{R}^m$, we can also represent an arbitrary matrix $B \in \mathbb{R}^m$ as $B = \int_0^\tau e^{A^\top \tau}\left(A^\top B + BA\right) e^{A\tau} d\tau$.

Now, we prove by contradiction that the local solution is global. Suppose the maximum interval of existence is finite, $\epsilon_+ \in (0,\infty)$. Then, $\lim_{\epsilon\to\epsilon_+}\|\Pi(\epsilon)\| = \infty$ and in turn $\lim_{\epsilon\to\epsilon_+}\|\Pi_{\theta x}(\epsilon)\| = \infty$, as $\Pi_{\theta\theta}(\epsilon)$ and $\Pi_{xx}(\epsilon)$ would both remain bounded by (A.3) and (A.4) if $\Pi_{\theta x}(\epsilon)$ does not blow up.

To work towards a contradiction, we choose a sequence $\epsilon_n \to \epsilon_+$ such that $\|\Pi_{\theta x}(\epsilon_n)\| > 0$ and the following normalized limit exists:

$$\widehat{\Pi}_{\theta x} := \lim_{n\to\infty} \frac{\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}}{\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|}, \qquad \text{with } \left\|\widehat{\Pi}_{\theta x}\right\| = 1.$$

By the uniqueness of the positive semidefine square root of positive semidefinite matrices, it in turn follows that

$$\lim_{n\to\infty} \frac{\left((\sigma_x\sigma_x^\top)^{1/2}\Pi_{\theta x}(\epsilon_n)^\top\Lambda^{-1}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right)^{1/2}}{\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|} = \left(\widehat{\Pi}_{\theta x}^\top\widehat{\Pi}_{\theta x}\right)^{1/2}.$$

Since 1 is the operator norm of $\widehat{\Pi}_{\theta x}$, it is also one of its singular values. Let $\theta \in \mathbb{R}^m$, $x \in \mathbb{R}^d$ with $\|\theta\| = 1 = \|x\|$ be a pair of singular vectors of $\widehat{\Pi}_{\theta x}$ with respect to the singular value 1, i.e.,

$$\widehat{\Pi}_{\theta x}x = \theta, \qquad \widehat{\Pi}_{\theta x}^\top\theta = x. \tag{A.16}$$

Then, $\widehat{\Pi}_{\theta x}^\top\widehat{\Pi}_{\theta x}x = \widehat{\Pi}_{\theta x}^\top\theta = x$ and it in turn follows from the singular value decomposition as in [5, Chapter 3] that

$$\left(\widehat{\Pi}_{\theta x}^\top\widehat{\Pi}_{\theta x}\right)^{1/2} x = x.$$

Next, we show x is an eigenvector of $(\sigma_x\sigma_x^\top)^{1/2}\Pi_{xx}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}/\|(\sigma_x\sigma_x^\top)^{1/2}\Pi_{xx}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\|$ as $n\to\infty$. Multiplying $(\sigma_x\sigma_x^\top)^{1/2}$ on both sides and rearranging terms, (A.4) can be written as

$$\begin{aligned}
&\gamma\epsilon_n(\sigma_x\sigma_x^\top)^{1/2}\Pi_{\theta x}(\epsilon_n)^\top\Lambda^{-1}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\\
&= \left(\gamma(\sigma_x\sigma_x^\top)^{1/2}\Pi_{xx}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2} + \frac{r}{2}I_d - (\sigma_x\sigma_x^\top)^{-1/2}A_x(\sigma_x\sigma_x^\top)^{1/2}\right)^\top\\
&\qquad\times\left(\gamma(\sigma_x\sigma_x^\top)^{1/2}\Pi_{xx}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2} + \frac{r}{2}I_d - (\sigma_x\sigma_x^\top)^{-1/2}A_x(\sigma_x\sigma_x^\top)^{1/2}\right)\\
&\quad - \left(\frac{r}{2}I_d - (\sigma_x\sigma_x^\top)^{-1/2}A_x(\sigma_x\sigma_x^\top)^{1/2}\right)^\top\left(\frac{r}{2}I_d - (\sigma_x\sigma_x^\top)^{-1/2}A_x(\sigma_x\sigma_x^\top)^{1/2}\right).
\end{aligned}\tag{A.17}$$

After normalizing by $\epsilon_n\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|^2$ and sending $n\to\infty$, the constant matrices terms vanish and we end up with:

$$\begin{aligned}
\lim_{n\to\infty}\frac{\left(\gamma(\sigma_x\sigma_x^\top)^{1/2}\Pi_{xx}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right)^2}{\gamma\epsilon_n\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|^2} &= \lim_{n\to\infty}\epsilon_n\frac{(\sigma_x\sigma_x^\top)^{1/2}\Pi_{\theta x}(\epsilon_n)^\top\Lambda^{-1}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}}{\epsilon_n\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|^2}\\
&= \widehat{\Pi}_{\theta x}^\top\widehat{\Pi}_{\theta x}.
\end{aligned}$$

After taking the square root and multiplying by the singular vector x from the right, this leads to

$$\lim_{n\to\infty}\frac{(\sigma_x\sigma_x^\top)^{1/2}\Pi_{xx}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}x}{\left\|(\sigma_x\sigma_x^\top)^{1/2}\Pi_{xx}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|} = \left(\widehat{\Pi}_{\theta x}^\top\widehat{\Pi}_{\theta x}\right)^{1/2}x = x. \tag{A.18}$$

Now, we focus on $y(\epsilon_n)$ from (A.14). First notice that

$$\lim_{n\to\infty}\left\|y(\epsilon_n)\sigma_x^\top(\sigma_x\sigma_x^\top)^{-1/2}\right\| = \lim_{n\to\infty}\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2} - \Lambda^{-1/2}\sigma_p\sigma_x^\top(\sigma_x\sigma_x^\top)^{-1/2}\right\| = \infty.$$

Next, we show that $(\theta, \sigma_x^\top(\sigma_x\sigma_x^\top)^{-1/2}x)$ is a pair of singular vectors for $y(\epsilon_n)/\|y(\epsilon_n)\|$ as $n \to \infty$ with respect to the singular value 1. By definition,

$$\lim_{n\to\infty} \frac{\left\|y(\epsilon_n)\sigma_x^\top(\sigma_x\sigma_x^\top)^{-1/2}\right\|}{\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|} = \lim_{n\to\infty} \frac{\left\|\Lambda^{-1/2}\left(\Pi_{\theta x}(\epsilon_n)\sigma_x - \sigma_p\right)\sigma_x^\top(\sigma_x\sigma_x^\top)^{-1/2}\right\|}{\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|} = 1,$$

and therefore,

$$\lim_{n\to\infty} \frac{\|y(\epsilon_n)\|}{\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|} = \lim_{n\to\infty} \frac{\|y(\epsilon_n)\|}{\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)\sigma_x\right\|} = 1. \tag{A.19}$$

As a consequence, as $n \to \infty$,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\theta^\top y(\epsilon_n)}{\|y(\epsilon_n)\ \|} &= \lim_{n\to\infty} \frac{\theta^\top\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)\sigma_x\sigma_x^\top\left(\sigma_x\sigma_x^\top\right)^{-1}\sigma_x}{\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|} \\
&= \theta^\top\widehat{\Pi}_{\theta x}\left(\sigma_x\sigma_x^\top\right)^{-1/2}\sigma_x \\
&= x^\top(\sigma_x\sigma_x^\top)^{-1/2}\sigma_x. \qquad (A.20)
\end{aligned}$$

Likewise,

$$\lim_{n\to\infty} \frac{y(\epsilon_n)\sigma_x^\top(\sigma_x\sigma_x^\top)^{-1/2}x}{\|y(\epsilon_n)\ \|} = \lim_{n\to\infty} \frac{\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)\sigma_x\sigma_x^\top(\sigma_x\sigma_x^\top)^{-1/2}x}{\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|} = \widehat{\Pi}_{\theta x}x = \theta. \tag{A.21}$$

Therefore $(\theta, \sigma_x^\top\left(\sigma_x\sigma_x^\top\right)^{-1/2}x)$ is indeed a pair of singular vectors with respect to the singular value 1, and

$$\lim_{n\to\infty} \frac{y(\epsilon_n)y(\epsilon_n)^\top\theta}{\|y(\epsilon_n)\ \|^2} = \lim_{n\to\infty} \frac{y(\epsilon_n)\sigma_x^\top(\sigma_x\sigma_x^\top)^{-1/2}x}{\|y(\epsilon_n)\ \|} = \theta.$$

Thus,

$$\lim_{n\to\infty} \frac{\left(y(\epsilon_n)y(\epsilon_n)^\top\right)^{1/2}\theta}{\|y(\epsilon_n)\ \|} = \theta.$$

Furthermore, θ is a eigenvector for $\|y(\epsilon_n)\|\left(\epsilon_n y(\epsilon_n)y(\epsilon_n)^\top + \frac{r^2}{4}I_m\right)^{-1/2}$ as $n \to \infty$, because

$$\begin{aligned}
\theta &= \lim_{n\to\infty} \frac{\|y(\epsilon_n)\|\left(\epsilon_n y(\epsilon_n)y(\epsilon_n)^\top + \frac{r^2}{4}I_m\right)^{-1/2}\left(\epsilon_n y(\epsilon_n)y(\epsilon_n)^\top + \frac{r^2}{4}I_m\right)^{1/2}\theta}{\|y(\epsilon_n)\|} \\
&= \lim_{n\to\infty} \|y(\epsilon_n)\|\left(\epsilon_n y(\epsilon_n)y(\epsilon_n)^\top + \frac{r^2}{4}I_m\right)^{-1/2}\frac{\left(\epsilon_n y(\epsilon_n)y(\epsilon_n)^\top\right)^{1/2}\theta}{\|y(\epsilon_n)\ \|} \\
&= \lim_{n\to\infty} \sqrt{\epsilon_n}\|y(\epsilon_n)\|\left(\epsilon_n y(\epsilon_n)y(\epsilon_n)^\top + \frac{r^2}{4}I_m\right)^{-1/2}\theta. \qquad (A.22)
\end{aligned}$$

Finally, we show $(\theta, \sigma_x^\top\left(\sigma_x\sigma_x^\top\right)^{-1/2}x)$ is also a pair of singular vectors with respect to singular value 1 for $\sqrt{\epsilon_n}\left(\epsilon_n y(\epsilon_n)y(\epsilon_n)^\top + \frac{r^2}{4}I_m\right)^{-1/2}y(\epsilon_n)$ as $n \to \infty$. To this end, by (A.21) and (A.22), notice

$$
\begin{aligned}
&\lim_{n\to\infty} \sqrt{\epsilon_n} \left(\epsilon_n y(\epsilon_n) y(\epsilon_n)^\top + \frac{r^2}{4} I_m\right)^{-1/2} y(\epsilon_n)\sigma_x^\top (\sigma_x\sigma_x^\top)^{-1/2} x \\
&= \lim_{n\to\infty} \sqrt{\epsilon_n} \|y(\epsilon_n)\| \left(\epsilon_n y(\epsilon_n) y(\epsilon_n)^\top + \frac{r^2}{4} I_m\right)^{-1/2} \frac{y(\epsilon_n)\sigma_x^\top (\sigma_x\sigma_x^\top)^{-1/2} x}{\|y(\epsilon_n)\ \|} \\
&= \lim_{n\to\infty} \sqrt{\epsilon_n} \|y(\epsilon_n)\| \left(\epsilon_n y(\epsilon_n) y(\epsilon_n)^\top + \frac{r^2}{4} I_m\right)^{-1/2} \theta \\
&= \theta,
\end{aligned}
$$

and using (A.22) then (A.20), we have

$$
\begin{aligned}
&\lim_{n\to\infty} \theta^\top \sqrt{\epsilon_n} \left(\epsilon_n y(\epsilon_n) y(\epsilon_n)^\top + \frac{r^2}{4} I_m\right)^{-1/2} y(\epsilon_n) \\
&= \lim_{n\to\infty} \sqrt{\epsilon_n} \|y(\epsilon_n)\| \theta^\top \left(\epsilon_n y(\epsilon_n) y(\epsilon_n)^\top + \frac{r^2}{4} I_m\right)^{-1/2} \frac{y(\epsilon_n)}{\|y(\epsilon_n)\|} \\
&= \lim_{n\to\infty} \theta^\top \frac{y(\epsilon_n)}{\|y(\epsilon_n)\|} \\
&= x^\top (\sigma_x\sigma_x^\top)^{-1/2} \sigma_x^\top. \qquad \text{(A.23)}
\end{aligned}
$$

Then, $\theta^\top \left(\epsilon_n y(\epsilon_n) y(\epsilon_n)^\top + \frac{r^2}{4} I_m\right)^{-1/2}$ converges to 0 as $n \to \infty$, and for arbitrary n, we have

$$
\begin{aligned}
&\frac{\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)\left(\frac{r}{2} I_d - A_x\right)(\sigma_x\sigma_x^\top)^{1/2}\right\|}{\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|} \\
&= \frac{\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}(\sigma_x\sigma_x^\top)^{-1/2}\left(\frac{r}{2} I_d - A_x\right)(\sigma_x\sigma_x^\top)^{1/2}\right\|}{\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|} \\
&\le \frac{\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|}{\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|} \left\|(\sigma_x\sigma_x^\top)^{-1/2}\left(\frac{r}{2} I_d - A_x\right)(\sigma_x\sigma_x^\top)^{1/2}\right\| \\
&= \left\|(\sigma_x\sigma_x^\top)^{-1/2}\left(\frac{r}{2} I_d - A_x\right)(\sigma_x\sigma_x^\top)^{1/2}\right\| < \infty,
\end{aligned}
$$

and we can infer that

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{\left\|\theta^\top \left(\epsilon_n y(\epsilon_n) y(\epsilon_n)^\top + \frac{r^2}{4} I_m\right)^{-1/2} \Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)\left(\frac{r}{2} I_d - A_x\right)(\sigma_x\sigma_x^\top)^{1/2}\right\|}{\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|} \\
&\le \lim_{n\to\infty} \frac{\left\|\theta^\top \left(\epsilon_n y(\epsilon_n) y(\epsilon_n)^\top + \frac{r^2}{4} I_m\right)^{-1/2}\right\| \left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)\left(\frac{r}{2} I_d - A_x\right)(\sigma_x\sigma_x^\top)^{1/2}\right\|}{\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|} \\
&\le \lim_{n\to\infty} \left\|\theta^\top \left(\epsilon_n y(\epsilon_n) y(\epsilon_n)^\top + \frac{r^2}{4} I_m\right)^{-1/2}\right\| \left\|(\sigma_x\sigma_x^\top)^{-1/2}\left(\frac{r}{2} I_d - A_x\right)(\sigma_x\sigma_x^\top)^{1/2}\right\| = 0 \qquad \text{(A.24)}
\end{aligned}
$$

Finally, we multiply the right-hand side of (A.15) by $(\sigma_x\sigma_x^\top)^{1/2}$ from the right and normalize by $\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|$. Then by (A.24) in the first step, (A.23) in the second and (A.18) in the third, we have

$$
\begin{aligned}
&-\lim_{n\to\infty}\frac{\theta^\top\left(\epsilon_n y(\epsilon_n)y(\epsilon_n)^\top+\frac{r^2}{4}I_m\right)^{-1/2}\left(\gamma y(\epsilon_n)\sigma_x^\top\Pi_{xx}(\epsilon_n)+\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)\left(\frac{r}{2}I_d-A_x\right)\right)(\sigma_x\sigma_x^\top)^{1/2}}{\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|}\\
&=-\lim_{n\to\infty}\frac{\theta^\top\left(\frac{\epsilon_n}{r\gamma}y(\epsilon_n)y(\epsilon_n)^\top+\frac{r^2}{4}I_m\right)^{-1/2}y(\epsilon_n)\sigma_x^\top\Pi_{xx}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}}{\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|}\\
&=-\lim_{n\to\infty}\frac{x^\top(\sigma_x\sigma_x^\top)^{-1/2}\sigma_x\sigma_x^\top\Pi_{xx}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}}{\sqrt{\frac{\epsilon_n}{r\gamma}}\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|}=-x^\top.
\end{aligned}
$$

However, multiplying the left hand side of (A.15) by $(\sigma_x\sigma_x^\top)^{1/2}$ from the right, normalizing by $\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|$ and using that θ is a singular vector for $\widehat{\Pi}_{\theta x}$ (cf. (A.16)), we obtain

$$
\lim_{n\to\infty}\frac{\theta^\top\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}}{\left\|\Lambda^{-1/2}\Pi_{\theta x}(\epsilon_n)(\sigma_x\sigma_x^\top)^{1/2}\right\|}=\theta^\top\widehat{\Pi}_{\theta x}=x^\top.
$$

As the singular vector x has unit norm, this is the desired contradiction and we therefore have $\epsilon_+=\infty$. □

Remark A.2. By replacing the sequence converging to $\epsilon_+\in(0,\infty)$ with a sequence that tends to ∞, the argument from Step 5 of the the proof of Theorem A.1 in fact shows that $\Pi_{\theta x}(\epsilon)$ is not just uniformly bounded on compacts but even uniformly bounded on $[0,\infty)$.

A.5 Proof of Theorem 4.3

Let $(c,\dot\theta)$ be an admissible policy. For any $\tau>0$, Itô's formula yields

$$
\begin{aligned}
&J(\tau,X(\tau),W^{c,\dot\theta}(\tau),\theta^{c,\dot\theta}(\tau))\\
&=J(0,x_0,w_0,\theta_0)+\int_0^\tau\left(\partial_t J+\mathcal{L}^{c,\dot\theta}J\right)\left(t,X(t),W^{c,\dot\theta}(t),\theta^{c,\dot\theta}(t)\right)dt\\
&\quad+\gamma\int_0^\tau J\left(t,X(t),W^{c,\dot\theta}(t),\theta^{c,\dot\theta}(t)\right)\left(\left(\Pi_{\theta x}\sigma_x-\sigma_p\right)^\top\theta(t)-\sigma_x^\top\left(\Pi_{xx}X(t)-\Pi_x\right)\right)^\top dB(t),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{L}^{c,\dot\theta}J&=\theta^\top\sigma_p\sigma_x^\top\frac{\partial^2 J}{\partial w\partial x}+\frac{1}{2}\mathrm{tr}\left(\sigma_x\sigma_x^\top\frac{\partial^2 J}{\partial x^2}\right)+x^\top A_x^\top\frac{\partial J}{\partial x}+\frac{1}{2}\theta^\top\sigma_p\sigma_p^\top\theta\frac{\partial^2 J}{\partial w^2}+\dot\theta^\top\frac{\partial J}{\partial\theta}\\
&\quad+\left(rw+y-c+\theta^\top(\bar\mu+C_x x)-\frac{1}{2}\dot\theta^\top\Lambda\dot\theta\right)\frac{\partial J}{\partial w}.
\end{aligned}
$$

For admissible policies, the stochastic integral is a true martingale and thus has expectation zero. As a consequence,

$$
\begin{aligned}
&\mathbb{E}\left[J(\tau,X(\tau),W^{c,\dot\theta}(\tau),\theta^{c,\dot\theta}(\tau))\right]\\
&\quad=\mathbb{E}\left[\int_0^\tau\left(\partial_t J+\mathcal{L}^{c,\dot\theta}J\right)\left(t,X(t),W^{c,\dot\theta}(t),\theta^{c,\dot\theta}(t)\right)dt\right]+J\left(t,x_0,w_0,\theta_0\right)\\
&\quad\geq\mathbb{E}\left[\int_0^\tau e^{-\delta t-\beta c(t)}dt\right]+J(0,x_0,w_0,\theta_0),
\end{aligned}
$$

where the inequality holds because J solves the HJB equation (3.1). For $\tau \to \infty$, the transversality condition (4.2) and monotone convergence in turn yield

$$J\left(0, x_0, w_0, \theta_0\right) \geq \mathbb{E}\left[\int_0^{\infty} -e^{-\delta t-\beta c(t)} dt\right].$$

This inequality becomes an equality for the pointwise maximizers (3.3),(3.4) of the HJB equation, if we can show that the corresponding feedback control is admissible. This is established in Appendix C.

A.6 Proof of Corollary 4.2

As the solution Π_{xx} of (3.6) is positive semidefinite, $-\left(\frac{r}{2} I_d - A_x + \gamma \sigma_x \sigma_x^{\top} \Pi_{xx}\right)$ is stable. Hence $-\left(r I_d - A_x + \gamma \sigma_x \sigma_x^{\top} \Pi_{xx}\right)$ is stable as well and thus invertible. As a consequence, (3.9) can be solved for Π_x in terms of Π_θ:

$$\Pi_x = -\left(r I_d - A_x^{\top} + \gamma \Pi_{xx}\left(\sigma_x \sigma_x^{\top}\right)\right)^{-1} \Pi_{\theta x}^{\top} \Lambda^{-1} \Pi_\theta,$$

After plugging this result into (3.8) and multiplying by $\Lambda^{-1/2}$, it therefore remains to show that Π_θ is well defined as the solution of the following autonomous linear system:

$$\begin{aligned}
&-\Lambda^{-1/2} \bar{\mu} \\
&= \left(\Lambda^{-1/2}\left(\left(\Pi_{\theta x} \sigma_x - \sigma_p\right) \sigma_x^{\top}\left(r I_d - A_x^{\top} + \gamma \Pi_{xx} \sigma_x \sigma_x^{\top}\right)^{-1} \Pi_{\theta x}^{\top} + \Pi_{\theta\theta}\right) \Lambda^{-1/2} + r I_m\right) \Lambda^{-1/2} \Pi_\theta .
\end{aligned}$$

To this end, we need to show that the matrix multiplying $\Lambda^{-1/2} \Pi_\theta$ is invertible.

First focus on the relationship between $\frac{r}{2} I_d - A_x + \gamma \sigma_x \sigma_x^{\top} \Pi_{xx}$ and $\Pi_{\theta x}$. Using complete-a-square trick, we rewrite (3.6) as

$$\begin{aligned}
&\left(\frac{r}{2} I_d - A_x^{\top} + \gamma \Pi_{xx} \sigma_x \sigma_x^{\top}\right)\left(\gamma \sigma_x \sigma_x^{\top}\right)^{-1}\left(\frac{r}{2} I_d - A_x + \gamma \sigma_x \sigma_x^{\top} \Pi_{xx}\right) \\
&= \Pi_{\theta x}^{\top} \Lambda^{-1} \Pi_{\theta x} + \left(\frac{r}{2} I_d - A_x\right)^{\top}\left(\gamma \sigma_x \sigma_x^{\top}\right)^{-1}\left(\frac{r}{2} I_d - A_x\right),
\end{aligned}$$

hence

$$\begin{aligned}
&\left\|\left(\gamma \sigma_x \sigma_x^{\top}\right)^{1/2}\left(r I_d - A_x + \gamma \Pi_{xx} \sigma_x \sigma_x^{\top}\right)^{-1} \Pi_{\theta x}^{\top} \Lambda^{-1/2}\right\|^2 \\
&= \left\|\left(\gamma \sigma_x \sigma_x^{\top}\right)^{1/2}\left(r I_d - A_x + \gamma \Pi_{xx} \sigma_x \sigma_x^{\top}\right)^{-1} \Pi_{\theta x}^{\top} \Lambda^{-1} \Pi_{\theta x}\left(r I_d - A_x + \gamma \sigma_x \sigma_x^{\top} \Pi_{xx}\right)^{-1}\left(\gamma \sigma_x \sigma_x^{\top}\right)^{1/2}\right\| \\
&\leq \left\|\left(\gamma \sigma_x \sigma_x^{\top}\right)^{1/2}\left(r I_d - A_x + \gamma \Pi_{xx} \sigma_x \sigma_x^{\top}\right)^{-1}\left(\frac{r}{2} I_d - A_x + \gamma \Pi_{xx} \sigma_x \sigma_x^{\top}\right)\left(\gamma \sigma_x \sigma_x^{\top}\right)^{-1/2}\right\|^2 \leq 1.
\end{aligned}$$

We can infer that

$$\begin{aligned}
&\left\|\sigma_x^{\top}\left(r I_d - A_x + \gamma \Pi_{xx} \sigma_x \sigma_x^{\top}\right)^{-1} \Pi_{\theta x}^{\top} \Lambda^{-1/2}\right\| \\
&= \left\|\sigma_x^{\top}\left(\gamma \sigma_x \sigma_x^{\top}\right)^{-1/2}\left(\gamma \sigma_x \sigma_x^{\top}\right)^{1/2}\left(r I_d - A_x + \gamma \Pi_{xx} \sigma_x \sigma_x^{\top}\right)^{-1} \Pi_{\theta x}^{\top} \Lambda^{-1/2}\right\| \\
&\leq \frac{1}{\sqrt{\gamma}}\left\|\left(\sigma_x \sigma_x^{\top}\right)^{1/2}\left(r I_d - A_x + \gamma \Pi_{xx} \sigma_x \sigma_x^{\top}\right)^{-1} \Pi_{\theta x}^{\top} \Lambda^{-1/2}\right\| \leq \frac{1}{\sqrt{\gamma}} .
\end{aligned} \tag{A.25}$$

Similarly, using the complete-a-square trick, we rewrite (3.5) as

$$\gamma\Lambda^{-1/2}\left(\Pi_{\theta x}\sigma_x-\sigma_p\right)\left(\Pi_{\theta x}\sigma_x-\sigma_p\right)^\top\Lambda^{-1/2}+\frac{r^2}{4}I_m=\left(\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}+\frac{r}{2}I_m\right)^2.$$

By singular value decomposition, there exists a $k\times k$ orthonormal matrix U such that

$$\sqrt{\gamma}\Lambda^{-1/2}\left(\Pi_{\theta x}\sigma_x-\sigma_p\right)=\left[\left(\left(\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}+\frac{r}{2}I_m\right)^2-\frac{r^2}{4}I_m\right)^{1/2},0\right]U. \tag{A.26}$$

Hence for arbitrary $x\in\mathbb{R}^d$, using first the triangle inequality, then (A.25) and finally (A.26):

$$\begin{aligned}
&\left\|x^\top\left(\Lambda^{-1/2}\left(\left(\Pi_{\theta x}\sigma_x-\sigma_p\right)\sigma_x^\top\left(rI_d-A_x+\gamma\Pi_{xx}\left(\sigma_x\sigma_x^\top\right)\right)^{-1}\Pi_{\theta x}^\top+\Pi_{\theta\theta}\right)\Lambda^{-1/2}+rI_m\right)\right\|\\
&\geq\left\|x^\top\left(\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}+rI_m\right)\right\|-\left\|x^\top\Lambda^{-1/2}\left(\Pi_{\theta x}\sigma_x-\sigma_p\right)\right\|\\
&=\left\|x^\top\left(\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}+rI_m\right)\right\|-\left\|x^\top\left(\left(\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}+\frac{r}{2}I_m\right)^2-\frac{r^2}{4}I_m\right)^{1/2}\right\|\\
&=\frac{rx^\top\left(\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}+\frac{r}{2}I_m\right)x+\frac{r^2}{2}\|x\|^2}{\left\|x^\top\left(\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}+rI_m\right)\right\|+\left\|x^\top\left(\left(\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}+\frac{r}{2}I_m\right)^2-\frac{r^2}{4}I_m\right)^{1/2}\right\|}\\
&\geq\frac{r^2\|x\|^2}{2\left\|\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}+rI_m\right\|\|x\|}\\
&=\frac{r^2\|x\|}{2r+2\left\|\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}\right\|}.
\end{aligned}$$

Therefore, the smallest singular value $\sigma_{\min}$ of the matrix is strictly positive as

$$\begin{aligned}
&\sigma_{\min}\left(\Lambda^{-1/2}\left(\left(\Pi_{\theta x}\sigma_x-\sigma_p\right)\sigma_x^\top\left(rI_d-A_x+\gamma\Pi_{xx}\left(\sigma_x\sigma_x^\top\right)\right)^{-1}\Pi_{\theta x}^\top+\Pi_{\theta\theta}\right)\Lambda^{-1/2}+rI_m\right)\\
&=\min_{x\neq 0}\left\{\frac{\left\|x^\top\left(\Lambda^{-1/2}\left(\left(\Pi_{\theta x}\sigma_x-\sigma_p\right)\sigma_x^\top\left(rI_d-A_x+\gamma\Pi_{xx}\left(\sigma_x\sigma_x^\top\right)\right)^{-1}\Pi_{\theta x}^\top+\Pi_{\theta\theta}\right)\Lambda^{-1/2}+rI_m\right)\right\|}{\|x\|}\right\}\\
&\geq\frac{r^2}{2r+2\left\|\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}\right\|}>0.
\end{aligned}$$

This matrix therefore is invertible, and a unique solution Π_θ, Π_x of (3.8)–(3.9) therefore indeed exists.

B Properties of the Matrix Function $\mathcal{A}(\Pi;\epsilon)$

The monotonicity of the solution $\epsilon\mapsto\Pi(\epsilon)$ of the matrix ODE (A.10) (cf. Theorem A.1(iii)) is established using the following property of the matrix function $\mathcal{A}(\Pi;\epsilon)$ from (A.11):

Lemma B.1. *For given $\epsilon>0$ and $\Pi_{\theta x}\in\mathbb{R}^{m\times d}$, let $\Pi_{\theta\theta}$ and Π_{xx} be the unique stabilizing solutions of the Riccati equations* (A.3) *and* (A.4)*. Then the corresponding matrix $-\mathcal{A}\left(\begin{bmatrix}\Pi_{xx} & -\Pi_{\theta x}^\top\\ -\Pi_{\theta x} & -\Pi_{\theta\theta}\end{bmatrix};\epsilon\right)$ is stable.*

Proof. Fix $\epsilon > 0$ and $\Pi_{\theta x} \in \mathbb{R}^{m\times d}$. By [12, Theorem 13.7], there exist unique $\Pi_{\theta\theta}$ and Π_{xx} that are positive semidefinite solutions of (A.3) and (A.4). Set

$$y = \Lambda^{-1/2}\Pi_{\theta x}(\gamma\sigma_x\sigma_x^\top)^{1/2}, \quad y_1 = (\gamma\sigma_x\sigma_x^\top)^{1/2}\Pi_{xx}(\gamma\sigma_x\sigma_x^\top)^{1/2}, \quad y_3 = \Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}.$$

Then by [12, Theorem 13.7], $y_1, y_2 \in R^{d\times d}$ and $y_3 \in \mathbb{R}^{m\times m}$ are the unique positive semidefinite solutions of the following system:

$$y_1^2 + \left(\frac{r}{2}I_d - (\sigma_x\sigma_x^\top)^{-1/2}A_x(\sigma_x\sigma_x^\top)^{1/2}\right)^\top y_1 + y_1\left(\frac{r}{2}I_d - (\sigma_x\sigma_x^\top)^{-1/2}A_x(\sigma_x\sigma_x^\top)^{1/2}\right) = \epsilon y^\top y, \quad \text{(B.1)}$$

$$y_2^2 - \left(\frac{r}{2}I_d - (\sigma_x\sigma_x^\top)^{-1/2}A_x(\sigma_x\sigma_x^\top)^{1/2}\right)^\top y_2 - y_2\left(\frac{r}{2}I_d - (\sigma_x\sigma_x^\top)^{-1/2}A_x(\sigma_x\sigma_x^\top)^{1/2}\right) = 0, \quad \text{(B.2)}$$

$$\epsilon y_3^2 + ry_3 = \gamma\left(y(\gamma\sigma_x\sigma_x^\top)^{-1/2}\sigma_x - \Lambda^{-1/2}\sigma_p\right)\left(y(\gamma\sigma_x\sigma_x^\top)^{-1/2}\sigma_x - \Lambda^{-1/2}\sigma_p\right)^\top. \quad \text{(B.3)}$$

Since $A_x - \frac{r}{2}I_d$ is stable, $(\sigma_x\sigma_x^\top)^{-1/2}A_x(\sigma_x\sigma_x^\top)^{1/2} - \frac{r}{2}I_d$ is stable as well. Thus, by [12, Theorem 13.7], it follows that y_2 is positive definite. As a consequence, $y_1 + y_2$ and $\epsilon y_3 + \frac{r}{2}I_m$ are positive definite as well as sums of positive definite matrices and positive semidefinite matrices.

Since similar matrices have the same eigenvalues, the stability of $\mathcal{A}\left(\begin{bmatrix}\Pi_{xx} & -\Pi_{\theta x}^\top \\ -\Pi_{\theta x} & -\Pi_{\theta\theta}\end{bmatrix}, \epsilon\right)$ is equivalent to the stability of

$$\begin{aligned}\widetilde{\mathcal{A}} &= \begin{bmatrix}(\gamma\sigma_x\sigma_x^\top)^{-1/2} & 0 \\ 0 & \Lambda^{1/2}/\sqrt{\epsilon}\end{bmatrix}\mathcal{A}\left(\begin{bmatrix}\Pi_{xx} & -\Pi_{\theta x}^\top \\ -\Pi_{\theta x} & -\Pi_{\theta\theta}\end{bmatrix}, \epsilon\right)\begin{bmatrix}(\gamma\sigma_x\sigma_x^\top)^{1/2} & 0 \\ 0 & \sqrt{\epsilon}\Lambda^{-1/2}\end{bmatrix} \\ &= \begin{bmatrix} y_1 + \frac{r}{2}I_d - (\sigma_x\sigma_x^\top)^{-1/2}A_x(\sigma_x\sigma_x^\top)^{1/2} & \sqrt{\epsilon}\left(\sqrt{\gamma}\Lambda^{-1/2}\sigma_p\sigma_x^\top(\sigma_x\sigma_x^\top)^{-1/2} - y\right)^\top \\ \sqrt{\epsilon}y & \epsilon y_3 + \frac{r}{2}I_m\end{bmatrix}.\end{aligned}$$

In view of the Lyapunov Lemma as in [12, Lemma 3.18], it therefore suffices to show the positive definiteness of

$$\begin{bmatrix}y_1 + y_2 & 0 \\ 0 & \epsilon y_3 + \frac{r}{2}I_m\end{bmatrix}\widetilde{\mathcal{A}} + \widetilde{\mathcal{A}}^\top\begin{bmatrix}y_1 + y_2 & 0 \\ 0 & \epsilon y_3 + \frac{r}{2}I_m\end{bmatrix},$$

where

$$\begin{aligned}&\begin{bmatrix}y_1 + y_2 & 0 \\ 0 & \epsilon y_3 + \frac{r}{2}I_m\end{bmatrix}\widetilde{\mathcal{A}} \\ &= \begin{bmatrix}(y_1 + y_2)\left(y_1 + \frac{r}{2}I_d - (\sigma_x\sigma_x^\top)^{-1/2}A_x(\sigma_x\sigma_x^\top)^{1/2}\right) & \sqrt{\epsilon}(y_1 + y_2)\left(\sqrt{\gamma}\Lambda^{-1/2}\sigma_p\sigma_x^\top(\sigma_x\sigma_x^\top)^{-1/2} - y\right)^\top \\ \sqrt{\epsilon}\left(\epsilon y_3 + \frac{r}{2}I_m\right)y & \left(\epsilon y_3 + \frac{r}{2}I_m\right)^2\end{bmatrix}.\end{aligned}$$

To this end, first notice that, by definition of y_3 and y,

$$\begin{aligned}&\left(\epsilon y_3 + \frac{r}{2}I_m\right)^2 \\ &= \frac{r^2}{4}I_m + \epsilon\gamma\left(y(\gamma\sigma_x\sigma_x^\top)^{-1/2}\sigma_x - \Lambda^{-1/2}\sigma_p\right)\left(y(\gamma\sigma_x\sigma_x^\top)^{-1/2}\sigma_x - \Lambda^{-1/2}\sigma_p\right)^\top \\ &= \frac{r^2}{4}I_m + \epsilon\left(yy^\top - \sqrt{\gamma}\Lambda^{-1/2}\sigma_p\sigma_x^\top(\sigma_x\sigma_x^\top)^{-1/2}y^\top - \sqrt{\gamma}y(\sigma_x\sigma_x^\top)^{-1/2}\sigma_x\sigma_p^\top\Lambda^{-1/2}\right) \\ &\quad + \epsilon\gamma\Lambda^{-1/2}\sigma_p\sigma_x^\top(\sigma_x\sigma_x^\top)^{-1}\sigma_x\sigma_p^\top\Lambda^{-1/2} + \epsilon\gamma\Lambda^{-1/2}\sigma_p\left(I_k - \sigma_x^\top(\sigma_x\sigma_x^\top)^{-1}\sigma_x\right)\sigma_p^\top\Lambda^{-1/2} \\ &= \frac{r^2}{4}I_m + \epsilon\left(y - \sqrt{\gamma}\Lambda^{-1/2}\sigma_p\sigma_x^\top(\sigma_x\sigma_x^\top)^{-1/2}\right)\left(y - \sqrt{\gamma}\Lambda^{-1/2}\sigma_p\sigma_x^\top(\sigma_x\sigma_x^\top)^{-1/2}\right)^\top \\ &\quad + \epsilon\gamma\Lambda^{-1/2}\sigma_p\left(I_k - \sigma_x^\top(\sigma_x\sigma_x^\top)^{-1}\sigma_x\right)\sigma_p^\top\Lambda^{-1/2} \\ &> \epsilon\left(y - \sqrt{\gamma}\Lambda^{-1/2}\sigma_p\sigma_x^\top(\sigma_x\sigma_x^\top)^{-1/2}\right)\left(y - \sqrt{\gamma}\Lambda^{-1/2}\sigma_p\sigma_x^\top(\sigma_x\sigma_x^\top)^{-1/2}\right)^\top. \quad \text{(B.4)}\end{aligned}$$

Next, the matrix equations (B.1) and (B.2) satisfied by y_1 and y_2 imply that

$$
\begin{aligned}
&(y_1+y_2)\left(y_1+\frac{r}{2}I_d-(\sigma_x\sigma_x^\top)^{-1/2}A_x(\sigma_x\sigma_x^\top)^{1/2}\right)+\left(y_1+\frac{r}{2}I_d-(\sigma_x\sigma_x^\top)^{-1/2}A_x(\sigma_x\sigma_x^\top)^{1/2}\right)^\top(y_1+y_2)\\
&=y_1^2+y_2y_1+y_1y_2+y_2\left(\frac{r}{2}I_d-(\sigma_x\sigma_x^\top)^{-1/2}A_x(\sigma_x\sigma_x^\top)^{1/2}\right)+\left(\frac{r}{2}I_d-(\sigma_x\sigma_x^\top)^{-1/2}A_x(\sigma_x\sigma_x^\top)^{1/2}\right)^\top y_2\\
&\quad+y_1^2+y_1\left(\frac{r}{2}I_d-(\sigma_x\sigma_x^\top)^{-1/2}A_x(\sigma_x\sigma_x^\top)^{1/2}\right)+\left(\frac{r}{2}I_d-(\sigma_x\sigma_x^\top)^{-1/2}A_x(\sigma_x\sigma_x^\top)^{1/2}\right)^\top y_1\\
&=y_1^2+y_2y_1+y_1y_2+y_2^2+\epsilon y^\top y\\
&=(y_1+y_2)^2+\epsilon y^\top y>0.
\end{aligned}
\tag{B.5}
$$

This allows us to finally show the positive definiteness of

$$
\begin{bmatrix} y_1+y_2 & 0\\ 0 & \epsilon y_3+\frac{r}{2}I_m\end{bmatrix}\widetilde{\mathcal{A}}+\widetilde{\mathcal{A}}^\top\begin{bmatrix} y_1+y_2 & 0\\ 0 & \epsilon y_3+\frac{r}{2}I_m\end{bmatrix}.
$$

To this end, for every $x\in\mathbb{R}^d$ and $\theta\in\mathbb{R}^m$, first notice that

$$
\begin{aligned}
&\begin{bmatrix}x\\ \theta\end{bmatrix}^\top\left(\begin{bmatrix} y_1+y_2 & 0\\ 0 & \epsilon y_3+\frac{r}{2}I_m\end{bmatrix}\widetilde{\mathcal{A}}+\widetilde{\mathcal{A}}^\top\begin{bmatrix} y_1+y_2 & 0\\ 0 & \epsilon y_3+\frac{r}{2}I_m\end{bmatrix}\right)\begin{bmatrix}x\\ \theta\end{bmatrix}\\
&=2\sqrt{\epsilon}\theta^\top\left(\left(\epsilon y_3+\frac{r}{2}I_m\right)y+\left(\sqrt{\gamma}\Lambda^{-1/2}\sigma_p\sigma_x^\top(\sigma_x\sigma_x^\top)^{-1/2}-y\right)(y_1+y_2)\right)x\\
&\quad+x^\top\left((y_1+y_2)^2+\epsilon y^\top y\right)x+2\theta^\top\left(\epsilon y_3+\frac{r}{2}I_m\right)^2\theta\\
&\geq x^\top\left((y_1+y_2)^2+\epsilon y^\top y\right)x+2\theta^\top\left(\epsilon y_3+\frac{r}{2}I_m\right)^2\theta-2\left\|\left(\epsilon y_3+\frac{r}{2}I_m\right)\theta\right\|\left\|\sqrt{\epsilon}yx\right\|\\
&\quad-2\left\|\sqrt{\epsilon}\left(\sqrt{\gamma}\Lambda^{-1/2}\sigma_p\sigma_x^\top(\sigma_x\sigma_x^\top)^{-1/2}-y\right)^\top\theta\right\|\left\|(y_1+y_2)x\right\|.
\end{aligned}
$$

Then second by (B.4), we have that for every $\theta\in\mathbb{R}^m$

$$
\left\|\sqrt{\epsilon}\left(\sqrt{\gamma}\Lambda^{-1/2}\sigma_p\sigma_x^\top(\sigma_x\sigma_x^\top)^{-1/2}-y\right)^\top\theta\right\|\leq\left\|\left(\epsilon y_3+\frac{r}{2}I_m\right)\theta\right\|,
$$

and the equality is achieved if and only if $\theta=0$. Hence for $\theta\neq 0$, we can further estimate that

$$
\begin{aligned}
&\begin{bmatrix}x\\ \theta\end{bmatrix}^\top\left(\begin{bmatrix} y_1+y_2 & 0\\ 0 & \epsilon y_3+\frac{r}{2}I_m\end{bmatrix}\widetilde{\mathcal{A}}+\widetilde{\mathcal{A}}^\top\begin{bmatrix} y_1+y_2 & 0\\ 0 & \epsilon y_3+\frac{r}{2}I_m\end{bmatrix}\right)\begin{bmatrix}x\\ \theta\end{bmatrix}\\
&>x^\top\left((y_1+y_2)^2+\epsilon y^\top y\right)x+2\theta^\top\left(\epsilon y_3+\frac{r}{2}I_m\right)^2\theta-2\left\|\left(\epsilon y_3+\frac{r}{2}I_m\right)\theta\right\|\left(\left\|\sqrt{\epsilon}yx\right\|+\left\|(y_1+y_2)x\right\|\right)\\
&=\left(\left\|\left(\epsilon y_3+\frac{r}{2}I_m\right)\theta\right\|-\left\|\sqrt{\epsilon}yx\right\|\right)^2+\left(\left\|\left(\epsilon y_3+\frac{r}{2}I_m\right)\theta\right\|-\left\|(y_1+y_2)x\right\|\right)^2\geq 0.
\end{aligned}
$$

For $\theta=0$, by (B.5),

$$
\begin{aligned}
&\begin{bmatrix}x\\ 0\end{bmatrix}^\top\left(\begin{bmatrix} y_1+y_2 & 0\\ 0 & \epsilon y_3+\frac{r}{2}I_m\end{bmatrix}\widetilde{\mathcal{A}}+\widetilde{\mathcal{A}}^\top\begin{bmatrix} y_1+y_2 & 0\\ 0 & \epsilon y_3+\frac{r}{2}I_m\end{bmatrix}\right)\begin{bmatrix}x\\ 0\end{bmatrix}\\
&=x^\top\left((y_1+y_2)^2+\epsilon y^\top y\right)x\geq 0.
\end{aligned}
$$

In summary, together with the positive semidefiniteness of $(y_1+y_2)^2+\epsilon y^\top y$

$$\begin{bmatrix} x \\ \theta \end{bmatrix}^\top \left(\begin{bmatrix} y_1+y_2 & 0 \\ 0 & \epsilon y_3 + \frac{r}{2} I_m \end{bmatrix} \widetilde{\mathcal{A}} + \widetilde{\mathcal{A}}^\top \begin{bmatrix} y_1+y_2 & 0 \\ 0 & \epsilon y_3 + \frac{r}{2} I_m \end{bmatrix} \right) \begin{bmatrix} x \\ \theta \end{bmatrix} = 0$$

if and only if $\theta = 0$ and $x = 0$. Therefore, Lyapunov's Lemma as in [12, Lemma 3.19] yields the asserted stability of $-\mathcal{A}\left(\left[\begin{smallmatrix} \Pi_{xx} & -\Pi_{\theta x}^\top \\ -\Pi_{\theta x} & -\Pi_{\theta\theta} \end{smallmatrix} \right]; \epsilon \right)$. □

C Admissibility of the Optimal Control in Theorem 4.3

In this appendix, we establish the most difficult element for the proof of Theorem 4.3: the admissibility of the feedback controls (3.3), (3.4). The first step is to use the Riccati equations for the coefficients of the candidate value function (3.2), the definitions of the controls (3.3), (3.4), and tedious but elementary algebraic manipulations to obtain the following compact representation for the value function evaluated along the candidate controls. (Since only these controls appear in the present section, we drop the corresponding indices to ease notation.)

Proposition C.1. *For any $t > 0$, we have*

$$J(t, X(t), W(t), \theta(t)) = J(0, x_0, w_0, \theta_0) \exp\left(-rt + \gamma M_t - \frac{\gamma^2}{2} \langle M \rangle_t \right), \tag{C.1}$$

for the local martingale

$$dM_t = \left((\Pi_{\theta x}\sigma_x - \sigma_p)^\top \theta(t) - \sigma_x^\top (\Pi_{xx} X(t) - \Pi_x) \right)^\top dB(t). \tag{C.2}$$

Proof. After inserting the controls (3.3), (3.4), the corresponding wealth dynamics can be rewritten as

$$\begin{aligned}
&dW(t) \\
&= \left(\frac{r}{\gamma} \ln r J_0 + r\left(-\frac{1}{2} X(t)^\top \Pi_{xx} X(t) + \Pi_x^\top X(t) + \frac{1}{2}\theta(t)^\top \Pi_{\theta\theta}\theta(t) + \Pi_\theta^\top \theta(t) + \theta(t)^\top \Pi_{\theta x} X(t) \right) \right) dt \\
&\quad + \left(\theta(t)^\top (\bar{\mu} + C_x X(t)) - \frac{1}{2} (\Pi_{\theta\theta}\theta(t) + \Pi_{\theta x} X(t) + \Pi_\theta)^\top \Lambda^{-1} (\Pi_{\theta\theta}\theta(t) + \Pi_{\theta x} X(t) + \Pi_\theta) \right) dt \\
&\quad + d\int_0^t e^{r(t-u)} y(u) du + \theta(t)^\top \sigma_p dB(t) \\
&= \left(\theta(t)^\top (\bar{\mu} + C_x X(t)) - \frac{1}{2} (\Pi_{\theta\theta}\theta(t) + \Pi_{\theta x} X(t) + \Pi_\theta)^\top \Lambda^{-1} (\Pi_{\theta\theta}\theta(t) + \Pi_{\theta x} X(t) + \Pi_\theta) \right) dt \\
&\quad + r\left(-\frac{1}{2} X(t)^\top \Pi_{xx} X(t) + \Pi_x^\top X(t) + \frac{1}{2}\theta(t)^\top \Pi_{\theta\theta}\theta(t) + \Pi_\theta^\top \theta(t) + \theta(t)^\top \Pi_{\theta x} X(t) \right) dt \\
&\quad + \frac{1}{\gamma}\left(r - \delta - \frac{\gamma}{2} \Pi_\theta^\top \Lambda^{-1} \Pi_\theta + \frac{\gamma^2}{2} \Pi_x^\top \sigma_x \sigma_x^\top \Pi_x - \frac{\gamma}{2} \mathrm{tr}\left(\sigma_x^\top \Pi_{xx} \sigma_x \right) \right) dt \\
&\quad + d\int_0^t e^{r(t-u)} y(u) du + \theta(t)^\top \sigma_p dB(t).
\end{aligned} \tag{C.3}$$

Next, we calculate the dynamics for the other terms appearing in the exponential of J, cf. (3.2). By definition of the trading rate (3.4), we have

$$d\theta(t)^\top \Pi_{\theta\theta}\theta(t) = -2\theta(t)^\top \Pi_{\theta\theta}\Lambda^{-1} (\Pi_{\theta\theta}\theta(t) + \Pi_{\theta x} X(t) + \Pi_\theta)\, dt. \tag{C.4}$$

Likewise, the dynamics of the state process yield

$$dX(t)^\top \Pi_{xx} X(t) = 2X(t)^\top \Pi_{xx} A_x X(t)dt + 2X(t)^\top \Pi_{xx}\sigma_x dB(t) + \operatorname{tr}\left(\sigma_x^\top \Pi_{xx}\sigma_x\right)dt, \tag{C.5}$$

as well as

$$\begin{aligned} d\theta(t)^\top \Pi_{\theta x} X(t) = &\left(\theta(t)^\top \Pi_{\theta x} A_x X(t) - \left(\Pi_{\theta\theta}\theta(t) + \Pi_{\theta x}X(t) + \Pi_\theta\right)^\top \Lambda^{-1}\Pi_{\theta x}X(t)\right)dt \\ &+ \theta(t)^\top \Pi_{\theta x}\sigma_x dB(t). \end{aligned} \tag{C.6}$$

Finally,

$$d\Pi_\theta^\top \theta(t) = -\Pi_\theta^\top \Lambda^{-1}(\Pi_{\theta\theta}\theta(t) + \Pi_{\theta x}X(t) + \Pi_\theta)dt, \tag{C.7}$$

as well as

$$d\Pi_x^\top X(t) = \Pi_x^\top A_x X(t)dt + \Pi_x\sigma_x dB(t). \tag{C.8}$$

Putting (C.3) - (C.8) all together, we obtain

$$\begin{aligned} &d\left(\int_0^t e^{r(t-u)}y(u)du - W(t) - \frac{1}{2}X(t)^\top \Pi_{xx}X(t) + \Pi_x^\top X(t) + \frac{1}{2}\theta(t)^\top \Pi_{\theta\theta}\theta(t) + \Pi_\theta^\top\theta(t) + \theta(t)^\top\Pi_{\theta x}X(t)\right) \\ =& \left(\left(\Pi_{\theta x}^\top\theta(t) + \Pi_x - \Pi_{xx}X(t)\right)^\top A_x X(t) - \frac{1}{2}\left(\Pi_{\theta\theta}\theta(t) + \Pi_{\theta x}X(t) + \Pi_\theta\right)^\top \Lambda^{-1}\left(\Pi_{\theta\theta}\theta(t) + \Pi_{\theta x}X(t) + \Pi_\theta\right)\right)dt \\ &- r\left(\theta(t)^\top\left(\bar{\mu} + C_x X(t)\right) - \frac{1}{2}X(t)^\top\Pi_{xx}X(t) + \Pi_x^\top X(t) + \frac{1}{2}\theta(t)^\top\Pi_{\theta\theta}\theta(t) + \Pi_\theta^\top\theta(t) + \theta(t)^\top\Pi_{\theta x}X(t)\right)dt \\ &+ \left(-\frac{r-\delta}{\gamma} + \frac{1}{2}\Pi_\theta^\top\Lambda^{-1}\Pi_\theta - \frac{1}{2}\Pi_x^\top\sigma_x\sigma_x^\top\Pi_x\right)dt + \left(\sigma_x^\top\left(\Pi_x + \Pi_{\theta x}^\top\theta(t) - \Pi_{xx}X(t)\right) - \sigma_p^\top\theta(t)\right)^\top dB(t) \\ =& -\left(\frac{1}{2}\theta(t)^\top\left(\Pi_{\theta\theta}\Lambda^{-1}\Pi_{\theta\theta} + r\Pi_{\theta\theta}\right)\theta(t) + \frac{1}{2}X(t)^\top\left(\Pi_{\theta x}^\top\Lambda^{-1}\Pi_{\theta x} + \Pi_{xx}A_x + A_x^\top\Pi_{xx} - r\Pi_{xx}\right)X(t)\right)dt \\ &- \left(\theta(t)^\top\left(\Pi_{\theta\theta}\Lambda^{-1}\Pi_{\theta x} + C_x - \Pi_{\theta x}A_x\right)X(t) + \frac{r-\delta}{\gamma} + \frac{1}{2}\Pi_x^\top\sigma_x\sigma_x^\top\Pi_x\right)dt \\ &- \left(\left((rI_d - A_x)^\top\Pi_x + \Pi_{\theta x}^\top\Lambda^{-1}\Pi_\theta\right)^\top X(t) + \left(\left(\Lambda^{-1}\Pi_{\theta\theta} + rI_m\right)^\top\Pi_\theta + \bar{\mu}\right)^\top\theta(t)\right)dt \\ &+ \left(\left(\Pi_{\theta x}\sigma_x - \sigma_p\right)^\top\theta(t) - \sigma_x^\top\left(\Pi_{xx}X(t) - \Pi_x\right)\right)^\top dB(t). \end{aligned} \tag{C.9}$$

Using the Riccati equation (3.5) for $\Pi_{\theta\theta}$, the term quadratic term in θ can be rewritten as

$$\frac{1}{2}\theta(t)^\top\left(\Pi_{\theta\theta}\Lambda^{-1}\Pi_{\theta\theta} + r\Pi_{\theta\theta}\right)\theta(t) = \frac{\gamma}{2}\theta(t)^\top\left(\Pi_{\theta x}\sigma_x - \sigma_p\right)\left(\Pi_{\theta x}\sigma_x - \sigma_p\right)^\top\theta(t).$$

Likewise, in view of the Riccati equation (3.6) for Π_{xx}, the term quadratic in X becomes:

$$\frac{1}{2}X(t)^\top\left(\Pi_{\theta x}^\top\Lambda^{-1}\Pi_{\theta x} + \Pi_{xx}A_x + A_x^\top\Pi_{xx} - r\Pi_{xx}\right)X(t) = \frac{\gamma}{2}X(t)^\top\Pi_{xx}\sigma_x\sigma_x^\top\Pi_{xx}X(t),$$

With the equation (3.7) for $\Pi_{\theta x}$, the cross term for θ and X can be rewritten as

$$\theta(t)^\top\left(\Pi_{\theta\theta}\Lambda^{-1}\Pi_{\theta x} + C_x + r\Pi_{\theta x} - \Pi_{\theta x}A_x\right)X(t) = -\gamma\theta(t)^\top\left(\Pi_{\theta x}\sigma_x - \sigma_p\right)\sigma_x^\top\Pi_{xx}X(t).$$

Finally, in view of the equations (3.8) and (3.9) for Π_θ and Π_x the terms linear in θ and X, respectively, become

$$\Big((rI_d - A_x)^\top \Pi_x + \Pi_{\theta x}^\top \Lambda^{-1}\Pi_\theta\Big)^\top X(t) = \gamma \Pi_x^\top \sigma_x \sigma_x^\top \Pi_{xx} X(t),$$
$$\Big(\big(\Lambda^{-1}\Pi_{\theta\theta} + rI_m\big)^\top \Pi_\theta + \bar{\mu}\Big)^\top \theta(t) = \gamma \Pi_x^\top \sigma_x \left(\Pi_{\theta x}\sigma_x - \sigma_p\right)^\top \theta(t).$$

Altogether, we obtain

$$\begin{aligned} d\gamma\Big(& - W(t) + \int_0^t e^{r(t-u)} y(u)du - \frac{1}{2}X(t)^\top \Pi_{xx} X(t) + \Pi_x^\top X(t) \\ &+ \frac{1}{2}\theta(t)^\top \Pi_{\theta\theta}\theta(t) + \Pi_\theta^\top \theta(t) + \theta(t)^\top \Pi_{\theta x} X(t)\Big) = \gamma dM_t - \frac{\gamma^2}{2} d\langle M\rangle_t + (\delta - r)dt. \end{aligned}$$

For any $\tau > 0$, we therefore indeed have (C.1) as asserted. □

The crucial step lefty to establish the admissibility of (3.3), (3.4) now is to show that the exponential local martingale $\exp\left(M_t - \frac{1}{2}\langle M\rangle_t\right)$ is a true martingale. To wit, suppose this holds, so that this process has expectation 1. The representation (C.1) then immediately yields the transversality condition (4.2):

$$\mathbb{E}\left[J(\tau, X(\tau), W(\tau), \theta(\tau))\right] = J(0, x_0, w_0, \theta_0)e^{-r\tau} \to 0 \quad \text{as} \quad \tau \to \infty.$$

To establish admissibility, it then remains to verify the martingale property of (4.1). In view of (C.1) and Itô's formula,

$$\begin{aligned} &\int_0^\tau J(t, X(t), W(t), \theta(t))dM_t \\ &\quad = J(\tau, X(\tau), W(\tau), \theta(\tau)) - J(0, x_0, w_0, \theta_0) + r\int_0^\tau J(t, X(t), W(t), \theta(t))dt. \end{aligned}$$

If the process $\exp\left(M_t - \frac{1}{2}\langle M\rangle_t\right)$ is a martingale, it follows from (C.1) that this process is integrable and, moreover, satisfies

$$\begin{aligned} \mathbb{E}_t\left[\int_t^\tau J(s, X(s), W(s), \theta(s))dM_s\right] &= \mathbb{E}_t\left[J(\tau, X(\tau), W(\tau), \theta(\tau))\right] - J(t, X(t), W(t), \theta(t)) \\ &\quad + r\mathbb{E}_t\left[\int_t^\tau J(s, X(s), W(s), \theta(s))ds\right] \\ &= J(t, X(t), W(t), \theta(t))\left(e^{-r(\tau-t)} - 1 + r\int_t^\tau e^{-rs}ds\right) \\ &= 0. \end{aligned}$$

(Here, we have again used (C.1) and the martingale property of $\exp\left(M_t - \frac{1}{2}\langle M\rangle_t\right)$to compute the conditional expectations in the next-to-last step.) Whence, the local martingale (4.1) is indeed a true martingale as required for the admissibility of the policy (3.3), (3.4).

In summary, it therefore remains to show that the exponential local martingale $\exp\left(M_t - \frac{1}{2}\langle M\rangle_t\right)$ is indeed a true martingale. The first ingredient for establishing this is the following multivatiate extension of [9, Lemma 4.12].

Lemma C.2. *Let $\{X(t)\}_{t\geq 0}$ be an $\mathbb{R}^d$-valued Gaussian process with $X(t) \sim N(0, \Sigma_t)$ and suppose the covariance matrices $(\Sigma_t)_{t\geq 0}$ are uniformly bounded ($\sup_{t\geq 0} \Sigma_t \leq \Sigma$ for a positive definite matrix Σ). Then:*

1. *for every $\alpha \in \mathbb{R}^{k\times d}$ and every $t \geq 0$, there exists $h_0 > 0$ such that, for every $0 < h < h_0$:*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_t^{t+h} \left\|\alpha^\top X(u)\right\|^2 du\right)\right] < \infty; \tag{C.10}$$

2. *for every $\alpha \in \mathbb{R}^{k\times d}$, there exists $h_1 > 0$ such that, for every $0 < h < h_1$:*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_{t_{n-1}}^{t_n}\int_0^v \left\|\alpha^\top X(u)\right\|^2 dudv\right)\right] < \infty, \tag{C.11}$$

   where $t_0 = 0$ and $t_n := \sum_{m=1}^n \frac{h}{m}$.

Proof. First, we show (C.10) holds. By the submultiplicativity of the matrix operator norm and the convexity of the exponential function, we have

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\frac{1}{2}\int_t^{t+h} \left\|\alpha^\top X(u)\right\|^2 du\right)\right] &\leq \mathbb{E}\left[\exp\left(\frac{\|\alpha\|^2}{2}\int_t^{t+h} \|X(u)\|^2 du\right)\right] \\
&= \mathbb{E}\left[\exp\left(\frac{h\|\alpha\|^2}{2}\frac{1}{h}\int_t^{t+h} \|X(u)\|^2 du\right)\right] \\
&\leq \frac{1}{h}\int_t^{t+h} \mathbb{E}\left[\exp\left(\frac{h\|\alpha\|^2}{2}\|X(u)\|^2\right)\right] du \\
&\leq \sup_{t\leq u\leq t+h} \mathbb{E}\left[\exp\left(\frac{h\|\alpha\|^2}{2}\|X(u)\|^2\right)\right].
\end{aligned}$$

As $X(u) \sim N(0, \Sigma_u)$, it follows that

$$\mathbb{E}[\|X(u)\|^2] = \mathrm{tr}(\Sigma_u) \leq \mathrm{tr}(\Sigma),$$

A "complete-a-square argument" in turn shows that, for all $0 < h < 1/\|\alpha\|^2\mathrm{tr}(\Sigma)$:

$$\mathbb{E}[\exp\left(\frac{h\|\alpha\|^2}{2}\|X(u)\|^2\right)] = \frac{1}{1 - h\|\alpha\|^2\mathrm{tr}(\Sigma_u)} \leq \frac{1}{1 - h\|\alpha\|^2\mathrm{tr}(\Sigma)} < \infty.$$

Therefore choosing $h_0 = 1/\|\alpha\|^2\mathrm{tr}(\Sigma)$, (C.10) follows from

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_t^{t+h} \left\|\alpha^\top X(u)\right\|^2 du\right)\right] \leq \sup_{t\leq u\leq t+h} \mathbb{E}\left[\exp\left(\frac{h\|\alpha\|^2}{2}\|X(u)\|^2\right)\right] \leq \frac{1}{1 - h\|\alpha\|^2\mathrm{tr}(\Sigma)} < \infty.$$

Now we turn to the proof of (C.11). To this end, again using the submultiplicativity of the matrix operator norm and the convexity of the exponential function twice, we have

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\frac{1}{2}\int_{t_{n-1}}^{t_n}\int_0^v \left\|\alpha^\top X(u)\right\|^2 dudv\right)\right] &\le \mathbb{E}\left[\exp\left(\frac{\|\alpha\|^2 h}{2n}\frac{1}{t_n-t_{n-1}}\int_{t_{n-1}}^{t_n}\int_0^v \|X(u)\|^2 dudv\right)\right]\\
&\le \frac{1}{t_n-t_{n-1}}\int_{t_{n-1}}^{t_n}\mathbb{E}\left[\exp\left(\frac{\|\alpha\|^2 h}{2n}\int_0^v \|X(u)\|^2 du\right)\right]dv\\
&\le \mathbb{E}\left[\exp\left(\frac{\|\alpha\|^2 h t_n}{2n}\frac{1}{t_n}\int_0^{t_n}\|X(u)\|^2 du\right)\right]\\
&\le \frac{1}{t_n}\int_0^{t_n}\mathbb{E}\left[\exp\left(\frac{\|\alpha\|^2 h t_n}{2n}\|X(u)\|^2\right)\right]du\\
&\le \sup_{0\le u\le t_n}\mathbb{E}\left[\exp\left(\frac{\|\alpha\|^2 h t_n}{2n}\|X(u)\|^2\right)\right].
\end{aligned}$$

Next, notice that

$$t_n=\sum_{m=1}^{n}\frac{h}{m}<nh.$$

Now, choose $h_1=1/\|\alpha\|\sqrt{\mathrm{tr}(\Sigma)}$, then for $0<h<h_1$,

$$\mathbb{E}\left[\exp\left(\frac{\|\alpha\|^2 h t_n}{2n}\|X(u)\|^2\right)\right]\le\mathbb{E}\left[\exp\left(\frac{\|\alpha\|^2 h^2 n}{2n}\|X(u)\|^2\right)\right]=\frac{1}{1-h^2\|\alpha\|^2\mathrm{tr}(\Sigma_u)}.$$

For $0<h<h_1$, (C.11) then follows from

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\frac{1}{2}\int_{t_{n-1}}^{t_n}\int_0^v \left\|\alpha^\top X(u)\right\|^2 dudv\right)\right] &\le \sup_{0\le u\le t_n}\mathbb{E}\left[\exp\left(\frac{\|\alpha\|^2 h t_n}{2n}\|X(u)\|^2\right)\right]\\
&\le \sup_{0\le u\le t_n}\frac{1}{1-h^2\|\alpha\|^2\mathrm{tr}(\Sigma_u)}\\
&\le \frac{1}{1-h^2\|\alpha\|^2\mathrm{tr}(\Sigma)}<\infty.
\end{aligned}$$

□

The next ingredient we need is an estimate for matrix exponentials:

Lemma C.3. *Consider a symmetric positive semidefinite matrix $A\in\mathbb{R}^{d\times d}$ and let $a>0$ be its smallest non-zero singular value if $\|A\|>0$ and $a=1$ when $\|A\|=0$. Then, for all $t\ge 0$:*

$$\left\|Ae^{-A^2 t}\right\|\le\|A\|e^{-a^2 t}. \tag{C.12}$$

Proof. When $\|A\|=0$, the claim becomes trivial.

So suppose $\|A\|>0$. Since A is symmetric positive semidefinite, there exists an orthonormal matrix O such that

$$A=O\ \mathrm{diag}\left(a_{(d_0)},\cdots,a_{(1)},0,\cdots,0\right)O^\top, \tag{C.13}$$

where $d_0=\mathrm{rank}(A)\le d$, and

$$0<a=a_{(1)}\le a_{(2)}\le\cdots\le a_{(d_0)}=\|A\|.$$

By definition of the matrix exponential, we have

$$e^{-A^2t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} A^{2n};$$

in particular, A and e^{-A^2t} commute. Further, notice that

$$\begin{aligned}
O\ A\ e^{-A^2t} O^\top &= \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} O\ A^{2n+1} O^\top \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} \mathrm{diag}\left(a_{(d_0)}^{2n+1}, \cdots, a_{(1)}^{2n+1}, 0, \cdots, 0\right) \\
&= \mathrm{diag}\left(\sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} a_{(d_0)}^{2n+1}, \cdots, \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} a_{(1)}^{2n+1}, 0, \cdots, 0\right) \\
&= \mathrm{diag}\left(a_{(d_0)} e^{-a_{(d_0)}^2 t}, \cdots, a_{(1)} e^{-a_{(1)}^2 t}, 0, \cdots, 0\right),
\end{aligned}$$

As a consequence,

$$\begin{aligned}
\left\| A\ e^{-A^2t} \right\| = \left\| O\ A\ e^{-A^2t} O^\top \right\| &= \left\| \mathrm{diag}\left(a_{(d_0)} e^{-a_{(d_0)}^2 t}, \cdots, a_{(1)} e^{-a_{(1)}^2 t}, 0, \cdots, 0\right) \right\| \\
&\leq a_{(d_0)} e^{-a_{(1)}^2 t} = \|A\| e^{-a^2 t}
\end{aligned}$$

as asserted. □

After these preparations, we can now show that our exponential local martingale is indeed a true martingale by verifying the conditions of Corollary 5.14 in [6, Chapter 3]:

Corollary C.4. *The stochastic exponential* $\mathcal{E}(M) = \left(\exp(M_t - \frac{1}{2}\langle M \rangle_t)\right)_{t\geq 0}$ *is a true martingale.*

Proof. In order to apply the estimate from Lemma C.3, let c be the smallest non-zero singular value of $\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}$ when $\|\Pi_{\theta\theta}\| > 0$, and $c = 1$ when $\|\Pi_{\theta\theta}\| = 0$. Plugging in equation (3.5) for $\Pi_{\theta\theta}$ in the second step and then using Lemma C.3 we obtain the following bound, for every $t > 0$,

$$\begin{aligned}
&\gamma \int_0^t \left\| \left(\Pi_{\theta x}\sigma_x - \sigma_p\right)^\top \Lambda^{-1/2} e^{-\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}(t-u)} \right\|^2 du \\
&= \gamma \int_0^t \left\| e^{-\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}(t-u)} \Lambda^{-1/2} \left(\Pi_{\theta x}\sigma_x - \sigma_p\right) \left(\Pi_{\theta x}\sigma_x - \sigma_p\right)^\top \Lambda^{-1/2} e^{-\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}(t-u)} \right\| du \\
&= \int_0^t \left\| e^{-\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}(t-u)} \Lambda^{-1/2} \left(\Pi_{\theta\theta}\Lambda^{-1}\Pi_{\theta\theta} + r\Pi_{\theta\theta}\right) \Lambda^{-1/2} e^{-\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}(t-u)} \right\| du \\
&= \int_0^t \left\| \left(\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2} + rI_m\right)^{1/2} \left(\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}\right)^{1/2} e^{-\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}(t-u)} \right\|^2 du \\
&\leq \left\| \Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2} + rI_m \right\| \int_0^t \left\| \left(\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}\right)^{1/2} e^{-\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}(t-u)} \right\|^2 du \\
&\leq \left\| \Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2} + rI_m \right\| \int_0^t \left\| \Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2} \right\| e^{-2c(t-u)} du \\
&\leq \frac{1}{2c} \left\| \Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2} + rI_m \right\| \left\| \Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2} \right\|. \qquad \text{(C.14)}
\end{aligned}$$

Now for the application of Lemma C.2, notice that the multivariate Ornstein-Uhlenbeck process $X(t)$ satisfies

$$X(t) \sim N\left(e^{Axt}X_0, \int_0^t e^{A_x(t-u)}\sigma_x\sigma_x^\top e^{A_x^\top(t-u)}du\right).$$

As A_x is stable, there exists a symmetric positive semidefinite matrix Σ such that

$$\int_0^t e^{A_x(t-u)}\sigma_x\sigma_x^\top e^{A_x^\top(t-u)}du \leq \Sigma.$$

The process $\tilde{X}(t) = X(t) - e^{Axt}X_0$ then is also Gaussian with

$$\tilde{X}(t) \sim N\left(0, \int_0^t e^{A_x(t-u)}\sigma_x\sigma_x^\top e^{A_x^\top(t-u)}du\right).$$

Next, the definition (3.4) of the trading rate $\dot{\theta}(t)$ implies

$$\begin{aligned}
d\ e^{\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}t}\Lambda^{1/2}\theta(t) &= e^{\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}t}\left(\Lambda^{1/2}\dot{\theta}(t) + \Lambda^{-1/2}\Pi_{\theta\theta}\theta(t)\right)dt \\
&= -e^{\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}t}\Lambda^{-1/2}\left(\Pi_{\theta x}X(t) + \Pi_\theta\right)dt \\
&= -e^{\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}t}\Lambda^{-1/2}\left(\Pi_{\theta x}\tilde{X}(t) + \Pi_{\theta x}e^{A_x t}X_0 + \Pi_\theta\right)dt.
\end{aligned}$$

Integrating both sides of the equation and multiplying with $\Lambda^{-1/2}e^{-\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}t}$, this gives

$$\begin{aligned}
\theta(t) =& \Lambda^{-1/2}e^{-\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}t}\Lambda^{1/2}\theta_0 \\
&- \int_0^t \Lambda^{-1/2}e^{-\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}(t-u)}\Lambda^{-1/2}\left(\Pi_{\theta x}e^{A_x t}X_0 + \Pi_\theta\right)du \\
&- \int_0^t \Lambda^{-1/2}e^{-\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}(t-u)}\Lambda^{-1/2}\Pi_{\theta x}\tilde{X}(u)du.
\end{aligned}$$

This represents θ as an integral of the mean-zero Gaussian process $\tilde{X}$, and in turn allows to apply the second part of Lemma C.2.

Next, we focus on the quadratic variation process

$$d\langle M\rangle_t = \left\|\left(\Pi_{\theta x}\sigma_x - \sigma_p\right)^\top\theta(t) - \sigma_x^\top\left(\Pi_{xx}X(t) - \Pi_x\right)\right\|^2 dt.$$

Minkowski's inequality, Hölder's inequality and (C.14) yield the following series of inequalities:

$$\begin{aligned}
&\frac{1}{3}\left\|\left(\Pi_{\theta x}\sigma_x - \sigma_p\right)^\top\theta(t) - \sigma_x^\top\left(\Pi_{xx}X(t) - \Pi_x\right)\right\|^2 \\
=& \frac{1}{3}\left\|\left(\Pi_{\theta x}\sigma_x - \sigma_p\right)^\top\theta(t) - \sigma_x^\top\left(\Pi_{xx}\tilde{X}(t) + \Pi_{xx}e^{A_x t}X_0 - \Pi_x\right)\right\|^2 \\
\leq& \|C_t\|^2 + \|\sigma_x^\top\Pi_{xx}\tilde{X}(t)\|^2 + \left\|\left(\Pi_{\theta x}\sigma_x - \sigma_p\right)^\top\int_0^t \Lambda^{-1/2}e^{-\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}(t-u)}\Lambda^{-1/2}\Pi_{\theta x}\tilde{X}(u)du\right\|^2 \\
\leq& \|C_t\|^2 + \|\sigma_x^\top\Pi_{xx}\tilde{X}(t)\|^2 + \left(\int_0^t\left\|\left(\Pi_{\theta x}\sigma_x - \sigma_p\right)^\top\Lambda^{-1/2}e^{-\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}(t-u)}\Lambda^{-1/2}\Pi_{\theta x}\tilde{X}(u)\right\|du\right)^2 \\
\leq& \|C_t\|^2 + \|\sigma_x^\top\Pi_{xx}\tilde{X}(t)\|^2 + \int_0^t\left\|\left(\Pi_{\theta x}\sigma_x - \sigma_p\right)^\top\Lambda^{-1/2}e^{-\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}(t-u)}\right\|^2 du\int_0^t\|\Lambda^{-1/2}\Pi_{\theta x}\tilde{X}(u)\|^2 du \\
\leq& \|C_t\|^2 + \|\sigma_x^\top\Pi_{xx}\tilde{X}(t)\|^2 + \frac{1}{2\gamma c}\left\|\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2} + rI_m\right\|\left\|\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}\right\|\int_0^t\|\Lambda^{-1/2}\Pi_{\theta x}\tilde{X}(u)\|^2 du,
\end{aligned}$$

where

$$C_t = (\Pi_{\theta x}\sigma_x - \sigma_p)^\top \Lambda^{-1/2} e^{-\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}t}\Lambda^{1/2}\theta_0 - \sigma_x^\top \left(\Pi_{xx}e^{A_x t}X_0 - \Pi_x\right)$$
$$- (\Pi_{\theta x}\sigma_x - \sigma_p)^\top \int_0^t e^{-\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}(t-u)}\Lambda^{-1/2}\left(\Pi_{\theta x}e^{A_x t}X_0 + \Pi_\theta\right) du.$$

Now, choose

$$0 < h < \frac{1}{6\sqrt{\mathrm{tr}(\Sigma)}} \min\left\{\frac{1}{\|\Pi_{xx}\sigma_x\|^2\sqrt{\mathrm{tr}(\Sigma)}},\ \frac{2\gamma c}{\left\|\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2} + rI_m\right\| \left\|\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}\right\| \|\Lambda^{-1/2}\Pi_{\theta x}\|}\right\},$$

and define $t_0 = 0$, $t_n = \sum_{m=1}^n \frac{h}{m}$. They by (C.10), we have

$$\mathbb{E}\left[\exp\left(3\int_{t_{n-1}}^{t_n} \|\sigma_x^\top \Pi_{xx}\tilde{X}(t)\|^2 dt\right)\right] \le \frac{1}{1 - 6h\|\Pi_{xx}\sigma_x\|^2\sqrt{\mathrm{tr}(\Sigma)}} < \infty, \tag{C.15}$$

and by (C.11),

$$\mathbb{E}\left[\exp\left(\frac{3}{2c}\left\|\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2} + rI_m\right\| \left\|\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}\right\| \int_{t_{n-1}}^{t_n}\int_0^t \|\Lambda^{-1/2}\Pi_{\theta x}\tilde{X}(u)\|^2 du dt\right)\right]$$
$$\le \frac{1}{1 - \frac{3\left\|\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2} + rI_m\right\| \left\|\Lambda^{-1/2}\Pi_{\theta\theta}\Lambda^{-1/2}\right\| \|\Lambda^{-1/2}\Pi_{\theta x}\| h^2 \mathrm{tr}(\Sigma)}{c}} < \infty. \tag{C.16}$$

Furthermore, notice that by Hölder's inequality, for some large constant K,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\frac{1}{2}\int_{t_{n-1}}^{t_n} d\langle M\rangle_t\right)\right] \\
&= \mathbb{E}\left[\exp\left(\frac{1}{2}\int_{t_{n-1}}^{t_n} \left\|(\Pi_{\theta x}\sigma_x - r\gamma\sigma_p)^\top \theta(t) - \sigma_x^\top (\Pi_{xx}X(t) - \Pi_x)\right\|^2 dt\right)\right] \\
&\le e^{\frac{3}{2}\|C_t\|^2 h}\mathbb{E}\left[\exp\left(\frac{3}{2}\int_{t_{n-1}}^{t_n} \|\sigma_x^\top \Pi_{xx}\tilde{X}(t)\|^2 dt\right)\exp\left(\frac{K}{2}\int_{t_{n-1}}^{t_n}\int_0^t \|\Lambda^{-1/2}\Pi_{\theta x}\tilde{X}(u)\|^2 du dt\right)\right] \\
&\le e^{\frac{3}{2}\|C_t\|^2 h}\mathbb{E}\left[\exp\left(3\int_{t_{n-1}}^{t_n} \|\sigma_x^\top \Pi_{xx}\tilde{X}(t)\|^2 dt\right)\right]^{1/2}\mathbb{E}\left[\exp\left(K\int_{t_{n-1}}^{t_n}\int_0^t \|\Lambda^{-1/2}\Pi_{\theta x}\tilde{X}(u)\|^2 du dt\right)\right]^{1/2} \\
&< \infty.
\end{aligned}$$

Finally, since $\sum_{m=1}^\infty \frac{h}{m} = \infty$, we have $t_n \to \infty$ as $n \to \infty$. The asserted true martingale property therefore follows from Corollary 5.14 in [6, Chapter 3]. □

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