Gaussian SLR: Hypothesis Testing

GR 5205 / GU 4205
Section 2/ Section 3

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Simple Linear Model

$$Y = \beta_0 + x\beta_1 + \epsilon, \qquad \text{where} \quad \mathbb{E}[\epsilon] = 0, \mathrm{Var}(\epsilon) = \sigma^2 I_n$$

- Since this is linear models, we can treat all indicators as known
- Model parameters: $\beta_0, \beta_1 (, \sigma^2)$
- With experiments data $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ $b_1 = \frac{(x - \bar{x}\mathbb{1}_n)^\top (y - \bar{y}\mathbb{1}_n)}{\|x - \bar{x}\mathbb{1}_n\|^2}, \quad b_0 = \bar{y} - \bar{x}b_1$
- Analyzing statistical properties: $\widehat{\beta}_1 = \frac{(x - \bar{x}\mathbb{1}_n)^\top (Y - \bar{Y}\mathbb{1}_n)}{\|x - \bar{x}\mathbb{1}_n\|^2}, \quad \widehat{\beta}_0 = \bar{Y} - \bar{x}\widehat{\beta}_1$

Gaussian Linear Model

$$Y = \beta_0 + x\beta_1 + \epsilon, \qquad \text{where} \quad \epsilon \sim \mathcal{N}\left(0, \sigma^2 I_n\right)$$

- Since this is linear models, we can treat all indicators as known
- Model parameters: $\beta_0, \beta_1 (, \sigma^2)$
- With experiments data $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ $b_1 = \frac{(x - \bar{x}\mathbb{1}_n)^\top (y - \bar{y}\mathbb{1}_n)}{\|x - \bar{x}\mathbb{1}_n\|^2}, \quad b_0 = \bar{y} - \bar{x}b_1$
- Analyzing statistical properties: $\widehat{\beta}_1 = \frac{(x - \bar{x}\mathbb{1}_n)^\top (Y - \bar{Y}\mathbb{1}_n)}{\|x - \bar{x}\mathbb{1}_n\|^2}, \quad \widehat{\beta}_0 = \bar{Y} - \bar{x}\widehat{\beta}_1$

Last time:

$$\widehat{\beta}_1 = \frac{(x - \bar{x}\mathbb{1}_n)^\top (Y - \bar{Y}\mathbb{1}_n)}{\|x - \bar{x}\mathbb{1}_n\|^2}, \quad \widehat{\beta}_0 = \bar{Y} - \bar{x}\widehat{\beta}_1$$

$$\widehat{\sigma}^2 = \mathbf{MSE} = \frac{\sum_{i=1}^n e_i^2}{n-2} = \frac{\sum_{i=1}^n (y_i - \hat{y}_i)^2}{n-2} = \frac{\|y - \hat{y}\|^2}{n-2}$$

Sampling distribution

$$\widehat{\beta}_1 \sim \mathcal{N}\left(\beta_1, \frac{\sigma^2}{\|x - \bar{x}1_n\|^2}\right)$$

$$\widehat{\beta}_0 \sim \mathcal{N}\left(\beta_0, \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{\|x - \bar{x}1_n\|^2}\right)\right)$$

$$\widehat{\sigma}^2_{\mathrm{LS}} \sim \frac{\sigma^2}{n-2}\chi^2(n-2)$$

$$(\widehat{\beta}_0, \widehat{\beta}_1) \perp \widehat{\sigma}^2_{\mathrm{LS}}$$

Hand-waving explanations for:

$$(\widehat{\beta}_0, \widehat{\beta}_1) \perp \widehat{\sigma}^2_{\mathrm{LS}}$$

- n i.i.d normal distributed error term
- using 2 degree of freedom to estimate $\widehat{\beta}_0, \widehat{\beta}_1$
- extra n-2 degree of freedom is in $\widehat{\sigma}^2_{\mathrm{LS}}$

Confidence interval for coefficient:

$$(\widehat{\beta}_0, \widehat{\beta}_1) \perp \widehat{\sigma}^2_{\mathrm{LS}}$$

$$\widehat{\beta}_1 \sim \mathcal{N}\left(\beta_1, \frac{\sigma^2}{\|x-\bar{x}1_n\|^2}\right)$$

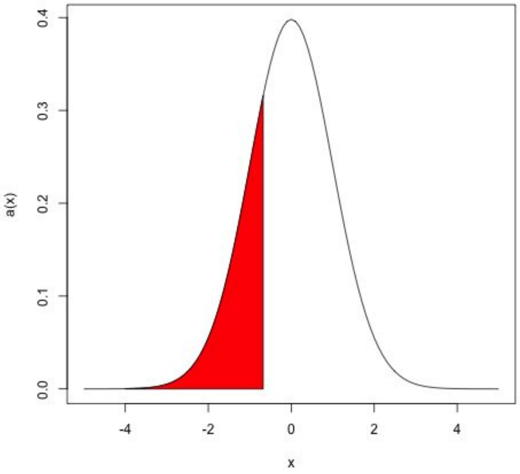
$$\widehat{\beta}_0 \sim \mathcal{N}\left(\beta_0, \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{\|x-\bar{x}1_n\|^2}\right)\right)$$

$$\widehat{\sigma}^2_{\mathrm{LS}} \sim \frac{\sigma^2}{n-2}\chi^2(n-2)$$

- Choice of statistics:

- (100-α)% confidence intervals for ground truth:

- where define $t(q; n-2)$ as $P(Z \leq t(q; n-2)) = q$



Confidence interval for coefficient:

$$(\widehat{\beta}_0, \widehat{\beta}_1) \perp \widehat{\sigma}^2_{\mathrm{LS}}$$

$$\widehat{\beta}_1 \sim \mathcal{N}\left(\beta_1, \frac{\sigma^2}{\|x-\bar{x}1_n\|^2}\right)$$

$$\widehat{\beta}_0 \sim \mathcal{N}\left(\beta_0, \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{\|x-\bar{x}1_n\|^2}\right)\right)$$

$$\widehat{\sigma}^2_{\mathrm{LS}} \sim \frac{\sigma^2}{n-2}\chi^2(n-2)$$

- Choice of statistics:

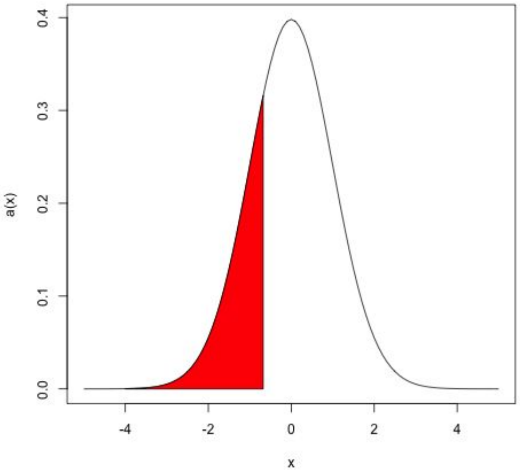
$$\sqrt{\frac{\|x-\bar{x}1_n\|^2}{\widehat{\sigma}^2}}\left(\widehat{\beta}_1 - \beta_1\right) \sim t(n-2), \quad \sqrt{\frac{\left(\frac{1}{n} + \frac{\bar{x}^2}{\|x-\bar{x}1_n\|^2}\right)^{-1}}{\widehat{\sigma}^2}}\left(\widehat{\beta}_0 - \beta_0\right) \sim t(n-2)$$

- (100-$\boldsymbol{\alpha}$)% confidence intervals for ground truth:

$$\left[\widehat{\beta}_1 - \sqrt{\frac{\widehat{\sigma}^2}{\|x-\bar{x}1_n\|^2}}t(1-\tfrac{\alpha}{2}; n-2), \widehat{\beta}_1 + \sqrt{\frac{\widehat{\sigma}^2}{\|x-\bar{x}1_n\|^2}}t(1-\tfrac{\alpha}{2}; n-2)\right]$$

$$\left[\widehat{\beta}_0 - \sqrt{\widehat{\sigma}^2\left(\frac{1}{n} + \frac{\bar{x}^2}{\|x-\bar{x}1_n\|^2}\right)}t(1-\tfrac{\alpha}{2}; n-2), \widehat{\beta}_0 + \sqrt{\widehat{\sigma}^2\left(\frac{1}{n} + \frac{\bar{x}^2}{\|x-\bar{x}1_n\|^2}\right)}t(1-\tfrac{\alpha}{2}; n-2)\right]$$

- where define $t(q; n-2)$ as $P(Z \le t(q; n-2)) = q$



Confidence interval for estimation:

$$(\widehat{\beta}_0, \widehat{\beta}_1) \perp \widehat{\sigma}^2_{\mathrm{LS}}$$

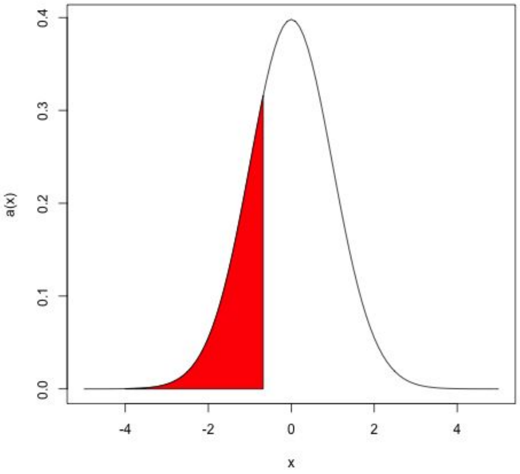
$$\widehat{\beta}_1 \sim \mathcal{N}\left(\beta_1, \frac{\sigma^2}{\|x-\bar{x}1_n\|^2}\right)$$

$$\widehat{\beta}_0 \sim \mathcal{N}\left(\beta_0, \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{\|x-\bar{x}1_n\|^2}\right)\right)$$

$$\widehat{\sigma}^2_{\mathrm{LS}} \sim \frac{\sigma^2}{n-2}\chi^2(n-2)$$

- Choice of statistics:

- (100-α)% confidence intervals for ground truth:



Confidence interval for prediction:

$$(\widehat{\beta}_0, \widehat{\beta}_1) \perp \widehat{\sigma}^2_{\text{LS}}$$

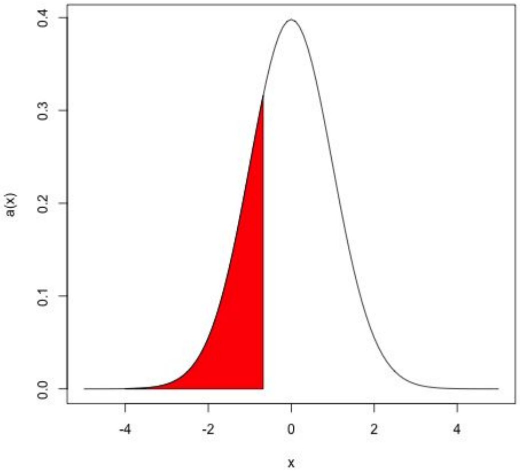
$$\widehat{\beta}_1 \sim \mathcal{N}\left(\beta_1, \frac{\sigma^2}{\|x - \bar{x}1_n\|^2}\right)$$

$$\widehat{\beta}_0 \sim \mathcal{N}\left(\beta_0, \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{\|x - \bar{x}1_n\|^2}\right)\right)$$

$$\widehat{\sigma}^2_{\text{LS}} \sim \frac{\sigma^2}{n-2}\chi^2(n-2)$$

- Choice of statistics:

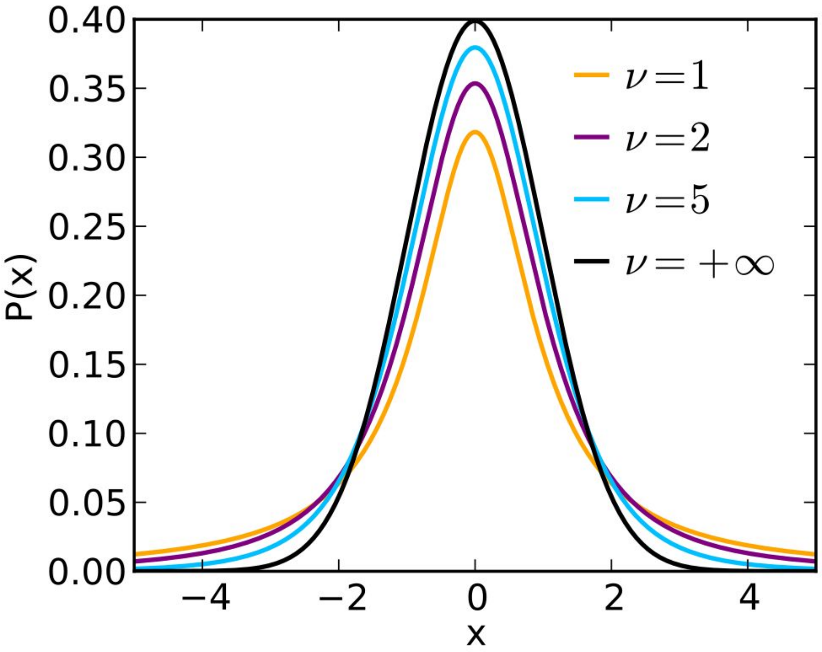
- (100-α)% confidence intervals for ground truth:



Summary of Gaussian SLR:
distribution of estimator, confidence interval

	distribution	1- α confidence interval
slop β_1	$\widehat{\beta}_1 \sim \mathcal{N}\left(\beta_1, \frac{\sigma^2}{\lVert x-\bar{x}1_n \rVert^2}\right)$	$\left[\widehat{\beta}_1 \pm \frac{\widehat{\sigma}_{\text{LS}}}{\lVert x-\bar{x}1_n \rVert} t(\frac{\alpha}{2}; n-2)\right]$
intercept β_0	$\widehat{\beta}_0 \sim \mathcal{N}\left(\beta_0, \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{\lVert x-\bar{x}1_n \rVert^2}\right)\right)$	$\left[\widehat{\beta}_0 \pm \widehat{\sigma}_{\text{LS}} \sqrt{\frac{1}{n} + \frac{\bar{x}^2}{\lVert x-\bar{x}1_n \rVert^2}} t(\frac{\alpha}{2}; n-2)\right]$
noise level σ^2	$\widehat{\sigma}^2_{\text{LS}} \sim \frac{\sigma^2}{n-2} \chi^2(n-2)$	$\left[\frac{\widehat{\sigma}^2_{\text{LS}}}{\chi^2\left(\frac{\alpha}{2}; n-2\right)}, \frac{\widehat{\sigma}^2_{\text{LS}}}{\chi^2\left(1-\frac{\alpha}{2}; n-2\right)}\right]$
mean of Y_0 at x_0 $\mathbb{E}[Y_0] = \beta_0 + x_0\beta_1$	$\widehat{\beta}_0 + x_0\widehat{\beta}_1 \sim \mathcal{N}\left(\mathbb{E}[Y_0], \sigma^2\left(\frac{1}{n} + \frac{(x_0-\bar{x})^2}{\lVert x-\bar{x}1_n \rVert^2}\right)\right)$	$\left[\left(\widehat{\beta}_0 + x_0\widehat{\beta}_1\right) \pm \widehat{\sigma}_{\text{LS}} \sqrt{\frac{1}{n} + \frac{(x_0-\bar{x})^2}{\lVert x-\bar{x}1_n \rVert^2}} t(\frac{\alpha}{2}; n-2)\right]$
new observation at x_0 $Y_0 = \beta_0 + x_0\beta_1 + \epsilon_0$	$\widehat{\beta}_0 + x_0\widehat{\beta}_1 \sim \mathcal{N}\left(\mathbb{E}[Y_0], \sigma^2\left(1 + \frac{1}{n} + \frac{(x_0-\bar{x})^2}{\lVert x-\bar{x}\mathbb{1}_n \rVert}\right)\right)$	$\left[\left(\widehat{\beta}_0 + x_0\widehat{\beta}_1\right) \pm \widehat{\sigma}_{\text{LS}} \sqrt{1 + \frac{1}{n} + \frac{(x_0-\bar{x})^2}{\lVert x-\bar{x}1_n \rVert^2}} t(\frac{\alpha}{2}; n-2)\right]$

Asymptotically normality



t Table

cum. prob	$t_{.50}$	$t_{.75}$	$t_{.80}$	$t_{.85}$	$t_{.90}$	$t_{.95}$	$t_{.975}$	$t_{.99}$	$t_{.995}$	$t_{.999}$	$t_{.9995}$
one-tail	**0.50**	**0.25**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	**0.001**	**0.0005**
two-tails	**1.00**	**0.50**	**0.40**	**0.30**	**0.20**	**0.10**	**0.05**	**0.02**	**0.01**	**0.002**	**0.001**
df											
1	0.000	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	318.31	636.62
2	0.000	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	22.327	31.599
3	0.000	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.000	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.000	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.000	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.000	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.000	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.000	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	0.000	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.000	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	0.000	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.000	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.000	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	0.000	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.000	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.000	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	0.000	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	0.000	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.000	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.000	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.000	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.000	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.000	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	0.000	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	0.000	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.000	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.000	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.000	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.000	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	0.000	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	0.000	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	3.232	3.460
80	0.000	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	3.195	3.416
100	0.000	0.677	0.845	1.042	1.290	1.660	1.984	2.364	2.626	3.174	3.390
1000	0.000	0.675	0.842	1.037	1.282	1.646	1.962	2.330	2.581	3.098	3.300
z	0.000	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	3.090	3.291
	0%	50%	60%	70%	80%	90%	95%	98%	99%	99.8%	99.9%
					Confidence Level						

When the number of samples goes to infty

$$(\widehat{\beta}_1 - \beta_1)/\mathrm{se}(\widehat{\beta}_1) \sim \mathcal{N}(0,1)$$

$$(\widehat{\beta}_0 - \beta_0)/\mathrm{se}(\widehat{\beta}_0) \sim \mathcal{N}(0,1)$$

What is hypothesis testing?

- Null hypothesis H_0

  Alternative hypothesis H_1
- Type I error:

  rejection of a true null hypothesis;
- Type II error:

  failure to reject a false null hypothesis.
- Can we control both?

Table of error types		Null hypothesis (H_0) is True	Null hypothesis (H_0) is False
Decision about null hypothesis (H_0)	Don't reject	Correct inference (true negative) (probability = $1-\alpha$)	Type II error (false negative) (probability = β)
	Reject	Type I error (false positive) (probability = α)	Correct inference (true positive) (probability = $1-\beta$)

Pipeline to design a test

1) State the statistical assumptions;
2) State the relevant null hypothesis and alternative hypothesis;
3) Set a threshold α;
4) Choosing the test statistics T and test methods;
5) Under the null hypothesis, derive the distribution p of the test statistics T
6) Insert data into T and get t_{obs};
7) Under the null hypothesis, calculate the p-value by $p\left(T \geq t_{\text{obs}}\right)$;
8) Reject the null hypothesis if and only if the p-value is less than or equal to the threshold.

In Linear Regression Models

1) State the statistical assumptions; $Y = \beta_0 + x\beta_1 + \epsilon, \quad \text{where} \quad \epsilon \sim \mathcal{N}\left(0, \sigma^2 I_n\right)$

2) State the relevant null hypothesis and alternative hypothesis;

$$H_0 : \beta_1 = 0 \quad \text{versus} \quad H_1 : \beta_1 \neq 0.$$

3) Typical choice: $\alpha = 5\%$

Wald Test: $T = \frac{\widehat{\beta}_1 - 0}{\widehat{\sigma}_{LS}}$ $\left[\widehat{\beta}_1 \pm \frac{\widehat{\sigma}_{\mathrm{LS}}}{\|x - \bar{x}1_n\|} t(\frac{\alpha}{2}; n-2)\right]$

Hypothesis testing based on confidence interval

- The upper bound, lower bound and the length of the confidence interval are all random variables!
- As α shrinks, the interval widens. (High confidence comes at the price of big margins of error.)
- As sample size grows, the interval shrinks. (Large samples mean precise estimates.)
- As noise level increases, the interval widens. (The more noise there is around the regression line, the less precisely we can measure the line.)
- As $\frac{1}{n-1}\|x - \bar{x}\mathbb{1}_n\|^2$ grows, the interval shrinks. (Widely-spread measurements give us a precise estimate of the slope.)

Statistical significance: p-value

The test statistic for the Wald test,

$$T = \frac{\widehat{\beta}_1 - \beta_1^*}{\widehat{\mathrm{se}}\left[\widehat{\beta}_1\right]}$$

has the nice, intuitive property that it ought to be close to zero when the null hypothesis $\beta_1 = \beta_1^*$ is true, and take large values (either positive or negative) when the null hypothesis is false. When a test statistic works like this, it makes sense to summarize just how bad the data looks for the null hypothesis in a ***p*-value**: when our observed value of the test statistic is T_{obs}, the p-value is

$$P = \mathbb{P}\left(|T| \geq |T_{obs}|\right)$$

When our test lets us calculate a p-value, we can form a $1-\alpha$ confidence set by taking all the β's where the p-value is $\geq \alpha$. Conversely, if we have some way of making confidence sets already, we can get a p-value for the hypothesis $\beta = \beta^*$; it's the largest α such that β^* is in the $1-\alpha$ confidence set.

Statistically significant

If we test a hypothesis and reject it, then it means the difference of the hypothesis and the reality is statistically significant.

- Model checking
- Actual scientific interest

Last time

Linear regression models with iid Gaussian noise

- The sampling distribution of the estimators for the regression coefficients

$$\widehat{\beta}_1 \sim \mathcal{N}\left(\beta_1, \frac{\sigma^2}{\|x-\bar{x}1_n\|^2}\right)$$

$$\widehat{\beta}_0 \sim \mathcal{N}\left(\beta_0, \sigma^2\left(\frac{1}{n}+\frac{\bar{x}^2}{\|x-\bar{x}1_n\|^2}\right)\right)$$

$$\widehat{\sigma}^2_{\mathrm{LS}} \sim \frac{\sigma^2}{n-2}\chi^2(n-2)$$

$$(\widehat{\beta}_0, \widehat{\beta}_1) \perp \widehat{\sigma}^2_{\mathrm{LS}}$$

- Estimation for mean

  prediction for the next observation

- Confidence interval and Wald test for regression coefficients

ANOVA: ANalysis Of VAriance

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- Decomposition: $\|Y - \bar{Y}1_n\|^2 = \|Y - \widehat{Y}\|^2 + \|\widehat{Y} - \bar{Y}1_n\|^2$
- Residual sum of squares: $RSS = \|Y - \widehat{Y}\|^2$
- Total sum of squares: $SS_{\text{total}} = \|Y - \bar{Y}1_n\|^2$
- The sum of squares due to regression: $SS_{\text{reg}} = \|\widehat{Y} - \bar{Y}\mathbb{1}_n\|^2 = SS_{total} - RSS$
- RSS and SS_{reg} are independent
- F test / ANOVA

ANOVA

Source	df	SS	MS	F	p-value
Regression	1	$\mathrm{SS}_{\mathrm{reg}}$	$\mathrm{MS}_{\mathrm{reg}} = \frac{\mathrm{SS}_{\mathrm{reg}}}{1}$	$F = \frac{\mathrm{MS}_{\mathrm{reg}}}{\mathrm{MS}_{\mathrm{res}}}$	
Residual	n-2	RSS	$\widehat{\sigma}^2 = \frac{\mathrm{RSS}}{n-2}$		
Total	n-1	$\mathrm{SS}_{\mathrm{total}}$			

F test: What are we really testing?

- An F test for whether the simple linear regression model “explains” (really, predicts) a “significant” amount of the variance in the response.
- Compare two versions of the simple linear regression model.

About the regression coefficient

For a set of data $(x_1, y_1), \ldots, (x_n, y_n)$, we fit to the linear regression model of Y with respect X:

$$y = b_1 x + b_0.$$

What is the linear regression model of X with respect to Y?

About R

As the number of samples $n \to \infty$,

$$R^2 = \frac{\beta_1^2 \mathrm{Var}(X)}{\beta_1^2 \mathrm{Var}(X) + \sigma^2}$$

- R does not measure goodness of fit
- R is also useless as measure of predictability
- R cannot be compared across dataset
- R cannot be compared between different transformations of the response variable Y
- MSE is more preferable than R when comparing the models

References and further reading

- Kutner, Nachtsheim, Neter: *Applied Linear Regression Models* Chapter 2
- Agresti: *Foundations of Linear and Generalized Linear Models* Chapter 2&3
- CMU 36-401 Lecture notes