Multivariate Regression

GR 5205 / GU 4205
Section 3

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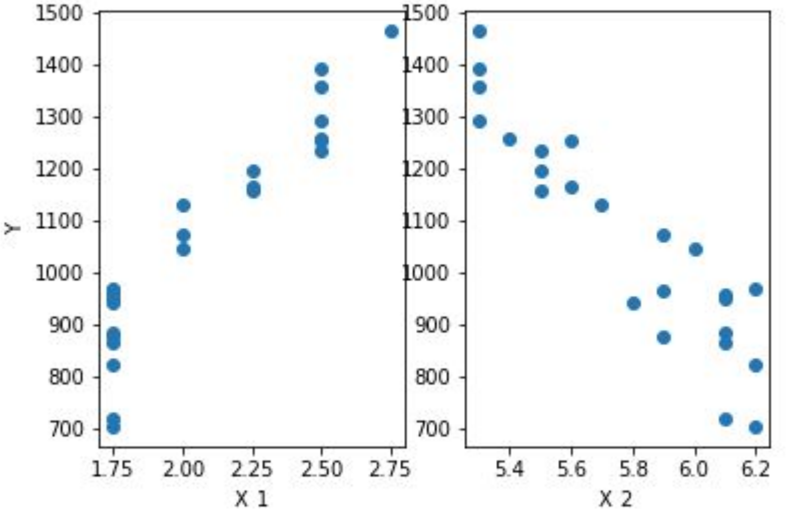
With more predictors



2.75	5.3	1464
2.50	5.3	1394
2.50	5.3	1357
2.50	5.3	1293
2.50	5.4	1256
2.50	5.6	1254
2.50	5.5	1234
2.25	5.5	1195
2.25	5.5	1159
2.25	5.6	1167
2.00	5.7	1130
2.00	5.9	1075
2.00	6.0	1047
1.75	5.9	965
1.75	5.8	943
1.75	6.1	958
1.75	6.2	971
1.75	6.1	949
1.75	6.1	884
1.75	6.1	866
1.75	5.9	876
1.75	6.2	822
1.75	6.2	704
1.75	6.1	719

Generally speaking...

$$Y = \beta_0 + x^{(1)}\beta_1 + x^{(2)}\beta_2 + \cdots + x^{(p-1)}\beta_{p-1} + \epsilon$$

$$x = \begin{bmatrix} 1 & x_1^{(1)} & \cdots & x_1^{(p-1)} \\ 1 & x_2^{(1)} & \cdots & x_2^{(p-1)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n^{(1)} & \cdots & x_n^{(p-1)} \end{bmatrix} = \begin{bmatrix} 1 & x_1^\top \\ 1 & x_2^\top \\ \vdots & \vdots \\ 1 & x_n^\top \end{bmatrix} \in \mathbb{R}^{n \times p}$$

Recall our road map...

x - predictor (random) variables **Y - response random variable**

- Build your model:

  1) relationship: $Y = x\beta + \epsilon, \qquad \mathbb{E}[\epsilon] = 0, \ \mathrm{Var}[\epsilon] = \sigma^2 I_n.$

  2) preference: choose $\widehat{\beta}$ to minimize mean squared error

- Estimate your model parameters: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

  1) using observed data to express your preference: $\min_{\beta_0,\beta_1} Q := \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$

  2) get parameters estimation for your model:

- Understand your model:

  1) properties of estimations:

  2) predictions: $\widehat{Y}_0 = x_0\widehat{\beta}, \qquad \widehat{y}_0 = x_0 b.$

Taking partial derivatives wrt a vector

Linear forms. If $f(\mathbf{X}) = \mathbf{X}^T\mathbf{a}$, with $\mathbf{a}$ not a function of $\mathbf{X}$, then

$$\nabla(\mathbf{X}^T\mathbf{a}) = \mathbf{a}$$

Quadratic forms. Let $\mathbf{C}$ be a $p \times p$ matrix which is not a function of $\mathbf{X}$, and consider the **quadratic form** $\mathbf{X}^T\mathbf{C}\mathbf{X}$. (You can check that this is scalar.) The gradient is

$$\nabla(\mathbf{X}^T\mathbf{C}\mathbf{X}) = (\mathbf{C} + \mathbf{C}^T)\mathbf{X}.$$

Back to optimize $Q(\beta) = \|y - x\beta\|^2$

$$\widehat{\beta} = (x^\top x)^{-1} x^\top Y$$

$$\widehat{\sigma}_{LS}^2 = \frac{1}{n-p}\|Y - \widehat{Y}\|^2 = \frac{1}{n-p}\|Y - x\widehat{\beta}\|^2$$

Collinear

In order to invert $x^\top x$

- Its determinant is non-zero.
- It is of "full column rank", meaning all of its columns are linearly independent.
- It is of "full row rank", meaning all of its rows are linearly independent.

Collinear in terms of data:

- If $n < p$.
- If one of the predictor variables is constant.
- If two of the predictor variables are proportional to each other.
- If two of the predictor variables are otherwise linearly related.

Multiple Regression Isn't Just a Bunch of Simple Regressions

Suppose the real model is $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \epsilon$. (Nothing turns on $p = 2$, it just keeps things short.) What would happen if we did a simple regression of Y on just X_1? We know that the optimal (population) slope on X_1 is

$$\frac{\mathrm{Cov}[X_1, Y]}{\mathrm{Var}[X_1]}$$

Let's substitute in the model equation for Y:

$$\begin{aligned}
\frac{\mathrm{Cov}[X_1, Y]}{\mathrm{Var}[X_1]} &= \frac{\mathrm{Cov}[X_1, \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \epsilon]}{\mathrm{Var}[X_1]} \\
&= \frac{\beta_1 \mathrm{Var}[X_1] + \beta_2 \mathrm{Cov}[X_1, X_2] + \mathrm{Cov}[X_1, \epsilon]}{\mathrm{Var}[X_1]} \\
&= \beta_1 + \frac{\beta_2 \mathrm{Cov}[X_1, X_2] + 0}{\mathrm{Var}[X_1]} \\
&= \beta_1 + \beta_2 \frac{\mathrm{Cov}[X_1, X_2]}{\mathrm{Var}[X_1]}
\end{aligned}$$

Multiple Regression
Isn't Just a Bunch of Simple Regressions

```
==============================================================================
Dep. Variable:                      y   R-squared:                       0.851
Model:                            OLS   Adj. R-squared:                  0.844
Method:                 Least Squares   F-statistic:                     125.4
Date:                Fri, 02 Oct 2020   Prob (F-statistic):           1.49e-10
Time:                        15:43:15   Log-Likelihood:                -139.14
No. Observations:                  24   AIC:                             282.3
Df Residuals:                      22   BIC:                             284.6
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const       4471.3393    304.254     14.696      0.000    3840.354    5102.324
x1          -588.9621     52.602    -11.196      0.000    -698.053    -479.871
==============================================================================
```

```
Dep. Variable:                      y   R-squared:                       0.876
Model:                            OLS   Adj. R-squared:                  0.870
Method:                 Least Squares   F-statistic:                     155.0
Date:                Fri, 02 Oct 2020   Prob (F-statistic):           1.95e-11
Time:                        20:28:29   Log-Likelihood:                -136.94
No. Observations:                  24   AIC:                             277.9
Df Residuals:                      22   BIC:                             280.2
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const        -99.4643     95.210     -1.045      0.308    -296.918      97.990
x_2          564.2039     45.317     12.450      0.000     470.221     658.186
==============================================================================
```

```
==============================================================================
Dep. Variable:                      y   R-squared:                       0.898
Model:                            OLS   Adj. R-squared:                  0.888
Method:                 Least Squares   F-statistic:                     92.07
Date:                Fri, 02 Oct 2020   Prob (F-statistic):           4.04e-11
Time:                        20:31:04   Log-Likelihood:                -134.61
No. Observations:                  24   AIC:                             275.2
Df Residuals:                      21   BIC:                             278.8
Df Model:                           2
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const       1798.4040    899.248      2.000      0.059     -71.685    3668.493
x_1         -250.1466    117.950     -2.121      0.046    -495.437      -4.856
x_2          345.5401    111.367      3.103      0.005     113.940     577.140
==============================================================================
```

Something more about the coding

- Multivariate linear regression in R:

```
lm(y ~ x_1 + x_2 + x_3, data=dataset)
```

- Multivariate linear regression in Python:
  - ```
    import statsmodels.api as sm
    import statsmodels
    ```
  - Intercept is not set as default!
    ```
    X = sm.add_constant(X)
    ```
  - ```
    model = sm.OLS(Y, X).fit()
    ```