Gaussian SLR: Inference and Prediction

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Classical inference steps:

Suppose iid $Y_i \sim \mathcal{N}(\mu, \sigma^2)$

- Estimation of parameters of observed data:

$$\widehat{\mu} = \bar{y} = \tfrac{1}{n}\textstyle\sum_{i=1}^{n} y_i \qquad \widehat{\sigma}^2 = \frac{1}{n-1}\textstyle\sum_{i=1}^{n}(y_i - \bar{y})^2$$

- "Distribution" of the statistics:

$$\widehat{\mu} = \bar{Y} = \tfrac{1}{n}\textstyle\sum_{i=1}^{n} Y_i \sim \mathcal{N}(\mu, \tfrac{\sigma^2}{n}) \qquad \widehat{\sigma}^2 = \tfrac{1}{n-1}\textstyle\sum_{i=1}^{n}(Y_i - \bar{Y})^2 \sim \tfrac{\sigma^2}{n-1}\chi^2(n-1)$$

- Relationship:

$$\bar{Y} \perp \frac{1}{n-1}\textstyle\sum_{i=1}^{n}(Y_i - \bar{Y})^2 \Rightarrow \dfrac{\sqrt{n}(\bar{Y}-\mu)}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(Y_i-\bar{Y})^2}} \sim t(n-1)$$

- Continue with confidence interval and hypothesis testing, etc.

(Simple) linear regression procedures:

X - predictor (random) variable **Y - response random variable**

- Build your model:

  1) relationship: $Y = \beta_0 + X\beta_1 + \epsilon,\ \mathbb{E}[\epsilon] = 0,\ \mathrm{Var}[\epsilon] = \sigma^2 I_n$

  2) preference: choose $\widehat{\beta}_0, \widehat{\beta}_1$ to minimize $\mathbb{E}\left[\|Y - \beta_0 - X\beta_1\|^2\right]$

- Estimate your model parameters: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

  1) using observed data to express your preference: $\min_{\beta_0,\beta_1} Q := \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2$

  2) get parameters estimation for your model: $b_1 = \frac{(x - \bar{x}1_n)^\top (y - \bar{y}1_n)}{\|x - \bar{x}1_n\|^2}$ $b_0 = \bar{y} - \bar{x}b_1$

- Understand your model: both $\widehat{\beta}_0, \widehat{\beta}_1$ and b_0, b_1

  1) properties of estimations: $\widehat{\beta}_0 = \bar{Y} - \bar{X}\widehat{\beta}_1,\ \widehat{\beta}_1 = \frac{(X - \bar{X}1_n)^\top (Y - \bar{Y}1_n)}{\|X - \bar{X}1_n\|^2}$

  2) predictions: $\widehat{Y}_0 = \widehat{\beta}_0 + X_0\widehat{\beta}_1$ $\hat{y}_0 = b_0 + x_0 b_1$

Prediction and residual

$$b_1 = \frac{(x - \bar{x}1_n)^\top (y - \bar{y}1_n)}{\|x - \bar{x}1_n\|^2} \qquad b_0 = \bar{y} - \bar{x}b_1$$

- Prediction: $\hat{y}_i = b_0 + x_i b_1$
- Residual: $e_i = y_i - \hat{y}_i = y_i - b_0 - x_i b_1$
- Residual can be viewed as the estimation of unobservable error terms

$$\hat{\epsilon}_i = e_i = y_i - \hat{y}_i = y_i - b_0 - x_i b_1$$

- Estimation of $\widehat{\sigma}^2 = \mathrm{MSE} = \frac{\sum_{i=1}^n e_i^2}{n-2} = \frac{\sum_{i=1}^n (y_i - \hat{y}_i)^2}{n-2} = \frac{\|y - \hat{y}\|^2}{n-2}$

n-2?

SLR model with Gaussian errors:

$$Y = \beta_0 + X\beta_1 + \epsilon, \quad \epsilon \sim \mathcal{N}(0, \sigma^2 I_n)$$

- Given $X_1, \ldots, X_n$
- $Y_i | X_i = \beta_0 + X_i\beta_1 + \epsilon_i \sim \mathcal{N}(\beta_0 + X_i\beta_1, \sigma^2)$
- It is enough to model the error term in the linear relationship!
- Let's first focus on $\widehat{\beta}_0 = \bar{Y} - \bar{X}\widehat{\beta}_1, \; \widehat{\beta}_1 = \frac{(X - \bar{X}1_n)^\top (Y - \bar{Y}1_n)}{\|X - \bar{X}1_n\|^2}$

SLR model with Gaussian errors:

$$Y = \beta_0 + X\beta_1 + \epsilon, \;\; \epsilon \sim \mathcal{N}(0, \sigma^2 I_n)$$

- Express $\widehat{\beta}_1$

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- Express $\widehat{\sigma}^2_{\mathrm{LS}}$

SLR model with Gaussian errors:

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- Express $\widehat{\sigma}^2_{\mathrm{LS}}$

Hand-waving explanations for:

$$(\widehat{\beta}_0, \widehat{\beta}_1) \perp \widehat{\sigma}^2_{\mathrm{LS}}$$

- n i.i.d normal distributed error term
- using 2 degree of freedom to estimate $\widehat{\beta}_0, \widehat{\beta}_1$
- extra n-2 degree of freedom is in $\widehat{\sigma}^2_{\mathrm{LS}}$

Confidence interval for coefficient:

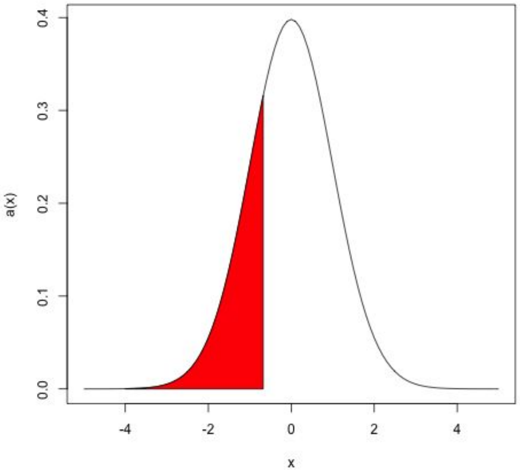
$$(\widehat{\beta}_0, \widehat{\beta}_1) \perp \widehat{\sigma}^2_{\mathrm{LS}}$$

$$\widehat{\beta}_1 \sim \mathcal{N}\left(\beta_1, \frac{\sigma^2}{\|x-\bar{x}1_n\|^2}\right)$$

$$\widehat{\beta}_0 \sim \mathcal{N}\left(\beta_0, \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{\|x-\bar{x}1_n\|^2}\right)\right)$$

$$\widehat{\sigma}^2_{\mathrm{LS}} \sim \frac{\sigma^2}{n-2}\chi^2(n-2)$$

- What to choose as a statistics?



Confidence interval for coefficient:

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$$\widehat{\sigma}^2_{\text{LS}} \sim \frac{\sigma^2}{n-2}\chi^2(n-2)$$

- Choice of statistics:

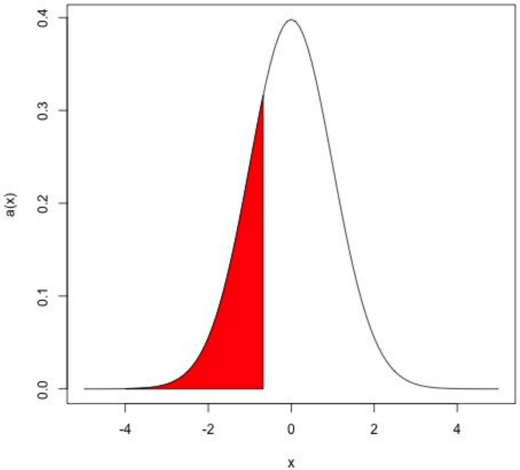
$$\sqrt{\frac{\|x-\bar{x}1_n\|^2}{\widehat{\sigma}^2}}\left(\widehat{\beta}_1 - \beta_1\right) \sim t(n-2), \quad \sqrt{\frac{\left(\frac{1}{n} + \frac{\bar{x}^2}{\|x-\bar{x}1_n\|^2}\right)^{-1}}{\widehat{\sigma}^2}}\left(\widehat{\beta}_0 - \beta_0\right) \sim t(n-2)$$

- (100-α)% confidence intervals for ground truth:

$$\left[\widehat{\beta}_1 - \sqrt{\frac{\widehat{\sigma}^2}{\|x-\bar{x}1_n\|^2}}\, t(1-\tfrac{\alpha}{2}; n-2), \widehat{\beta}_1 + \sqrt{\frac{\widehat{\sigma}^2}{\|x-\bar{x}1_n\|^2}}\, t(1-\tfrac{\alpha}{2}; n-2)\right]$$

$$\left[\widehat{\beta}_0 - \sqrt{\widehat{\sigma}^2\left(\frac{1}{n} + \frac{\bar{x}^2}{\|x-\bar{x}1_n\|^2}\right)}\, t(1-\tfrac{\alpha}{2}; n-2), \widehat{\beta}_0 + \sqrt{\widehat{\sigma}^2\left(\frac{1}{n} + \frac{\bar{x}^2}{\|x-\bar{x}1_n\|^2}\right)}\, t(1-\tfrac{\alpha}{2}; n-2)\right]$$

- where define $t(q; n-2)$ as $P(Z \le t(q; n-2)) = q$



Confidence interval for estimation:

$$(\widehat{\beta}_0, \widehat{\beta}_1) \perp \widehat{\sigma}^2_{\mathrm{LS}}$$

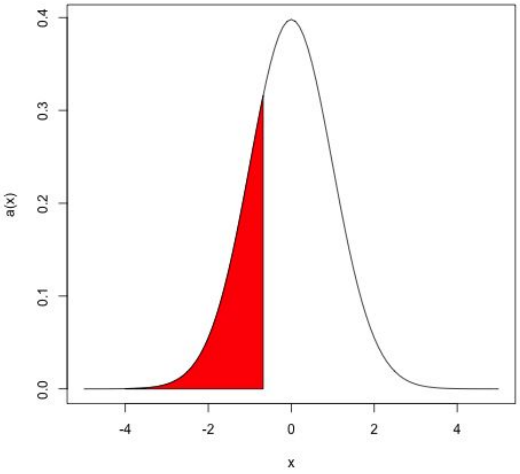
$$\widehat{\beta}_1 \sim \mathcal{N}\left(\beta_1, \frac{\sigma^2}{\|x-\bar{x}1_n\|^2}\right)$$

$$\widehat{\beta}_0 \sim \mathcal{N}\left(\beta_0, \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{\|x-\bar{x}1_n\|^2}\right)\right)$$

$$\widehat{\sigma}^2_{\mathrm{LS}} \sim \frac{\sigma^2}{n-2}\chi^2(n-2)$$

- Choice of statistics:

- (100-α)% confidence intervals for ground truth:



Confidence interval for prediction:

$$(\widehat{\beta}_0, \widehat{\beta}_1) \perp \widehat{\sigma}^2_{\mathrm{LS}}$$

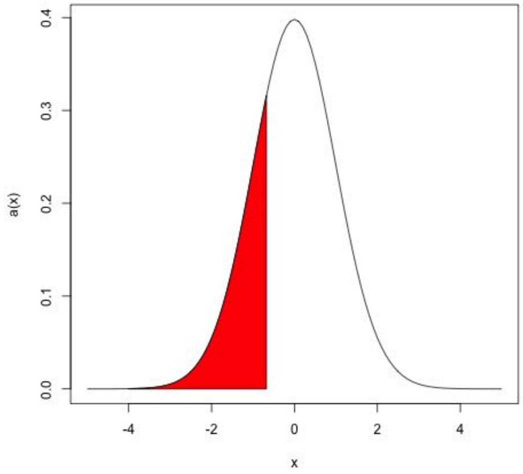
$$\widehat{\beta}_1 \sim \mathcal{N}\left(\beta_1, \frac{\sigma^2}{\|x - \bar{x}1_n\|^2}\right)$$

$$\widehat{\beta}_0 \sim \mathcal{N}\left(\beta_0, \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{\|x - \bar{x}1_n\|^2}\right)\right)$$

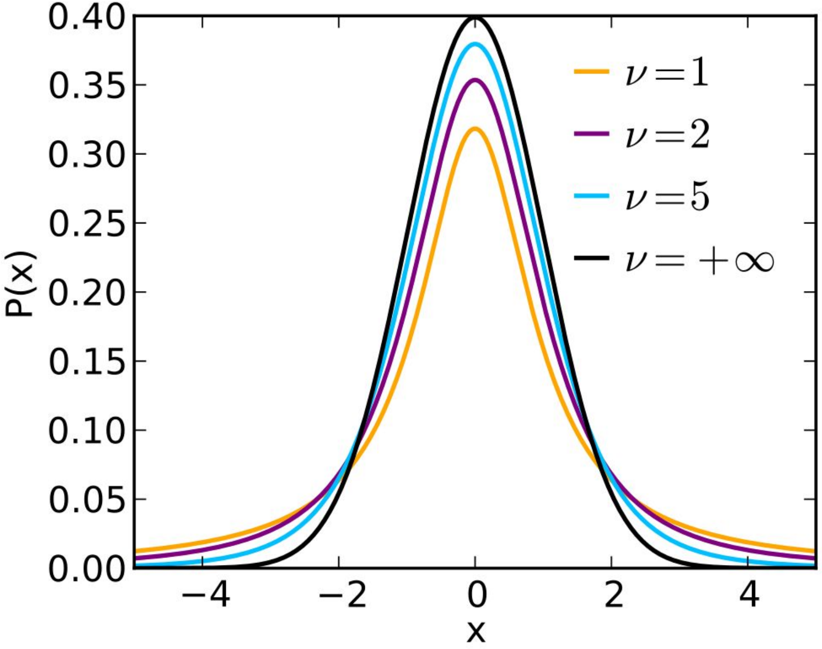
$$\widehat{\sigma}^2_{\mathrm{LS}} \sim \frac{\sigma^2}{n-2}\chi^2(n-2)$$

- Choice of statistics:

- (100-***α***)% confidence intervals for ground truth:



Asymptotically normality



t Table

cum. prob	$t_{.50}$	$t_{.75}$	$t_{.80}$	$t_{.85}$	$t_{.90}$	$t_{.95}$	$t_{.975}$	$t_{.99}$	$t_{.995}$	$t_{.999}$	$t_{.9995}$
one-tail	**0.50**	**0.25**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	**0.001**	**0.0005**
two-tails	**1.00**	**0.50**	**0.40**	**0.30**	**0.20**	**0.10**	**0.05**	**0.02**	**0.01**	**0.002**	**0.001**
df											
1	0.000	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	318.31	636.62
2	0.000	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	22.327	31.599
3	0.000	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.000	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.000	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.000	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.000	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.000	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.000	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	0.000	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.000	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	0.000	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.000	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.000	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	0.000	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.000	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.000	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	0.000	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	0.000	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.000	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.000	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.000	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.000	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.000	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	0.000	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	0.000	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.000	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.000	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.000	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.000	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	0.000	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	0.000	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	3.232	3.460
80	0.000	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	3.195	3.416
100	0.000	0.677	0.845	1.042	1.290	1.660	1.984	2.364	2.626	3.174	3.390
1000	0.000	0.675	0.842	1.037	1.282	1.646	1.962	2.330	2.581	3.098	3.300
z	0.000	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	3.090	3.291
	0%	50%	60%	70%	80%	90%	95%	98%	99%	99.8%	99.9%
	Confidence Level										

References and further reading

- Kutner, Nachtsheim, Neter: *Applied Linear Regression Models* Chapter 2
- Agresti: *Foundations of Linear and Generalized Linear Models* Chapter 2&3