第七章 带电粒子和电磁场的相互作用

（1）已知粒子运动求电磁场

（2）粒子激发电磁场对粒子的反作用

（3）粒子与外电磁场的相互作用

§1 运动带电粒子的势和辐射电磁场

1. 任意运动带电粒子的势

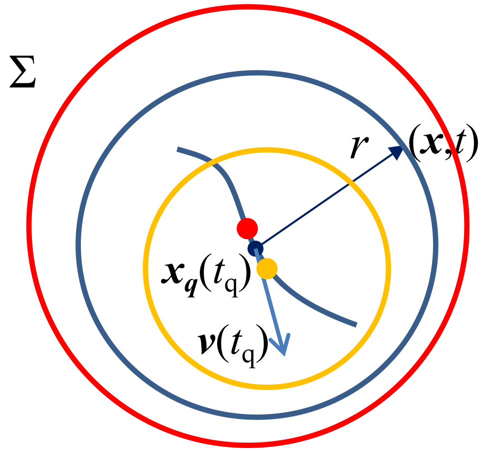


图1. 电荷运动轨迹$\boldsymbol{x}_q$和场点$(\boldsymbol{x},t)$.
红圈、蓝圈、黄圈分别表示电荷在红点、蓝点、黄点激发的电磁场在t时刻所到达的球面. 三个球面不可能有交点。

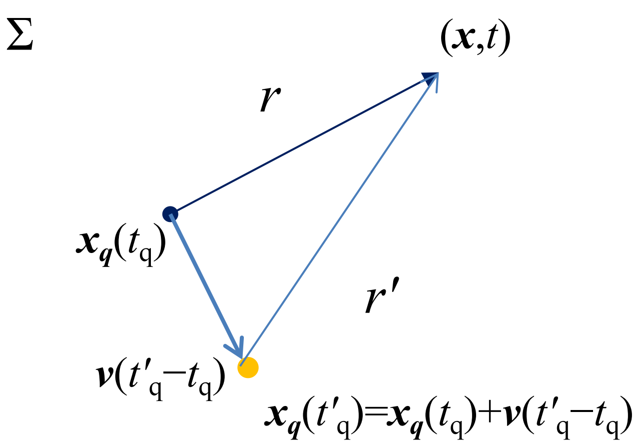
电荷在t_q激发的电磁场局限在以$\boldsymbol{x}_q(t_q)$为圆心、以光速扩张的无穷薄球面上

$$r \equiv |\boldsymbol{x} - \boldsymbol{x}_q(t_q)| = c(t - t_q)$$

因为光速与发光电荷的运动无关，而电荷的速度小于光速，所以三个球面不可能有交点，即对给定的$\boldsymbol{x}$和t只有唯一的t_q 满足下式

$$|\boldsymbol{x}-\boldsymbol{x}_q(t_q)|=c(t-t_q) \qquad (1)$$

证明：设 $r=c(t-t_q)$ ，$r'=|\boldsymbol{x}-\boldsymbol{x}_q(t'_q)|$ ， $t'_q>t_q$.



因为三角形两边和大于第三边，

$$r'+v(t'_q-t_q)>r=c(t-t_q)$$

所以

$$r'>c(t-t_q)-v(t'_q-t_q)$$
$$=c(t-t'_q)+(c-v)(t'_q-t_q)$$
$$>c(t-t'_q)$$

所以，$\boldsymbol{x}_q(t'_q)$发出的电磁波不可能在t 时刻达到$\boldsymbol{x}$点。

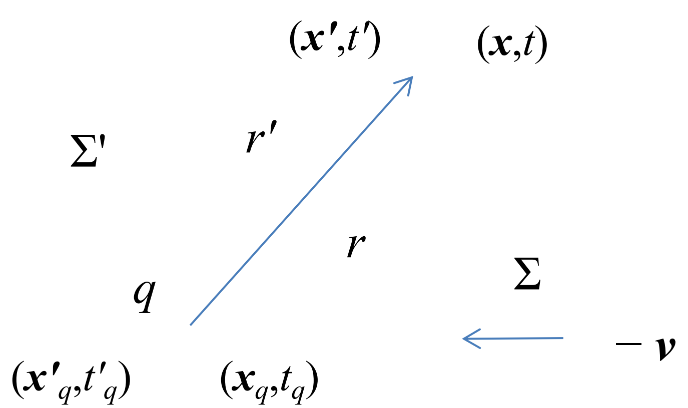
场方程

$$(\partial_\nu \partial^\nu) A^\mu - \partial^\mu (\partial_\nu A^\nu) = -\mu_0 j^\mu$$

带电物质仅通过四维电流密度与电磁势发生相互作用，所以带电物质激发的电磁势与电荷加速度无关。从而在电荷瞬时静止的惯性系Σ'中，电磁势等于静电荷激发的静电势（已取无穷远处为零的边界条件）

$$A'_a(r') = \left(0,0,0,\frac{i}{c}\frac{q}{4\pi\varepsilon_0 r'}\right) \tag{2}$$

q可以有加速度



注意：$r' \neq r$

设在实验室惯性系Σ的t_q时刻电荷具有瞬时速度$\boldsymbol{v}(t_q)$，则Σ相对Σ′以速度$-\boldsymbol{v}(t_q)$运动. 不妨设运动沿X轴方向. 由典型洛伦兹变换得Σ系中的电磁势

$$\begin{pmatrix} A^1 \\ A^2 \\ A^3 \\ \frac{i}{c}\varphi \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\frac{iv}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{iv}{c}\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{i}{c}\frac{q}{4\pi\varepsilon_0 r'} \end{pmatrix} = \gamma(v) \begin{pmatrix} \frac{qv}{4\pi\varepsilon_0 c^2 r'} \\ 0 \\ 0 \\ \frac{i}{c}\frac{q}{4\pi\varepsilon_0 r'} \end{pmatrix} \tag{3}$$

其中 $v = v(t_q)$，和 $\gamma(v) = \dfrac{1}{\sqrt{1 - v^2(t_q)/c^2}}$

$$t' - t'_q = \frac{t - t_q - \frac{v}{c^2}(x^1 - x_q^1)}{\sqrt{1-(v/c)^2}}$$

根据光速不变原理，在Σ′系源点与场点的距离

$$r' = c(t' - t'_q) = \frac{c(t - t_q) - \frac{1}{c}v(x^1 - x_q^1)}{\sqrt{1 - v^2 / c^2}} = \frac{r - \frac{1}{c}v(x^1 - x_q^1)}{\sqrt{1 - v^2 / c^2}}$$

通过空间转动，可以从电荷沿X轴运动的结果得到电荷任意运动的结果。

存在空间转动三维正交矩阵 $\hat{O}$， $\hat{O}\hat{O}^T = \hat{O}^T\hat{O} = I$ ，使得

$$\begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} \xrightarrow{\hat{O}} \begin{pmatrix} v''^1 \\ 0 \\ 0 \end{pmatrix} = \hat{O}\begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix}, \qquad \begin{pmatrix} x^1 - x_q^1 \\ x^2 - x_q^2 \\ x^3 - x_q^3 \end{pmatrix} \xrightarrow{\hat{O}} \begin{pmatrix} x''^1 - x_q''^1 \\ x''^2 - x_q''^2 \\ x''^3 - x_q''^3 \end{pmatrix} = \hat{O}\begin{pmatrix} x^1 - x_q^1 \\ x^2 - x_q^2 \\ x^3 - x_q^3 \end{pmatrix}$$

从而

$$r' = \frac{r - \frac{1}{c}v''^1(x''^1 - x_q''^1)}{\sqrt{1 - v^2 / c^2}} = \frac{r - \frac{1}{c}\boldsymbol{v}^T\hat{O}^T\hat{O}\boldsymbol{r}}{\sqrt{1 - v^2 / c^2}} = \frac{r - \frac{1}{c}\boldsymbol{v}^T\boldsymbol{r}}{\sqrt{1 - v^2 / c^2}} = \frac{r - \frac{1}{c}\boldsymbol{v} \cdot \boldsymbol{r}}{\sqrt{1 - v^2 / c^2}} \qquad (4)$$

其中 $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}_q$和$r = \sqrt{(\boldsymbol{r}\cdot\boldsymbol{r})} = c(t - t_q)$. 右边表达式在空间坐标转动下不变，因此它对任意选取的笛卡尔坐标架普遍地成立.

同理可以通过空间转动，从电荷瞬时沿X轴运动的结果(3)式和(4)得到任意运动电荷辐射的电磁势。

李纳–维谢尔势

$$\boldsymbol{A} = \frac{q\boldsymbol{v}}{4\pi\varepsilon_0 c^2 (r - \frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{r})} \tag{5a}$$

$$\varphi = \frac{q}{4\pi\varepsilon_0 (r - \frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{r})} \tag{5b}$$

复杂性包含在$\boldsymbol{v}$和$\boldsymbol{r}$中，他们均依赖于t_q，而后者是$\boldsymbol{x}$和t 的隐函数。所以求$\boldsymbol{E}$和$\boldsymbol{B}$时涉及隐函数求导。

2. 连续分布带电体的势(推迟势)

考虑带电粒子集合$\{q_n|n=1,2,3...\}$，q_n的轨迹为$\boldsymbol{x}_n(t')$. 记$\boldsymbol{r}_n(t')=\boldsymbol{x}-\boldsymbol{x}_n(t')$.

$$r_n\frac{\partial r_n}{\partial t'}=-\frac{d\boldsymbol{x}_n}{dt'}\cdot\boldsymbol{r}_n=-\boldsymbol{v}_n(t')\cdot\boldsymbol{r}_n,\qquad (\text{给定}\boldsymbol{x}\text{和}t)$$

$$r_n-\tfrac{1}{c}\boldsymbol{v}_n\cdot\boldsymbol{r}_n=r_n+r_n\frac{\partial r_n}{\partial t'}=r_n\frac{\partial}{\partial t'}\left(t'-t+\tfrac{1}{c}r_n\right)\tag{6}$$

设 q_n 在 t_n 时刻发射的电磁波在t 时刻达到场点 $\boldsymbol{x}$ 即 $\boldsymbol{r}_n(t_n)=c(t-t_n)$.

可利用(6)式把q_n激发的李纳-维谢尔矢势写成

$$\boldsymbol{A}_n(\boldsymbol{x},t)=\frac{q_n\boldsymbol{v}_n(t_n)}{4\pi\varepsilon_0c^2(r_n-\frac{1}{c}\boldsymbol{v}_n(t_n)\cdot\boldsymbol{r}_n)}=\frac{q_n\boldsymbol{v}_n(t_n)}{4\pi\varepsilon_0c^2r_n\frac{\partial}{\partial t_n}(t_n-t+\frac{1}{c}r_n(t_n))}\tag{7}$$

为了方便对n求和，改写(7)式（目的是消去分母对n的依赖），

$$\boldsymbol{A}_n(\boldsymbol{x},t)=\int dV'\int dt'\frac{\delta(t'-t_n)}{\frac{\partial}{\partial t_n}(t_n-t+\frac{1}{c}r_n(t_n))}\frac{\delta(\boldsymbol{x}'-\boldsymbol{x}_n(t'))q_n\boldsymbol{v}_n(t')}{4\pi\varepsilon_0c^2|\boldsymbol{x}-\boldsymbol{x}'|}\tag{8}$$

记：$f(t')=t'-t+r_n(t')/c$

因为 $r_n(t_n)=c(t-t_n)$，所以 $f(t_n)=0$，即 t_n 的是方程 $f(t')=0$ 的根，而且前面已经分析过，这个根是唯一的。从而

$$\delta(f(t'))=\delta\left(\left.\frac{\partial f(t')}{\partial t'}\right|_{t'=t_n}(t'-t_n)\right)=\frac{1}{\left.\frac{\partial f(t')}{\partial t'}\right|_{t'=t_n}}\delta(t'-t_n) \tag{9}$$

利用(9)式并记 $r=|\boldsymbol{x}-\boldsymbol{x}'|$，(8)式写成

$$\begin{aligned}
\boldsymbol{A}_n&=\int dV'\int dt'\delta(t'-t+\tfrac{1}{c}r_n(t'))\frac{\delta(\boldsymbol{x}'-\boldsymbol{x}_n(t'))q_n\boldsymbol{v}_n(t')}{4\pi\varepsilon_0c^2|\boldsymbol{x}-\boldsymbol{x}'|}\\
&=\int dV'\int dt'\delta(t'-t+\tfrac{1}{c}|\boldsymbol{x}-\boldsymbol{x}'|)\frac{\delta(\boldsymbol{x}'-\boldsymbol{x}_n(t'))q_n\boldsymbol{v}_n(t')}{4\pi\varepsilon_0c^2|\boldsymbol{x}-\boldsymbol{x}'|}\\
&=\int dV'\frac{\delta(\boldsymbol{x}'-\boldsymbol{x}_n(t-\frac{1}{c}r))q_n\boldsymbol{v}_n(t-\frac{1}{c}r)}{4\pi\varepsilon_0c^2r}
\end{aligned} \tag{10}$$

回忆电流密度

$$\boldsymbol{J}(\boldsymbol{x},t)=\sum_{n}q_{n}\delta(\boldsymbol{x}-\boldsymbol{x}_{n}(t))\boldsymbol{v}_{n}(t) \tag{11}$$

从而总矢势

$$\boldsymbol{A}(\boldsymbol{x},t)=\sum_{n}\boldsymbol{A}_{n}=\int dV'\frac{\boldsymbol{J}(\boldsymbol{x}',t-\frac{1}{c}r)}{4\pi\varepsilon_{0}c^{2}r} \tag{12}$$

类似可得标势

$$\varphi(\boldsymbol{x},t)=\int dV'\frac{\rho(\boldsymbol{x}',t-\frac{1}{c}r)}{4\pi\varepsilon_{0}r} \tag{13}$$

公式(12)和(13)式正是推迟势公式，它们也可以在罗伦芝规范条件下直接从麦克斯韦方程方程组推导出来. 推迟势反映了真空中光信号传播速度恒等于c.

3. 辐射电磁场

数学准备。记

$$s \equiv r - \tfrac{1}{c}\boldsymbol{v}\cdot\boldsymbol{r} \tag{14}$$

则(4)式给出

$$r' = \gamma(v)s \tag{15}$$

对给定的$\boldsymbol{x}$和t，t_q 唯一，故存在单值函数$t_q = t_q(\boldsymbol{x},t)$. 可证下面偏导数(附录1），

$$\frac{\partial t_q}{\partial t} = \frac{r}{s}, \qquad \left.\frac{\partial \boldsymbol{r}}{\partial t}\right|_{\boldsymbol{x}} = -\frac{r}{s}\boldsymbol{v}, \qquad \left.\frac{\partial r}{\partial t}\right|_{\boldsymbol{x}} = -\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{s}, \tag{16a}$$

$$\frac{\partial t_q}{\partial x^j} = -\frac{r^j}{cs}, \qquad \left.\frac{\partial \boldsymbol{r}}{\partial x^j}\right|_t = \hat{\boldsymbol{e}}_j + \frac{r^j}{cs}\boldsymbol{v}, \qquad \left.\frac{\partial r}{\partial x^j}\right|_t = \frac{r^j}{s} \tag{16b}$$

利用上述数学关系式，计算李纳-维谢尔势对应的场强，可得(附录2，3)

$$\boldsymbol{E} = -\nabla\varphi - \frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{E}_c + \boldsymbol{E}_{rad}, \qquad \boldsymbol{B} = \nabla\times\boldsymbol{A} = \boldsymbol{B}_c + \boldsymbol{B}_{rad} \tag{17}$$

非辐射场（最右表达式作了低速近似）

$$\boldsymbol{E}_c = \frac{q}{4\pi\varepsilon_0 r^2}\frac{\hat{\boldsymbol{e}}_r - \frac{1}{c}\boldsymbol{v}}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^3\gamma^2} \sim \frac{q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{e}}_r \tag{18}$$

$$\boldsymbol{B}_c = \frac{q}{4\pi\varepsilon_0 c^2 r^2}\frac{(1-\frac{v^2}{c^2})\boldsymbol{v}\times\hat{\boldsymbol{e}}_r}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^3} \sim \frac{q}{4\pi\varepsilon_0 c^2 r^2}\boldsymbol{v}\times\hat{\boldsymbol{e}}_r \tag{19}$$

辐射场（最右表达式作了低速近似）

$$\boldsymbol{E}_{rad} = \frac{q}{4\pi\varepsilon_0 c^2 r}\frac{\hat{\boldsymbol{e}}_r\times[(\hat{\boldsymbol{e}}_r - \frac{1}{c}\boldsymbol{v})\times\dot{\boldsymbol{v}}]}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^3} \sim \frac{q}{4\pi\varepsilon_0 c^2 r}\hat{\boldsymbol{e}}_r\times(\hat{\boldsymbol{e}}_r\times\dot{\boldsymbol{v}}) \tag{20}$$

$$\boldsymbol{B}_{rad} = \frac{q}{4\pi\varepsilon_0 c^3 r}\frac{\dot{\boldsymbol{v}}\times\hat{\boldsymbol{e}}_r + \frac{1}{c}[(\dot{\boldsymbol{v}}\cdot\hat{\boldsymbol{e}}_r)(\boldsymbol{v}\times\hat{\boldsymbol{e}}_r) - (\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)(\dot{\boldsymbol{v}}\times\hat{\boldsymbol{e}}_r)]}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^3} = \frac{1}{c}\hat{\boldsymbol{e}}_r\times\boldsymbol{E}_{rad} \sim \frac{q}{4\pi\varepsilon_0 c^3 r}\dot{\boldsymbol{v}}\times\hat{\boldsymbol{e}}_r \tag{21}$$

辐射场的特点：

辐射场强以$1/r$衰减，而非辐射场强以$1/r^2$衰减.

与 $|\dot{\boldsymbol{v}}|$ 成正比

$\boldsymbol{E}_{rad}$ 和 $\boldsymbol{B}_{rad}$ 互相正交并分别与 $\boldsymbol{r}$ 正交（横场。在自由空间的情形）

可直接验证

$$\boldsymbol{B}_{rad} = \frac{1}{c}\hat{\boldsymbol{e}}_r \times \boldsymbol{E}_{rad} \tag{22}$$

§ 2 高速运动带电粒子的辐射

1. 高速运动带电粒子的辐射功率和角分布

辐射场的能量密度

$$\boldsymbol{S}=\boldsymbol{E}\times\boldsymbol{H}=\frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B}=\frac{1}{\mu_0 c}\boldsymbol{E}\times(\hat{\boldsymbol{e}}_r\times\boldsymbol{E})=\varepsilon_0 cE^2\hat{\boldsymbol{e}}_r \tag{23}$$

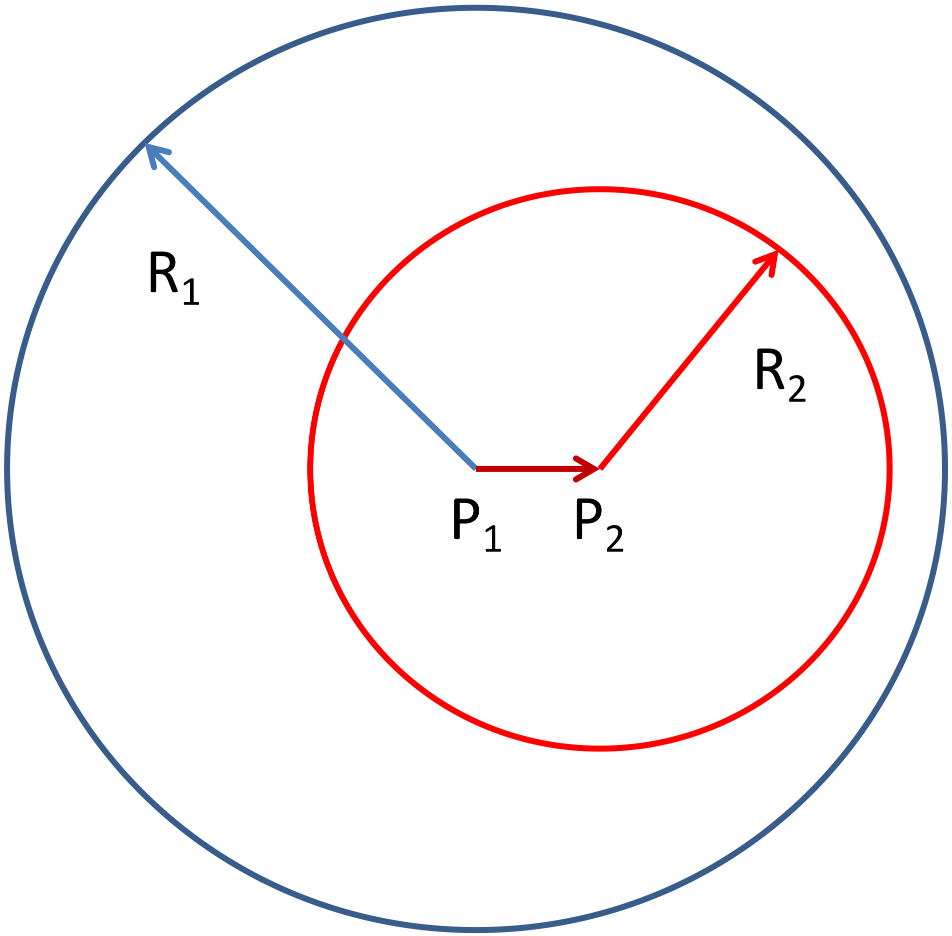
辐射场

$$\boldsymbol{E}=\frac{q}{4\pi\varepsilon_0 c^2 r}\frac{\hat{\boldsymbol{e}}_r\times[(\hat{\boldsymbol{e}}_r-\frac{1}{c}\boldsymbol{v})\times\dot{\boldsymbol{v}}]}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^3} \tag{24}$$

故 t 时刻 $\boldsymbol{x}$ 处（场点）

$$\boldsymbol{S}=\frac{q^2}{16\pi^2\varepsilon_0 c^3 r^2}\frac{\left|\hat{\boldsymbol{e}}_r\times[(\hat{\boldsymbol{e}}_r-\frac{1}{c}\boldsymbol{v})\times\dot{\boldsymbol{v}}]\right|^2}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^6}\hat{\boldsymbol{e}}_r \tag{25}$$

在源点附近S很复杂，所以一般不在源点附近求辐射功率.



P_1: 电荷在t'时刻所在位置

P_2: 电荷在t'+dt'时刻所在位置

在时刻t，P_1处发出的辐射达到半径为R_1的球面

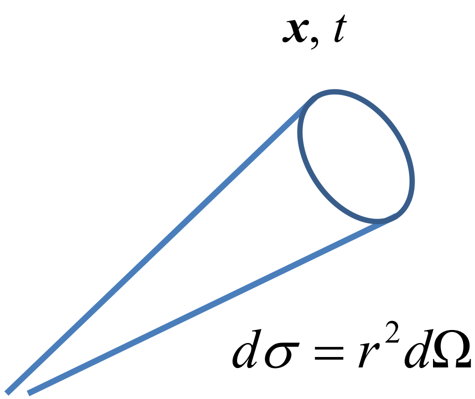
在时刻t，P_2处发出的辐射达到半径为R_2的球面

$$R_1 = c(t-t') \qquad R_2 = c(t-t'-dt')$$

$$\overline{P_1P_2} = v(t')dt'$$

电荷在dt'时间辐射的能量等于两球面之间的辐射场能量

辐射功率角分布 [方法一]



因为$\boldsymbol{x}_q(t')$已知，从 $|\boldsymbol{x}-\boldsymbol{x}_q(t')| = c(t-t')$ 可解出

$$t = t' + \tfrac{1}{c}|\boldsymbol{x}-\boldsymbol{x}_q(t')| \equiv t(\boldsymbol{x}, t')$$

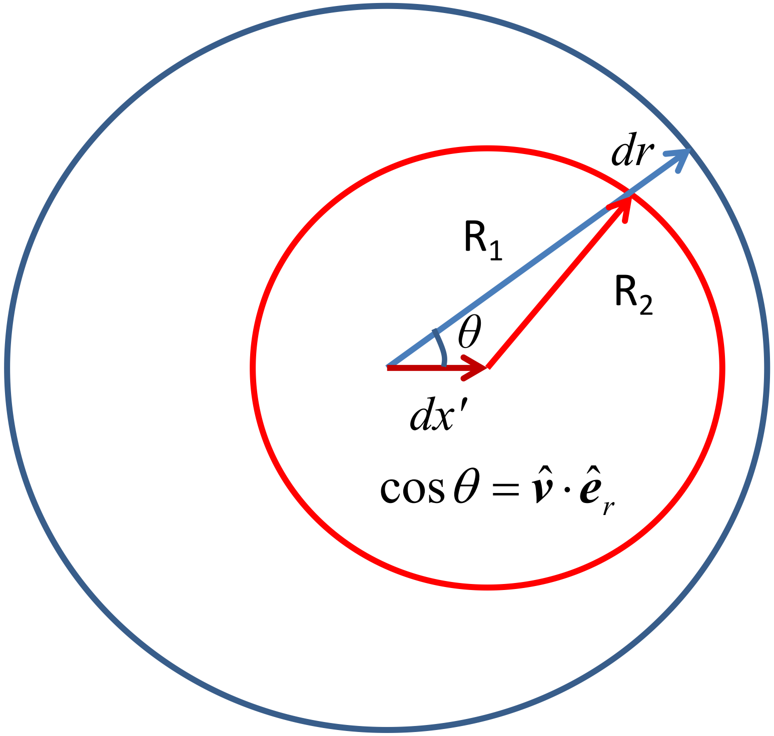
电荷在dt'时间产生的辐射经过$\boldsymbol{x}$处面元$d\sigma$的时间为

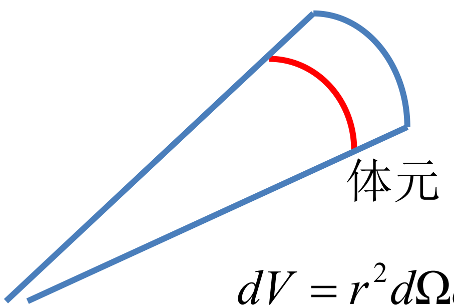
$$dt = \left.\frac{\partial t}{\partial t'}\right|_{\boldsymbol{x}} dt' = \left(\left.\frac{\partial t'}{\partial t}\right|_{\boldsymbol{x}}\right)^{-1} dt' = (1-\tfrac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)dt' \qquad (26)$$

记电荷辐射功率角分布为$P(t',\Omega)$，即在dt'时间内辐射到dΩ立体角方向的能量为$P(t',\Omega)dt'd\Omega$ （课本省略了变量Ω）. 电荷在dt' 辐射到dΩ的能量在t~$t+dt$通过$\boldsymbol{x}$处的dσ面元，

$$P(t',\Omega)dt'd\Omega = Sdtd\sigma$$

$$= Sr^2(1-\tfrac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)dt'd\Omega = \frac{q^2}{16\pi^2\varepsilon_0 c^3}\frac{\left|\hat{\boldsymbol{e}}_r\times[(\hat{\boldsymbol{e}}_r-\tfrac{1}{c}\boldsymbol{v})\times\dot{\boldsymbol{v}}]\right|^2}{(1-\tfrac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^5}dt'd\Omega \qquad (27)$$





$$dV = r^2 d\Omega dr$$

辐射功率角分布[方法二]

当 $dt' \to 0$ 时， $dx' = vdt'$

设dt'辐射到θ方向的场到达$\boldsymbol{x}$处厚为dr的壳层.

$$R_1 = c(t-t') = r,$$

$$R_2 = c(t-t'-dt') = r - cdt'$$

$$R_2^2 = (R_1 - dr)^2 + dx'^2 - 2(R_1 - dr)dx'\cos\theta$$

从而（只需比较一阶无穷小量）

$$dr = (c - v\cos\theta)dt' = c(1 - \tfrac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)dt'$$

在 t 时刻 $\boldsymbol{x}$ 处的辐射能量密度 $w = \varepsilon_0 E^2$

立体角$d\Omega$内dr厚体元dV的辐射能量即

$$P(t',\Omega)dt'd\Omega = wdV = \varepsilon_0 E^2(\boldsymbol{x},t)r^2 d\Omega dr$$

$$= \varepsilon_0 c E^2(\boldsymbol{x},t)r^2(1 - \tfrac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)dt'd\Omega$$

它等于dt'辐射到$d\Omega$的能量.

与(27)一致

辐射功率

记电荷辐射功率为$P(t')$.

$$P(t') = \int P(t',\Omega)d\Omega \tag{28}$$

辐射功率角分布为

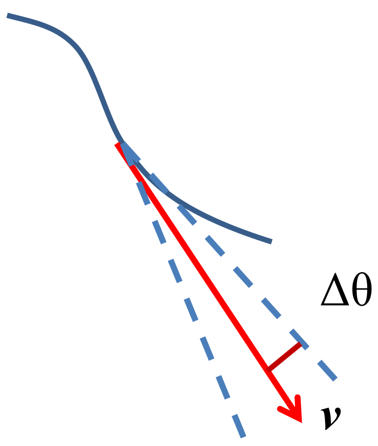
$$\frac{dP(t')}{d\Omega} \equiv P(t',\Omega) = \frac{q^2}{16\pi^2\varepsilon_0 c^3}\frac{\left|\hat{\boldsymbol{e}}_r \times [(\hat{\boldsymbol{e}}_r - \frac{1}{c}\boldsymbol{v})\times\dot{\boldsymbol{v}}]\right|^2}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^5} \tag{29}$$

故辐射功率为

$$P(t') = \frac{q^2}{16\pi^2\varepsilon_0 c^3}\int\frac{\left|\hat{\boldsymbol{e}}_r \times [(\hat{\boldsymbol{e}}_r - \frac{1}{c}\boldsymbol{v})\times\dot{\boldsymbol{v}}]\right|^2}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^5}d\Omega \tag{30}$$

高速运动: $v/c \sim 1$

$$1-\tfrac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r = 1-\tfrac{v}{c}\cos\theta$$



当$\theta \sim 0$时 $1-\tfrac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r \approx \tfrac{1}{2}(\gamma^{-2}+\theta^2)$

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0 c^3}\frac{\left|\hat{\boldsymbol{e}}_r\times[(\hat{\boldsymbol{e}}_r-\tfrac{1}{c}\boldsymbol{v})\times\dot{\boldsymbol{v}}]\right|^2}{(1-\tfrac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^5}$$

上式分母很小

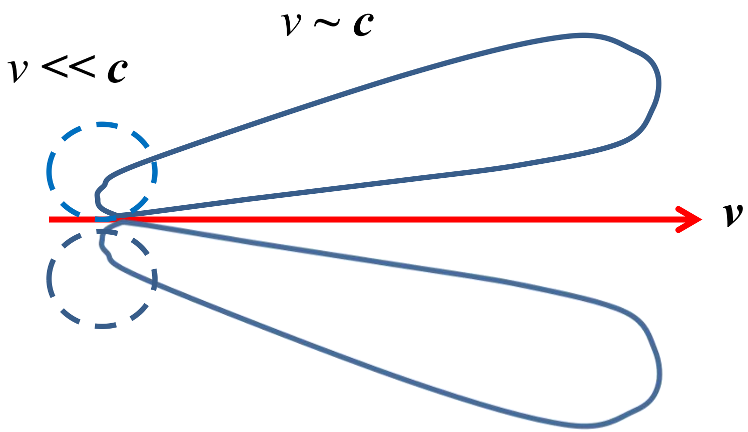
$$\frac{1}{(1-\tfrac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^5} \approx \frac{32}{(\gamma^{-2}+\theta^2)^5} \tag{31}$$

当$\theta > \gamma^{-1}$，辐射功率迅速减少，从而射功率集中在粒子运动方向（此因子不涉及加速度方向）

$$\Delta\theta \sim \frac{1}{\gamma} \tag{32}$$

2. 加速度平行速度情形

例如直线加速器



辐射功率角分布

$$\frac{dP(t')}{d\Omega}=\frac{q^2\dot{\boldsymbol{v}}^2}{16\pi^2\varepsilon_0 c^3}\frac{\sin^2\theta}{\left(1-\frac{v}{c}\cos\theta\right)^5} \qquad (33)$$

辐射功率

$$P(t')=\frac{q^2\dot{\boldsymbol{v}}^2}{16\pi^2\varepsilon_0 c^3}\int\frac{\sin^2\theta}{\left(1-\frac{v}{c}\cos\theta\right)^5}d\Omega$$

$$=\frac{q^2\dot{v}^2}{6\pi\varepsilon_0 c^3}\gamma^6 \qquad (34)$$

电荷受力（总力）

$$\boldsymbol{F}(t') = \frac{d(\gamma m\boldsymbol{v})}{dt'}$$

$$= m\frac{\dot{\boldsymbol{v}}(1-(\frac{v}{c})^2) + \frac{1}{c^2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\boldsymbol{v}}{(1-(\frac{v}{c})^2)^{3/2}} = \gamma^3 m\dot{\boldsymbol{v}} \tag{35}$$

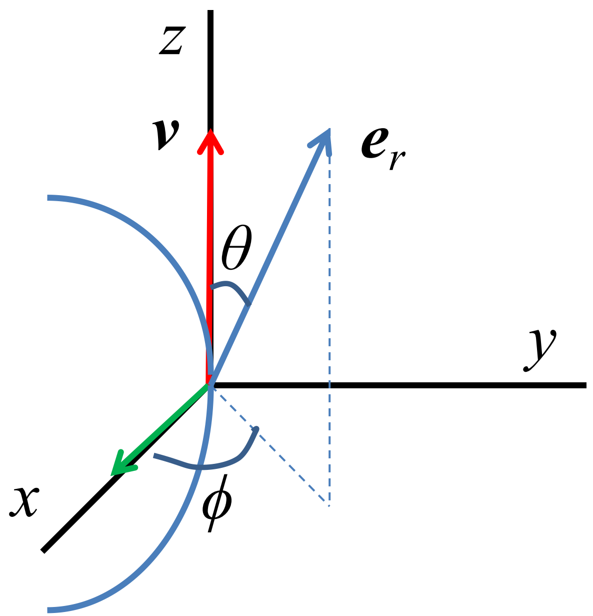
从而

$$P(t') = \frac{q^2}{6\pi\varepsilon_0 m^2 c^3}F^2 \tag{36}$$

直线加速运动电荷辐射功率只与其受力有关，而与粒子能量无关，由此可产生高能粒子.

3. 加速度垂直速度情形

例如回旋加速器的同步辐射



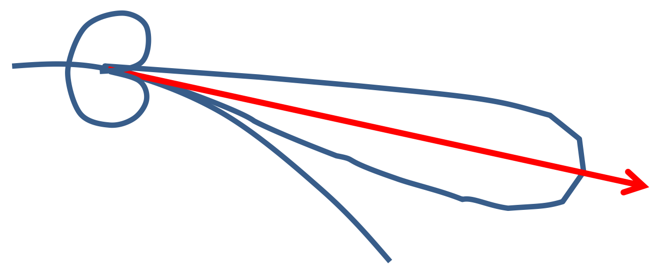
$$\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r = v\cos\theta$$

$$\dot{\boldsymbol{v}}\cdot\hat{\boldsymbol{e}}_r = \dot{v}\sin\theta\cos\phi$$

$$\dot{\boldsymbol{v}}\cdot\boldsymbol{v} = 0$$

$$\begin{aligned}&\hat{\boldsymbol{e}}_r\times[(\hat{\boldsymbol{e}}_r-\tfrac{1}{c}\boldsymbol{v})\times\dot{\boldsymbol{v}}]\\&=\dot{v}\sin\theta\cos\phi(\hat{\boldsymbol{e}}_r-\tfrac{1}{c}\boldsymbol{v})-(1-\tfrac{v}{c}\cos\theta)\dot{\boldsymbol{v}}\end{aligned}$$

$$\begin{aligned}&\left|\hat{\boldsymbol{e}}_r\times[(\hat{\boldsymbol{e}}_r-\tfrac{1}{c}\boldsymbol{v})\times\dot{\boldsymbol{v}}]\right|^2\\&=\dot{v}^2\left[(1-\tfrac{v}{c}\cos\theta)^2-(1-\tfrac{v^2}{c^2})\sin^2\theta\cos^2\phi\right]\end{aligned}$$



辐射功率角分布

$$\frac{dP(t')}{d\Omega}=\frac{q^2\dot{v}^2}{16\pi^2\varepsilon_0 c^3}\frac{(1-\frac{v}{c}\cos\theta)^2-(1-\frac{v^2}{c^2})\sin^2\theta\cos^2\phi}{(1-\frac{v}{c}\cos\theta)^5} \tag{37}$$

辐射功率

$$P(t')=\frac{q^2}{16\pi^2\varepsilon_0 c^3}\int\frac{dP(t')}{d\Omega}d\Omega=\frac{q^2\dot{v}^2}{6\pi\varepsilon_0 c^3}\gamma^4 \tag{38}$$

电荷受力（总力）

$$\boldsymbol{F}(t') = \frac{d(\gamma m\boldsymbol{v})}{dt'} = m\frac{\dot{\boldsymbol{v}}(1-(\frac{v}{c})^2) + \frac{1}{c^2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\boldsymbol{v}}{(1-(\frac{v}{c})^2)^{3/2}} = \gamma m\dot{\boldsymbol{v}} \tag{39}$$

辐射功率写成

$$\begin{aligned} P(t') &= \frac{q^2}{6\pi\varepsilon_0 m^2 c^3}\gamma^2 F^2 \\ &= \frac{q^2}{6\pi\varepsilon_0 m^4 c^7} E_q^2 F^2, \end{aligned} \tag{40}$$

电荷能量

$$E_q = \gamma mc^2. \tag{41}$$

当F给定后，辐射功率与电荷能量平方成正比. 当辐射功率等于输入功率时，粒子能量不能继续增加. 但由此产生的“同步辐射”是很有用的.

§ 3 辐射的频谱分析

1. 频谱分析的一般公式

傅里叶变换

$$f(t)=\int_{-\infty}^{\infty} f_\omega e^{-i\omega t}d\omega \qquad (42)$$

$$f_\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt \qquad (43)$$

若$f(t)$为实函数，

$$f_\omega^*=f_{-\omega} \qquad (44)$$

$$\int_{-\infty}^{\infty} f^2(t)dt=4\pi\int_0^{\infty}\left|f_\omega\right|^2 d\omega \qquad (45)$$

例

（1）电流密度

$$\boldsymbol{J}(\boldsymbol{x},t)=\int_{-\infty}^{\infty}\boldsymbol{J}_{\omega}(\boldsymbol{x})e^{-i\omega t}d\omega,\qquad \boldsymbol{J}_{\omega}(\boldsymbol{x})=\frac{1}{2\pi}\int_{-\infty}^{\infty}\boldsymbol{J}(\boldsymbol{x},t)e^{i\omega t}dt \tag{46}$$

（2）推迟势

$$\begin{aligned}\boldsymbol{A}(\boldsymbol{x},t)&=\frac{\mu_0}{4\pi}\int\frac{\boldsymbol{J}(\boldsymbol{x}',t-\frac{r}{c})}{r}dV'\\&=\frac{\mu_0}{4\pi}\int\frac{dV'}{r}\int_{-\infty}^{\infty}\boldsymbol{J}_{\omega}(\boldsymbol{x}')e^{-i\omega t+i\omega\frac{r}{c}}d\omega\\&=\frac{\mu_0}{4\pi}\int_{-\infty}^{\infty}e^{-i\omega t}d\omega\left[\int\frac{\boldsymbol{J}_{\omega}(\boldsymbol{x}')}{r}e^{i\omega\frac{r}{c}}dV'\right]\end{aligned} \tag{47}$$

可见

$$\boldsymbol{A}_{\omega}(\boldsymbol{x})=\frac{\mu_0}{4\pi}\int\frac{\boldsymbol{J}_{\omega}(\boldsymbol{x}')}{r}e^{i\omega\frac{r}{c}}dV'=\frac{\mu_0}{8\pi^2}\int e^{i\omega(t'+\frac{r}{c})}dt'\int\frac{\boldsymbol{J}(\boldsymbol{x}',t')}{r}dV' \tag{48}$$

类似可得

$$\varphi_{\omega}(\boldsymbol{x})=\frac{1}{8\pi^2\varepsilon_0}\int e^{i\omega(t'+\frac{r}{c})}dt'\int\frac{\rho(\boldsymbol{x}',t')}{r}dV' \tag{49}$$

直接用电流密度计算势的频谱分量更方便.

一般推迟势的频谱分量公式

$$A_\omega(\boldsymbol{x}) = \frac{\mu_0}{8\pi^2}\int e^{i\omega(t'+\frac{r}{c})}dt'\int \frac{\boldsymbol{J}(\boldsymbol{x}',t')}{r}dV', \tag{48}$$

$$\varphi_\omega(\boldsymbol{x}) = \frac{1}{8\pi^2\varepsilon_0}\int e^{i\omega(t'+\frac{r}{c})}dt'\int \frac{\rho(\boldsymbol{x}',t')}{r}dV' \tag{49}$$

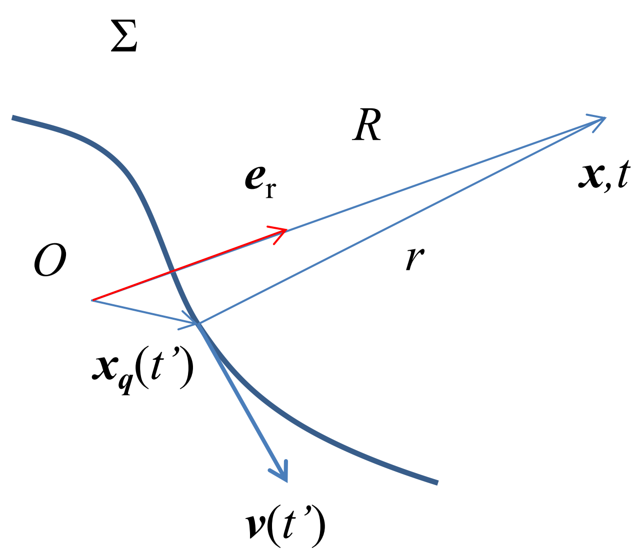
电流密度和电荷密度分别为

$$\boldsymbol{J}(\boldsymbol{x}',t') = \rho(\boldsymbol{x}',t')\boldsymbol{v}(t'), \tag{50}$$

$$\rho(\boldsymbol{x}',t') = q\delta(\boldsymbol{x}' - \boldsymbol{x}_q(t')) \tag{51}$$

故

$$A_\omega(\boldsymbol{x}) = \frac{\mu_0}{8\pi^2}\int_{-\infty}^{\infty}\frac{q\boldsymbol{v}(t')}{r}e^{i\omega(t'+\frac{r}{c})}dt' \tag{52}$$



远场近似(如同多极矩展开)

$$r \approx R - \hat{\boldsymbol{e}}_r \cdot \boldsymbol{x}_q(t') \tag{53}$$

$$\boldsymbol{A}_\omega(\boldsymbol{x}) = \frac{\mu_0 q}{8\pi^2} \frac{e^{ikR}}{R} \int_{-\infty}^{\infty} \boldsymbol{v}(t') e^{i\omega\left(t' - \frac{1}{c}\hat{\boldsymbol{e}}_r \cdot \boldsymbol{x}_q\right)} dt' \tag{54}$$

其中 $k=\omega/c$， $R=|\boldsymbol{x}|$.

由（54）： $\boldsymbol{A}_\omega(\boldsymbol{x}) = \dfrac{\mu_0 q}{8\pi^2}\dfrac{e^{ikR}}{R}\displaystyle\int_{-\infty}^{\infty}\boldsymbol{v}(t')e^{i\omega(t'-\frac{1}{c}\hat{\boldsymbol{e}}_r\cdot\boldsymbol{x}_q)}dt'$ 可得

磁场（不用对分母求导）

$$\begin{aligned}\boldsymbol{B}_\omega &= \nabla\times\boldsymbol{A}_\omega = ik\hat{\boldsymbol{e}}_r\times\boldsymbol{A}_\omega \\ &= \frac{iq\omega}{8\pi^2\varepsilon_0 c^3}\frac{e^{ikR}}{R}\int_{-\infty}^{\infty}\hat{\boldsymbol{e}}_r\times\boldsymbol{v}(t')e^{i\omega(t'-\frac{1}{c}\hat{\boldsymbol{e}}_r\cdot\boldsymbol{x}_q)}dt' \qquad (55)\end{aligned}$$

电场

$$\begin{aligned}\boldsymbol{E}_\omega &= -c\hat{\boldsymbol{e}}_r\times\boldsymbol{B}_\omega \\ &= -\frac{iq\omega}{8\pi^2\varepsilon_0 c^2}\frac{e^{ikR}}{R}\int_{-\infty}^{\infty}\hat{\boldsymbol{e}}_r\times(\hat{\boldsymbol{e}}_r\times\boldsymbol{v})e^{i\omega(t'-\frac{1}{c}\hat{\boldsymbol{e}}_r\cdot\boldsymbol{x}_q)}dt' \qquad (56)\end{aligned}$$

这两个式子和直接对辐射场$\boldsymbol{E}(t)$和$\boldsymbol{B}(t)$进行傅里叶变换的结果一致.

直接对辐射场$\boldsymbol{E}(t)$ 进行傅里叶变换:

$$\boldsymbol{E}_\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\boldsymbol{E}_{rad}e^{i\omega t}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\boldsymbol{E}_{rad}e^{i\omega t}\partial_{t'}t dt'$$

由 $\partial_{t'}t = 1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r$ 和 $\boldsymbol{E}_{rad} = \dfrac{q}{4\pi\varepsilon_0 c^2 r}\dfrac{\hat{\boldsymbol{e}}_r\times[(\hat{\boldsymbol{e}}_r-\frac{1}{c}\boldsymbol{v})\times\dot{\boldsymbol{v}}]}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^3}$ 得

$$\boldsymbol{E}_\omega = \frac{q}{8\pi^2\varepsilon_0 c^2}\int\frac{\hat{\boldsymbol{e}}_r\times[(\hat{\boldsymbol{e}}_r-\frac{1}{c}\boldsymbol{v})\times\dot{\boldsymbol{v}}]}{r(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^2}e^{i\omega t}dt'$$

利用 $\dfrac{d}{dt'}\left[\dfrac{\hat{\boldsymbol{e}}_r\times(\hat{\boldsymbol{e}}_r\times\boldsymbol{v})}{1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r}\right] = \dfrac{\hat{\boldsymbol{e}}_r\times[(\hat{\boldsymbol{e}}_r-\frac{1}{c}\boldsymbol{v})\times\dot{\boldsymbol{v}}]}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^2}$ 得

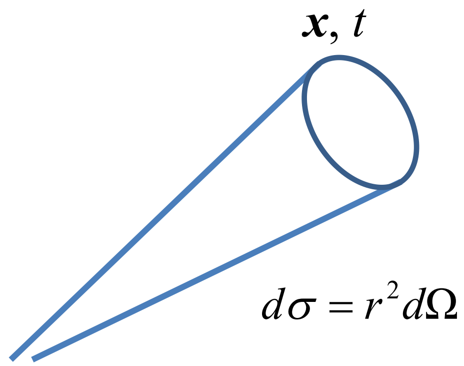
$$\boldsymbol{E}_\omega = \frac{q}{8\pi^2\varepsilon_0 c^2}\int\frac{e^{i\omega t}}{r}\frac{d}{dt'}\left[\frac{\hat{\boldsymbol{e}}_r\times(\hat{\boldsymbol{e}}_r\times\boldsymbol{v})}{1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r}\right]dt' = -\frac{q}{8\pi^2\varepsilon_0 c^2}\int\frac{\hat{\boldsymbol{e}}_r\times(\hat{\boldsymbol{e}}_r\times\boldsymbol{v})}{1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r}\frac{d}{dt'}\left[\frac{e^{i\omega t}}{r}\right]dt'$$

略去$1/r^2$项,

$$\boldsymbol{E}_\omega = -\frac{iq\omega}{8\pi^2\varepsilon_0 c^2}\int\frac{\hat{\boldsymbol{e}}_r\times(\hat{\boldsymbol{e}}_r\times\boldsymbol{v})}{r}e^{i\omega(t'-\frac{r}{c})}dt'$$

与(56)一致

辐射能量的角分布



$$\frac{dW}{d\Omega} = \int \frac{dP}{d\Omega} dt = \int_{-\infty}^{\infty} \boldsymbol{S} \cdot \hat{\boldsymbol{e}}_r R^2 dt \tag{57}$$

代入 $\boldsymbol{S} = \varepsilon_0 c E^2 \hat{\boldsymbol{e}}_r$, 利用 $\int_{-\infty}^{\infty} f^2(t) dt = 4\pi \int_0^{\infty} |f_\omega|^2 d\omega$

$$\frac{dW}{d\Omega} = \varepsilon_0 c R^2 \int_{-\infty}^{\infty} E^2 dt = 4\pi\varepsilon_0 c R^2 \int_0^{\infty} |E_\omega|^2 d\omega \equiv \int_0^{\infty} \frac{dW_\omega}{d\Omega} d\omega \tag{58}$$

单位频率间隔辐射能量角分布 （角频率在ω）

$$\frac{dW_\omega}{d\Omega} = 4\pi\varepsilon_0 cR^2 |E_\omega|^2 . \tag{59}$$

$$\left(\frac{dW_\omega}{d\Omega} = \frac{d^2W}{d\Omega d\omega}\right)$$

对Ω积分得**单位频率间隔辐射能量**

$$W_\omega = 4\pi\varepsilon_0 cR^2 \oint |E_\omega|^2 d\Omega \tag{60}$$

先求E_ω，然后通过上面两式进行辐射能量的频谱分析.

电场傅里叶分量（复习）

$$\boldsymbol{E}_\omega = \frac{q}{8\pi^2\varepsilon_0 c^2}\int \frac{\hat{\boldsymbol{e}}_r \times [(\hat{\boldsymbol{e}}_r - \frac{1}{c}\boldsymbol{v})\times\dot{\boldsymbol{v}}]}{r(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^2} e^{i\omega t} dt'$$

匀速运动，$\boldsymbol{v}$为常矢量， 故 $\boldsymbol{E}_\omega$=0.

远场近似 $r \approx R - \hat{\boldsymbol{e}}_R \cdot \boldsymbol{x}_q(t')$,

$$\boldsymbol{E}_\omega = -\frac{iq\omega}{8\pi^2\varepsilon_0 c^2}\frac{e^{ikR}}{R}\hat{\boldsymbol{e}}_R \times \left[\hat{\boldsymbol{e}}_R \times \int_{-\infty}^{\infty} \boldsymbol{v} e^{i\omega(t'-\frac{1}{c}\hat{\boldsymbol{e}}_R\cdot\boldsymbol{x}_q)} dt'\right] \tag{61}$$

周期直线运动 $\boldsymbol{x}_q(t') = \boldsymbol{x}_{q0}\cos(\omega_0 t')$

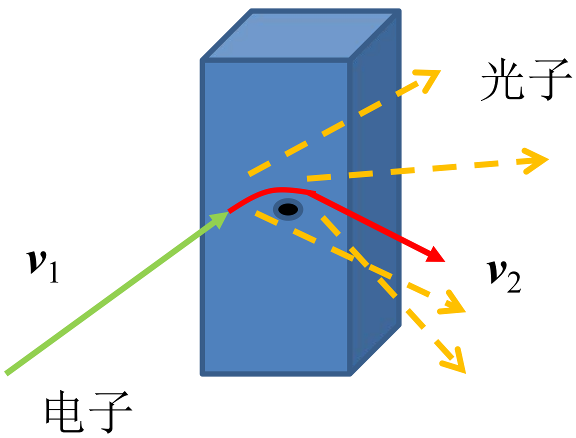
$$\boldsymbol{E}_\omega \approx -\frac{q\omega\omega_0}{16\pi^2\varepsilon_0 c^2}\frac{e^{ikR}}{R}\hat{\boldsymbol{e}}_R \times \left[\hat{\boldsymbol{e}}_R \times \boldsymbol{x}_{q0}\int_{-\infty}^{\infty} e^{i(\omega-\omega_0)t' - i\frac{\omega}{c}x_{q0R}\cos(\omega_0 t')} dt'\right] \tag{62}$$

低速， $\omega_0 x_{q0R}/c << 1$

$$\boldsymbol{E}_\omega \approx -\frac{q\omega\omega_0}{8\pi\varepsilon_0 c^2}\frac{e^{ikR}}{R}\hat{\boldsymbol{e}}_R \times \left(\hat{\boldsymbol{e}}_R \times \boldsymbol{x}_{q0}\right)\delta(\omega-\omega_0) \tag{63}$$

可见，辐射频率集中在粒子运动的频率附近。

2. 低速运动带电粒子在碰撞过程中的辐射频谱



韧致辐射, $\Delta\boldsymbol{v} = \boldsymbol{v}_2 - \boldsymbol{v}_1$

$$\boldsymbol{E}_\omega = \frac{q}{8\pi^2\varepsilon_0 c^2}\int \frac{\hat{\boldsymbol{e}}_r \times [(\hat{\boldsymbol{e}}_r - \frac{1}{c}\boldsymbol{v})\times\dot{\boldsymbol{v}}]}{r(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^2} e^{i\omega t} dt'$$

设 $v << c$。采用偶极近似

$$r \approx R - \boldsymbol{e}_r \cdot \boldsymbol{x}_q \approx R, \quad t \approx t' + \frac{1}{c}(R - \boldsymbol{e}_r \cdot \boldsymbol{x}_q) \approx t' + \frac{1}{c}R$$

$$\boldsymbol{E}_\omega = \frac{q}{8\pi^2\varepsilon_0 c^2}\frac{e^{ikR}}{R}\int \hat{\boldsymbol{e}}_r \times (\hat{\boldsymbol{e}}_r \times \dot{\boldsymbol{v}}) e^{i\omega t'} dt' \tag{64}$$

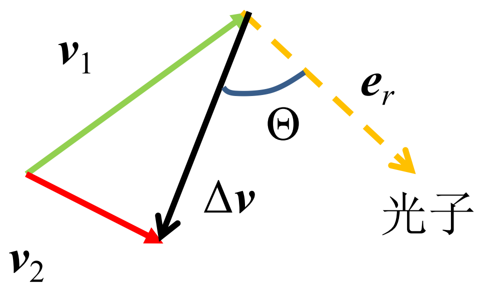
物理过程特点： $\dot{\boldsymbol{v}}$ 只在很短的时间 τ 内非零，从而

(1)若 $\omega\tau << 1$，因为$e^{i\omega t'} \sim 1$, 所以

$$\boldsymbol{E}_\omega \approx \frac{q}{8\pi^2\varepsilon_0 c^2}\frac{e^{ikR}}{R}\hat{\boldsymbol{e}}_r \times (\hat{\boldsymbol{e}}_r \times \Delta\boldsymbol{v}) \tag{65}$$

(2)若 $\omega\tau >> 1$，因为$e^{i\omega t'}$ 快速振荡，所以

$$\boldsymbol{E}_\omega \approx 0 \tag{67}$$



对(1): $\omega\tau<<1$ 的情形,

$$|\boldsymbol{E}_\omega|\approx\frac{q}{8\pi^2\varepsilon_0c^2}\frac{1}{R}|\hat{\boldsymbol{e}}_r\times(\hat{\boldsymbol{e}}_r\times\Delta\boldsymbol{v})|$$

$$=\frac{q}{8\pi^2\varepsilon_0c^2}\frac{1}{R}|\Delta v\sin\Theta| \tag{68}$$

单位频率间隔辐射能量角分布为

$$\frac{dW_\omega}{d\Omega}=4\pi\varepsilon_0cR^2|E_\omega|^2=\frac{q^2}{16\pi^3\varepsilon_0c^3}|\Delta v|^2\sin^2\Theta \tag{69}$$

单位频率间隔辐射能量

$$W_\omega=\int\frac{dW_\omega}{d\Omega}d\Omega=\frac{q^2}{6\pi^2\varepsilon_0c^3}|\Delta v|^2 \tag{70}$$

与ω无关!

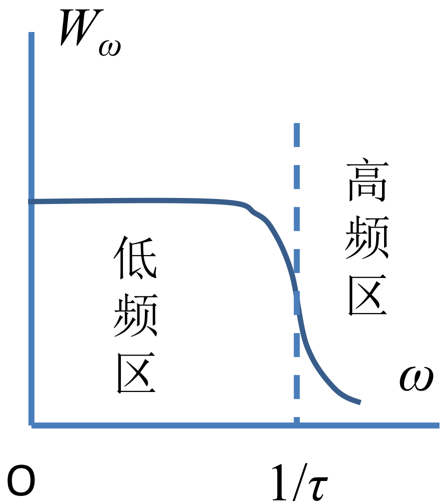
辐射能量按波长的分布

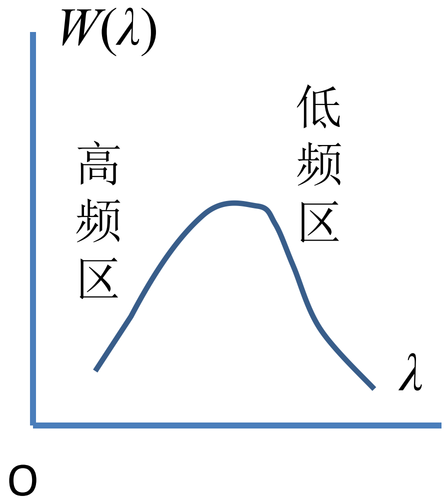
对(1)：$\omega\tau \ll 1$，有 $c\tau/\lambda \ll 1$.

$$W(\lambda) = W_\omega \left|\frac{d\omega}{d\lambda}\right| = \frac{q^2}{3\pi\varepsilon_0}\left|\frac{\Delta v}{c}\right|^2 \frac{1}{\lambda^2} \tag{71}$$

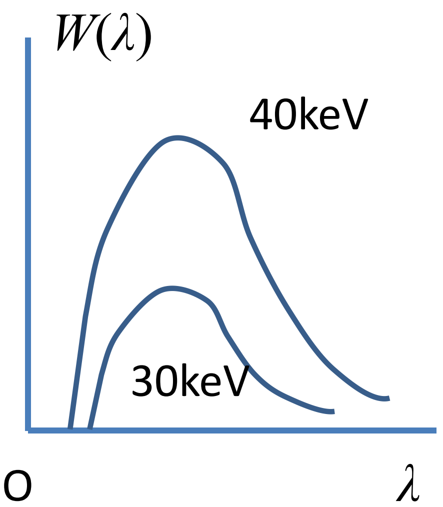
对(2)：$\omega\tau \gg 1$，有 $c\tau/\lambda \gg 1$

$$W(\lambda) = 0 \tag{72}$$





实验



短波区的截止波长是量子效应

$$E_e = \frac{1}{2} m v_1^2 = \hbar \omega_c \qquad (73)$$

记光子数分布为N(ω)。对 $\omega\tau << 1$,

$$N(\omega) d\omega = \frac{W_\omega}{\hbar\omega} d\omega = \frac{2\alpha}{3\pi}\left(\frac{\Delta v}{c}\right)^2 \frac{d\omega}{\omega}, \qquad (74)$$

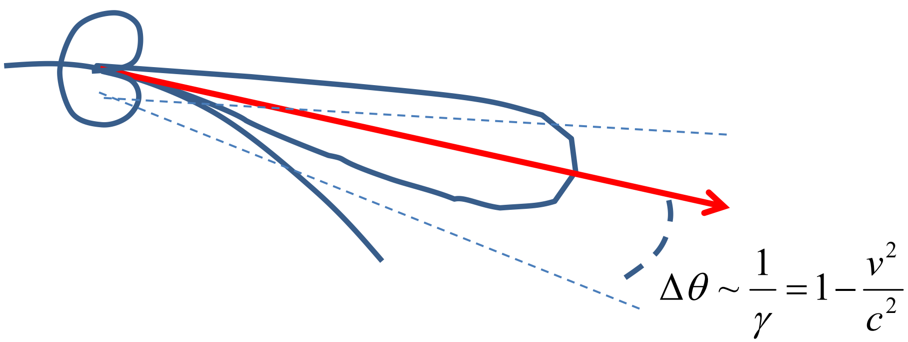
$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = \frac{1}{137.0359895(61)} \qquad (75)$$

3. 高速圆周运动带电粒子的辐射频谱

加速度沿垂直速度方向

$$\frac{dP(t')}{d\Omega}=\frac{q^2\dot{v}^2}{16\pi^2\varepsilon_0c^3}\frac{(1-\frac{v}{c}\cos\theta)^2-(1-\frac{v^2}{c^2})\sin^2\theta\cos^2\phi}{(1-\frac{v}{c}\cos\theta)^5} \tag{76}$$

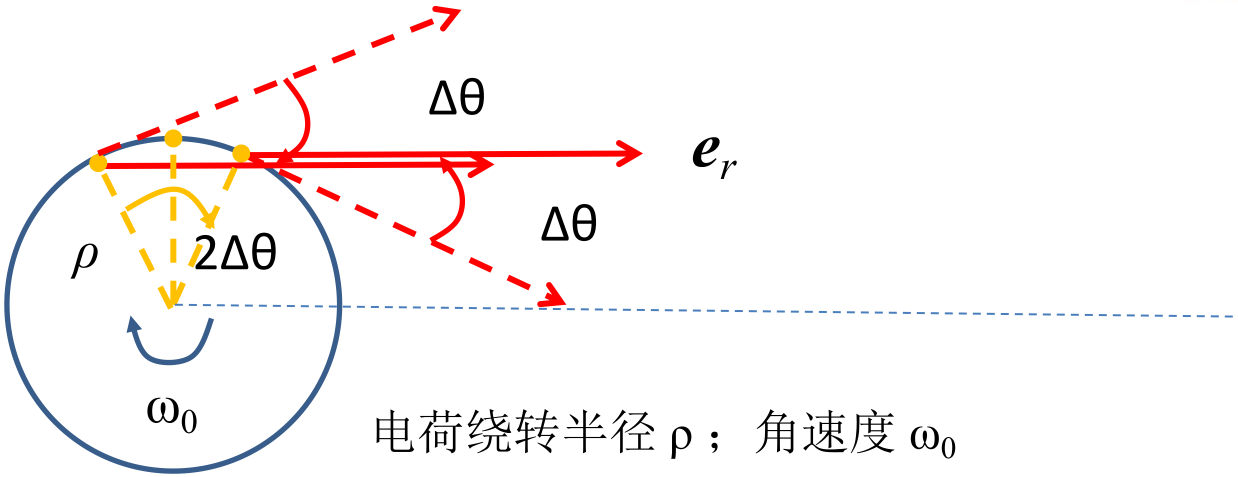
$$\frac{1}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^5}\approx\frac{32}{(\gamma^{-2}+\theta^2)^5} \tag{77}$$



当$v\sim c$时，$\Delta\theta$非常小

高速圆周运动的带电粒子产生的辐射像一个旋转的探照灯一样

中子星



电荷绕转半径 ρ； 角速度 ω_0

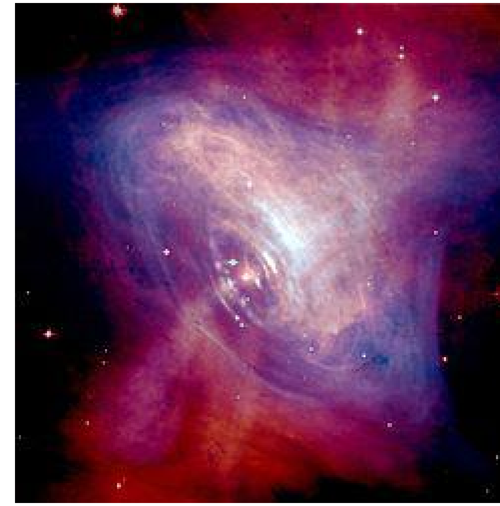
脉冲星转过角度Δθ的时间 $\Delta t'=2\Delta\theta/\omega_0$

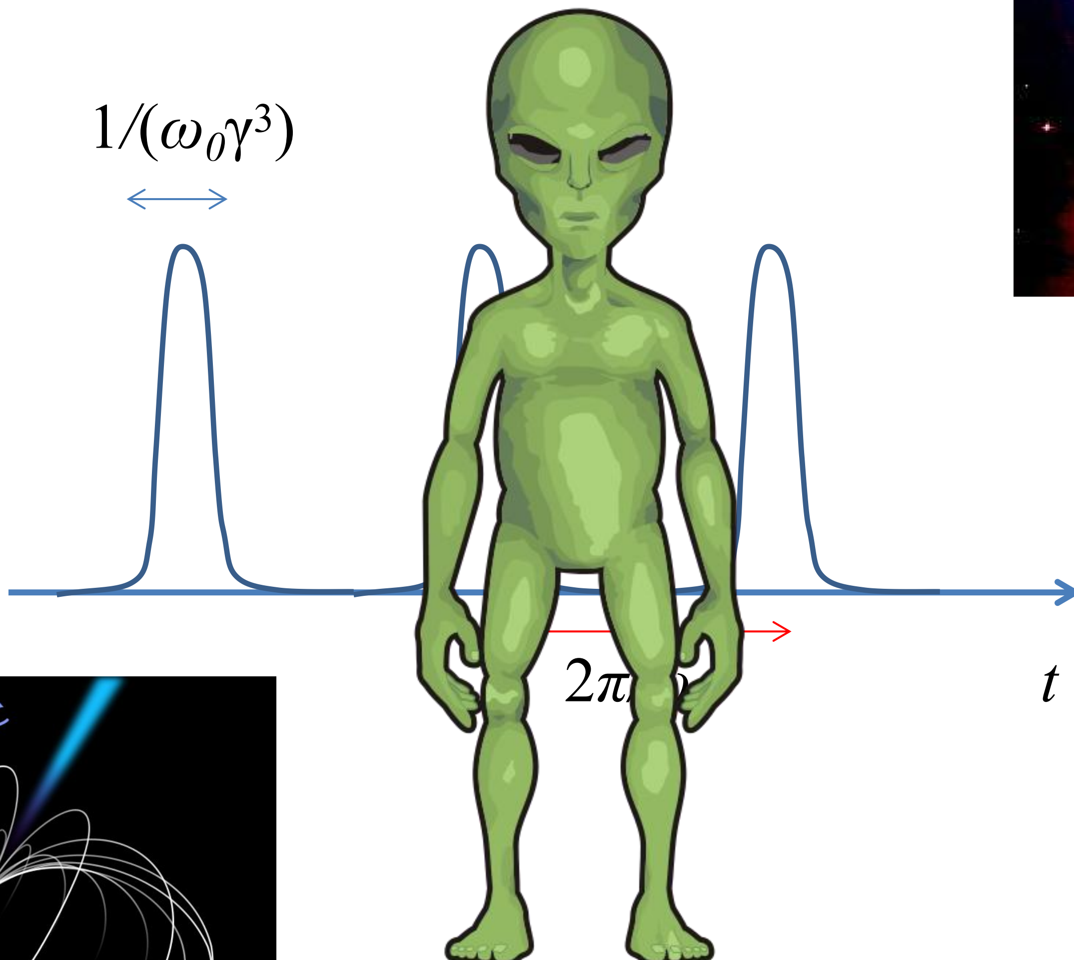
收到脉冲星辐射的持续时间$dt=(dt/dt')\,dt'$， $\dfrac{dt}{dt'}=1-\dfrac{1}{c}\boldsymbol{v}\cdot\boldsymbol{e}_r$

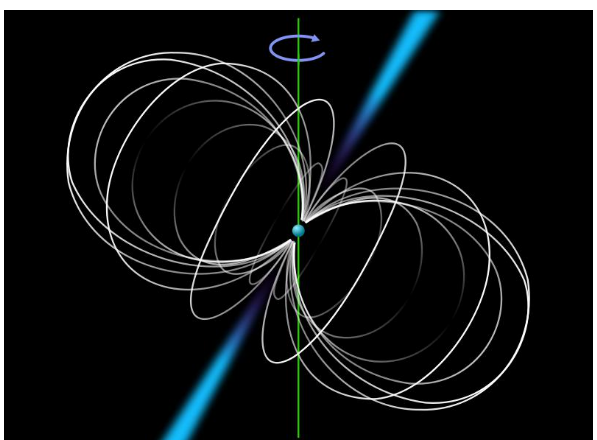
$$\Delta t=\int_0^{\Delta t'}\frac{dt}{dt'}dt'=\int_0^{\Delta t'}\left(1-\frac{v}{c}\cos(\Delta\theta-\omega_0 t')\right)dt'$$

$$=\Delta t'-\frac{2v}{c\omega_0}\sin\frac{\omega_0\Delta t'}{2}\approx\left(1-\frac{v}{c}\right)\Delta t'\approx\frac{\Delta t'}{2\gamma^2}=\frac{\Delta\theta}{\omega_0\gamma^2}=\frac{1}{\omega_0\gamma^3}\approx\frac{\rho}{c\gamma^3}$$

$$\Delta t = \frac{1}{\omega_0 \gamma^3} \approx \frac{\rho}{c\gamma^3} \qquad (78)$$



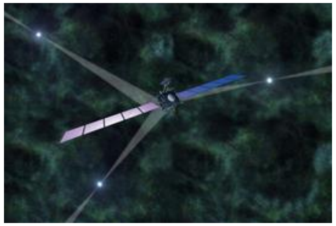




“小绿人”

1967.8 Susan Jocelyn Bell Burnell

中国于2016年11月发射首颗脉冲星导航试验卫星（XPNAV－1）



“中国天眼”500米口径球面射电望远镜（FAST）

上海光源是一台高性能的中能第三代同步辐射光源，它的英文全名为Shanghai Synchrotron Radiation facility，简称SSRF。

单位立体角内辐射能量

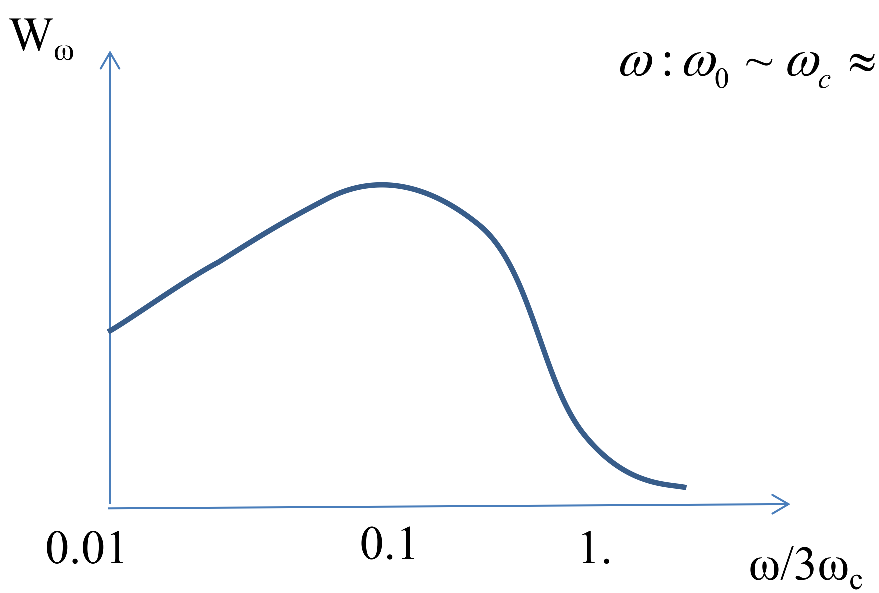
$$\frac{dW}{d\Omega} = \varepsilon_0 c R^2 \int_{-\infty}^{\infty} E^2 dt = 4\pi\varepsilon_0 c R^2 \int_0^{\infty} |E_\omega|^2 d\omega \equiv \int_0^{\infty} \frac{dW_\omega}{d\Omega} d\omega \tag{79}$$

单位频率间隔辐射能量

$$W_\omega = 4\pi\varepsilon_0 c R^2 \oint |\boldsymbol{E}_\omega|^2 \, d\Omega \tag{80}$$

同步辐射：E_e =100MeV, 轨道半径 ρ =0.4m， $\omega_0 \sim 8\times10^8 \mathrm{s}^{-1}$， $\gamma \sim 200$

$$\omega : \omega_0 \sim \omega_c \approx \frac{1}{\Delta t} \sim \omega_0 \gamma^3 \tag{81}$$



作业

1. 证明(45)式 $$\int_{-\infty}^{\infty} f^2(t)dt = 4\pi\int_0^{\infty}\left|f_\omega\right|^2 d\omega$$

2. 从

$$\frac{dP(t')}{d\Omega}=\frac{q^2\dot{v}^2}{16\pi^2\varepsilon_0 c^3}\frac{(1-\frac{v}{c}\cos\theta)^2-(1-\frac{v^2}{c^2})\sin^2\theta\cos^2\phi}{(1-\frac{v}{c}\cos\theta)^5}$$

推导$P(t')$.

3. p272第2题

附录1 推导偏导数关系

对给定的$\boldsymbol{x}$和t，t_q 唯一，因此存在函数关系$t_q = t_q(\boldsymbol{x},t)$. 对 $\boldsymbol{r} = \boldsymbol{x}- \boldsymbol{x}_q$ 求t的偏导数，

$$\left.\frac{\partial \boldsymbol{r}}{\partial t}\right|_x = -\left.\frac{\partial \boldsymbol{x}_q(t_q)}{\partial t}\right|_x = -\left.\frac{\partial t_q}{\partial t}\right|_x \frac{d\boldsymbol{x}_q}{dt_q} = -\left.\frac{\partial t_q}{\partial t}\right|_x \boldsymbol{v} \qquad (附1)$$

对 $r =\sqrt{(\boldsymbol{r}\cdot\boldsymbol{r})}$ 求t的偏导数，利用(附1)，

$$\left.\frac{\partial r}{\partial t}\right|_x = \frac{\boldsymbol{r}}{r}\cdot\left.\frac{\partial \boldsymbol{r}}{\partial t}\right|_x = -\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{r}\left.\frac{\partial t_q}{\partial t}\right|_x \qquad (附2)$$

此外，由 $r = c(t- t_q)$，

$$\left.\frac{\partial r}{\partial t}\right|_x = c\left(1-\left.\frac{\partial t_q}{\partial t}\right|_x\right) \qquad (附3)$$

结合(附2)和(附3)，

$$\frac{\partial t_q}{\partial t} = \frac{r}{r-\frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{r}} = \frac{r}{s} \qquad (附4)$$

代回(附1)和(附2)，

$$\left.\frac{\partial \boldsymbol{r}}{\partial t}\right|_x = -\frac{r}{s}\boldsymbol{v}, \qquad \left.\frac{\partial r}{\partial t}\right|_x = -\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{s} \qquad (附5)$$

对 $\boldsymbol{r}=\boldsymbol{x}-\boldsymbol{x}_q$ 求x^j的偏导数，考虑到$\boldsymbol{x}_q$是t_q的函数，

$$\left.\frac{\partial \boldsymbol{r}}{\partial x^j}\right|_t=\hat{\boldsymbol{e}}_j-\left.\frac{\partial \boldsymbol{x}_q}{\partial x^j}\right|_t=\hat{\boldsymbol{e}}_j-\left.\frac{\partial t_q}{\partial x^j}\right|_t\dot{\boldsymbol{x}}_q=\hat{\boldsymbol{e}}_j-\left.\frac{\partial t_q}{\partial x^j}\right|_t\boldsymbol{v} \tag{附6}$$

对 $r=\sqrt{(\boldsymbol{r}\cdot\boldsymbol{r})}$ 求x^j的偏导数，利用(附6)，

$$\left.\frac{\partial r}{\partial x^j}\right|_t=\frac{\boldsymbol{r}}{r}\cdot\left.\frac{\partial \boldsymbol{r}}{\partial x^j}\right|_t=\frac{r^j}{r}-\left.\frac{\partial t_q}{\partial x^j}\right|_t\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{r} \tag{附7}$$

此外，由 $r=c(t-t_q)$，

$$\left.\frac{\partial r}{\partial x^j}\right|_t=-c\left.\frac{\partial t_q}{\partial x^j}\right|_t \tag{附8}$$

结合(附7)和(附8)，

$$\left.\frac{\partial t_q}{\partial x^j}\right|_t=-\frac{r^j}{c(r-\frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{r})}=-\frac{r^j}{cs} \tag{附9}$$

代回(附6)和(附8)，

$$\left.\frac{\partial \boldsymbol{r}}{\partial x^j}\right|_t=\hat{\boldsymbol{e}}_j+\frac{r^j}{cs}\boldsymbol{v},\qquad \left.\frac{\partial r}{\partial x^j}\right|_t=\frac{r^j}{s} \tag{附10}$$

附录2 电场强度

$$\boldsymbol{E}=-\nabla\varphi-\frac{\partial \boldsymbol{A}}{\partial t} \tag{附11}$$

以李纳-维谢尔势代入

$$\begin{aligned}\boldsymbol{E}&=-\nabla\varphi-\frac{\partial \boldsymbol{A}}{\partial t}=-\nabla\frac{q}{4\pi\varepsilon_0(r-\frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{r})}-\frac{\partial}{\partial t}\frac{q\boldsymbol{v}}{4\pi\varepsilon_0 c^2(r-\frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{r})}\\&=\frac{q}{4\pi\varepsilon_0(r-\frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{r})^2}\left[\nabla r-\tfrac{1}{c}\nabla(\boldsymbol{v}\cdot\boldsymbol{r})+\tfrac{\boldsymbol{v}}{c^2}\left(\partial_t r-\tfrac{1}{c}\partial_t(\boldsymbol{v}\cdot\boldsymbol{r})\right)-(r-\tfrac{1}{c}\boldsymbol{v}\cdot\boldsymbol{r})\frac{\partial_t\boldsymbol{v}}{c^2}\right]\end{aligned} \tag{附12}$$

应用(附4,5)式和(附9,10)式，

$$\begin{aligned}&\nabla r=\frac{\hat{\boldsymbol{e}}_r}{1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r},\qquad \nabla(\boldsymbol{v}\cdot\boldsymbol{r})=\boldsymbol{v}+\frac{\frac{1}{c}v^2\hat{\boldsymbol{e}}_r}{1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r}-\frac{\frac{1}{c}(\hat{\boldsymbol{e}}_r\cdot\dot{\boldsymbol{v}})\boldsymbol{r}}{1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r},\\&\partial_t r=-\frac{\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r}{1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r},\qquad \partial_t(\boldsymbol{v}\cdot\boldsymbol{r})=\frac{\boldsymbol{r}\cdot\dot{\boldsymbol{v}}-v^2}{1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r},\qquad \partial_t\boldsymbol{v}=\frac{\dot{\boldsymbol{v}}}{1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r}\end{aligned} \tag{附13}$$

代回(附12)式，得到$\boldsymbol{E}$为非辐射电场$\boldsymbol{E}_c$和辐射电场$\boldsymbol{E}_{rad}$之和

$$\boldsymbol{E}_c=\frac{q}{4\pi\varepsilon_0 r^2}\frac{\hat{\boldsymbol{e}}_r-\frac{1}{c}\boldsymbol{v}}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^3\gamma^2}\sim\frac{q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{e}}_r \tag{附14}$$

$$\boldsymbol{E}_{rad}=\frac{q}{4\pi\varepsilon_0 c^2 r}\frac{\hat{\boldsymbol{e}}_r\times[(\hat{\boldsymbol{e}}_r-\frac{1}{c}\boldsymbol{v})\times\dot{\boldsymbol{v}}]}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^3}\sim\frac{q}{4\pi\varepsilon_0 c^2 r}\hat{\boldsymbol{e}}_r\times(\hat{\boldsymbol{e}}_r\times\dot{\boldsymbol{v}}) \tag{附15}$$

附录3 磁感应强度

$$\boldsymbol{B}=\nabla\times\boldsymbol{A} \tag{附16a}$$

在笛卡尔坐标中的分量形式(重复空间指标，j和k，隐含从1到3求和)

$$B^i=\varepsilon^{ijk}\partial_j A_k \tag{附16b}$$

以李纳-维谢尔势代入

$$\begin{aligned}
B^i&=\varepsilon^{ijk}\partial_j\frac{qv_k}{4\pi\varepsilon_0c^2s}=\frac{q\varepsilon^{ijk}}{4\pi\varepsilon_0c^2s}\partial_j v_k-\frac{q\varepsilon^{ijk}v_k}{4\pi\varepsilon_0c^2s^2}\partial_j s\\
&=\frac{q\varepsilon^{ijk}}{4\pi\varepsilon_0c^2s}\dot{v}_k\frac{\partial t_q}{\partial x^j}-\frac{q\varepsilon^{ijk}v_k}{4\pi\varepsilon_0c^2s^2}\left[\frac{\partial r}{\partial x^j}-\frac{\boldsymbol{r}}{c}\cdot\frac{\partial\boldsymbol{v}}{\partial x^j}-\frac{\boldsymbol{v}}{c}\cdot\frac{\partial\boldsymbol{r}}{\partial x^j}\right]\\
&=\frac{q\varepsilon^{ijk}}{4\pi\varepsilon_0c^2s}\dot{v}_k\frac{\partial t_q}{\partial x^j}-\frac{q\varepsilon^{ijk}v_k}{4\pi\varepsilon_0c^2s^2}\left[\frac{\partial r}{\partial x^j}-\frac{\boldsymbol{r}}{c}\cdot\dot{\boldsymbol{v}}\frac{\partial t_q}{\partial x^j}-\frac{\boldsymbol{v}}{c}\cdot\frac{\partial\boldsymbol{r}}{\partial x^j}\right]
\end{aligned} \tag{附17}$$

其中 $s=r-\frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{r}$. 应用(附9、10)式，

$$\begin{aligned}
\boldsymbol{B}&=\frac{q\dot{\boldsymbol{v}}\times\boldsymbol{r}}{4\pi\varepsilon_0c^3s^2}-\frac{q}{4\pi\varepsilon_0c^2s^2}\left[\frac{\boldsymbol{r}\times\boldsymbol{v}}{s}+\frac{(\dot{\boldsymbol{v}}\cdot\boldsymbol{r})(\boldsymbol{r}\times\boldsymbol{v})}{c^2s}-\frac{v^2(\boldsymbol{r}\times\boldsymbol{v})}{c^2s}\right]\\
&=\frac{q}{4\pi\varepsilon_0c^2r^2}\frac{(1-\frac{v^2}{c^2})\boldsymbol{v}\times\hat{\boldsymbol{e}}_r}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^3}+\frac{q}{4\pi\varepsilon_0c^3r}\frac{\dot{\boldsymbol{v}}\times\hat{\boldsymbol{e}}_r+\frac{1}{c}[(\dot{\boldsymbol{v}}\cdot\hat{\boldsymbol{e}}_r)(\boldsymbol{v}\times\hat{\boldsymbol{e}}_r)-(\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)(\dot{\boldsymbol{v}}\times\hat{\boldsymbol{e}}_r)]}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^3}
\end{aligned} \tag{附18}$$

$\boldsymbol{B}$的第一项是非辐射项

$$\boldsymbol{B}_c = \frac{q}{4\pi\varepsilon_0 c^2 r^2}\frac{(1-\frac{v^2}{c^2})\boldsymbol{v}\times\hat{\boldsymbol{e}}_r}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^3} \tag{附19}$$

第二项是辐射项

$$\begin{aligned}\boldsymbol{B}_{rad} &= \frac{q}{4\pi\varepsilon_0 c^3 r}\frac{\dot{\boldsymbol{v}}\times\hat{\boldsymbol{e}}_r + \frac{1}{c}[(\dot{\boldsymbol{v}}\cdot\hat{\boldsymbol{e}}_r)(\boldsymbol{v}\times\hat{\boldsymbol{e}}_r)-(\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)(\dot{\boldsymbol{v}}\times\hat{\boldsymbol{e}}_r)]}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^3} \\ &= \frac{q}{4\pi\varepsilon_0 c^3 r}\frac{\hat{\boldsymbol{e}}_r\times\hat{\boldsymbol{e}}_r\times(\hat{\boldsymbol{e}}_r\times\dot{\boldsymbol{v}}) + \frac{1}{c}[(\dot{\boldsymbol{v}}\cdot\hat{\boldsymbol{e}}_r)\boldsymbol{v}-(\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)\dot{\boldsymbol{v}}]\times\hat{\boldsymbol{e}}_r}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^3} \\ &= \frac{q}{4\pi\varepsilon_0 c^3 r}\frac{\hat{\boldsymbol{e}}_r\times\hat{\boldsymbol{e}}_r\times(\hat{\boldsymbol{e}}_r\times\dot{\boldsymbol{v}}) - \frac{1}{c}\hat{\boldsymbol{e}}_r\times[\hat{\boldsymbol{e}}_r\times(\boldsymbol{v}\times\dot{\boldsymbol{v}})]}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^3}\end{aligned} \tag{附20}$$

与(附15)比较，

$$\boldsymbol{B}_{rad} = \frac{1}{c}\hat{\boldsymbol{e}}_r\times\boldsymbol{E}_{rad} \tag{附21}$$

低速，$v/c << 1$，

$$\boldsymbol{B}_c \sim \frac{q}{4\pi\varepsilon_0 c^2 r^2}\boldsymbol{v}\times\hat{\boldsymbol{e}}_r, \qquad \boldsymbol{B}_{rad} \sim \frac{q}{4\pi\varepsilon_0 c^3 r}\dot{\boldsymbol{v}}\times\hat{\boldsymbol{e}}_r \qquad \frac{|B_c|}{|B_{rad}|}\sim\frac{\lambda}{r} \tag{附22}$$