

09 级一期 B 卷参考解答

一.(每小题 6 分,共 12 分)求极限:

$$(1) \lim_{x \rightarrow 0} \frac{\sqrt{1+\tan x} - \sqrt{1+\sin x}}{2x^3};$$

$$\begin{aligned} \text{解} \quad \text{原式} &= \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{2x^3(\sqrt{1+\tan x} + \sqrt{1+\sin x})} = \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{4x^3} \\ &= \lim_{x \rightarrow 0} \frac{\sec^2 x - \cos x}{12x^2} = \lim_{x \rightarrow 0} \frac{1 - \cos^3 x}{12x^2 \cos^2 x} = \lim_{x \rightarrow 0} \frac{1 - \cos^3 x}{12x^2} = \lim_{x \rightarrow 0} \frac{3\cos^2 x \sin x}{24x} = \frac{1}{8}. \end{aligned}$$

$$(2) \lim_{x \rightarrow 0} (\cos x)^{\frac{1}{\sin^2 x}}.$$

$$\text{解} \quad \lim_{x \rightarrow 0} (\cos x)^{\frac{1}{\sin^2 x}} = \lim_{x \rightarrow 0} e^{\frac{1}{\sin^2 x} \ln \cos x}. \text{ 而}$$

$$\lim_{x \rightarrow 0} \frac{\ln \cos x}{\sin^2 x} = \lim_{x \rightarrow 0} \frac{-\sin x / \cos x}{2 \sin x \cos x} = -\frac{1}{2} \lim_{x \rightarrow 0} \frac{1}{\cos^2 x} = -\frac{1}{2}.$$

$$\text{故} \quad \lim_{x \rightarrow 0} (\cos x)^{\frac{1}{\sin^2 x}} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}.$$

二. (每小题 6 分,共 24 分)求下列积分:

$$(1) \int \frac{dx}{x(2+x^{10})};$$

$$\begin{aligned} \text{解} \quad \int \frac{dx}{x(2+x^{10})} &= \int \frac{x^9 dx}{x^{10}(2+x^{10})} = \frac{1}{10} \int \frac{dx^{10}}{x^{10}(2+x^{10})}; \\ &= \frac{1}{20} \left(\int \frac{dx^{10}}{x^{10}} - \int \frac{dx^{10}}{2+x^{10}} \right) = \frac{1}{20} \ln \left| \frac{x^{10}}{2+x^{10}} \right| + C. \end{aligned}$$

$$(2) \int \cos(\ln x) dx;$$

$$\begin{aligned} \text{解} \quad \int \cos(\ln x) dx &\stackrel{u=\ln x}{=} \int e^u \cos u du = \int e^u d \sin u = e^u \sin u - \int \sin u de^u \\ &= e^u \sin u + \int e^u d \cos u = e^u \sin u + e^u \cos u - \int e^u \cos u du \end{aligned}$$

$$\text{所以} \quad \int \cos(\ln x) dx = \frac{e^u}{2} [\sin u + \cos u] + C = \frac{x}{2} [\sin \ln x + \cos \ln x] + C.$$

$$(3) \int_1^e \frac{dx}{x(2+\ln^2 x)};$$

$$\begin{aligned} \text{解} \quad \int_1^e \frac{dx}{x(2+\ln^2 x)} & \stackrel{u=\ln x}{=} \int_0^1 \frac{du}{2+u^2} = \frac{1}{\sqrt{2}} \int_0^1 \frac{d(u/\sqrt{2})}{1+(u/\sqrt{2})^2} \\ & = \frac{1}{\sqrt{2}} \arctan \frac{u}{\sqrt{2}} \Big|_0^1 = \frac{1}{\sqrt{2}} \arctan \frac{1}{\sqrt{2}}. \end{aligned}$$

$$(4) \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t + \cos t} dt$$

$$\begin{aligned} \text{解} \quad \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t + \cos t} dt & \stackrel{s=\frac{\pi}{2}-t}{=} \int_{\frac{\pi}{2}}^0 \frac{\cos\left(\frac{\pi}{2}-s\right)}{\sin\left(\frac{\pi}{2}-t\right) + \cos\left(\frac{\pi}{2}-t\right)} d\left(\frac{\pi}{2}-t\right) \\ & = \int_0^{\frac{\pi}{2}} \frac{\sin s}{\sin s + \cos s} ds = \int_0^{\frac{\pi}{2}} \frac{\sin t}{\sin t + \cos t} dt \end{aligned}$$

$$\text{而} \quad \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t + \cos t} dt + \int_0^{\frac{\pi}{2}} \frac{\sin t}{\sin t + \cos t} dt = \int_0^{\frac{\pi}{2}} dt = \frac{\pi}{2},$$

$$\text{于是} \quad \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t + \cos t} dt = \frac{\pi}{4}.$$

三. (每小题 7 分, 共 21 分)

$$(1) \text{ 设 } z(x, y) = \frac{x}{\sqrt{x^2 + y^2}}, \text{ 求 } dz \Big|_{(0,1)};$$

$$\text{解} \quad \frac{\partial z}{\partial x} = \frac{\sqrt{x^2 + y^2} - x \frac{x}{\sqrt{x^2 + y^2}}}{x^2 + y^2} = \frac{y^2}{(x^2 + y^2)^{3/2}}, \quad \frac{\partial z}{\partial y} = \frac{-x \frac{y}{\sqrt{x^2 + y^2}}}{x^2 + y^2} = \frac{-xy}{(x^2 + y^2)^{3/2}},$$

$$\text{于是} \quad \frac{\partial z}{\partial x} \Big|_{(0,1)} = 1, \quad \frac{\partial z}{\partial y} \Big|_{(0,1)} = 0, \quad \text{故} \quad dz \Big|_{(0,1)} = \frac{\partial z}{\partial x} \Big|_{(0,1)} dx + \frac{\partial z}{\partial y} \Big|_{(0,1)} dy = dx.$$

$$(2) \text{ 已知 } f(x, y, z) = \ln(x + \sqrt{y^2 + z^2}) \text{ 及点 } A(1, 0, 1), B(3, -2, -2), \text{ 求函数 } f(x, y, z) \text{ 在点 } A \text{ 处}$$

沿由 A 到 B 的方向导数, 并求此函数在点 A 处方向导数的最大值.

解 $\overline{AB} = \boldsymbol{l} = (2, -2, -3)$, 故 $(\cos \alpha, \cos \beta, \cos \gamma) = \left(\frac{2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{-3}{\sqrt{17}} \right)$, $\frac{\partial f}{\partial x} = \frac{1}{x + \sqrt{y^2 + z^2}}$,

$$\frac{\partial f}{\partial y} = \frac{\frac{y}{\sqrt{y^2 + z^2}}}{x + \sqrt{y^2 + z^2}} = \frac{y}{x\sqrt{y^2 + z^2} + y^2 + z^2}, \quad \frac{\partial f}{\partial y} = \frac{\frac{z}{\sqrt{y^2 + z^2}}}{x + \sqrt{y^2 + z^2}} = \frac{z}{x\sqrt{y^2 + z^2} + y^2 + z^2}$$

于是 $\left. \frac{\partial f}{\partial x} \right|_{(1,0,1)} = \frac{1}{2}, \left. \frac{\partial f}{\partial y} \right|_{(1,0,1)} = 0, \left. \frac{\partial f}{\partial z} \right|_{(1,0,1)} = \frac{1}{2},$

$$\left. \frac{\partial f}{\partial \boldsymbol{l}} \right|_{(1,0,1)} = \frac{1}{2} \times \frac{2}{\sqrt{17}} + 0 \times \frac{(-2)}{\sqrt{17}} + \frac{1}{2} \times \frac{(-3)}{\sqrt{17}} = -\frac{1}{2\sqrt{17}}.$$

在 A 点的方向导数最大值为 $\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2} = \sqrt{\left(\frac{1}{2}\right)^2 + 0 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}.$

(3) 设函数 $z = z(x, y)$ 由方程 $x + y + z = e^z$ 给出, 求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ 及 $\frac{\partial^2 z}{\partial x^2}.$

解 令 $F(x, y, z) = e^z - x - y - z$, 则 $\frac{\partial F}{\partial x} = -1, \frac{\partial F}{\partial y} = -1, \frac{\partial F}{\partial z} = e^z - 1$. 于是

$$\frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x} / \frac{\partial F}{\partial z} = \frac{1}{e^z - 1}, \quad \frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y} / \frac{\partial F}{\partial z} = \frac{1}{e^z - 1},$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{1}{e^z - 1} \right) = \frac{-e^z}{(e^z - 1)^2} \frac{\partial z}{\partial x} = \frac{-e^z}{(e^z - 1)^3}.$$

四. (第一小题 4 分, 第二小题 6 分, 共 10 分)

(1) 给定空间三点: $A(1, 2, 0), B(-1, 3, 1), C(2, -1, 2)$, 求 $\triangle ABC$ 的面积 S .

解 根据向量积的定义, 可知三角形 ABC 的面积

$$2S_{\triangle ABC} = |\overline{AB}| |\overline{AC}| \sin \angle A = |\overline{AB} \times \overline{AC}|$$

由于 $\overline{AB} = (-2, 1, 1), \overline{AC} = (1, -3, 1)$, 故

$$\overline{AB} \times \overline{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -2 & 1 & 1 \\ 1 & -3 & 2 \end{vmatrix} = 5\boldsymbol{i} + 5\boldsymbol{j} + 5\boldsymbol{k}, \quad |\overline{AB} \times \overline{AC}| = \sqrt{5^2 + 5^2 + 5^2} = 5\sqrt{3}$$

$$S_{\triangle ABC} = \frac{1}{2} |\overline{AB} \times \overline{AC}| = \frac{5\sqrt{3}}{2}.$$

(2)求经过直线 $L_1: \frac{x-1}{2} = \frac{y+2}{3} = \frac{z+3}{4}$ 且平行于直线 $L_2: x = y = \frac{z}{2}$ 的平面方程.

解 平面的法向量为 $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ 1 & 1 & 2 \end{vmatrix} = 2\mathbf{i} + 0\mathbf{j} - \mathbf{k}$, 又显然所求平面过点 $(1, -2, -3)$

故所求平面方程为 $2(x-1) + 0(y+2) - (z+3) = 0$, 即 $2x - z - 5 = 0$.

五. (7 分) 求函数 $f(x) = x^{\frac{1}{x}}, x > 0$ 的极值.

解 $f'(x) = \frac{d}{dx}(e^{\frac{\ln x}{x}}) = e^{\frac{\ln x}{x}} \cdot \frac{1 - \ln x}{x^2} = x^{\frac{1}{x}} \cdot \frac{1 - \ln x}{x^2},$

令 $f'(x) = 0$, 得驻点 $x = e$. 又 当 $0 < x < e$ 时, $f'(x) > 0$, 当 $x > e$ 时, $f'(x) < 0$, 故此点

为极大值点, 极大值为 $f(e) = e^{\frac{1}{e}}$.

六. (12 分) 设函数 $f(x) = \frac{(x-1)^3}{(x+1)^2}$, 求(1)此函数的单调区间与极值点;(2)此函数

的凹凸区间与拐点;(3)此函数的渐近线.

解 $f'(x) = \frac{3(x-1)^2(x+1)^2 - 2(x-1)^3(x+1)}{(x+1)^4}$
 $= \frac{(x-1)^2[3(x+1) - 2(x-1)]}{(x+1)^3} = \frac{(x-1)^2(x+5)}{(x+1)^3},$
 $f''(x) = \frac{[2(x-1)(x+5) + (x-1)^2](x+1)^3 - 3(x-1)^2(x+5)(x+1)^2}{(x+1)^6}$
 $= \frac{(x-1)[(3x+9)(x+1) - 3(x-1)(x+5)]}{(x+1)^4} = \frac{24(x-1)}{(x+1)^4}.$

| | | | | | | |
|----------|-----------------|-------------|------------|-----------|-------------|----------------|
| x | $(-\infty, -5)$ | -5 | $(-5, -1)$ | $(-1, 1)$ | 1 | $(1, +\infty)$ |
| $f'(x)$ | + | 0 | - | + | 0 | + |
| $f''(x)$ | - | - | - | - | 0 | + |
| $f(x)$ | 凸, ↗ | 极大值 $-27/2$ | 凸, ↘ | 凸, ↗ | 拐点 $(1, 0)$ | 凹, ↗ |

单调增加区间为 $(-\infty, -5)$, $(-1, 1)$ 和 $(1, +\infty)$, 单调减少区间为 $(-5, -1)$ 函数在点

$x=-5$ 处取到极大值, 极大值为 $f(-5)=-27/2$.

曲线的凸区间为 $(-\infty, -1)$ 和 $(-1, 1)$, 凹区间为 $(1, +\infty)$, 曲线拐点为 $(1, 0)$.

显然曲线有垂直渐近线 $x=-1$, 曲线无水平渐近线. 由于

$$k = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{(x-1)^3}{x(x+1)^2} = 1,$$

$$b = \lim_{x \rightarrow \infty} [f(x) - kx] = \lim_{x \rightarrow \infty} \left[\frac{(x-1)^3}{(x+1)^2} - x \right] = \lim_{x \rightarrow \infty} \frac{-5x^2 + 2x - 1}{(x+1)^2} = -5,$$

因此曲线有斜渐近线 $y = x - 5$.

(七. (每小题 7 分, 共 14 分))

1. 求证不等式 $\sin x + \tan x > 2x$, $0 < x < \frac{\pi}{2}$;

证 令 $f(x) = \sin x + \tan x - 2x$, $0 < x < \frac{\pi}{2}$, 则 $f'(x) = \cos x + \sec^2 x - 2$,

$$f''(x) = -\sin x + 2 \sec x \cdot \sec x \tan x = \frac{\sin x(2 - \cos^3 x)}{\cos^3 x} \geq 0,$$

故 $f'(x) = \cos x + \sec^2 x - 2$, 在区间 $0 < x < \frac{\pi}{2}$ 内单调增加, 而 $f'(0) = 0$, 于是

在区间 $0 < x < \frac{\pi}{2}$ 内 $f'(x) > 0$, 从而 $f(x) = \sin x + \tan x - 2x$ 在区间 $0 < x < \frac{\pi}{2}$ 内单调增加,

于是 $f(x) = \sin x + \tan x - 2x > f(0) = 0$. 证毕.

2. 设函数 $f(x)$ 在闭区间 $[a, b]$ 上二阶可导, 且 $f(a) = f(b) = 0$, $f''(x) \neq 0, x \in (a, b)$.

求证: $f(x) \neq 0, x \in (a, b)$.

证 用反证法. 设存在 $c \in (a, b)$, 使得 $f(c) = 0$. 于是由罗尔定理,

$$\exists \xi \in (a, c), \exists \eta \in (c, b), \text{使得 } f'(\xi) = f'(\eta) = 0.$$

同样由罗尔定理, $\exists \zeta \in (\xi, \eta) \subset (a, b)$, 使得 $f''(\zeta) = 0$. 矛盾.