§1 复数

P6 习题 1: 求 $z = x + iy$ 所对应的复球面上的点 P 上的坐标 (x_1, x_2, x_3)。设球面的半径为 R。

[解] z 的坐标为 $(x, y, 0)$，P 的坐标为 (x_1, x_2, x_3)，N 的坐标为 $(0, 0, 2R)$。

于是由 z, P, N 共线的条件 $\overrightarrow{NP} = \lambda \overrightarrow{Nz} (\lambda > 0)$：$\begin{cases} x_1 - 0 = \lambda(x - 0) \\ x_2 - 0 = \lambda(y - 0) \\ x_3 - 2R = \lambda(0 - 2R) \end{cases}$，或 $\begin{cases} x_1 = \lambda x \\ x_2 = \lambda y \\ x_3 = 2R(1 - \lambda) \end{cases}$

又由：$x_1{}^2 + x_2{}^2 + (x_3 - R)^2 = R^2$，于是：$\lambda^2 x^2 + \lambda^2 y^2 + R^2 (1 - 2\lambda)^2 = R^2$

即：$[\lambda(x^2 + y^2 + 4R^2) - 4R^2]\lambda = 0$，于是：$\lambda = \dfrac{4R^2}{x^2 + y^2 + 4R^2}$

因此：
$$(x_1, x_2, x_3) = \left(\frac{4R^2 x}{x^2 + y^2 + 4R^2}, \frac{4R^2 y}{x^2 + y^2 + 4R^2}, \frac{2R(x^2 + y^2)}{x^2 + y^2 + 4R^2} \right)$$

P6 习题 2： 考虑多项式 $P_n(z) = (z+1)^n - 1$ 的 $n-1$ 个非零根的乘积，证明：

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}$$

[证明] 由 $(z+1)^n - 1 = 0$，得：$(z+1)^n = 1 = e^{i2k\pi}$，

于是 $z = z_k = e^{i2k\pi/n} - 1$（$k = 1, 2, \cdots, n-1$）是 $P_n(z)$ 的 $n-1$ 个非零根，其乘积为：

$$\begin{aligned}
\prod_{k=1}^{n-1} \left(e^{i2k\pi/n} - 1 \right) &= \prod_{k=1}^{n-1} \left[e^{ik\pi/n} \left(e^{ik\pi/n} - e^{-ik\pi/n} \right) \right] = \prod_{k=1}^{n-1} \left[e^{ik\pi/n} 2i \sin\left(\frac{k\pi}{n}\right) \right] \\
&= (2i)^{n-1} e^{\frac{i\pi}{n} \sum_{k=1}^{n-1} k} \prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = 2^{n-1} e^{\frac{i\pi}{2}(n-1)} e^{i\pi \frac{(n-1)}{2}} \prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) \\
&= 2^{n-1} (-1)^{n-1} \prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right)
\end{aligned}$$

这里用到了 $\prod_{k=m}^{n} \exp(a_k) = \exp(\sum_{k=m}^{n} a_k)$。

又 $P_n(z) = (z+1)^n - 1 = z^n + nz^{n-1} + \cdots + nz \equiv z(z - z_1)(z - z_2) \cdots (z - z_{n-1})$

前一个等号用了牛顿二项式定理，后一个等号用了代数基本定理。

于是：$z^{n-1} + nz^{n-2} + \cdots + n = (z - z_1)(z - z_2) \cdots (z - z_{n-1})$

于是在两边令 $z \to 0$ 得到： $n = (-1)^{n-1}\prod_{k=1}^{n-1}\left(e^{i2k\pi/n}-1\right)$

结合上面的结论，于是： $n = (-1)^{n-1}2^{n-1}(-1)^{n-1}\prod_{k=1}^{n-1}\sin\left(\frac{k\pi}{n}\right)$

即：
$$\frac{n}{2^{n-1}} = \prod_{k=1}^{n-1}\sin\left(\frac{k\pi}{n}\right)$$

§3 复变函数

P12 习题：已知一解析函数 $f(z)$ **的虚部为** $v(x,y)=\dfrac{y}{x^2+y^2}$**，且** $f(2)=0$**，求该解析函数。**

[解] 由 CR 条件：该解析函数的实部应该满足：

$$\begin{cases}\dfrac{\partial u(x,y)}{\partial x}=\dfrac{\partial v(x,y)}{\partial y}=\dfrac{x^2-y^2}{(x^2+y^2)^2}\\[2ex]\dfrac{\partial u(x,y)}{\partial y}=-\dfrac{\partial v(x,y)}{\partial x}=\dfrac{2xy}{(x^2+y^2)^2}\end{cases}$$

于是由 CR 条件的第二式（把 x 当做常数看待）得：

$$u(x,y)=\int\frac{2xydy}{(x^2+y^2)^2}=x\int\frac{d(y^2)}{(x^2+y^2)^2}=x\int\frac{d(x^2+y^2)}{(x^2+y^2)^2}=-\frac{x}{x^2+y^2}+C(x)$$

代入 CR 条件的第一式（把 y 当做常数看待）得：

$$-\frac{y^2-x^2}{(x^2+y^2)^2}+\frac{dC(x)}{dx}=\frac{x^2-y^2}{(x^2+y^2)^2}$$

于是：$\dfrac{dC(x)}{dx}=0$，$C(x)=C$（C 是任意常数）

于是：$u(x,y)=-\dfrac{x}{x^2+y^2}+C$， $f(z)=u(x,y)+iv(x,y)=-\dfrac{x}{x^2+y^2}+C+\dfrac{iy}{x^2+y^2}$

代入 $f(2)=0$，得到常数 C 的值：$C=\dfrac{1}{2}$，于是：

$$f(z)=\frac{1}{2}-\frac{x-iy}{x^2+y^2}=\frac{1}{2}-\frac{1}{x+iy}=\frac{1}{2}-\frac{1}{z}$$

P14 习题：计算下列数值（其中 a**、**b **为实数）：(1)** $\sin(a+ib)$；**(2)** $\cos(a+ib)$；

(3) $\cosh^2(z)-\sinh^2(z)$；**(4)** $\left|\exp(iaz-ib\sin z)\right|$。

[解] (1)

$$\begin{aligned}\sin(a+ib) &= \sin(a)\cos(ib)+\cos(a)\sin(ib)\\ &= \sin(a)\cosh(b)+i\cos(a)\sinh(b)\end{aligned}$$

(2)

$$\begin{aligned}\cos(a+ib) &= \cos(a)\cos(ib)-\sin(a)\sin(ib)\\ &= \cos(a)\cosh(b)-i\sin(a)\sinh(b)\end{aligned}$$

(3)

$$\cosh^2(z)-\sinh^2(z)=\cos^2(iz)-\frac{1}{i^2}\sin^2(iz)=\cos^2(iz)+\sin^2(iz)=1$$

(4)令 $z=x+iy$

$$\begin{aligned}\left|\exp(iaz-ib\sin z)\right| &= \left|\exp\left[ia(x+iy)-ib\sin(x+iy)\right]\right|\\ &= \left|\exp\left\{(iax-ay)-ib\left[\sin(x)\cosh(y)+i\cos(x)\sinh(y)\right]\right\}\right|\\ &= \left|\exp\left\{i\left[ax-b\sin(x)\cosh(y)\right]\right\}\exp\left[-ay+b\cos(x)\sinh(y)\right]\right|\\ &= \exp\left[-ay+b\cos(x)\sinh(y)\right]\end{aligned}$$

§6 解析函数的物理意义

P20 习题 1：已知一解析函数 $f(z)$ 的实部为 $u(x,y)=\dfrac{\sin(2x)}{\cosh(2y)-\cos(2x)}$，且 $f\left(\dfrac{\pi}{2}\right)=0$，求该解析函数。

[解] 由 CR 条件得到：

$$\begin{cases}\dfrac{\partial v(x,y)}{\partial x}=-\dfrac{\partial u(x,y)}{\partial y}=\dfrac{2\sin(2x)\sinh(2y)}{\left[\cosh(2y)-\cos(2x)\right]^2}\\[2ex] \dfrac{\partial v(x,y)}{\partial y}=\dfrac{\partial u(x,y)}{\partial x}=\dfrac{2\cos(2x)\cosh(2y)-2}{\left[\cosh(2y)-\cos(2x)\right]^2}\end{cases}$$

于是：

$$dv(x,y)=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy=\frac{2\sin(2x)\sinh(2y)}{\left[\cosh(2y)-\cos(2x)\right]^2}dx+\frac{2\cos(2x)\cosh(2y)-2}{\left[\cosh(2y)-\cos(2x)\right]^2}dy$$

于是由全微分的曲线积分与路径无关得到（积分路径都是直接取连接上下两点的直线）：

$$v(x,y)-v(\frac{\pi}{2},0)=\int_{(\frac{\pi}{2},0)}^{(x,y)}dv(x,y)$$

$$=\int_{(\frac{\pi}{2},0)}^{(\frac{\pi}{2},y)}\left\{\frac{2\cos(\pi)\cosh(2y)-2}{\left[\cosh(2y)-\cos(\pi)\right]^2}dy\right\}+\int_{(\frac{\pi}{2},y)}^{(x,y)}\left\{\frac{2\sin(2x)\sinh(2y)}{\left[\cosh(2y)-\cos(2x)\right]^2}dx\right\}$$

$$=-2\int_0^y\frac{\cosh(2y)+1}{\left[\cosh(2y)+1\right]^2}dy-\sinh(2y)\int_{\frac{\pi}{2}}^{x}\frac{d\cos(2x)}{\left[\cosh(2y)-\cos(2x)\right]^2}$$

由于：$\int_0^y\frac{\cosh(2y)+1}{\left[\cosh(2y)+1\right]^2}dy=\int_0^y\frac{dy}{\cosh(2y)+1}=\int_0^y\frac{dy}{2\cosh^2(y)}=\frac{\tanh(y)}{2}$

这里，利用了：$\frac{d}{dz}\tanh(z)=\frac{d}{dz}\frac{\sinh(z)}{\cosh(z)}=\frac{\cosh^2(z)-\sinh^2(z)}{\cosh^2(z)}=\frac{1}{\cosh^2(z)}$

$$\int_{\frac{\pi}{2}}^{x}\frac{d\cos(2x)}{\left[\cosh(2y)-\cos(2x)\right]^2}=\frac{1}{\cosh(2y)-\cos(2x)}\bigg|_{\frac{\pi}{2}}^{x}=\frac{1}{\cosh(2y)-\cos(2x)}-\frac{1}{\cosh(2y)+1}$$

于是：

$$v(x,y)-v(\frac{\pi}{2},0)=-\tanh(y)-\frac{\sinh(2y)}{\cosh(2y)-\cos(2x)}+\frac{\sinh(2y)}{\cosh(2y)+1}$$

由于：$\frac{\sinh(2y)}{\cosh(2y)+1}=\frac{2\sinh(y)\cosh(y)}{2\cosh^2(y)}=\frac{\sinh(y)}{\cosh(y)}=\tanh(y)$

$$v(x,y)=v(\frac{\pi}{2},0)-\frac{\sinh(2y)}{\cosh(2y)-\cos(2x)}$$

由 $f(\frac{\pi}{2})=u(\frac{\pi}{2},0)+iv(\frac{\pi}{2},0)=0$ 得：　$v(\frac{\pi}{2},0)=0$

于是：

$$\begin{aligned}f(z)&=\frac{\sin(2x)}{\cosh(2y)-\cos(2x)}-i\frac{\sinh(2y)}{\cosh(2y)-\cos(2x)}\\&=\frac{\sin(2x)-i\sinh(2y)}{\cosh(2y)-\cos(2x)}=\frac{\sin(2x)-\sin(i2y)}{\cos(i2y)-\cos(2x)}\\&=-\frac{\sin(2x)-\sin(i2y)}{\cos(2x)-\cos(i2y)}=-\frac{2\cos(x+iy)\sin(x-iy)}{-2\sin(x+iy)\sin(x-iy)}\\&=\frac{\cos(x+iy)}{\sin(x+iy)}=\cot(x+iy)\\&=\cot(z)\end{aligned}$$

P20 习题 2：已知某静电场的等势线方程为$x^2+y^2=c_1$，用复变函数方法求电力线方程。

[解] 令$v(x,y)=F(x^2+y^2)$，$F(t)$是t的函数，令v满足调和函数的条件，因为这样v才可以作为一个解析函数$f(z)$的虚部，然后才能求实部$u(x,y)$。下面用调和函数的特性来确定$F(t)$。由：$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$，$v(x,y)=F(t),t=x^2+y^2$

即：$\frac{\partial v}{\partial x}=2x\frac{dF}{dt},\frac{\partial^2 v}{\partial x^2}=2\frac{dF}{dt}+4x^2\frac{d^2F}{dt^2}$；$\frac{\partial v}{\partial y}=2y\frac{dF}{dt},\frac{\partial^2 v}{\partial y^2}=2\frac{dF}{dt}+4y^2\frac{d^2F}{dt^2}$

于是：$\frac{dF}{dt}+\left(x^2+y^2\right)\frac{d^2F}{dt^2}=0$，即：$\frac{dF}{dt}+t\frac{d^2F}{dt^2}=\frac{d}{dt}\left(t\frac{dF}{dt}\right)=0$

故：$t\frac{dF}{dt}=C$，$\frac{dF}{dt}=\frac{C}{t}$，于是：$F=C\ln t+C'$

取一个特解（$C=1,C'=0$）得到：$F=\ln t$，于是：$v(x,y)=\ln(x^2+y^2)$

于是由 CR 条件得到：

$$\begin{cases}\dfrac{\partial u(x,y)}{\partial x}=\dfrac{\partial v(x,y)}{\partial y}=\dfrac{2y}{x^2+y^2}\\ \dfrac{\partial u(x,y)}{\partial y}=-\dfrac{\partial v(x,y)}{\partial x}=-\dfrac{2x}{x^2+y^2}\end{cases}$$

于是：

$$\begin{aligned}du&=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=\frac{2y}{x^2+y^2}dx-\frac{2x}{x^2+y^2}dy\\&=\frac{2ydx-2xdy}{x^2+y^2}=\frac{2y^2}{x^2+y^2}\frac{ydx-xdy}{y^2}\\&=\frac{2}{\left(\frac{x}{y}\right)^2+1}d\left(\frac{x}{y}\right)=d\left[2\arctan\left(\frac{x}{y}\right)\right]\end{aligned}$$

于是：$u=2\arctan\left(\frac{x}{y}\right)+c_1$，即：$x=y\tan\frac{u-c_1}{2}$，令$\frac{u-c_1}{2}=C$，

得到电力线方程为：$x=y\tan C$，$\left(C\in\mathbb{R}\right)$

由于C可取尽一切实数，故电力线为通过原点的所有直线。

补充习题

P21 习题 1：求解方程 $\sin z = 2$ 。

[解] 令 $z = x + iy$，则：

$$\sin(x+iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y) = 2$$

于是：$\sin(x)\cosh(y) = 2$， $\cos(x)\sinh(y) = 0$

由后一个式子：$\cos(x) = 0$ 或者 $\sinh(y) = 0$

若 $\sinh(y) = 0$ 则：$\cosh(y) = 1$， $\sin(x) = 2$，这不可能。

因此 $\cos(x) = 0$，此时由于 $\cosh(y) > 0$，故：$\sin(x) > 0$，因此 $\sin(x) = 1$

因此 $x = \dfrac{\pi}{2} + 2k\pi$（$k \in \mathbb{Z}$）， $\cosh(y) = 2$，即 $\dfrac{e^{y} + e^{-y}}{2} = 2$， $e^{2y} - 4e^{y} + 1 = 0$，

于是：$e^{y} = 2 \pm \sqrt{3}$， $y = \ln(2 \pm \sqrt{3})$

于是解为：$z = \dfrac{\pi}{2} + 2k\pi + i\ln(2 \pm \sqrt{3})$，（$k \in \mathbb{Z}$）

或：$z = \dfrac{\pi}{2} + 2k\pi - i\ln(2 \pm \sqrt{3})$，（$k \in \mathbb{Z}$）

P21 习题 2：试由 Cauchy-Riemann 条件(18)中消去 v**，由此所得的** u **所满足的方程即为二维 Laplace 方程在极坐标中的形式。**

[解] 由**(18)**：$\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}, \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} = -\dfrac{\partial v}{\partial r}$

于是

$$\begin{cases} \dfrac{\partial v}{\partial \theta} = r\dfrac{\partial u}{\partial r} \\ \dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta} \end{cases} \Rightarrow \begin{cases} \dfrac{\partial^2 v}{\partial \theta \partial r} = \dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) \\ \dfrac{\partial^2 v}{\partial r \partial \theta} = -\dfrac{1}{r}\dfrac{\partial^2 u}{\partial \theta^2} \end{cases} \Rightarrow \dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) = -\dfrac{1}{r}\dfrac{\partial^2 u}{\partial \theta^2}$$

于是，二维 Laplace 方程在极坐标中的形式为：

$$\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} = 0$$

或者：

$$r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} = 0$$

即：

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

P21 习题 3： 已知函数 $f(z)$ 在区域 D 内解析，且在区域 D 内满足下列条件之一，试证 $f(z)$ 在 D 内为常数。

(a) $f'(z)=0$。**(b)** $v(x,y)$ 为常数。**(c)** $|f(z)|$ 为常数。**(d)** $\arg f(z)$ 为常数。

[证明] 首先：$f=u+iv$ 是常数等价于 u,v 都是常数，后者又等价于 $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}$ 都为零。因此只要对每种情况证明以上四个偏导数都是零即可。

这里先写出 CR 条件，$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

(a) 由于：$f'=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=0$，于是 $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial x}=0$，再由 CR 条件，$\frac{\partial u}{\partial y}=\frac{\partial v}{\partial y}=0$

(b) 显然 $\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}=0$，于是由 CR 条件，$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=0$

(c) $|f(z)|=\sqrt{u^2+v^2}=C$，则：$u^2+v^2=C^2$

两边对 x 偏导：$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}=0$

两边对 y 偏导：$u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}=0$

于是结合 CR 条件后有：$\begin{cases} u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}=0 \\ -u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}=0 \end{cases}$ 和 $\begin{cases} u\frac{\partial v}{\partial y}-v\frac{\partial u}{\partial y}=0 \\ u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}=0 \end{cases}$

前者消去 $\frac{\partial v}{\partial x}$ 得：$\left(u^2+v^2\right)\frac{\partial u}{\partial x}=0$

后者消去 $\frac{\partial v}{\partial y}$ 得：$\left(u^2+v^2\right)\frac{\partial u}{\partial y}=0$

若 $u^2+v^2=0$，则此时，$|f(z)|=\sqrt{u^2+v^2}=C\equiv 0$，于是 $f(z)\equiv 0$ 显然为常数。

若 $u^2+v^2\neq 0$，则：$\frac{\partial u}{\partial x}=0$，$\frac{\partial u}{\partial y}=0$，再由 CR 条件 $\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}=0$

(d) $\arg f(z)$ 为常数 $\Rightarrow \tan(\arg f(z))$ 为常数 $\Rightarrow \frac{v}{u}$ 为常数 $\Rightarrow v=Cu$（C 可为任意常数）

于是代入 CR 条件得到：$\frac{\partial u}{\partial x}=C\frac{\partial u}{\partial y},\frac{\partial u}{\partial y}=-C\frac{\partial u}{\partial x}$

即：$(C^2+1)\dfrac{\partial u}{\partial y}=0$，于是：$\dfrac{\partial u}{\partial y}=0$，$\dfrac{\partial u}{\partial x}=C\dfrac{\partial u}{\partial y}=0$，再由 CR 条件：$\dfrac{\partial v}{\partial x}=\dfrac{\partial v}{\partial y}=0$

P21 习题 4: 已知一解析函数 $f(z)$ 的实部 $u(x,y)$ 或虚部 $v(x,y)$ 和附加条件，求该解析函数。

(a) $u(x,y)=e^x(x\cos y-y\sin y)$，$f(0)=0$

(b) $u(x,y)=\dfrac{x^2-y^2}{(x^2+y^2)^2}$，$f(\infty)=0$

(c) $u(x,y)=x^4-6x^2y^2+y^4$，$f(0)=0$

(d) $v(x,y)=\sqrt{-x+\sqrt{x^2+y^2}}$，$f(0)=0$

[解] (a) 由 $u(x,y)=e^x(x\cos y-y\sin y)$

$$\begin{cases}\dfrac{\partial v}{\partial x}=-\dfrac{\partial u}{\partial y}=e^x(x\sin y+\sin y+y\cos y)\\ \dfrac{\partial v}{\partial y}=\dfrac{\partial u}{\partial x}=e^x(x\cos y+\cos y-y\sin y)\end{cases}$$

于是；

$$dv=e^x(x\sin y+\sin y+y\cos y)dx+e^x(x\cos y+\cos y-y\sin y)dy$$

$$\begin{aligned}v(x,y)-v(0,0)&=\int_{(0,0)}^{(x,y)}dv=\int_{(0,0)}^{(x,0)}dv+\int_{(x,0)}^{(x,y)}dv\\&=\int_{(0,0)}^{(x,0)}e^x(x\sin y+\sin y+y\cos y)dx+\int_{(x,0)}^{(x,y)}e^x(x\cos y+\cos y-y\sin y)dy\\&=e^x\int_0^y(x\cos y+\cos y-y\sin y)dy\\&=e^x(x\sin y+y\cos y)\end{aligned}$$

于是由 $f(0)=0$ 得：$v(0,0)=0$

$$\begin{aligned}f(z)&=e^x(x\cos y-y\sin y)+ie^x(x\sin y+y\cos y)\\&=e^x\left[x(\cos y+i\sin y)+iy(\cos y+i\sin y)\right]\\&=e^x\left(xe^{iy}+iye^{iy}\right)=e^{x+iy}\left(x+iy\right)\\&=ze^z\end{aligned}$$

(b) 由 $u(x,y)=\dfrac{x^2-y^2}{(x^2+y^2)^2}$

$$\begin{cases} \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} = -\dfrac{2y(y^2-3x^2)}{(x^2+y^2)^3} \\ \dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} = -\dfrac{2x(x^2-3y^2)}{(x^2+y^2)^3} \end{cases}$$

于是；

$$dv = -\frac{2y(y^2-3x^2)}{(x^2+y^2)^3}dx - \frac{2x(x^2-3y^2)}{(x^2+y^2)^3}dy$$

$$\begin{aligned} v(x,y)-v(0,0) &= \int_{(0,0)}^{(x,y)} dv = \int_{(0,0)}^{(x,0)} dv + \int_{(x,0)}^{(x,y)} dv \\ &= -\int_{(0,0)}^{(x,0)} \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}dx - \int_{(x,0)}^{(x,y)} \frac{2x(x^2-3y^2)}{(x^2+y^2)^3}dy \\ &= -2x\int_0^y \frac{x^2-3y^2}{(x^2+y^2)^3}dy \\ &= -\frac{2xy}{(x^2+y^2)^2} \end{aligned}$$

于是：

$$f(z) = \frac{x^2-y^2}{(x^2+y^2)^2} - i\frac{2xy}{(x^2+y^2)^2} + iv(0,0)$$

由于： $f(\infty) = \lim\limits_{\substack{x\to\infty \\ y\to\infty}} \left[\dfrac{x^2-y^2}{(x^2+y^2)^2} - i\dfrac{2xy}{(x^2+y^2)^2} \right] + iv(0,0) = iv(0,0) = 0$

于是：

$$\begin{aligned} f(z) &= \frac{x^2-y^2}{(x^2+y^2)^2} - i\frac{2xy}{(x^2+y^2)^2} = \frac{x^2-y^2-i2xy}{(x^2+y^2)^2} \\ &= \frac{(x-iy)^2}{(x^2+y^2)^2} = \frac{(x-iy)^2}{(x-iy)^2(x+iy)^2} = \frac{1}{(x+iy)^2} \\ &= \frac{1}{z^2} \end{aligned}$$

(c) 由 $u(x,y) = x^4 - 6x^2y^2 + y^4$

$$\begin{cases} \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} = 12x^2y - 4y^3 \\ \dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} = 4x^3 - 12xy^2 \end{cases}$$

于是；

$$dv = (12x^2y - 4y^3)dx + (4x^3 - 12xy^2)dy$$

$$
\begin{aligned}
v(x,y)-v(0,0) &= \int_{(0,0)}^{(x,y)} dv = \int_{(0,0)}^{(x,0)} dv + \int_{(x,0)}^{(x,y)} dv \\
&= \int_{(0,0)}^{(x,0)} (12x^2y-4y^3)dx + \int_{(x,0)}^{(x,y)} (4x^3-12xy^2)dy \\
&= \int_0^y (4x^3-12xy^2)dy \\
&= 4x^3y-4xy^3
\end{aligned}
$$

由 $f(0)=u(0,0)+iv(0,0)=0$ 得， $v(0,0)=0$

于是：

$$
\begin{aligned}
f(z) &= x^4-6x^2y^2+y^4+i(4x^3y-4xy^3) \\
&= x^4+4x^3(iy)+6x^2(iy)^2+4x(iy)^3+y^4 \\
&= (x+iy)^4 \\
&= z^4
\end{aligned}
$$

(d) 由 $v(x,y)=\sqrt{-x+\sqrt{x^2+y^2}}$

$$
\begin{cases}
\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = \dfrac{y}{2\sqrt{x^2+y^2}\sqrt{-x+\sqrt{x^2+y^2}}} \\[2ex]
\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} = \dfrac{-x+\sqrt{x^2+y^2}}{2\sqrt{x^2+y^2}\sqrt{-x+\sqrt{x^2+y^2}}}
\end{cases}
$$

于是；

$$
du = \frac{y}{2\sqrt{x^2+y^2}\sqrt{-x+\sqrt{x^2+y^2}}}dx + \frac{-x+\sqrt{x^2+y^2}}{2\sqrt{x^2+y^2}\sqrt{-x+\sqrt{x^2+y^2}}}dy
$$

$$
\begin{aligned}
u(x,y)-u(0,0) &= \int_{(0,0)}^{(x,y)} du = \int_{(0,0)}^{(x,0)} du + \int_{(x,0)}^{(x,y)} du \\
&= \int_{(0,0)}^{(x,0)} \frac{y}{2\sqrt{x^2+y^2}\sqrt{-x+\sqrt{x^2+y^2}}}dx + \int_{(x,0)}^{(x,y)} \frac{-x+\sqrt{x^2+y^2}}{2\sqrt{x^2+y^2}\sqrt{-x+\sqrt{x^2+y^2}}}dy \\
&= \int_0^y \frac{-x+\sqrt{x^2+y^2}}{2\sqrt{x^2+y^2}\sqrt{-x+\sqrt{x^2+y^2}}}dy \\
&= \frac{y}{\sqrt{-x+\sqrt{x^2+y^2}}} = \sqrt{x+\sqrt{x^2+y^2}}
\end{aligned}
$$

于是：

$$f(z)=\sqrt{x+\sqrt{x^2+y^2}}+u(0,0)+i\sqrt{-x+\sqrt{x^2+y^2}}$$

由于： $f(0)=u(0,0)=0$

于是：

$$\begin{aligned}
f(z)&=\sqrt{x+\sqrt{x^2+y^2}}+i\sqrt{-x+\sqrt{x^2+y^2}}\\
&=\sqrt{\left(\sqrt{x+\sqrt{x^2+y^2}}+i\sqrt{-x+\sqrt{x^2+y^2}}\right)^2}\\
&=\sqrt{x+\sqrt{x^2+y^2}+x-\sqrt{x^2+y^2}+2i\sqrt{\left(x+\sqrt{x^2+y^2}\right)\left(-x+\sqrt{x^2+y^2}\right)}}\\
&=\sqrt{2x+2iy}\\
&=\sqrt{2z}
\end{aligned}$$

P21 习题 5：记 $f(z,\overline{z})=u(x,y)+iv(x,y)$，试证 **Cauchy-Riemann** 条件等价于：

$$\frac{\partial f(z,\overline{z})}{\partial \overline{z}}=0$$

[证明] 首先，由于： $x=\dfrac{z+\overline{z}}{2},y=\dfrac{z-\overline{z}}{2i}$

$$f(z,\overline{z})=u(x,y)+iv(x,y)=u\left(\frac{z+\overline{z}}{2},\frac{z-\overline{z}}{2i}\right)+iv\left(\frac{z+\overline{z}}{2},\frac{z-\overline{z}}{2i}\right)$$

$\dfrac{\partial f(z,\overline{z})}{\partial \overline{z}}=0$ 等价于： $\dfrac{\partial u\left(\frac{z+\overline{z}}{2},\frac{z-\overline{z}}{2i}\right)}{\partial \overline{z}}+i\dfrac{\partial v\left(\frac{z+\overline{z}}{2},\frac{z-\overline{z}}{2i}\right)}{\partial \overline{z}}=0$

等价于： $\dfrac{\partial u}{\partial x}\dfrac{1}{2}+\dfrac{\partial u}{\partial y}\left(-\dfrac{1}{2i}\right)+i\left[\dfrac{\partial v}{\partial x}\dfrac{1}{2}+\dfrac{\partial v}{\partial y}\left(-\dfrac{1}{2i}\right)\right]=0$

等价于： $\left(\dfrac{\partial u}{\partial x}-\dfrac{\partial v}{\partial y}\right)+i\left(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\right)=0$

等价于： $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y},\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$，此即 CR 条件。

Lamberto Qin
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QQ:397968240
Tel:15975454463