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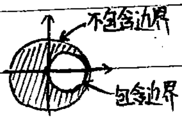
第六章 多元函数微分学

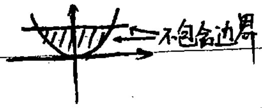
1 多元函数

1.解 (1) $\{(x,y)\mid x^2+y^2-2x\geq 0$ 且 $x^2+y^2<4\}$. 图为:

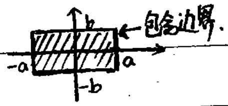


(2) $\{(x,y)\mid y\neq \pm x\}$. 图为:

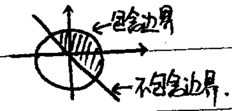
(3) $\{(x,y)\mid y>x^2$ 且 $y<1\}$ 图为:



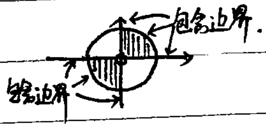
(4) $\{(x,y)\mid |x|\leq a$ 且 $|y|\leq b\}$ 图为:



(5) $\{(x,y)\mid x^2+y^2\leq 1$ 且 $x+y>0\}$ 图为:



(6) $\{(x,y)\mid x^2+y^2\leq 1$ 且 $xy\geq 0\}$ 图为:



2.解 (1)区域 (2)有界区域 (3)有界闭区域 (4)开集

思考题: E_4的边界点集合为. $\{(x,y)\mid y=\sin\frac{1}{x}$ 且 $x\neq 0\}\cup\{(0,y)\mid -1\leq y\leq 1\}$

3.证明: 若E为闭集合. 则 $E\cup\partial E=E$. 也即是 $\bar{E}=E\cup\partial E=E$ 为闭集合.

若E不是闭集合. $\bar{E}=E\cup\partial E$ 知 E的边界点也为 $\bar{E}$ 的边界点. 且 $E\subset\bar{E}$. 可知 $\bar{E}$ 为闭集合

综上述: $\bar{E}=E\cup\partial E$ 是一个闭集合.

4.证明: (1) 由 $|\alpha+\lambda\beta|^2\geq 0$ 得 $\beta^2\lambda^2+2\alpha\beta\lambda+\alpha^2\geq 0$ …… ※

① $|\beta|\neq 0$ 时. 方程※对 $\lambda\in R$ 均成立. 应有 $\Delta=(2\alpha\beta)^2-4\alpha^2\beta^2\leq 0$. 即 $|\alpha\cdot\beta|\leq|\alpha|\cdot|\beta|$

② $|\beta|=0$ 时. 显然 $\beta=\vec{0}$ 故 $|\alpha\cdot\beta|\leq|\alpha|\cdot|\beta|$ 成立.

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综上述 $|\alpha\cdot\beta|\leq|\alpha|\cdot|\beta|$, $\forall\alpha,\beta\in R^n$

(2) $\cancel{\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}\leq\sqrt{\sum_{j=1}^{n}(x_j-z_j)^2}+\sqrt{\sum_{k=1}^{n}(y_k-z_k)^2}}$

由(1)中结论可知有 $|(\alpha-\gamma)\cdot(\gamma-\beta)|\leq|\alpha-\gamma||\gamma-\beta|$

即 $|\alpha\gamma+\gamma\beta-\alpha\beta-\gamma^2|\leq|\alpha-\gamma|\cdot|\gamma-\beta|$

$\therefore$ 有 $\alpha\gamma+\gamma\beta-\alpha\beta-\gamma^2\leq|\alpha-\gamma|\cdot|\gamma-\beta|\cdots$ ④

而由结论. $|\alpha-\beta|\leq|\alpha-\gamma|+|\gamma-\beta|\Leftrightarrow\alpha^2-2\alpha\beta+\beta^2\leq\alpha^2-2\alpha\gamma+\gamma^2+\gamma^2-2\gamma\beta+\beta^2+2|\alpha-\gamma||\gamma-\beta|$

$\Leftrightarrow\alpha\gamma+\gamma\beta-\alpha\beta-\gamma^2\leq|\alpha-\gamma|\cdot|\gamma-\beta|$. 即 由于④式成立. 故结论成立.

也即是 $|\alpha-\beta|\leq|\alpha-\gamma|+|\gamma-\beta|$, $\forall\alpha,\beta,\gamma\in R^n$

(3) $\because d(P,Q)=\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}=\sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}$

而 $\alpha-\beta=(x_1-y_1,x_2-y_2,\cdots x_n-y_n)$ $\therefore|\alpha-\beta|=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2}$

$\therefore d(P,Q)=|\alpha-\beta|$

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2 多元函数的极限

1 解：(1) $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2-y^2+5}{x^2+y^2+2} = \dfrac{\lim\limits_{(x,y)\to(0,0)}(3x^2-y^2+5)}{\lim\limits_{(x,y)\to(0,0)}(x^2+y^2+2)} = \dfrac{3\cdot 0^2-0^2+5}{0^2+0^2+2} = \dfrac{5}{2}$

(2) $\because 0 \leq \left|\dfrac{\sin(x^3+y^4)}{x^2+y^2}\right| \leq \left|\dfrac{x^3+y^4}{x^2+y^2}\right| \leq \left|\dfrac{x^3}{x^2+y^2}\right| + \left|\dfrac{y^4}{x^2+y^2}\right| \leq |x|+|y^2|$

而 $\lim\limits_{(x,y)\to(0,0)} 0 = 0$，$\lim\limits_{(x,y)\to(0,0)} (|x|+|y^2|) = 0$

由夹逼定理知 $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^3+y^4)}{x^2+y^2} = 0$

(3) 由于 $z=\sin\dfrac{1}{x^2+y^2}$ 在其定义域上有界，且有 $\lim\limits_{(x,y)\to(0,0)}(x^2+y^2)=0$

$\therefore \lim\limits_{(x,y)\to(0,0)} (x^2+y^2)\cdot\sin\dfrac{1}{x^2+y^2} = 0$

(4). $\lim\limits_{(x,y)\to(0,1)} \left|\dfrac{x^3+(y-1)^3}{x^2+(y-1)^2}\right| \leq \lim\limits_{(x,y)\to(0,1)} \left|\dfrac{x^3}{x^2} + \lim\limits_{(x,y)\to(0,1)} \dfrac{(y-1)^3}{(y-1)^2}\right| = \lim\limits_{(x,y)\to(0,1)} |x+y-1| = 0$

而 $\lim\limits_{(x,y)\to(0,1)} \left|\dfrac{x^3+(y-1)^3}{x^2+y-1}\right| \geq 0$

故 $\lim\limits_{(x,y)\to(0,1)} \dfrac{x^3+(y-1)^3}{x^2+(y-1)^2} = 0$

(5) $\lim\limits_{(x,y)\to(1,1)} \dfrac{xy-y-2x+2}{x-1} = \lim\limits_{(x,y)\to(1,1)} \dfrac{y(x-1)-2(x-1)}{x-1} = \lim\limits_{(x,y)\to(1,1)} (y-2) = -1$

(6) $\lim\limits_{(x,y,z)\to(1,-2,0)} \ln\sqrt{x^2+y^2+z^2} = \ln\left(\lim\limits_{(x,y,z)\to(1,-2,0)} \sqrt{x^2+y^2+z^2}\right) = \ln\sqrt{\lim\limits_{(x,y,z)\to(1,-2,0)}(x^2+y^2+z^2)} = \ln\sqrt{5}$

2. 证明：(1) 设 $y=kx^2$. 当 (x,y) 沿曲线 $y=kx^2$ 趋向于 $(0,0)$ 时，$f(x,y)=\dfrac{1-k^2}{1+k^2}$

由于 k 的任意性，函数 $f(x,y)$ 趋向于不同的常数. 由 $f(x,y)$ 当 $(x,y)\to(0,0)$ 时无极限

(2) $\because \lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{x+y}{x-y}$ 令 $y=kx$. 则有 ($k\neq 1$ 时)

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{1+k}{1-k}$. 由于 k 的任意性. $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ 不存在.

(当 $k=1$ 时有 $\lim\limits_{(x,y)\to(0,0)} f(x,y)=0$. 此时才有极限存在)

综上述. 当 $(x,y)\to(0,0)$ 时. $f(x,y)$ 无极限.

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3 解: (1) $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \frac{(x+2y)\cdot(x^2+y^2)}{x^2+y^2}\cdot \ln(x^2+y^2) = \lim\limits_{(x,y)\to(0,0)} \frac{\ln(x^2+y^2)}{x^2+y^2} \lim\limits_{(x,y)\to(0,0)} (x+2y)(x^2+y^2)$

$= 1\cdot(0+2\cdot 0)(0^2+0^2) = 0$

(2) 显然 $\frac{1-\cos(x^2+y^2)}{(x^2+y^2)x^2\cdot y^2} \leq \frac{2}{(x^2+y^2)x^2\cdot y^2}$ 而 $\lim\limits_{(x,y)\to(0,0)} \frac{2}{(x^2+y^2)x^2y^2} = \infty$

令 $u=\frac{x^2+y^2}{2}$. 则当$(x,y)\to(0,0)$时. $u\to 0$.

$\frac{1-\cos(x^2+y^2)}{(x^2+y^2)\cdot x^2 y^2} \geq \frac{2\sin\frac{x^2+y^2}{2}}{2\cdot\frac{x^2+y^2}{2}\cdot(\frac{x^2+y^2}{2})^2} = \frac{\sin u}{u^3}$ 而 $\lim\limits_{u\to 0}\frac{\sin u}{u^3} = \infty$

故 当$(x,y)\to(0,0)$时. $f(x,y)$无极限.

(3) $f(x,y) = (x^2+y^2)^{x^2y^2} = e^{x^2y^2\ln(x^2+y^2)}$. 考虑 $\lim\limits_{(x,y)\to(0,0)} x^2y^2\ln(x^2+y^2)$

$\because\ x^2y^2 \leq (\frac{x^2+y^2}{2})^2,\ x^2+y^2 \geq (4x^2y^2)^{\frac{1}{2}}$

$\therefore\ x^2y^2|\ln(4x^2y^2)^{\frac{1}{2}}| \leq |x^2y^2\ln(x^2+y^2)| \leq \frac{1}{4}(x^2+y^2)^2|\ln(x^2+y^2)|$

令 $t=4x^2y^2$. $u=x^2+y^2$. 当$(x,y)\to(0,0)$时. $t\to 0$, $u\to 0$

once有 $\frac{1}{8}|t\ln t| \leq |x^2y^2\ln(x^2+y^2)| \leq \frac{1}{4}u^2|\ln u|$

而 $\lim\limits_{t\to 0}\frac{1}{8}t|\ln t| = \frac{1}{8}\lim\limits_{t\to 0}\frac{|\ln t|}{\frac{1}{t}}$ ($\frac{\infty}{\infty}$未定型) $= \frac{1}{8}\lim\limits_{t\to 0}\frac{\frac{1}{t}}{-\frac{1}{t^2}} = -\frac{1}{8}\lim\limits_{t\to 0} t = 0$

$\lim\limits_{u\to 0}\frac{1}{4}u^2|\ln u| = \frac{1}{4}\lim\limits_{u\to 0}\frac{|\ln u|}{\frac{1}{u^2}}$ ($\frac{\infty}{\infty}$未定型) $= \frac{1}{4}\lim\limits_{u\to 0}\frac{\frac{1}{u}}{-2\frac{1}{u^3}} = -\frac{1}{8}\lim\limits_{u\to 0} u^2 = 0$

由夹逼定理知 $\lim\limits_{(x,y)\to(0,0)} x^2y^2\ln(x^2+y^2) = 0$

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y) = e^{\lim\limits_{(x,y)\to(0,0)} x^2y^2\ln(x^2+y^2)} = e^0 = 1$

(4) 设 $P_n(x,y)$中任意一项为 $x^t y^{n-t}$ ($t=0,1,2,\cdots,n$)

只需证明 $\lim\limits_{(x,y)\to(0,0)} \left|\frac{x^t y^{n-t}}{\rho^{n-1}}\right| = 0$. 即可知 $\lim\limits_{(x,y)\to(0,0)} \frac{P_n(x,y)}{\rho^{n-1}} = 0$

令 $y=kx$. 当 (x,y)沿直线$y=kx$趋向于$(0,0)$时.

$\lim\limits_{(x,y)\to(0,0)} \left|\frac{x^t y^{n-t}}{\rho^{n-1}}\right| = \lim\limits_{(x,y)\to(0,0)} \left|\frac{x^t\cdot k^{n-t}x^{n-t}}{\sqrt{x^2+k^2x^2}^{\,n-1}}\right| = \lim\limits_{(x,y)\to(0,0)} \left|\frac{k^{n-t}}{\sqrt{1+k^2}^{\,n-1}}\right||x| = 0.$

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故有 $\lim\limits_{(x,y)\to(0,0)}\dfrac{p_n(x,y)}{\rho^{n-1}}=0$. 也即是当 $(x,y)\to(0,0)$ 时，函数极限为0.

4.解 (1) $\lim\limits_{x\to0}\lim\limits_{y\to0}f(x,y)=\lim\limits_{x\to0}\lim\limits_{y\to0}\dfrac{|x|-|y|}{|x|+|y|}=\lim\limits_{x\to0}1=1$

$\lim\limits_{y\to0}\lim\limits_{x\to0}f(x,y)=\lim\limits_{y\to0}\lim\limits_{x\to0}\dfrac{|x|-|y|}{|x|+|y|}=\lim\limits_{y\to0}\dfrac{-|y|}{|y|}=\lim\limits_{y\to0}(-1)=-1$

(2) $\lim\limits_{x\to0}\lim\limits_{y\to0}f(x,y)=\lim\limits_{x\to0}\lim\limits_{y\to0}\dfrac{y^3+\sin x^2}{x^2+y^2}=\lim\limits_{x\to0}\dfrac{\sin x^2}{x^2}=\lim\limits_{x^2\to0}\dfrac{\sin x^2}{x^2}=1$

$\lim\limits_{y\to0}\lim\limits_{x\to0}f(x,y)=\lim\limits_{y\to0}\lim\limits_{x\to0}\dfrac{y^3+\sin x^2}{x^2+y^2}=\lim\limits_{y\to0}\dfrac{y^3}{y^2}=\lim\limits_{y\to0}y=0$

(3) $\lim\limits_{x\to0}\lim\limits_{y\to0}f(x,y)=\lim\limits_{x\to0}\lim\limits_{y\to0}(1+x)^{y/x}=\lim\limits_{x\to0}(1+x)^0=1$

$\lim\limits_{y\to0}\lim\limits_{x\to0}f(x,y)=\lim\limits_{y\to0}\lim\limits_{x\to0}(1+x)^{y/x}=\lim\limits_{y\to0}f(0,y)=\lim\limits_{y\to0}1=1$.（连续性）

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3 多元函数的连续性

1 解：(1) $\{(x,y)\mid x\neq 0$ 或 $y\neq 0\}$　　(2) $\{(x,y)\mid x\neq k\pi$ ~~且~~ $y\neq \frac{\pi}{2}+t\pi,\ k\in Z, t\in Z\}$

(3) $\{(x,y)\mid y^2\neq 2x\}$

2. 证明：设 $P\in\bar{D}$. $f(P,P_0)=d(P,P_0)$ 其在 $\bar{D}$ 上是连续的.

故 $f(P,P_0)$ 必有界. 即存在 $M,N>0$ 使 $N\leq f(P,P_0)\leq M$.

此时满足 $f(P,P_0)=N$ 的 P 点为与 P_0 距离最短的点. 满足 $f(P,P_0)=M$ 的 P 点为与 P_0 距离最长的点. 证毕。

3. 证明：由于函数 $f(x,y)$ 在区域 D 上连续. 所以 $f(x,y)$ 必有界. 设 $N\leq f(x,y)\leq M$.

则 $f(x_i,y_i)\in[N,M]$. $i=1,2,\cdots,n$.

故 $N=\frac{1}{n}\cdot N\cdot n\leq$ ~~f(x,y)~~ $\frac{1}{n}[f(x_1,y_1)+f(x_2,y_2)+\cdots+f(x_n,y_n)]\leq\frac{1}{n}\cdot M\cdot n=M$

故必∃点 (ξ,y) 使 $f(\xi,y)=\frac{1}{n}[f(x_1,y_1)+f(x_2,y_2)+\cdots+f(x_n,y_n)]$ (介值定理)

4. 证明：由于 $f(x,y)$ 在 (x_0,y_0) 处连续. ~~因此~~对 $\forall\varepsilon$. $|f(x,y)-f(x_0,y_0)|<\varepsilon$. ~~都~~∃$\delta>0$ 使

$|x-x_0|<\delta$. $|y-y_0|<\delta$. 显然当 $y=y_0$ 时. $|y-y_0|<\delta$ 恒成立. 故

$|f(x,y_0)-f(x_0,y_0)|<\varepsilon$ 时. 有∃$\delta>0$ 使 $|x-x_0|<\delta$. 证毕.

5 证明：显然 $[a_n,b_n]$ 是一个区间套. 对 $\forall n=1,2,3\cdots$. ~~……~~

要找到 R^2 中的 (x,y) 中的 x $(a_n\leq x\leq b_n)$. 必定这样的 x 应在 $[a_n,b_n]$ $(n\to\infty)$ 中

而 $0<b_n-a_n\to 0$ 说明了 $\lim\limits_{n\to\infty}b_n=\lim\limits_{n\to\infty}a_n$. 故此时仅存在这样一个 x 满足条件

其中 $x=\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}b_n$. 令 $\xi=\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}b_n$. 即此时 x 惟一值为 ξ

同理知仅存在一个 $y=\lim\limits_{n\to\infty}c_n=\lim\limits_{n\to\infty}d_n$. 令 $R^2(x,y)$ 中的 y 在 R_n 内.

令 $y=\lim\limits_{n\to\infty}c_n=\lim\limits_{n\to\infty}d_n$. 此时 y 惟一值为 y.

综上 (x,y) 中的 x,y 仅有惟一值 ξ,y. 即存在惟一的一个点 $(\xi,y)\in R_n$.

对 $\forall n=1,2,3,\cdots$.

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6解：如函数 $f(x,y)=\ln[1-(x^2+y^2)]$；$f(x,y)=\frac{1}{1-(x^2+y^2)}$ 等.

7证明：在区域D中找其一个子区域 D'. 其中 D' 为有界的闭区域

显然这样 D' 是存在的. 并且令 P_1, P_2 在 D' 内. 也即是在D中找一个有界

闭区域的子区域 D' 包含 P_1, P_2.

此时，对于 D' 区域中. $f(P)$ 是连续的. 对于 $f(P_1) \leq y \leq f(P_2)$.

由介值定理得 $\exists P_0 \in D' \subset D$. 即 $P_0 \in D$ 使 $f(P_0)=y$. 证毕

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4 偏导数与全微分

1解：(1) $\frac{\partial z}{\partial x}=\frac{1}{x+\sqrt{x^2+y^2}}\cdot\left(1+\frac{2x}{2\sqrt{x^2+y^2}}\right)=\frac{1}{\sqrt{x^2+y^2}}$

$\frac{\partial z}{\partial y}=\frac{1}{x+\sqrt{x^2+y^2}}\cdot\left(0+\frac{2y}{2\sqrt{x^2+y^2}}\right)=\frac{y}{\sqrt{x^2+y^2}\,(x+\sqrt{x^2+y^2})}$

(2) $\frac{\partial z}{\partial x}=\frac{1\cdot\sqrt{x^2+y^2}-x\cdot\frac{2x}{2\sqrt{x^2+y^2}}}{(\sqrt{x^2+y^2})^2}=\frac{x^2+y^2-x^2}{(x^2+y^2)^{3/2}}=\frac{y^2}{(x^2+y^2)^{3/2}}$

$\frac{\partial z}{\partial y}=x\cdot-\frac{1}{2}\cdot\frac{2y}{(x^2+y^2)^{3/2}}=\frac{-xy}{(x^2+y^2)^{3/2}}$

(3) 函数两边取对数 $\ln z=x^y\cdot\ln x$.

$\therefore\ \frac{1}{z}\cdot\frac{\partial z}{\partial x}=x^y\cdot\frac{1}{x}+y\cdot x^{(y-1)}\ln x\quad\therefore\ \frac{\partial z}{\partial x}=z\cdot(x^{y-1}+y\cdot x^{y-1}\ln x)$

即 $\frac{\partial z}{\partial x}=x^{x^y}\cdot x^{y-1}(y\ln x+1)$

$\frac{1}{z}\cdot\frac{\partial z}{\partial y}=\ln x\cdot x^y\cdot\ln x\quad\therefore\ \frac{\partial z}{\partial y}=x^{x^y}\cdot x^y\cdot(\ln x)^2$

(4). $\frac{\partial z}{\partial x}=\frac{y(x-y)-xy}{(x-y)^2}=\frac{-y^2}{(x-y)^2}$

$\frac{\partial z}{\partial y}=\frac{x(x-y)+xy}{(x-y)^2}=\frac{x^2}{(x-y)^2}$

(5) $\frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-(x\sqrt{y})^2}}\cdot\sqrt{y}=\frac{\sqrt{y}}{\sqrt{1-x^2y}}$

$\frac{\partial z}{\partial y}=\frac{1}{\sqrt{1-(x\sqrt{y})^2}}\cdot x\cdot\frac{1}{2\sqrt{y}}=\frac{x}{2\sqrt{y\cdot(1-x^2y)}}$

(6) $\frac{\partial z}{\partial x}=e^{-xy}+(-y)\cdot x\cdot e^{-xy}=(1-xy)\cdot e^{-xy}$

$\frac{\partial z}{\partial y}=x\cdot(-x)\cdot e^{-xy}=-x^2\cdot e^{-xy}$

(7) $\frac{\partial u}{\partial x}=-\frac{y}{x^2}-\frac{1}{z}\qquad\frac{\partial u}{\partial y}=\frac{1}{x}-\frac{z}{y^2}\qquad\frac{\partial u}{\partial z}=\frac{1}{y}+\frac{x}{z^2}$

(8) $\frac{\partial u}{\partial x}=z\cdot y\cdot(xy)^{z-1}$

$\frac{\partial u}{\partial y}=z\cdot x(xy)^{z-1}\qquad\frac{\partial u}{\partial z}=\ln(xy)\cdot(xy)^z$

2解：(1) $\frac{\partial z}{\partial x}\Big|_{(0,1)}=\left(\frac{x\cos 0-0}{1+\sin x+\sin 0}\right)'\Big|_{x=0}=\left(\frac{x}{1+\sin x}\right)'\Big|_{x=0}=\frac{1+\sin x-\cos x\cdot x}{(1+\sin x)^2}\Big|_{x=0}=1$

$\frac{\partial z}{\partial y}\Big|_{(0,1)}=\left(\frac{0-(y-1)\cos 0}{1+\sin 0+\sin(y-1)}\right)'\Big|_{y=1}=\left(\frac{1-y}{1+\sin(y-1)}\right)'\Big|_{y=1}=\frac{-(1+\sin(y-1))-(1-y)(\cos(y-1))}{(1+\sin(y-1))^2}\Big|_{y=1}$

$=\frac{-(1+0)-0}{(1+0)^2}=-1$

(2) $\frac{\partial z}{\partial x}\Big|_{(\frac{\pi}{2},1)}=\left(\frac{2}{1+\cos x}\right)'\Big|_{x=\frac{\pi}{2}}=-\frac{2}{(1+\cos x)^2}\cdot(-\sin x)\Big|_{x=\frac{\pi}{2}}=\frac{2\cdot 1}{(1+0)^2}=2,\ \frac{\partial z}{\partial y}\Big|_{(\frac{\pi}{2},1)}=0$

(3) $f_x(2,1,0) = [\ln(xy+z)]'|_{(2,1,0)} = (\ln x)'|_{x=2} = \frac{1}{2}$

$f_y(2,1,0) = \frac{df(2,y,0)}{dy}|_{y=1} = \frac{d(\ln 2y)}{dy}|_{y=1} = \frac{2}{2y}|_{y=1} = 1$

$f_z(2,1,0) = \frac{d(2,1,z)}{dz}|_{z=0} = \frac{d(\ln(2+z))}{dz}|_{z=0} = \frac{1}{2+z}|_{z=0} = \frac{1}{2}$

3.证明:

$0 < \left|\frac{x^2+y^2}{|x|+|y|}\right| \le \left|\frac{x^2}{|x|+0} + \frac{y^2}{|y|+0}\right| = |x|+|y|$

而 $\lim_{(x,y)\to 0} 0 = 0$, $\lim_{(x,y)\to(0,0)} (|x|+|y|) = 0$

∴ $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$. 故$f(x,y)$在$(0,0)$处连续.

而$f_x(0,0) = \frac{df(x,0)}{dx}|_{x=0} = \frac{d|x|}{dx}|_{x=0}$

易知函数 $u=|x|$在$x=0$处不可导. 故 $\frac{d|x|}{dx}$不存在. 即$f_x(0,0)$不存在

4.证明: $\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x}}\cdot\sin\frac{y}{x} + \sqrt{x}\cdot\frac{-y}{x^2}\cdot\cos\frac{y}{x}$. $\frac{\partial z}{\partial y} = \sqrt{x}\cos\frac{y}{x}\cdot\frac{1}{x}$

∴ $x\cdot\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y} = \frac{\sqrt{x}}{2}\sin\frac{y}{x} - \frac{y\sqrt{x}}{x}\cos\frac{y}{x} + \frac{\sqrt{x}y}{x}\cos\frac{y}{x} = \frac{\sqrt{x}}{2}\sin\frac{y}{x} = \frac{z}{2}$. 证毕

5.解 (1). $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{2}{2x+3y}\right) = -3\cdot\frac{2}{(2x+3y)^2} = -\frac{6}{(2x+3y)^2}$

(2). $f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right) = \frac{\partial}{\partial y}(y\cos x + e^x) = \cos x$

(3) $f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right) = \frac{\partial}{\partial y}\left(1+y^2+12x^2-\frac{2x}{x^2+1}\right) = 2y$

(4) $f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right) = \frac{\partial}{\partial y}\left(x\cdot\frac{y}{xy}+\ln(xy)\right) = \frac{\partial}{\partial y}(1+\ln(xy)) = \frac{x}{xy} = \frac{1}{y}$

6 证明. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(-3e^{-3y}\sin 3x) = -9e^{-3y}\cos 3x$

$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(-3\cos 3x\cdot e^{-3y}) = 9\cos 3x\cdot e^{-3y}$

∴ $\Delta u \equiv \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u = \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$

7.证明: $\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(ce^{x+ct} + 4\cdot 3c\cdot[-\sin(3x+3ct)]\right) = c^2e^{x+ct} - 36c^2\cos(3x+3ct)$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(e^{x+ct} - 12\cos(3x+3ct)\right) = e^{x+ct} - 36\cos(3x+3ct)$

∴ $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$

8.证明: $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$

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$= V_{yx}(x,y) - V_{xy}(x,y) = 0$ （$\because V(x,y)$在区域D内有连续二阶偏导数）

同理有 $\Delta V = 0$. 证毕.

9解. $z(x,y) = \int \frac{\partial z}{\partial x} dx = -\sin y \cdot x - \frac{1}{y}\ln|1-xy| + C$

而 $z(0,y) = 0 - 0 + C = 2\sin y + y^2$ 即 $C = 2\sin y + y^2$

$\therefore z(x,y) = -x\cdot\sin y - \frac{1}{y}\ln|1-xy| + 2\sin y + y^2$

10解. (1) $dz = \frac{d\, e^{y/x}}{dx}\cdot dx + \frac{d(e^{y/x})}{dy}\cdot dy$

$= (-\frac{y}{x^2}\cdot e^{y/x})dx + (\frac{1}{x}\cdot e^{y/x})dy$

(2). $dz = \frac{d\frac{x+y}{x-y}}{dx}\cdot dx + \frac{d\frac{x+y}{x-y}}{dy}\cdot dy = -\frac{2y}{(x-y)^2}dx + \frac{2x}{(x-y)^2}dy$

(3) $dz = \frac{d(\arctan\frac{y}{x} + \arctan\frac{x}{y})}{dx}dx + \frac{d(\arctan\frac{y}{x} + \arctan\frac{x}{y})}{dy}\cdot dy$

$= \left[\frac{1}{1+(\frac{y}{x})^2}\cdot(-\frac{y}{x^2}) + \frac{1}{1+(\frac{x}{y})^2}\cdot\frac{1}{y}\right]dx + \left[\frac{1}{1+(\frac{y}{x})^2}\cdot\frac{1}{x} + \frac{1}{1+(\frac{x}{y})^2}\cdot(-\frac{x}{y^2})\right]dy$

$= 0\cdot dx + 0\cdot dy = 0$

(4). $du = \frac{\partial u(x,y,z)}{\partial x}dx + \frac{\partial u(x,y,z)}{\partial y}dy + \frac{\partial u(x,y,z)}{\partial z}dz$

$= \frac{2x}{2\sqrt{x^2+y^2+z^2}}dx + \frac{2y}{2\sqrt{x^2+y^2+z^2}}dy + \frac{2z}{2\sqrt{x^2+y^2+z^2}}dz$

$= (x\,dx + y\,dy + z\cdot dz)/\sqrt{x^2+y^2+z^2}$

11解. $\because dz = f_x(x,y)dx + f_y(x,y)dy$

$\therefore f_x(x,y) = 4x^3 + 10xy^3 - 3y^4$ $\therefore f(x,y) = \int f_x(x,y)dx = x^4 + 5x^2y^3 - 3y^4x + C_1 + C$

且 $f_y(x,y) = 15x^2y^2 - 12xy^3 + 5y^4$ $\therefore f(x,y) = \int f_y(x,y)dy = 5x^2y^3 - 3xy^4 + y^5 + C_2 + C$

其中 C_1 不含 x, C_2 不含 y. 且 $x^4 + 5x^2y^3 - 3y^4x + C_1 = 5x^2y^3 - 3xy^4 + y^5 + C_2$

故 $C_1 = y^5$, $C_2 = x^4$.

$\therefore f(x,y) = x^4 + 5x^2y^3 - 3xy^4 + y^5 + C$. (C为任意常数)

12.解. 依题意有: $z_x(x,y) = x - \frac{y}{x^2+y^2}$. $\therefore z(x,y) = \frac{1}{2}x^2 - \arctan\frac{x}{y} + C_1 + C$

$z_y(x,y) = y + \frac{x}{x^2+y^2}$ $\therefore z(x,y) = \frac{1}{2}y + \arctan\frac{y}{x} + C_2 + C$

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其中 C_1 是不含 x 的含 y 多项式. C_2 为不含 y 含 x 多项式.

而 $\frac{1}{2}x^2-\arctan\frac{x}{y}+C_1+C=\frac{1}{2}y^2+\arctan\frac{y}{x}+C_2+C$

$\therefore C_1=\frac{1}{2}y^2,\ C_2=\frac{1}{2}x^2$ ($\because -\arctan\frac{x}{y}=\arctan\frac{y}{x}$)

故 $z(x,y)=\frac{1}{2}(x^2+y^2)+\arctan\frac{y}{x}+C$. ($C$ 为任意常数)

13. 证明: 对 D 内任意一点 (x,y), (x_0,y_0) 也在 D 内.

$\therefore f(x,y)-f(x_0,y_0)=[f(x,y)-f(x,y_0)]+[f(x,y_0)-f(x_0,y_0)]$

$=\frac{\partial f}{\partial x}\Big|_{x=C_1}\cdot(y-y_0)+\frac{\partial f}{\partial y}\Big|_{y=C_2}(x-x_0)\equiv 0$

$\therefore f(x,y)\equiv f(x_0,y_0)$. 显然 $f(x_0,y_0)$ 为一个常数. 证毕.

14. 证明: $\lim\limits_{(x,y)\to(0,0)}f(x,y)=\lim\limits_{(x,y)\to(0,0)}\sqrt{|xy|}=\sqrt{\left|\lim\limits_{(x,y)\to(0,0)}xy\right|}=0=f(0,0)$

故函数 $f(x,y)$ 在点 $(0,0)$ 处连续.

$\because \lim\limits_{x\to 0}f_x(x,0)=\lim\limits_{x\to 0}0=0$, 故 $f_x(0,0)$ 存在. 同理 $f_y(0,0)$ 存在.

$\because d|xy|$ 在 $x=0$ 或 $y=0$ 时均不存在.

故 $f(x,y)$ 在 $(0,0)$ 处不可微.

15. 证明: 设 $z(x,y)=P(x,y)dx+Q(x,y)dy$ 则 $\frac{\partial z(x,y)}{\partial x}=P(x,y)$, $\frac{\partial z(x,y)}{\partial y}=Q(x,y)$

$\therefore \frac{\partial P}{\partial y}=\frac{\partial}{\partial y}\cdot\frac{\partial z(x,y)}{\partial x}=z_{xy}$. 同理 $\frac{\partial Q}{\partial x}=z_{yx}$.

$\because P,Q\in C^1$. 则 $z_{xy}=z_{yx}$. 故 $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$.

16 (1) 解 $f_x(0,y)=\frac{d}{dx}\left[\frac{(x^2-y^2)xy}{x^2+y^2}\right]\Big|_{x=0}=\frac{(3yx^2-y^3)(x^2+y^2)-2x^2y(x^2-y^2)}{(x^2+y^2)^2}\Big|_{x=0}=-y$

(2) 证明: $f_x(0,0)=\frac{d}{dx}\left(\frac{(x^2-0^2)\cdot x\cdot 0}{x^2+0^2}\right)\Big|_{x=0}=\frac{d}{dx}(0)\Big|_{x=0}=0$. 证毕.

(3) 证明: $f_{xy}(0,0)=\frac{d}{dy}[f_x(0,y)]\Big|_{y=0}=\frac{d}{dy}(-y)\Big|_{y=0}=-1$

(4) 解. $f_y(x,0)=\frac{d}{dy}\left[\frac{(x^2-y^2)xy}{x^2+y^2}\right]\Big|_{y=0}=\frac{(x^3-3xy^2)(x^2+y^2)-2y^2x(x^2-y^2)}{(x^2+y^2)^2}\Big|_{y=0}=x$

$\therefore f_{yx}(0,0)=\frac{d}{dx}[f_y(x,0)]\Big|_{x=0}=\frac{dx}{dx}\Big|_{x=0}=1$

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17解：$\frac{\partial^3 z}{\partial x^3}=\frac{\partial}{\partial x}\cdot\left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\right)=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\ln(xy)+x\cdot\frac{1}{xy}\cdot y\right)\right)=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(\ln(xy)+1)\right)$

$=\frac{\partial}{\partial x}\left(\frac{y}{xy}\right)=\frac{\partial}{\partial x}\left(\frac{1}{x}\right)=-\frac{1}{x^2}$

$\frac{\partial^3 z}{\partial x\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\right)=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(\ln(xy)+1)\right)=\frac{\partial}{\partial y}\left(\frac{x}{xy}\right)=\frac{\partial}{\partial y}\left(\frac{1}{y}\right)=-\frac{1}{y^2}$

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5 复合函数与隐函数微分法.

1解：(1) $\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{2u}{2\sqrt{u^2+v^2}}\cdot y+\frac{2v}{2\sqrt{u^2+v^2}}\cdot 0=\frac{xy}{\sqrt{x^2y^2+y^4}}\cdot y=\frac{x|y|}{\sqrt{x^2+y^2}}$

$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}=\frac{2u}{2\sqrt{u^2+v^2}}\cdot x+\frac{2v}{2\sqrt{u^2+v^2}}\cdot 2y=\frac{xu+2vy}{\sqrt{u^2+v^2}}=\frac{\operatorname{sgn}y\cdot(x^2+2y^2)}{\sqrt{x^2+y^2}}$

(2) $\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{2u}{v}\cdot ye^x+(-\frac{u^2}{v^2})\cdot\ln y$

$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}=\frac{2u}{v}\cdot e^x+(-\frac{u^2}{v^2})\frac{x}{y}$

(3). $\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{y}{2\sqrt{xy}}\cdot\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}$.

$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=\frac{x}{2\sqrt{xy}}\cdot\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}$

(4) $\frac{\partial z}{\partial x}=f_1'\cdot y+f_2'\cdot\frac{1}{y}$　　$\frac{\partial z}{\partial y}=f_1'\cdot x+f_2'\cdot(-\frac{x}{y^2})$

(5) $\frac{\partial z}{\partial x}=f_1'\cdot 2x+f_2'\cdot e^{xy}\cdot y$　　$\frac{\partial z}{\partial y}=f_1'(-2y)+f_2'\cdot e^{xy}\cdot x$

2解：$\frac{\partial u}{\partial x}=f_1'\cdot 1+f_2'\cdot 2x$.

则 $\frac{\partial}{\partial x}(\frac{\partial u}{\partial x})=\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}(f_1'+f_2'\cdot 2x)=\frac{\partial}{\partial x}(f_1')+\frac{\partial}{\partial x}(f_2')\cdot 2x+\frac{\partial}{\partial x}(2x)\cdot f_2'$

$=\frac{\partial}{\partial x}(f_1')+\frac{\partial}{\partial x}(f_2')\cdot 2x+2f_2'$　　--- ※

而 $\frac{\partial}{\partial x}(f_1')=f_{11}''\cdot 1+f_{12}''\cdot 2x$　　$\frac{\partial}{\partial x}(f_2')=f_{21}''\cdot 1+f_{22}''\cdot 2x$. 代入※得

$\frac{\partial^2 u}{\partial x^2}=f_{11}''+f_{12}''\cdot 2x+f_{21}''\cdot 2x+4x^2\cdot f_{22}''+2f_2'$

根据 x, y, z 对称性知 [redacted]

$\therefore\ \Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=3f_{11}''+2(x+y+z)\cdot(f_{12}''+f_{21}'')+4(x^2+y^2+z^2)f_{22}''+6f_2'$

3解：①. $\frac{\partial u}{\partial x}=f_1'\cdot\frac{\partial \xi}{\partial x}+f_2'\cdot\frac{\partial \eta}{\partial x}=f_1'\cdot e^x\cos y+f_2' e^x\sin y$

$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}(\frac{\partial u}{\partial x})=\frac{\partial}{\partial x}(f_1')\cdot e^x\cos y+f_1'\cdot\frac{\partial}{\partial x}(e^x\cos y)+\frac{\partial}{\partial x}(f_2')\cdot e^x\sin y+f_2'\cdot\frac{\partial}{\partial x}(e^x\sin y)$

$=\frac{\partial}{\partial x}(f_1')\cdot e^x\cos y+f_1' e^x\cos y+\frac{\partial}{\partial x}(f_2')e^x\sin y+f_2'\cdot e^x\sin y$

$=(f_{11}''\cdot e^x\cos y+f_{12}'' e^x\sin y)e^x\cos y+f_1' e^x\cos y+(f_{21}'' e^x\cos y+f_{22}'' e^x\sin y)\cdot e^x\sin y$

$+f_2' e^x\sin y$

$=e^{2x}(f_{11}''\cos^2 y+f_{12}''\sin y\cos y+f_{21}''\sin y\cos y+f_{22}''\sin^2 y)+e^x(f_1'\cos y+f_2'\sin y)$.

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② $\frac{\partial u}{\partial y}=f_1'\cdot\frac{\partial \xi}{\partial y}+f_2'\cdot\frac{\partial \eta}{\partial y}=f_1'(-\sin y)e^x+f_2'\cdot e^x\cos y.$

$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}(\frac{\partial u}{\partial y})=\frac{\partial}{\partial y}(f_1')\cdot(-\sin y\cdot e^x)+\frac{\partial}{\partial y}(-\sin y\, e^x)\cdot f_1'+\frac{\partial}{\partial y}(f_2')\cdot e^x\cos y+\frac{\partial}{\partial y}(e^x\cos y)\cdot f_2'$

$=(f_{11}''\cdot\frac{\partial \xi}{\partial y}+f_{12}''\cdot\frac{\partial \eta}{\partial y})(-\sin y\cdot e^x)+(-\cos y)e^x f_1'+(f_{21}''\cdot\frac{\partial \xi}{\partial y}+f_{22}''\frac{\partial \eta}{\partial y})\cdot e^x\cos y$

$+(-e^x\sin y)f_2'$

$=-e^x(f_1'\cos y+f_2'\sin y)+e^{2x}(f_{11}''\sin^2 y-\sin y\cos y(f_{12}''+f_{21}'')+f_{22}''\cos^2 y)$

4.证明：$\frac{\partial z}{\partial x}=nx^{n-1}\cdot f(\frac{y}{x^2})+x^n\cdot f'(\frac{y}{x^2})\cdot\frac{-2y}{x^3}$.　　$\frac{\partial z}{\partial y}=x^n\cdot f'(\frac{y}{x^2})\cdot\frac{1}{x^2}$

$\therefore x\cdot\frac{\partial z}{\partial x}+2y\cdot\frac{\partial z}{\partial y}=nx^n\cdot f(\frac{y}{x^2})+x^{n-2}\cdot f'(\frac{y}{x^2})\cdot(-2y)+x^{n-2}\cdot f'(\frac{y}{x^2})\cdot 2y$

$=nx^n f(\frac{y}{x^2})=n\cdot z$. 证毕.

5.证明：$\frac{\partial z}{\partial x}=\frac{-F'(x^2-y^2)\cdot 2xy}{F^2(x^2-y^2)}$　　$\frac{\partial z}{\partial y}=\frac{F(x^2-y^2)-y\cdot F'(x^2-y^2)\cdot(-2y)}{F^2(x^2-y^2)}$

$\therefore\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\cdot\frac{\partial z}{\partial y}=\frac{-F'(x^2-y^2)\cdot 2y}{F^2(x^2-y^2)}+\frac{F(x^2-y^2)+2y^2F'(x^2-y^2)}{yF^2(x^2-y^2)}=\frac{1}{yF(x^2-y^2)}=\frac{z}{y^2}$

6.证明：$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f_{11}''+f_{22}''=0$

$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}=f_1'\cdot e^s\cos t+f_2'\cdot e^s\sin t\cdot$

$\therefore\frac{\partial}{\partial s}(\frac{\partial u}{\partial s})=\frac{\partial}{\partial s}(f_1')\cdot e^s\cos t+e^s\cos t\cdot f_1'+e^s\cdot\sin t\cdot f_2'+\frac{\partial}{\partial s}(f_2')\cdot e^s\sin t$

$=e^s(\cos t\cdot f_1'+\sin t\cdot f_2')+e^{2s}(f_{11}''\cos^2 t+\sin t\cos t(f_{12}''+f_{21}'')+f_{22}''\cdot\sin^2 t)$

同理求得 $\frac{\partial}{\partial t}\cdot(\frac{\partial u}{\partial t})=-e^s(f_1'\cdot\cos t+f_2'\sin t)+(-e^{2s})(f_{11}''\sin^2 t+\sin t\cos t(f_{12}''+f_{21}'')-f_{22}''\cos^2 t)$

$\therefore\frac{\partial u^2}{\partial s^2}+\frac{\partial u^2}{\partial t^2}=e^{2x}(f_{11}''+f_{22}'')=0$. 证毕.

7.解：(1) $\frac{\partial u}{\partial x}=F'(x^2+y^2)\cdot 2x$，$\frac{\partial u}{\partial y}=F'(x^2+y^2)\cdot 2y$.　$\therefore y\frac{\partial u}{\partial x}-x\frac{\partial u}{\partial y}=0$

(2) $\frac{\partial u}{\partial t}=F'(x-ct)\cdot(-c)$，$\frac{\partial u}{\partial x}=F'(x-ct)$　$\therefore$　$\frac{\partial u}{\partial t}+c\cdot\frac{\partial u}{\partial x}=0$

8 证明：由于 $f(tx,ty,tz)=t^n f(x,y,z)$ 对任意 t 都成立，∴该式两端可看成关于 x,y,z,t 这四个变量的函数. 令 $u=tx$，$v=ty$，$w=tz$. 则有 $f(u,v,w)=t^n f(x,y,z)$.

两边对 t 求偏微商.有 $x\cdot f_u+y\cdot f_v+z f_w=n\cdot t^{n-1}\cdot f(x,y,z)$. 令 $t=1$ 得.

$u=x$，$v=y$，$w=z$　∴上式化为 $x\cdot f_x+y f_y+z f_z=n f(x,y,z)$. 证毕

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9. 证明：$\frac{\partial f(x,y)}{\partial x} = f_x(x,y) \equiv 0$.　$\therefore f(x,y) = C$.（C为与x无关的表达式）

可以令 $F(y) = C$.　$\therefore f(x,y) = F(y)$. 证毕.

10 证明：令 $x = r\cos\theta,\ y = r\cdot\sin\theta$.　$\therefore x\cdot f_x(x,y) + y\cdot f_y(x,y) = r\cos\theta\cdot f_1' + r\sin\theta\cdot f_2' \equiv 0$

所以 $\frac{\partial f(r\cdot\cos\theta,\ r\cdot\sin\theta)}{\partial r} = f_1'\cdot\cos\theta + f_2'\cdot\sin\theta \equiv 0$.

由9题知，存在一个函数 $F(\theta)$，使得 $f(r\cos\theta, r\sin\theta) = F(\theta)$

11 证明：令 $x = r\cos\theta,\ y = r\sin\theta$.　$\therefore y\cdot f_x(x,y) - x f_y(x,y) = r\sin\theta f_1' - r\cos\theta f_2' \equiv 0$

$\therefore \frac{\partial f(r\cos\theta,\ r\sin\theta)}{\partial \theta} = -r\sin\theta f_1' + r\cos\theta\cdot f_2' \equiv 0$.

由9题知，存在一个函数 $G(r)$ 使得 $f(r\cos\theta, r\sin\theta) = G(r)$

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6 方向导数与梯度

1. 解 函数$f(x,y)$在P_0处沿l方向导数为 $\frac{\partial f}{\partial l}\Big|_{P_0}=3\cos\alpha+3\sqrt{3}\cos\beta=3\cos\theta+3\sqrt{3}\sin\theta$

即 $\frac{\partial f}{\partial l}\Big|_{P_0}=6\sin(\theta+\frac{\pi}{6})$

(1) 当$\theta=\frac{\pi}{3}$时. $\frac{\partial f}{\partial l}\Big|_{P_0}$取到最大值.

(2) 当$\theta=\frac{4}{3}\pi$时, $\frac{\partial f}{\partial l}\Big|_{P_0}$取到最小值.

(2) 当$\theta=\frac{11}{6}\pi$或$\frac{5\pi}{6}$时. $\frac{\partial f}{\partial l}\Big|_{P_0}=0$.

2解. P_0P的方向向量为$(3,4)$. 其方向余弦为$(\frac{3}{5},\frac{4}{5})$

$\therefore \frac{\partial f}{\partial P_0P}\Big|_{P_0}=(3x_0^2-[illegible]+3y_0^2)\cdot\frac{3}{5}+(-3x_0^2+6x_0y_0)\frac{4}{5}=0$

3解. 抛物线在该点的切线斜率 $k=4x|_{x=1}=4$. $\therefore$ 其方向余弦为$(\frac{1}{\sqrt{17}},\frac{4}{\sqrt{17}})$或$(-\frac{1}{\sqrt{17}},-\frac{4}{\sqrt{17}})$

$\therefore \frac{\partial f}{\partial l}\Big|_{P_0}=\frac{1}{x_0+y_0}\cdot\frac{1}{\sqrt{17}}+\frac{1}{x_0+y_0}\frac{4}{\sqrt{17}}=\frac{5}{3\sqrt{17}}$ 或 $\frac{\partial f}{\partial l}\Big|_{P_0}=-\frac{5}{3\sqrt{17}}$

4解: 根据 $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$ 和 $\cos\alpha=\cos\beta=\cos\gamma$ 得 $\cos\alpha=\cos\beta=\cos\gamma=\pm\frac{\sqrt{3}}{3}$

$\therefore \frac{\partial u}{\partial l}\Big|_{P_0}=(y_0+z_0)\cos\alpha+(x_0+z_0)\cos\beta+(x_0+y_0)\cdot\cos\gamma=2(x_0+y_0+z_0)\cdot\cos\alpha=\pm4\sqrt{3}$

5解: $f_x(1,2)=2\cdot1+2\cdot2=6$　$f_y(x,y)=2\cdot1+2\cdot2=6$

$\therefore z$在$(1,2)$处的梯度为$(6,6)$

6解: $f_x(x_0,y_0)=\frac{1}{1+(\frac{y_0}{x_0})^2}\cdot\frac{-y_0}{x_0^2}=\frac{-y_0}{x_0^2+y_0^2}$　$f_y(x_0,y_0)=\frac{1}{1+(\frac{y_0}{x_0})^2}\cdot\frac{1}{x_0}=\frac{x_0}{x_0^2+y_0^2}$

$\therefore z=f(x,y)$在点(x_0,y_0)处的梯度为 $\frac{1}{x_0^2+y_0^2}(-y_0,x_0)$

沿向量(x_0,y_0)的方向导数为 $\frac{\partial f}{\partial l}\Big|_{P_0}=-\frac{y_0}{x_0^2+y_0^2}\cdot\frac{x_0}{\sqrt{x_0^2+y_0^2}}+\frac{x_0}{x_0^2+y_0^2}\cdot\frac{y_0}{\sqrt{x_0+y_0}}=0$.

7解. $f_x(x,y)=\frac{x}{y}\cdot\frac{-y}{x^2}=-\frac{1}{x}$.　$f_y(x,y)=\frac{x}{y}\cdot\frac{1}{x}=\frac{1}{y}$

$\therefore$ 函数z在点A的梯度为$(-3,10)$. 在点B的梯度为$(-1,6)$

两个梯度之间的夹角余弦 $\cos\theta=$ [illegible] $\frac{(-3,10)\cdot(-1,6)}{\sqrt{109}\cdot\sqrt{37}}=\frac{63}{\sqrt{109}\cdot\sqrt{37}}$

8解: $f_x(1,1)=2$. $f_y(1,1)=0$. $\therefore$ 函数$f(x,y)$沿方向$(\cos\alpha,\cos\beta)$方向导数为$2\cos\alpha$

$\therefore$ 其最大值为2. 沿方向$(1,0)$. 最小值为-2. 沿方向$(-1,0)$

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9 证明：椭圆周在 (x_0, y_0) 处的切线方向为 (a, b).

对椭圆方程两边进行求导. $2x + 4y \cdot y' = 0$.

$\therefore$ 椭圆在 (x_0, y_0) 处的切线斜率为 $y' = -\frac{x_0}{2y_0}$. $\therefore$ 其法方向为 $(x_0, 2y_0)$

$f_x(x, y) = \frac{-2y}{x^3}$, $f_y(x, y) = \frac{1}{x^2}$

$\frac{\partial f}{\partial \ell}\Big|_{P_0} = -\frac{2y_0}{x_0^3} \cdot \frac{2x_0}{\sqrt{4y_0^2 + x_0^2}} + \frac{1}{x_0^3} \cdot \frac{2y_0}{\sqrt{4y_0^2 + x_0^2}} = 0$. 证毕

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7 多元函数的微分中值定理与泰勒公式

1解：$f(1,1)=1-1=0$. $\frac{\partial f}{\partial x}(1,1)=1$. $\frac{\partial f}{\partial y}(1,1)=0$. $\frac{\partial^2 f}{\partial x^2}(1,1)=0$

$\frac{\partial^2 f}{\partial x\partial y}(1,1)=1$ $\frac{\partial^2 f}{\partial y^2}(1,1)=0$.

令 $\Delta x=x-1$, $\Delta y=y-1$. 则有

$f(1+\Delta x,1+\Delta y)=0+\frac{1}{1!}(1\cdot\Delta x+0\cdot\Delta y)+\frac{1}{2!}(0\cdot\Delta x^2+2\cdot1\cdot\Delta x\cdot\Delta y+0\cdot\Delta y^2)$

即 $f(x,y)=x-1+\frac{1}{2}\cdot2(x-1)\cdot(y-1)$

即 $f(x,y)=xy-y=x-1+(x-1)(y-1)=y(x-1)$

2解：(1) $f(0,0)=1$ $\frac{\partial f}{\partial x}(0,0)=\frac{1}{\cos y}(-\sin x)\Big|_{(0,0)}=0$. $\frac{\partial f}{\partial y}(0,0)=\frac{-\cos x}{\cos^2 y}\cdot(-\sin y)\Big|_{(0,0)}=0$

$\frac{\partial^2 f}{\partial x^2}(0,0)=-\frac{1}{\cos y}\cdot\cos x\Big|_{(0,0)}=-1$ $\frac{\partial^2 f}{\partial y^2}=\frac{\cos x\cos^2 y+2\sin^2 y\cos x}{\cos^3 y}\Big|_{(0,0)}=1$

$\frac{\partial^2 f}{\partial x\partial y}(0,0)=-\sin x\cdot\frac{-1}{\cos^2 y}\cdot(-\sin y)\Big|_{(0,0)}=0$

令 $\Delta x=x$, $\Delta y=y$. 则有

$f(\Delta x+0,\Delta y+0)=f(0,0)+\frac{1}{1!}(0\cdot\Delta x+0\cdot\Delta y)+\frac{1}{2!}(-1\cdot\Delta x^2+2\cdot0\cdot\Delta x\cdot\Delta y+1\cdot\Delta y^2)+o(\rho^2)$

即 $f(x,y)=1-\frac{1}{2}(x^2-y^2)+o(\rho^2)$, $\rho\to0$. 其中 $\rho=\sqrt{x^2+y^2}$.

(2) $f(0,0)=0$, $\frac{\partial f}{\partial x}(0,0)=\frac{1}{1+x+y}\Big|_{(0,0)}=1$ $\frac{\partial f}{\partial y}(0,0)=\frac{1}{1+x+y}\Big|_{(0,0)}=1$

$\frac{\partial^2 f}{\partial x^2}(0,0)=-\frac{1}{(1+x+y)^2}\Big|_{(0,0)}=-1$. $\frac{\partial^2 f}{\partial y^2}=-\frac{1}{(1+x+y)^2}\Big|_{(0,0)}=-1$.

$\frac{\partial^2 f}{\partial x\partial y}=-\frac{1}{(1+x+y)^2}\Big|_{(0,0)}=-1$

令 $\Delta x=x$, $\Delta y=y$. 则

$f(\Delta x+0,\Delta y+0)=f(0)+\frac{1}{1!}(1\cdot\Delta x+1\cdot\Delta y)+\frac{1}{2!}(-1\cdot\Delta x^2+(-1)2\Delta x\Delta y+(-1)\cdot\Delta y^2)+o(\rho^2)$

即 $f(x,y)=x+y-\frac{1}{2}(x^2+2xy+y^2)+o(\rho^2)$, $\rho\to0$ ($\rho=\sqrt{x^2+y^2}$)

(3) $f(0,0)=1$. $\frac{\partial f}{\partial x}(0,0)=\frac{-2x}{2\sqrt{1-x^2-y^2}}\Big|_{(0,0)}=0$, $\frac{\partial f}{\partial y}(0,0)=\frac{-2y}{2\sqrt{1-x^2-y^2}}\Big|_{(0,0)}=0$

$\frac{\partial^2 f}{\partial x^2}(0,0)=-1$, $\frac{\partial^2 f}{\partial y^2}(0,0)=-1$, $\frac{\partial^2 f}{\partial x\partial y}(0,0)=0$.

$\therefore f(x,y)=f(0,0)+\frac{1}{1!}(0\cdot x+0\cdot y)+\frac{2}{2!}(-1\cdot x^2+2\cdot0\cdot x\cdot y+(-1)\cdot y^2)+o(\rho^2)$

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即 $f(x,y)=1-\frac{1}{2}x^2-\frac{1}{2}y^2+o(\rho^2),\ \rho\to 0$

(4) $f(0,0)=0,\quad \frac{\partial f}{\partial x}(0,0)=\cos(x^2+y^2)\cdot 2x\Big|_{(0,0)}=0 \quad \frac{\partial f}{\partial y}(0,0)=0$

$\frac{\partial^2 f}{\partial x^2}=\left[-\sin(x^2+y^2)\cdot 2x\right]\cdot 2x+2\cos(x^2+y^2)\Big|_{(0,0)}=2 \quad \frac{\partial^2 f}{\partial y^2}=2$

$\frac{\partial^2 f}{\partial x\partial y}=2x\cdot\left[-\sin(x^2+y^2)\right]\cdot 2y\Big|_{(0,0)}=0$

令 $\Delta x=x,\ \Delta y=y$. 则有

$f(\Delta x+0,\Delta y+0)=f(0,0)+\frac{1}{1!}\cdot(0\cdot\Delta x+0\cdot\Delta y)+\frac{1}{2!}(2\cdot\Delta x^2+0\cdot 2\cdot\Delta x\cdot\Delta y+2\cdot\Delta y^2)+o(\rho^2)$

即 $f(x,y)=x^2+y^2+o(\rho^2),\ \rho\to 0$

3.解. $f(0,0)=0 \quad \frac{\partial f}{\partial x}(0,0)=\frac{1}{1+x+y}\Big|_{(0,0)}=1 \quad \frac{\partial f}{\partial y}(0,0)=\frac{1}{1+x+y}\Big|_{(0,0)}=1$

$d^2f(x,y)=d\left(\frac{\partial f}{\partial x}\cdot dx+\frac{\partial f}{\partial y}dy\right)=d\left[\frac{1}{1+x+y}(dx+dy)\right]=-\frac{(dx+dy)^2}{(1+x+y)^2}$

令 $\Delta x=x,\ \Delta y=y$.

则 $d^2f(0+\theta\cdot\Delta x,0+\theta\cdot\Delta y)=-\frac{(x+y)^2}{(1+\theta\cdot x+\theta y)^2}$.

$\therefore f(x,y)=f(0,0)+\frac{1}{1!}(1\cdot x+1\cdot y)+\frac{1}{2!}\cdot d^2f(0+\theta\Delta x,0+\theta\cdot\Delta y)=x+y-\frac{(x+y)^2}{2(1+\theta x+\theta y)^2}$

4.证明: 依题意, 当 $x,y,z\to 0$ 时结论成立为所需证明.

令 $f(x,y,z)=\cos(x+y+z)-\cos x\cos y\cos z$.

则 $\frac{\partial f(x,y,z)}{\partial x}(0,0,0)=-\sin(x+y+z)+\cos y\cos z\sin x\Big|_{(0,0,0)}=0$

$\frac{\partial^2 f(x,y,z)}{\partial x^2}(0,0,0)=-\cos(x+y+z)+\cos x\cos y\cos z\Big|_{(0,0,0)}=0$

$\frac{\partial^2 f(x,y,z)}{\partial x\partial y}(0,0,0)=-\cos(x+y+z)+\sin x\cos z\cdot(-\sin y)\Big|_{(0,0,0)}=-1$

根据 x,y,z 的对称性知. $\frac{\partial f}{\partial y}(0,0,0)=0,\ \frac{\partial f}{\partial z}(0,0,0)=0,\ \frac{\partial^2 f}{\partial y^2}(0,0,0)=0$

$\frac{\partial^2 f}{\partial z^2}(0,0,0)=0 \quad \frac{\partial^2 f}{\partial y\partial z}(0,0,0)=-1,\ \frac{\partial^2 f}{\partial x\partial z}(0,0,0)=-1$

令 $\Delta x=x,\ \Delta y=y,\ \Delta z=z$. 则

$f(0+\Delta x,0+\Delta y,0+\Delta z)=f(0,0,0)+\frac{1}{1!}(0\cdot\Delta x+0\cdot\Delta y+0\cdot\Delta z)+\frac{1}{2!}(0(\Delta x^2+\Delta y^2+\Delta z^2)-2(\Delta x\Delta y+\Delta y\Delta z$

$+\Delta x\cdot\Delta z))+o(\rho^2)=0+0-\Delta x\cdot\Delta y-\Delta x\cdot\Delta z-\Delta y\cdot\Delta z+o(\rho^2),\ \rho\to 0.$

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$\therefore f(x,y,z) = -xy - xz - yz + o(\rho^2)$，$\rho \to 0$.

当$|x|,|y|,|z|$足够小，即 $x,y,z \to 0$ 时. $o(\rho^2)$可看作0.

$\therefore f(x,y,z) \approx -xy - xz - yz$. 即 $\cos(x+y+z) - \cos x \cos y \cos z \approx -(xy+yz+xz)$.

5 证明：令$\Delta x = x$，$\Delta y = y$. 则由二元函数的拉格朗日公式得：

$$f(0+\Delta x, 0+\Delta y) - f(0,0) = \frac{\partial f}{\partial x}(0+\theta\Delta x, 0+\theta\Delta y)\cdot\Delta x + \frac{\partial f}{\partial y}(0+\theta\Delta x, 0+\theta\Delta y)\cdot\Delta y$$

即 $f(x,y) - f(0,0) = \left(\frac{\partial f}{\partial x}\cdot x + \frac{\partial f}{\partial y}\cdot y\right) \equiv 0$

$\therefore f(x,y) = f(0,0)$　$\because f(0,0)$为一常数. $\therefore f(x,y)$在D内是一个常数.

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8 隐函数存在定理

1解: (1). 对方程两边关于x求偏导数 得: $3x^2 z + x^3 z_x + 3z^2 z_x x + z^3 - 2y z_x = 0$

$\therefore z_x = -\frac{3x^2 z + z^3}{x^3+3xz^2-2y}$

对方程两边关于y求偏导数得. $x^3 z_y + 3xz^2 z_y - 2z - 2y z_y = 0$

$\therefore z_y = \frac{2z}{x^3+3xz^2-2y}$

(2). 对方程两边关于x求偏导数得 $y z_x - \frac{1}{z} z_x = 1$ $\therefore z_x = \frac{z}{yz-1}$

对方程两边关于y求偏导数得 $z + y z_y - \frac{1}{z} z_y = 1$ $\therefore z_y = \frac{z(1-z)}{yz-1}$

(3) 对方程两边关于x求偏导数得. $1 + z_x - \varepsilon\cos z\cdot z_x = 0$ $\therefore z_x = \frac{1}{\varepsilon\cos z - 1}$

对方程两边关于y求偏导数得. $0 + z_y - \varepsilon\cos z\cdot z_y = 1$ $\therefore z_y = \frac{1}{1-\varepsilon\cos z}$

(4) [illegible]

方程两边取对数得 $x\ln z = z\ln y$. 再对上式两边关于x求偏导数得:

$\ln z + x\cdot\frac{1}{z}\cdot z_x = \ln y\cdot z_x$ $\therefore z_x = \frac{z\ln z}{z\ln y - x}$

对两边关于y求偏导数得. $x\cdot\frac{1}{z}\cdot z_y = z\cdot\frac{1}{y} + \ln y\cdot z_y$ $\therefore z_y = \frac{z^2}{xy - zy\ln y}$

(5) 对方程两边关于x求偏导数得 $\cos y + y(-\sin z) z_x + z\cdot(-\sin x) + z_x\cdot\cos x = 0$

$\therefore z_x = \frac{z\sin x - \cos y}{\cos x - y\sin z}$.

对方程两边关于y求偏导数得 $x(-\sin y) + \cos z + y\cdot(-\sin z)\cdot z_y + \cos x\cdot z_y = 0$

$\therefore z_y = \frac{x\sin y - \cos z}{\cos x - y\sin z}$

2解: 对方程两边关于x求偏导数得 $f_1'\cdot(\frac{dx}{dx}\cdot y^2 + x\cdot\frac{dy^2}{dx}) + f_2'(1+\frac{dy}{dx}) = 0$

即 $f_1'(y^2 + 2xy\frac{dy}{dx}) + f_2'(1+\frac{dy}{dx}) = 0$ 解得 $\frac{dy}{dx} = \frac{-(y^2 f_1' + f_2')}{2xy f_1' + f_2'}$

3解: 方程两边对x求偏导数得: $\frac{\partial z}{\partial x} + (-\sin(xy))\cdot y = e^z\cdot\frac{\partial z}{\partial x}$ $\therefore \frac{\partial z}{\partial x} = \frac{y\sin(xy)}{1-e^z}$

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial x}) = \frac{\partial}{\partial x}(\frac{y\sin(xy)}{1-e^z}) = \frac{y^2\cos(xy)(1-e^z) - y\sin(xy)\cdot(-e^z\frac{\partial z}{\partial x})}{(1-e^z)^2}$

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即 $\frac{\partial^2 z}{\partial x^2}=y^2\frac{[(1-e^z)^2\cos(xy)+e^z\sin^2(xy)]}{(1-e^z)^3}$

4解. $F_x=F_1'+F_2'+F_3'\quad F_y=F_2'+F_3'\quad F_z=F_3'$

$\therefore\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{F_1'+F_2'+F_3'}{F_3'}\qquad\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{F_2'+F_3'}{F_3'}$

5.证明: $dz=\frac{\partial z}{\partial x}\cdot dx+\frac{\partial z}{\partial y}\cdot dy$ 又 $\because\frac{\partial z}{\partial x}=\frac{-F_x}{F_z},\ \frac{\partial z}{\partial y}=\frac{-F_y}{F_z}$

$\therefore dz=-\frac{F_x}{F_z}dx-\frac{F_y}{F_z}dy$

函数 $F(x^2+y^2+z^2,xy-z^2)=0$. 有 $F_x=F_1'\cdot 2x+F_2'\cdot y\quad F_y=F_1'\cdot 2y+F_2'\cdot x$

$F_z=F_1'\cdot 2z-2F_2'\cdot z$.

$\therefore dz=-\frac{(F_1'\cdot 2x+F_2'\cdot y)dx+(F_1'\cdot 2y+F_2'\cdot x)\cdot dy}{F_1'\cdot 2z-F_2'\cdot 2z}$

6.证明: $J=\frac{D(x,y,z)}{D(r,\theta,\varphi)}=\begin{vmatrix}x_r & x_\theta & x_\varphi\\ y_r & y_\theta & y_\varphi\\ z_r & z_\theta & z_\varphi\end{vmatrix}=x_r(y_\theta\cdot z_\varphi-y_\varphi\cdot z_\theta)-x_\theta(y_r\cdot z_\varphi-y_\varphi\cdot z_r)$
$+x_\varphi(y_r\cdot z_\theta-y_\theta\cdot z_r)$

$=\cos\theta\cdot\sin\varphi[r\cdot\sin\varphi\cdot\cos\theta\cdot(-r\sin\varphi)-r\cdot\sin\theta\cdot\cos\varphi\cdot 0]$

$+r\sin\varphi\cdot\sin\theta[\sin\theta\cdot\sin\varphi\cdot(-r\cdot\sin\varphi)-r\cdot\sin\theta\cdot\cos\varphi\cdot\cos\varphi]$

$+r\cos\theta\cdot\cos\varphi(\sin\theta\cdot\sin\varphi\cdot 0-r\cdot\sin\varphi\cdot\cos\theta\cdot\cos\varphi)$

$=-r^2\cos^2\theta\sin^3\varphi-r^2\sin^2\theta\cdot\sin^3\varphi-r^2\sin^2\theta\cdot\sin\varphi\cdot\cos^2\varphi-r^2\cdot\cos^2\theta\cdot\sin\varphi$

$=-r^2\cos^2\theta\cdot\sin\varphi(\sin^2\varphi+\cos^2\varphi)-r^2\cdot\sin^2\theta\cdot\sin\varphi(\sin^2\varphi+\cos^2\varphi)$

$=-r^2\cdot\sin\varphi$

7解. 设 $x=x(u,v),\ y=y(u,v)$. 则 $z=z(x,y)=z(x(u,v),y(u,v))$

$\therefore\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\Rightarrow 3u^2=\frac{\partial z}{\partial x}\cdot 1+\frac{\partial z}{\partial y}(2u)$.

$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\Rightarrow 3v^2=\frac{\partial z}{\partial x}\cdot 1+\frac{\partial z}{\partial y}\cdot 2v$ 把 $u=\frac{1}{2},v=-\frac{1}{2}$ 代入

解得 $\frac{\partial z}{\partial x}=\frac{3}{4},\ \frac{\partial z}{\partial y}=0$

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8.解：对 $xu+yv=0$ 两边关于 x 求偏导，有 $u+\frac{\partial u}{\partial x}\cdot x+y\cdot\frac{\partial v}{\partial x}=0$ …①

再对以上方程关于 x 和 y 分别求偏导，有 $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x^2}\cdot x+y\cdot\frac{\partial^2 v}{\partial x^2}=0$ …②

$\frac{\partial u}{\partial y}+\frac{\partial^2 u}{\partial x\partial y}\cdot x+\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial x\partial y}\cdot y=0$ …③

对 $uv-xy=5$ 两边关于 x 求偏导有 $\frac{\partial u}{\partial x}\cdot v+\frac{\partial v}{\partial x}\cdot u-y=0$ …④

再对④方程关于 x 和 y 分别求偏导有 $\frac{\partial^2 u}{\partial x^2}\cdot v+\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial x^2}\cdot u+\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial x}=0$ …⑤

$\frac{\partial u}{\partial x\partial y}\cdot v+\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}+\frac{\partial^2 v}{\partial x\cdot\partial y}\cdot u+\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}-1=0$ …⑥

对方程组关于 y 求偏导得 $x\cdot\frac{\partial u}{\partial y}+v+\frac{\partial v}{\partial y}\cdot y=0$ …⑦，$\frac{\partial u}{\partial y}\cdot v+\frac{\partial v}{\partial y}\cdot u-x=0$ …⑧

把 $x=1,y=-1,u=v=2$ 代入①-⑧解得 $\frac{\partial^2 u}{\partial x^2}=\frac{55}{32}$，$\frac{\partial^2 v}{\partial x\partial y}=\frac{25}{32}$

9.解：设 $x=x(z)$，$y=y(z)$，故分别对两个方程求导得

$2x\cdot\frac{dx}{dz}+2y\cdot\frac{dy}{dz}=\frac{1}{2}\cdot 2\cdot z$，$\frac{dx}{dz}+\frac{dy}{dz}+1=0$

把 $x=1,y=-1,z=2$ 代入以上两个式子解得 $\frac{dx}{dz}=0$，$\frac{dy}{dz}=-1$

10.解：设 $z=z(x,y)=z(x(\theta,\varphi),y(\theta,\varphi))$

$\therefore\frac{\partial z}{\partial\theta}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial\theta}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial\theta}\Rightarrow 0=\frac{\partial z}{\partial x}\cdot(\cos\varphi(-\sin\theta))+\frac{\partial z}{\partial y}\cdot(\cos\varphi\cdot\cos\theta)$ …①

$\frac{\partial z}{\partial\varphi}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial\varphi}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial\varphi}\Rightarrow\cos\varphi=\frac{\partial z}{\partial x}\cdot(\cos\theta\cdot(-\sin\varphi))+\frac{\partial z}{\partial y}\cdot(\sin\theta\cdot(-\sin\varphi))$ …②

由①②解得 $\frac{\partial z}{\partial x}=-\frac{\cos\theta\cdot\cos\varphi}{\sin\varphi}=-\frac{x}{z}$

另解：由于 $F(x,y,z)=x^2+y^2+z^2=1$，∴设 $G(x,y,z)=x^2+y^2+z^2-1$，

则 $G(x,y,z)=0$ ∴ $G_x=2x$，$G_y=2y$，$G_z=2z$。

$\therefore\frac{\partial z}{\partial x}=-\frac{G_x}{G_z}=-\frac{x}{z}$

11.证明：$\frac{D(u,v)}{D(\xi,\eta)}=\begin{vmatrix}u_\xi & u_\eta\\ v_\xi & v_\eta\end{vmatrix}=\begin{vmatrix}u_x x_\xi+u_y y_\xi & u_x x_\eta+u_y\cdot y_\eta\\ v_x x_\xi+v_y\cdot y_\xi & v_x x_\eta+v_y\cdot y_\eta\end{vmatrix}$

$=(u_x x_\xi+u_y\cdot y_\xi)(v_x\cdot x_\eta+v_y\cdot y_\eta)-(u_x\cdot x_\eta+u_y\cdot y_\eta)\cdot(v_x\cdot x_\xi+v_y y_\xi)$

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$$= x_\xi \cdot y_\eta \cdot (u_x \cdot v_y - v_x u_y) - x_\eta y_\xi (v_x u_y - u_x v_y)$$

$$+ u_x x_\xi v_x \cdot x_\eta + u_y v_y \cdot y_\eta \cdot y_\xi - u_x v_x \cdot x_\eta \cdot x_\xi - u_y y_\eta \cdot v_y \cdot y_\xi$$

$$= (x_\xi \cdot y_\eta - x_\eta \cdot y_\xi)(u_x \cdot v_y - v_x u_y)$$

$$= \begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix} \cdot \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \frac{D(u,v)}{D(x,y)} \cdot \frac{D(x,y)}{D(\xi,\eta)}$$

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9 极值问题

1解:(1) $\frac{\partial z}{\partial x}=2x(x-1)(2x-1)$. $\frac{\partial z}{\partial y}=2y$. 由 $\begin{cases}\frac{\partial z}{\partial x}=0\\ \frac{\partial z}{\partial y}=0\end{cases}$ 解得 $\begin{cases}x=0\\ y=0\end{cases}$ 或 $\begin{cases}x=1\\ y=0\end{cases}$ 或 $\begin{cases}x=\frac{1}{2}\\ y=0\end{cases}$

$A=\frac{\partial^2 z}{\partial x^2}=12x^2-12x+2$. $B=\frac{\partial^2 z}{\partial x\partial y}=0$　$C=\frac{\partial^2 z}{\partial y^2}=2$

当 $\begin{cases}x=0\\ y=0\end{cases}$ 时　$A=2$. $B=0$. $C=2$. 此时 $B^2<AC$ 且 $A>0$. ∴ $z(x,y)$ 在 $(0,0)$ 处取得极小值为 $z(0,0)=0$

当 $\begin{cases}x=1\\ y=0\end{cases}$ 时　$A=2$. $B=0$. $C=2$. 此时 $B^2<AC$ 且 $A>0$. ∴ $z(x,y)$ 在 $(1,0)$ 处取得极小值为 $z(1,0)=0$.

当 $\begin{cases}x=\frac{1}{2}\\ y=0\end{cases}$ 时. $A=-1$. $B=0$, $C=2$ 此时 $B^2>AC$. $z(x,y)$ 在 $(\frac{1}{2},0)$ 取不到极值

综上述. $z(x,y)$ 在 $(0,0)$ 处和 $(1,0)$ 处均取得极小值 0.

(2). $\frac{\partial z}{\partial x}=2y-10x+4$. $\frac{\partial z}{\partial y}=2x-4y+4$ 由 $\begin{cases}\frac{\partial z}{\partial x}=0\\ \frac{\partial z}{\partial y}=0\end{cases}$ 解得 $\begin{cases}x=\frac{2}{3}\\ y=\frac{4}{3}\end{cases}$

$A=\frac{\partial^2 z}{\partial x^2}=-10$. $B=\frac{\partial^2 z}{\partial x\partial y}=2$. $C=\frac{\partial^2 z}{\partial y^2}=-4$.

当 $\begin{cases}x=2/3\\ y=4/3\end{cases}$ 时　$B^2<AC$ 且 $A<0$. ∴ $z(x,y)$ 在 $(\frac{2}{3},\frac{4}{3})$ 取得极大值 $z(\frac{2}{3},\frac{4}{3})=3$.

(3) $\frac{\partial z}{\partial x}=12x-6x^2+6y$　$\frac{\partial z}{\partial y}=6y+6x$. 由 $\begin{cases}\frac{\partial z}{\partial x}=0\\ \frac{\partial z}{\partial y}=0\end{cases}$ 解得 $\begin{cases}x=0\\ y=0\end{cases}$ 或 $\begin{cases}x=1\\ y=-1\end{cases}$

$A=\frac{\partial^2 z}{\partial x^2}=12-12x$　$B=\frac{\partial^2 z}{\partial x\partial y}=6$　$C=\frac{\partial^2 z}{\partial y^2}=6$

当 $\begin{cases}x=0\\ y=0\end{cases}$ 时 $A=12$. $B=C=6$ 此时 $B^2<AC$ 且 $A>0$. ∴ $z(x,y)$ 在 $(0,0)$ 处取得极小值 $z(0,0)=-1$.

当 $\begin{cases}x=1\\ y=-1\end{cases}$ 时 $A=0$. $B=C=6$. 此时 $B^2>AC$. $z(x,y)$ 在 $(1,-1)$ 处不取到极值.

综上述. $z(x,y)$ 在 $(0,0)$ 处取到极小值 1.

(4). $\frac{\partial z}{\partial x}=4y-4x^3$　$\frac{\partial z}{\partial y}=4x-4y^3$. 由 $\begin{cases}\frac{\partial z}{\partial x}=0\\ \frac{\partial z}{\partial y}=0\end{cases}$ 解得 $\begin{cases}x=1\\ y=1\end{cases}$ 或 $\begin{cases}x=-1\\ y=-1\end{cases}$ 或 $\begin{cases}x=0\\ y=0\end{cases}$

$A=\frac{\partial^2 z}{\partial x^2}=-12x^2$. $B=\frac{\partial^2 z}{\partial x\partial y}=4$. $C=\frac{\partial^2 z}{\partial y^2}=-12y^2$.

当 $\begin{cases}x=1\\ y=1\end{cases}$ 时 $A=-12$. $B=4$. $C=-12$. 此时 $B^2<AC$ 且 $A<0$. ∴ $z(x,y)$ 在 $(1,1)$ 处取得极大值 $z(1,1)=7$.

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当$\begin{cases}x=-1\\y=-1\end{cases}$时. 同理知$z(x,y)$在$(-1,-1)$处取得极大值7

当$\begin{cases}x=0\\y=0\end{cases}$时 $A=C=0$. $B=4$. 此时$B^2>AC$. $z(x,y)$在$(0,0)$处取不到极值

综上知. $z(x,y)$在$(1,1)$处和$(-1,-1)$取得极大值均为7.

(5) $\frac{\partial z}{\partial x}=18x^2y^2-4x^3y^2-3x^2y^3$. $\frac{\partial z}{\partial y}=12x^3y-2x^4y-3x^3y^2$. 由$\begin{cases}\frac{\partial z}{\partial x}=0\\\frac{\partial z}{\partial y}=0\end{cases}$解得

$\begin{cases}x=0\\y\in R\end{cases}$或$\begin{cases}x\in R\\y=0\end{cases}$或$\begin{cases}x=3\\y=2\end{cases}$, $A=\frac{\partial^2 z}{\partial x^2}=36xy^2-12x^2y^2-6xy^3$. $B=\frac{\partial^2 z}{\partial x\partial y}=36x^2y-8x^3y-9x^2y^2$

$C=\frac{\partial^2 z}{\partial y^2}=12x^3-2x^4-6x^3y$.

当$\begin{cases}x=0\\y\in R\end{cases}$. $A=B=C=0$. 当$\begin{cases}x\in R\\y=0\end{cases}$时. $A=B=0$, $C\in R$

对于以上两种情况有$B^2=AC$. 无法确定在$\begin{cases}x=0\\y\in R\end{cases}$和$\begin{cases}x\in R\\y=0\end{cases}$处取得极值.

当$\begin{cases}x=3\\y=2\end{cases}$时 $A=-144$. $B=-108$. $C=-162$. $\therefore B^2<AC$且$A<0$. $\therefore z(x,y)$在$(3,2)$处

取得极大值$z(3,2)=108$

综上述. $z(x,y)$在$(3,2)$处取得极大值108.

2解: (1) 构造函数$F(x,y,\lambda)=x^2+y^2+\lambda(\frac{x}{2}+\frac{y}{3}-1)$.

解方程组. $\begin{cases}F_x=2x+\frac{1}{2}\lambda=0\\F_y=2y+\frac{1}{3}\lambda=0\\F_\lambda=\frac{x}{2}+\frac{y}{3}-1=0\end{cases}$ 得 $\begin{cases}x=\frac{18}{13}\\y=\frac{12}{13}\\\lambda=\frac{-72}{13}\end{cases}$

当$x\to\infty,y\to\infty$. $z(x,y)\to\infty$. 即$z(x,y)$无最大值. 又$(\frac{18}{13},\frac{12}{13})$为$z(x,y)$唯一的极值

点. $\therefore z(x,y)$在$(\frac{18}{13},\frac{12}{13})$处取得最小值$z(\frac{18}{13},\frac{12}{13})=\frac{36}{13}$

综上述. $z(x,y)$在$(\frac{18}{13},\frac{12}{13})$处取得最小值$\frac{36}{13}$. 无最大值.

(2). 构造函数$F(x,y,\lambda)=3x+4y+\lambda(x^2+y^2-1)$

解方程组 $\begin{cases}F_x=0\\F_y=0\\F_\lambda=0\end{cases}$ 得 $\begin{cases}x=\frac{3}{5}\\y=\frac{4}{5}\end{cases}$或$\begin{cases}x=-\frac{3}{5}\\y=-\frac{4}{5}\end{cases}$. 由其图象知$z(x,y)$在$(\frac{3}{5},\frac{4}{5})$处取

得最大值5. 在$(-\frac{3}{5},-\frac{4}{5})$处取得最小值$-5$.

3解:

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构造函数 $F(x)=6x-y^2+xz+60+\lambda(x^2+y^2+z^2-36)$.

解方程组 $\begin{cases} F_x=6+z+2\lambda x=0 \\ F_y=-2y+2\lambda y=0 \\ F_z=x+2\lambda z=0 \\ F_\lambda=x^2+y^2+z^2-36=0 \end{cases}$ 得 $\begin{cases} x=-4 \\ y=\pm4 \\ z=2 \\ \lambda=1 \end{cases}$ 或 $\begin{cases} x=3\sqrt{3} \\ y=0 \\ z=3 \\ \lambda=-\frac{\sqrt{3}}{2} \end{cases}$ 或 $\begin{cases} x=0 \\ y=0 \\ z=-6 \\ \lambda=0 \end{cases}$

算且 $H(-4,\pm4,2)=44$. $H(3\sqrt{3},0,3)=27\sqrt{3}+60$. $H(0,0,-6)=60$

而在其边界 $x=0$ 和 $y=0$ 和 $z=0$ 所在的曲线上点所对应的值都大于44.

故应将望远镜安装在 $(-4,\pm4,2)$ 处.

4.设三角形三边分别为 x,y,z. 则 $x+y+z=2p$. $(x,y,z>0)$. 旋转一周体积为:(绕长为 z 的边转)

$V=\frac{1}{3}\pi\cdot\frac{-(x^4+y^4+z^4)+2x^2y^2+2x^2z^2+2y^2z^2}{4z}$　构造函数 $F(x,y,z,\lambda)=V'+\lambda(x+y+z-2p)$

解方程组 $\begin{cases} F_x=0 \\ F_y=0 \\ F_z=0 \\ F_\lambda=0 \end{cases}$ 化为 $\begin{cases} \frac{-x^3+y^2x+xz^2}{z}+\lambda=0 \\ \frac{yx^2-y^3+z^2y}{z}+\lambda=0 \\ \frac{(x^2-y^2)^2}{4z^2}-\frac{3}{4}z^2+\frac{x^2+y^2}{2}+\lambda=0 \\ x+y+z-2p=0 \end{cases}$ 解得. $\begin{cases} x=3p/4 \\ y=3p/4 \\ z=p/2 \end{cases}$

上述 $V'=\frac{-(x^4+y^4+z^4)+2x^2y^2+2x^2z^2+2y^2z^2}{4z}$.

易判断 $V(x,y,z)$ 在 $(\frac{3p}{4},\frac{3p}{4},\frac{p}{2})$ 处取最大值.

即三角形三边长分别为 $\frac{p}{2},\frac{3p}{4},\frac{3p}{4}$. 绕边长为 $\frac{1}{2}p$ 的边线转.

5解. 设交线上到原点距离为 d. 设 $D(x,y,z)=d^2=x^2+y^2+z^2$.

构造函数 $F=x^2+y^2+z^2-\lambda_1(y+2z-12)-\lambda_2(x+y-6)$

解方程组. $\begin{cases} F_x=2x-\lambda_2=0 \\ F_y=2y-\lambda_1-\lambda_2=0 \\ F_z=2z-2\lambda_1=0 \\ F_{\lambda_1}=y+2z-12=0 \\ F_{\lambda_2}=x+y-6=0 \end{cases}$ 得 $\begin{cases} x=2 \\ y=4 \\ z=4 \end{cases}$. 易验证 $D(x,y,z)$ 在 $(2,4,4)$ 处取最小值36

$\therefore$ 所求点为 $(2,4,4)$

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6解. 设 $D(x,y,z)=x^2+y^2+z^2$. 构造函数 $F(x,y,z,\lambda_1,\lambda_2)=D(x,y,z)-\lambda_1(x^2+y^2+\frac{z^2}{4}-1)-\lambda_2(x+y+z)$

解方程组 $\begin{cases} F_x=2x-2\lambda_1 x-\lambda_2=0 \\ F_y=2y-2\lambda_1 y-\lambda_2=0 \\ F_z=2z-\frac{1}{2}\lambda_1 z-\lambda_2=0 \\ F_{\lambda_1}=x^2+y^2+\frac{z^2}{4}-1=0 \\ F_{\lambda_2}=x+y+z=0 \end{cases}$ 得. $\begin{cases} x=\pm\frac{1}{\sqrt{3}} \\ y=\pm\frac{1}{\sqrt{3}} \\ z=\mp\frac{2}{\sqrt{3}} \\ \lambda_1=\cancel{2} \\ \lambda_2=-2/\sqrt{3} \end{cases}$ 或 $\begin{cases} x=\pm\frac{\sqrt{2}}{2} \\ y=\mp\frac{\sqrt{2}}{2} \\ z=0 \\ \lambda_1=1 \\ \lambda_2=0 \end{cases}$

易验证 $D(x,y,z)$ 在 $(\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}},\mp\frac{2}{\sqrt{3}})$ 处有最大值2. 在 $(\pm\frac{\sqrt{2}}{2},\mp\frac{\sqrt{2}}{2},0)$ 处有最小值1

即交线上的点到原点距离最远的点为 $(\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}},\mp\frac{2}{\sqrt{3}})$. 最远距离为 $\sqrt{2}$. 到原点距离最近的点为 $(\pm\frac{\sqrt{2}}{2},\mp\frac{\sqrt{2}}{2},0)$. 最近距离为1.

7.解. 设长方体长.宽高分别为 x,y,z. 圆锥体底面半径为 R. 高为 h.

则有. $1-\frac{z}{h}=\frac{\sqrt{x^2+y^2}}{2R}$. $V(x,y,z)=xyz$.

构造函数 $F(x,y,z,\lambda)=xyz-\lambda(\frac{\sqrt{x^2+y^2}}{2R}+\frac{z}{h}-1)$

解方程组 $\begin{cases} F_x=yz-\frac{\lambda}{2R}\cdot\frac{2x}{2\sqrt{x^2+y^2}}=0 \\ F_y=xz-\frac{\lambda}{2R}\cdot\frac{2y}{2\sqrt{x^2+y^2}}=0 \\ F_z=xy-\frac{\lambda}{h}=0 \\ F_\lambda=\frac{\sqrt{x^2+y^2}}{2R}+\frac{z}{h}-1=0 \end{cases}$ 解得 $\begin{cases} x=\frac{2\sqrt{2}}{3}R \\ y=\frac{2\sqrt{2}}{3}R \\ z=h/3 \\ \lambda=\frac{8}{9}R^2h \end{cases}$

易验证 $V(x,y,z)$ 在 $(\frac{2\sqrt{2}}{3}R.\frac{2\sqrt{2}}{3}R,\frac{h}{3})$ 处取得最大值.

$\therefore$ 所求长方体的边长分别为 $\frac{2\sqrt{2}}{3}R$. $\frac{2\sqrt{2}}{3}R$. $\frac{h}{3}$

8.①解. 设 $f(x_1,x_2,\cdots x_n)=x_1\cdot x_2\cdot\cdots\cdot x_n$, 构造函数 $F(x_1,x_2,\cdots x_n)=f(x_1,x_2,\cdots x_n)+\lambda(\sum x_i-L)$

解方程组 $\begin{cases} F_{x_1}=x_2\cdot x_3\cdots x_n+\lambda=0 \\ F_{x_2}=x_1\cdot x_3\cdots x_n+\lambda=0 \\ \vdots \\ F_{x_n}=x_1 x_2\cdots x_{n-1}+\lambda=0 \\ F_\lambda=x_1+x_2+\cdots+x_n-L=0 \end{cases}$ 得 $x_1=x_2=\cdots=x_n=\frac{L}{n}$

易证得 $f(x_1,x_2,\cdots x_n)$ 在 $(\frac{L}{n},\frac{L}{n},\cdots\frac{L}{n})$ 处取得最大值. $(\frac{L}{n})^n$

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②证明：由①知 $x_1\cdot x_2\cdots x_n \le (\frac{1}{n})^n = (\frac{x_1+x_2+\cdots+x_n}{n})^n$. 用 a_i 代替 x_i 得

$$a_1a_2\cdots a_n \le (\frac{a_1+a_2+\cdots+a_n}{n})^n.\ 即\ \sqrt[n]{a_1\cdot a_2\cdots a_n} \le \frac{a_1+a_2+\cdots+a_n}{n}$$

9.解. 设椭圆上一点 (x,y). 根据对称性设 $x,y>0$. 易求所求三角形面积 $S(x,y)$ 与 (x,y) 关系为 $S(x,y)=\frac{1}{2}\cdot\frac{(a^2y^2+b^2x^2)^2}{a^2b^2xy}$ 构造函数 $F(x,y)=S(x,y)+\lambda(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1)$

解方程组
$$\begin{cases} F_x=0 \\ F_y=0 \\ F_\lambda=0 \end{cases}$$
即
$$\begin{cases} \frac{(a^2y^2+b^2x^2)^2\cdot y-2(a^2y^2+b^2x^2)\cdot 2b^2x\cdot xy}{2a^2b^2x^2y^2}+2\lambda\cdot\frac{x}{a^2}=0 \\ \frac{(a^2y^2+b^2x^2)^2\cdot x-2(a^2y^2+b^2x^2)\cdot 2a^2y\cdot xy}{2a^2b^2x^2y^2}+2\lambda\frac{y}{b^2}=0 \\ \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0 \end{cases}$$

得：$\begin{cases} x=a/\sqrt{2} \\ y=b/\sqrt{2} \end{cases}$. 易验证 $S(a/\sqrt{2}, b/\sqrt{2})$ 为 $S(x,y)$ 在 $x>0, y>0$ 区域内的最小值.

根据对称性知：$\frac{1}{\sqrt{2}}(a,b)$, $\frac{1}{\sqrt{2}}(a,-b)$, $\frac{1}{\sqrt{2}}(-a,-b)$, $\frac{1}{\sqrt{2}}(-a,b)$ 为所求点.

10.①解：构造函数 $F(x,y)=f(x,y)-\lambda(x+y-A)$

解方程组
$$\begin{cases} F_x=\frac{1}{2}n\cdot x^{n-1}-\lambda=0 \\ F_y=\frac{1}{2}n\cdot y^{n-1}-\lambda=0 \\ F_\lambda=x+y-A=0 \end{cases}$$
得 $\begin{cases} x=\frac{A}{2} \\ y=\frac{A}{2} \end{cases}$ $(x>0, y>0)$

易验证 $f(x,y)$ 在 $(\frac{A}{2},\frac{A}{2})$ 取得最小值 $f(\frac{A}{2},\frac{A}{2})=(\frac{A}{2})^n$.

②证明：由①知 $\frac{1}{2}(x^n+y^n)\ge(\frac{A}{2})^n=(\frac{x+y}{2})^n$. 证毕.

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10 曲面论初步

1 解：(1) 曲面在 $(0,\frac{b}{\sqrt{2}},\frac{c}{\sqrt{2}})$ 处的法向量为 $\vec{n}=(F_x,F_y,F_z)\big|_{(0,\frac{b}{\sqrt{2}},\frac{c}{\sqrt{2}})}=(0,\frac{\sqrt{2}}{b},\frac{\sqrt{2}}{c})$

∴所求切平面方程为 $0\cdot(x-0)+\frac{\sqrt{2}}{b}(y-\frac{b}{\sqrt{2}})+\frac{\sqrt{2}}{c}(z-\frac{c}{\sqrt{2}})=0$

(2) 曲面在 $(2,1,3)$ 处的法向量为 $\vec{n}=(F_x,F_y,F_z)\big|_{(2,1,3)}=(4,-2,-1)$

∴所求切平面方程为 $4(x-2)-2(y-1)-(z-3)=0$

(3) 在 $\rho=1,\theta=\frac{\pi}{2}$ 对应的点为 $(0,\frac{1}{2}(e+e^{-1}),1)$.

$x_\rho=\frac{1}{2}\cos\theta(e^\rho-e^{-\rho})$ $y_\rho=\frac{1}{2}\sin\theta(e^\rho-e^{-\rho})$ $z_\rho=1$

$x_\theta=ch\rho(-\sin\theta)$. $y_\theta=ch\rho\cdot\cos\theta$ $z_\theta=0$

∴ $r_\rho\big|_{\rho=1,\theta=\frac{\pi}{2}}=(0,\frac{1}{2}(e-e^{-1}),1)$ $r_\theta\big|_{\rho=1,\theta=\frac{\pi}{2}}=(-\frac{1}{2}(e+e^{-1}),0,0)$

所求切平面方程为：

$$\begin{vmatrix} x-0 & y-\frac{1}{2}(e+e^{-1}) & z-1 \\ 0 & \frac{1}{2}(e-e^{-1}) & 1 \\ -\frac{1}{2}(e+e^{-1}) & 0 & 0 \end{vmatrix}=0$$

化为 $-ch1(y-ch1)+sh1\cdot ch1(z-1)=0$

(4) 曲面在 $(2,1,0)$ 处的法向量 $\vec{n}=(F_x,F_y,F_z)\big|_{(2,1,0)}=(1,2,-1)$

所求切平面方程为 $(x-2)+2(y-1)-(z-0)=0$

2. 证明：设切点坐标为 (x_0,y_0,z_0) 在该点的切平面的法向量 $\vec{n}=(F_x,F_y,F_z)\big|_{(x_0,y_0,z_0)}$

又 $F_x=\frac{1}{2\sqrt{x}}$ $F_y=\frac{1}{2\sqrt{y}}$, $F_z=\frac{1}{2\sqrt{z}}$. ∴ $\vec{n}=(\frac{1}{2\sqrt{x_0}},\frac{1}{2\sqrt{y_0}},\frac{1}{2\sqrt{z_0}})$

切平面方程为 $\frac{1}{2\sqrt{x_0}}(x-x_0)+\frac{1}{2\sqrt{y_0}}(y-y_0)+\frac{1}{2\sqrt{z_0}}(z-z_0)=0$

∴在x轴的截距为 $x_0'=\sqrt{x_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})$. 在y轴截距为 $y_0'=\sqrt{y_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})$

在z轴的截距为 $z_0'=\sqrt{z_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})$.

结合 $\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}=\sqrt{a}$ 得 $x_0'+y_0'+z_0'=(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})^2=\sqrt{a}^2=a$.

即曲面 $\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}\ (a>0)$ 上任一点处的切平面在各坐标轴上的截距之和等于a.

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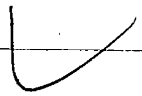
第六章总练习题

1解. 令 $t=\frac{y}{x}$. 则 $y=tx$. $\therefore f(t)=\frac{x^3}{[x^2+(tx)^2]^{3/2}}=\frac{x^3}{(1+t^2)^{3/2}\cdot x^3}=\frac{1}{(1+t^2)^{3/2}}$

$\therefore f(x)=\frac{1}{(1+t^2)^{3/2}}$



2.解: 当 $y=4$ 时. $z=2+f(\sqrt{x}-1)=x+1$ 即 $f(\sqrt{x}-1)=x-1=(\sqrt{x}-1+1)^2-1$

$\therefore f(x)=(x+1)^2-1=x^2+2x$. $\quad z(x,y)=\sqrt{y}+f(\sqrt{x}-1)=\sqrt{y}+x-1$

3.解: 令 $u=x+y$. $v=\frac{y}{x}$. 解得. $x=\frac{u}{v+1}$, $y=\frac{uv}{v+1}$ 其中 $v\neq-1$

$\therefore f(x+y,\frac{y}{x})=f(u,v)=(\frac{u}{v+1})^2-(\frac{uv}{v+1})^2=\frac{u^2(1-v)}{v+1}$ (其中 $v\neq-1$)

即有 $f(x,y)=\frac{x^2(1-y)}{y+1}$ $(y\neq-1)$.

4解: 考虑 (x,y) 沿曲线 $y=x-x^3$ 趋于 $(0,0)$ 时. $f(x,y)=\frac{1-x^2}{x}\to\infty$. 说明 $f(x,y)$ 此时无极限

5证明: $\lim\limits_{x\to0}\{\lim\limits_{y\to0}f(x,y)\}=\lim\limits_{x\to0}\frac{x^2\cdot0^2}{x^2\cdot0^2+(x-0)^2}=0$. 同理. $\lim\limits_{y\to0}\{\lim\limits_{x\to0}f(x,y)\}=0$.

考虑 (x,y) 沿曲线 $y=x-x^2$ 趋于 $(0,0)$ 时

$f(x,y)=\frac{x^4-2x^5+x^6}{2x^4-2x^5+x^6}\to\frac{1}{2}$.

考虑 (x,y) 沿直线 $y=x$ 趋于 $(0,0)$ 时 $f(x,y)=\frac{x^2y^2}{x^2y^2}=1$.

由上知. (x,y) 沿不同曲线趋于 $(0,0)$ 时. $f(x,y)$ 趋于不同的常数. 故当 $(x,y)\to(0,0)$ 时

$f(x,y)$ 无极限. 也即 $\lim\limits_{(x,y)\to(0,0)}f(x,y)$ 不存在. 证毕.

6解 (1). $\{(x,y)\mid x=0,\forall y\in R$ 或 $x\neq0, y\in Z\}$. (2) $\{(x,y)\mid y=x^2\}$

7.证明: 当 $(x,y)\to0$ 时 $t\to0$. $f(x,y)=g(t)=\frac{t\cos\alpha\cdot t^2\sin^2\alpha}{t^2\cos^2\alpha+t^4\sin^4\alpha}=\frac{t\cos\alpha\cdot\sin^2\alpha}{\cos^2\alpha+t^2\sin^4\alpha}$

当 $\alpha=k\cdot\frac{\pi}{2}$ 时. $g(t)=0$. 即 $\lim\limits_{(x,y)\to(0,0)}f(x,y)=\lim\limits_{t\to0}g(t)=0=f(0,0)$. $(k\in Z)$

当 $\alpha\neq k\cdot\frac{\pi}{2}$ 时. $g(t)\to\frac{0}{\cos^2\alpha}=0$. 即 $\lim\limits_{(x,y)\to(0,0)}f(x,y)=\lim\limits_{t\to0}g(t)=0=f(0,0)$ $(k\in Z)$

综上知. 任意取定 $\alpha\in[0,2\pi]$, 函数 $f(x,y)$ 沿射线 $x=t\cos\alpha$, $y=t\sin\alpha$ $(0<t<+\infty)$

在点 $(0,0)$ 处连续.

考虑 (x,y) 沿曲线 $y=x$ 和 $y=\sqrt{x}$ 趋向 $(0,0)$ 时. $f(x,y)$ 分别趋于两个不同常数 $0,\frac{1}{2}$.

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$\therefore f(x,y)$ 作为二元函数在 $(0,0)$ 处并不连续.

证毕.

8. 证明: $f(x,y)$ 在此区域内对 x 连续. 则对 $\forall \varepsilon_1'>0$. 都 $\exists \delta_1>0$ 使 $|x-x_0|<\delta_1$ 时

$|f(x,y_0)-f(x_0,y_0)|<\varepsilon_1'$ (x_0,y_0) 为给定的一点

对 $\forall \varepsilon>0$. 要使 $|f(x,y)-f(x_0,y_0)|<\varepsilon$. 而 $|f(x,y)-f(x_0,y_0)|<|f(x,y)-f(x_0,y)|+|f(x_0,y)-f(x_0,y_0)|$. 只需 $|f(x,y)-f(x_0,y)|+|f(x_0,y)-f(x_0,y_0)|<\varepsilon$ 即可. ——①

由上知. 只需找到 δ_2 使 $|y-y_0|<\delta_2$ 时. $|f(x_0,y)-f(x_0,y_0)|<\varepsilon_2$ 即可.

又当 $|x-x_0|<\delta_1$ 时 $|f(x,y)-f(x_0,y)|<\varepsilon_1'$. 故只需 $|f(x_0,y)-f(x_0,y_0)|<\varepsilon-\varepsilon_1'$ 即可

显然只需令 $\varepsilon_2=\varepsilon-\varepsilon_1'$. 只要找到 δ_2 即可完成证明.

由李普希茨条件知. $\exists L$ 使 $|f(x_0,y)-f(x_0,y_0)|\le L|y-y_0|$.

令 $\delta_2=\frac{\varepsilon_2}{L}$. 显然满足上述结论.

综上. 对 $\forall \varepsilon>0$. $\exists \delta_1,\delta_2$ 使 $|x-x_0|<\delta_1$, $|y-y_0|<\delta_2$ 时有 $|f(x,y)-f(x_0,y_0)|<\varepsilon$.

即二元函数 $f(x,y)$ 在区域 D 内是连续的.

9. 证明: 令 $z=xy$. 则 $|z|=|xy|\le\sqrt{\frac{x^2+y^2}{2}}<\frac{1}{\sqrt2}$.

(1) ① 当 $z>0$ 时. 设 $F_1(z)=\arctan z-z+\frac{z^3}{3}$. $F_2(z)=\arctan z-z$.

$F_1'(z)=\frac{1}{1+z^2}-1+z^2=\frac{z^4}{1+z^2}>0$. $\therefore F_1(z)$ 在 $(0,\frac{1}{\sqrt2})$ 递增.

$\therefore F_1(z)=\arctan z-z+\frac{z^3}{3}>F_1(0)=0$ 即 $\frac{\arctan z}{z}>1-\frac{z^2}{3}$.

$F_2'(z)=\frac{1}{1+z^2}-1=-\frac{z^2}{1+z^2}<0$. $\therefore F_2(z)$ 在 $(0,\frac{1}{\sqrt2})$ 上递减.

$\therefore F_2(z)=\arctan z-z<F(0)=0$. 即 $\frac{\arctan z}{z}<1$

② 当 $z<0$ 时. 由 $F_1'(z)>0$. $\therefore F_1(z)$ 在 $(-\frac{1}{\sqrt2},0)$ 上递增. $\therefore F_1(z)<F(0)=0$

即 $\arctan z-z+\frac{z^3}{3}<0$. $\therefore \arctan z/z>1-\frac{z^2}{3}$. 而 $F_2(z)$ 在 $(-\frac{1}{\sqrt2},0)$ 上递减

$\therefore F_2(z)>F(0)$ 即 $\arctan z-z>0$. 即 $\frac{\arctan z}{z}<1$.

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由①②知. $1-\frac{x^2y^2}{3}<\frac{\arctan(xy)}{xy}<1$.（其中 $xy\neq0$）

(2) 不等式等价于 $2z-\frac{z^2}{6}<4-4\cos\sqrt{z}<2z$.（取定 $0<z<\frac{1}{\sqrt{2}}$）

而 $4-4\cos\sqrt{z}=4\cdot(1-\cos\sqrt{z})=8\sin^2\frac{\sqrt{z}}{2}$. 故只需证明:

$2z-\frac{z^2}{6}<8\sin^2\frac{\sqrt{z}}{2}<2z$　$(0<z<\frac{1}{\sqrt{2}})$. ---- ※

利用不等式 $x-\frac{x^3}{6}<\sin x<x$ 有 $\frac{\sqrt{z}}{2}-\frac{z\sqrt{z}}{48}<\sin\frac{\sqrt{z}}{2}<\frac{\sqrt{z}}{2}$

显然 $\frac{\sqrt{z}}{2}-\frac{z\sqrt{z}}{48}>0$, $\frac{\sqrt{z}}{2}>0$　$\therefore(\frac{\sqrt{z}}{2}-\frac{z\sqrt{z}}{48})^2<\sin^2\frac{\sqrt{z}}{2}<\frac{z}{4}$.

即 $\frac{z}{4}-\frac{z^2}{48}+\frac{z^3}{(48)^2}<\sin^2\frac{\sqrt{z}}{2}<\frac{z}{4}$. 即 $\frac{z}{4}-\frac{z^2}{48}<\frac{z}{4}-\frac{z^2}{48}+\frac{z^3}{(48)^2}<\sin^2\frac{\sqrt{z}}{2}<\frac{z}{4}$

两边同时乘以8得 $2z-\frac{z^2}{6}<8\sin^2\frac{\sqrt{z}}{2}<2z$.

即 $2|xy|-\frac{x^2y^2}{6}<4-4\cos\sqrt{|xy|}<2|xy|$.

10. 解: (1) $\because 1-\frac{x^2y^2}{3}<\frac{\arctan(xy)}{xy}<1$. 而 $\lim\limits_{(x,y)\to(0,0)}(1-\frac{x^2y^2}{3})=\lim\limits_{(x,y)\to(0,0)}1=1$

故由夹逼定理知 $\lim\limits_{(x,y)\to(0,0)}\frac{\arctan(xy)}{xy}=1$.

(2) $\because 2|xy|-\frac{x^2y^2}{6}<4-4\cos\sqrt{|xy|}<2|xy|$ $\therefore 2-\frac{|xy|}{6}<\frac{4-4\cos\sqrt{|xy|}}{|xy|}<2$

而 $\lim\limits_{(x,y)\to(0,0)}(2-\frac{|xy|}{6})=\lim\limits_{(x,y)\to(0,0)}2=2$. $\therefore\lim\limits_{(x,y)\to(0,0)}\frac{4-4\cos\sqrt{|xy|}}{|xy|}=2$.

(3) $\because\sin\frac{1}{x^2+y^2}$ 有界. 而 $\lim\limits_{(x,y)\to(0,0)}(2x^2-y^2)=0$

$\therefore\lim\limits_{(x,y)\to(0,0)}(2x^2-y^2)\cdot\sin\frac{1}{x^2+y^2}=0$

11. 证明: 对方程 $f(tx,ty,tz)=t^nf(x,y,z)$ 两边对t求二阶导数得. 令 $u=tx$, $v=ty$, $w=tz$

$\frac{\partial}{\partial t}(f_u\cdot t)+\frac{\partial}{\partial t}(f_v\cdot t)+\frac{\partial}{\partial t}(f_w t)=t^{n-2}\cdot n(n-1)f(x,y,z)$

即 $f_{uu}\cdot t+f_u+f_{vv}t+f_v+f_{ww}\cdot t+f_w=t^{n-2}\cdot n(n-1)f(x,y,z)$.

令 $t=1$. 则 $u=x$, $v=y$, $w=z$. ∴有 $\frac{\partial^2 f}{\partial x^2}+\frac{\partial f}{\partial x}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial f}{\partial y}+\frac{\partial^2 f}{\partial z^2}+\frac{\partial f}{\partial z}=n(n-1)f(x,y,z)$

即 $(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z})^2f=n(n-1)f$

12. 证明 (1). $f_x(x,y)=2x\cdot\sin\frac{1}{x^2+y^2}\cdot$ [redacted] $+(x^2+y^2)\cancel{\cos}\frac{1}{x^2+y^2}\cdot\frac{-2x}{(x^2+y^2)^2}$

即 $f_x(x,y)=2x\cdot\sin\frac{1}{x^2+y^2}-2x\cdot\frac{1}{x^2+y^2}\cos\frac{1}{x^2+y^2}$

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同理求得 $f_y(x,y)=2y\sin\frac{1}{x^2+y^2}-2y\frac{1}{x^2+y^2}\cdot\cos\frac{1}{x^2+y^2}$. ∴ $f_x(x,y)$ 与 $f_y(x,y)$ 存在.

显然两个偏导数在 $(0,0)$ 处不连续 ($\because x^2+y^2\neq 0$)

(2). $\because f(x,y)=0,\ (x,y)=(0,0)$ ∴ $f_x(0,0)=0,\quad f_y(0,0)=0$

∴ $df(x,y)=f_x\cdot dx+f_y\cdot dy+o(\rho)=o(\rho)$. 故 $f(x,y)$ 在 $(0,0)$ 处可微.

13. 证明. $\frac{\partial^3 t}{\partial x\partial y\partial z}=\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\frac{\partial t}{\partial x}\right)\right)=\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(f'(xyz)\cdot yz\right)\right)=\frac{\partial}{\partial z}\left(f''(xyz)xyz^2+f'(xyz)\cdot z\right)$

$=f'''(xyz)x^2y^2z^2+2f''(xyz)xyz+f''(xyz)\cdot xyz+f'(x,y,z)$.

$=f'''(u)\cdot u^2+2f''(u)\cdot u+f''(u)\cdot u+f'(u)=f'(u)+3uf''(u)+u^2f'''(u)$

令 $F(u)=f'(u)+3uf''(u)+u^2f'''(u)$. 证毕

14. 证明: $\frac{\partial z}{\partial x}=\frac{-y^2}{3x^2}+\varphi'(xy)\cdot y$. $\quad\frac{\partial z}{\partial y}=\frac{2y}{3x}+\varphi'(xy)\cdot x$

$\therefore x^2\frac{\partial z}{\partial x}-xy\frac{\partial z}{\partial y}+y^2=x^2\cdot\left(-\frac{y^2}{3x^2}+\varphi'(xy)\cdot y\right)-xy\cdot\left(\frac{2y}{3x}+\varphi'(xy)\cdot x\right)+y^2=0$ 证毕

15. 证明: $\frac{\partial u}{\partial t}=\varphi'(x-at)\cdot(-a)+\psi'(x+at)\cdot a$.

$\therefore\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)=\varphi''(x-at)\cdot(-a)^2+\psi''(x+at)\cdot a^2=a^2\left(\varphi''(x-at)+\psi''(x+at)\right)$

$\frac{\partial u}{\partial x}=\varphi'(x-at)+\psi'(x+at)\quad\therefore\frac{\partial^2 u}{\partial x^2}=\varphi''(x-at)+\psi''(x+at)$

$\therefore\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$

16. 解: $\frac{\partial x}{\partial r}=\cos\theta$. $\frac{\partial x}{\partial\theta}=-r\sin\theta$ $\quad\frac{\partial x}{\partial z}=0$. $\quad\frac{\partial y}{\partial r}=\sin\theta$. $\quad\frac{\partial y}{\partial\theta}=r\cos\theta$. $\quad\frac{\partial y}{\partial z}=0$

$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial\theta}=0\quad\frac{\partial z}{\partial z}=1$

$$\therefore\frac{D(x,y,z)}{D(r,\theta,z)}=\begin{vmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}=\cos\theta\cdot r\cos\theta-(-r\sin\theta)\cdot\sin\theta=r$$

17. 解 $\frac{\partial x}{\partial\rho}=\sin\varphi\cos\theta\quad\frac{\partial x}{\partial\theta}=\rho\sin\varphi(-\sin\theta)\quad\frac{\partial x}{\partial\varphi}=\rho\cos\theta\cos\varphi$

$\frac{\partial y}{\partial\rho}=\sin\varphi\cdot\sin\theta\quad\frac{\partial y}{\partial\theta}=\rho\sin\varphi\cdot\cos\theta\quad\frac{\partial y}{\partial\varphi}=\rho\sin\theta\cdot\cos\varphi$

$\frac{\partial z}{\partial\rho}=\cos\varphi\quad\frac{\partial z}{\partial\theta}=0\quad\frac{\partial z}{\partial\varphi}=\rho(-\sin\varphi)$.

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$\therefore \dfrac{D(x,y,z)}{D(\rho,\varphi,\theta)}=\begin{vmatrix}\sin\varphi\cdot\cos\theta & -\rho\sin\theta\cdot\sin\varphi & \rho\cos\theta\cdot\cos\varphi\\ \sin\varphi\cdot\sin\theta & \rho\sin\varphi\cdot\cos\theta & \rho\sin\theta\cdot\cos\varphi\\ \cos\varphi & 0 & -\rho\cdot\sin\varphi\end{vmatrix}$

$=-\rho^2\sin^3\varphi\cos^2\theta+\rho\sin\theta\sin\varphi(-\rho\sin^2\varphi\sin\theta-\rho\cdot\sin\theta\cdot\cos^2\varphi)-\rho^2\sin\varphi\cos^2\theta\cos^2\varphi$

$=-\rho^2\sin\varphi.$

18. 证明(1). $r_1(t)\cdot r_2(t)=(x_1(t)\cdot x_2(t), y_1(t)\cdot y_2(t), z_1(t)\cdot z_2(t))$

而 $d[x_1(t)x_2(t)]=x_1(t)dx_2(t)+x_2(t)\cdot dx_1(t)\cdot$

同理 $d[y_1(t)y_2(t)]=y_1(t)dy_2(t)+y_2(t)dy_1(t)$

$d[z_1(t)z_2(t)]=z_1(t)dz_2(t)+z_2(t)dz_1(t).$

$\therefore d(r_1(t)\cdot r_2(t))=(d(x_1(t)x_2(t)), d(y_1(t)y_2(t)), d(z_1(t)z_2(t)))$

$=(x_1(t)dx_2(t), y_1(t)dy_2(t), z_1(t)dz_2(t))+(x_2(t)dx_1(t), y_2(t)dy_1(t), z_2(t)dz_1(t))$

$=(x_1(t), y_1(t), z_1(t))\cdot(dx_2(t), dy_2(t), dz_2(t))+(x_2(t), y_2(t), z_2(t))(dx_1(t), dy_1(t), dz_1(t))$

$=r_1(t)dr_2(t)+r_2(t)dr_1(t)$. 证毕.

(2). $\because d(r(t)\cdot r(t))=2r(t)\cdot dr(t)$. 即 $d(|r(t)|^2)=2r(t)\cdot dr(t)$

$\because |r(t)|\equiv1 \quad \therefore d(|r(t)|^2)\equiv0$ 即 $r(t)\cdot dr(t)\equiv0$.

几何意义: 球的任意一条半径的方向向量都为该点切平面的法向量.

19. 解. (1) $\dfrac{\partial f}{\partial x}=2x+y$, $\dfrac{\partial f}{\partial y}=2y+x$. $\therefore \dfrac{\partial f(-1,1)}{\partial x}=-1, \dfrac{\partial f(-1,1)}{\partial y}=1$.

设方向 l 的方向余弦为 $\cos\alpha$, $\sin\alpha$. 则 f 在方向 l 上的方向导数为

$\dfrac{\partial f}{\partial l}\Big|_{(-1,1)}=-\cos\alpha+\sin\alpha=\sqrt{2}\sin(\alpha-\frac{\pi}{4})$. 当 $\alpha=\frac{3}{4}\pi$ 时最大、当 $\alpha=-\frac{\pi}{4}$ 时最小

对应方向为 $(-1,1)$, $(1,-1)$. 函数沿这两个方向导数为 $\sqrt{2}$, $-\sqrt{2}$.

(2) $\dfrac{\partial f}{\partial x}\Big|_{M_0}=e^y\Big|_{M_0}=2$. $\dfrac{\partial f}{\partial y}\Big|_{M_0}=x\cdot e^y\Big|_{M_0}=2$. $\dfrac{\partial f}{\partial z}\Big|_{M_0}=2z\Big|_{M_0}=1$

设方向 l 的方向余弦为 $\cos\alpha$, $\cos\beta$, $\cos\gamma$. 则 f 在方向 l 上的方向导数为

$\dfrac{\partial f}{\partial l}\Big|_{M_0}=2\cos\alpha+2\cos\beta+\cos\gamma$.

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梯度 $grad\,f|_{M_0}=(2,2,1)$. 沿梯度方向函数值增加最快 沿负梯度方向函数值减小最快. 所求的方向为 $(2,2,1)$, $(-2,-2,-1)$ 对应的方向导数为 $3,-3$;

20解: (1) $r(\frac{\pi}{4})=(\frac{\sqrt2}{2},\frac{\sqrt2}{2},\frac{\pi}{4})$. 其切线方向为 $(-\frac{\sqrt2}{2},\frac{\sqrt2}{2},1)$. $\rho=\sqrt{(\frac{\sqrt2}{2})^2+(\frac{\sqrt2}{2})^2+(1)^2}$

$\therefore \frac{\partial f}{\partial l}\Big|_{r(\frac{\pi}{4})}=\sqrt2\cdot\frac{-\frac{\sqrt2}{2}}{\rho}+\sqrt2\cdot\frac{\frac{\sqrt2}{2}}{\rho}+\frac{\pi}{2}\cdot\frac{1}{\rho}=\frac{\sqrt2}{4}\pi$

(2) $r(0)=(1,0,0)$. 其切线方向为 $(0,1,1)$. $\rho=\sqrt{0^2+1^2+1^2}$

$\therefore \frac{\partial f}{\partial l}\Big|_{r(0)}=2\cdot\frac{0}{\rho}+0\cdot\frac{1}{\rho}+0\cdot\frac{1}{\rho}=0.$

(3). $r(-\frac{\pi}{4})=(\frac{\sqrt2}{2},-\frac{\sqrt2}{2},-\frac{\pi}{4})$ 其切线方向为 $(\frac{\sqrt2}{2},\frac{\sqrt2}{2},1)$. $\rho=\sqrt{(\frac{\sqrt2}{2})^2+(\frac{\sqrt2}{2})^2+1^2}$

$\therefore \frac{\partial f}{\partial l}\Big|_{r(-\frac{\pi}{4})}=\sqrt2\cdot\frac{\frac{\sqrt2}{2}}{\rho}-\sqrt2\cdot\frac{\frac{\sqrt2}{2}}{\rho}-\frac{\pi}{2}\cdot\frac{1}{\rho}=-\frac{\sqrt2\pi}{4}$

21解: 易知 $i=(1,0)$. $j=(0,1)$. $\therefore i+j$ 的方向余弦为 $(\frac{1}{\sqrt2},\frac{1}{\sqrt2})$. $-2j$ 的方向余弦为 $(0,-1)$

则有 $\begin{cases}\frac{1}{\sqrt2}f_x(2,3)+\frac{1}{\sqrt2}f_y(2,3)=2\sqrt2\\ 0\cdot f_x(2,3)-1\cdot f_y(2,3)=-3\end{cases}$ 解得 $\begin{cases}f_x(2,3)=1\\ f_y(2,3)=3\end{cases}$

$\therefore f(x,y)$ 在$(2,3)$处沿 $2i+j$ 的方向导数为: $(f_x(2,3),f_y(2,3))\cdot(2i+j)$

即为 $(1,3)\cdot(2,1)=5$.

$\therefore f(x,y)$ 在$(2,3)$处沿$2i+j$的方向导数为5.

22.证明: (1) $\frac{\partial f}{\partial x}=f_x(x,y)$, $\frac{\partial f}{\partial y}=f_y(x,y)$.

$\therefore \frac{\partial f}{\partial l_1}=f_x(x,y)\cdot\cos\alpha_1+f_y(x,y)\cos\beta_1$

$\therefore (\frac{\partial f}{\partial l_1})^2=f_x^2(x,y)\cos^2\alpha_1+f_y^2(x,y)\cos^2\beta_1+2f_x(x,y)f_y(x,y)\cos\alpha_1\cos\beta_1$ ……①

同理: $(\frac{\partial f}{\partial l_2})^2=f_x^2(x,y)\cdot\cos^2\alpha_2+f_y^2(x,y)\cos^2\beta_2+2f_x(x,y)\cdot f_y(x,y)\cos\alpha_2\cdot\cos\beta_2$ ……②

$\because \cos^2\alpha_1+\cos^2\beta_1=1$, $\cos^2\alpha_2+\cos^2\beta_2=1$ 且 $\cos\alpha_1\cdot\cos\alpha_2+\cos\beta_1\cdot\cos\beta_2=0$

$\therefore$ 有 $\cos^2\alpha_1+\cos^2\alpha_2=1$ $\cos^2\beta_1+\cos^2\beta_2=1$.

由①+②得 $(\frac{\partial f}{\partial l_1})^2+(\frac{\partial f}{\partial l_2})^2=f_x^2(x,y)+f_y^2(x,y)=(\frac{\partial f}{\partial x})^2+(\frac{\partial f}{\partial y})^2$. 证毕.

(2). $\frac{\partial f}{\partial l_1}=\frac{\partial f}{\partial x}(x,y)\cos\alpha_1+\frac{\partial f}{\partial y}(x,y)\cos\beta_1$

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$\frac{\partial^2 u}{\partial l_1^2}=\frac{\partial^2 u}{\partial x^2}\cos^2\alpha_1+\frac{\partial^2 u}{\partial y^2}\cos^2\beta_1$

同理：$\frac{\partial^2 u}{\partial l_2^2}=\frac{\partial^2 u}{\partial x^2}\cos^2\alpha_2+\frac{\partial^2 u}{\partial y^2}\cos^2\beta_2$

$\therefore \frac{\partial^2 u}{\partial l_1^2}+\frac{\partial^2 u}{\partial l_2^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$ $(\cos^2\alpha_1+\cos^2\alpha_2=1,\ \cos^2\beta_1+\cos^2\beta_2=1)$.

23.解：$f(x,y)=\int\frac{\partial z}{\partial x}dx=\int(x^2+y)dx=\frac{1}{3}x^3+xy+F(y)$.

当x=y时 $f(x,x)=\frac{1}{3}x^3+x^2+F(x)=x^2$　　$\therefore F(x)=-\frac{1}{3}x^3$.

$\therefore F(y)=-\frac{1}{3}y^3$

故 $z=f(x,y)=\frac{1}{3}x^3+xy-\frac{1}{3}y^3$

24解：$\frac{\partial z}{\partial y}=\int\frac{\partial^2 z}{\partial x\partial y}dy=\int(x+y+1)dy=xy+\frac{y^2}{2}+y+F(x)$.

$\therefore z=f(x,y)=\int\frac{\partial z}{\partial x}dx=\int[xy+\frac{y^2}{2}+y+F(x)]dx=\frac{1}{2}yx^2+\frac{y^2}{2}x+yx+\int F(x)dx+G(y)$

$\therefore f(x,0)=\int F(x)dx+G(0)=x^2$

$f(0,y)=\int F(0)dx+G(y)=2y$

设 $\int F(x)dx=H(x)+C_1$ (C_1为任意给定常数). $G(y)=I(y)+C_2$.

$\therefore$有 $\begin{cases}H(x)+C_1+I(0)+C_2=x^2 & \cdots① \\ H(0)+C_1+I(y)+C_2=2y & \cdots②\end{cases}$

令x=0. ①式化为 $C_1+C_2+H(0)+I(0)=0$.

$\therefore$①+②得 $H(x)+I(y)+C_1+C_2+C_2+C_1+H(0)+I(0)=x^2+2y$

$\therefore H(x)+I(y)+C_1+C_2=x^2+2y$. 即 $\int F(x)dx+G(y)=x^2+2y$

$\therefore z=f(x,y)=\frac{1}{2}x^2y+\frac{1}{2}xy^2+xy+x^2+2y$.

25解：$F(h)=\frac{1}{h^2}([f(x+h,y)-f(x,y)]+[f(x,y+h)-f(x,y)]+[f(x-h,y)-f(x,y)]+[f(x,y-h)-f(x,y)])$ (*)

而 $f(x+h,y)-f(x,y)=\frac{1}{1!}f_x h+\frac{1}{2!}\cdot f_{xx}\cdot h^2+\frac{1}{3!}\cdot f_{xxx}\cdot h^3+\frac{1}{4!}f_{xxxx}\cdot h^4+\cdots$

$f(x,y+h)-f(x,y)=\frac{1}{1!}\cdot f_y\cdot h+\frac{1}{2!}\cdot f_{yy}\cdot h^2+\frac{1}{3!}\cdot f_{yyy}\cdot h^3+\frac{1}{4!}f_{yyyy}\cdot h^4+\cdots$

$f(x-h,y)-f(x,y)=-\frac{1}{1!}f_x\cdot h+\frac{1}{2!}f_{xx}\cdot h^2-\frac{1}{3!}\cdot f_{xxx}\cdot h^3+\frac{1}{4!}f_{xxxx}\cdot h^4+\cdots$

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$$f(x,y-h)-f(x,y)=-\frac{1}{1!}f_y\cdot h+\frac{1}{2!}f_{yy}h^2-\frac{1}{3!}f_{yyy}\cdot h^3+\frac{1}{4!}f_{yyyy}\cdot h^4+\cdots$$

由以上四式代入(*)式得

$$F(h)=\frac{1}{4}\left[(f_{xx}+f_{yy})\cdot h^2+\frac{1}{12}(f_{xxxx}+f_{yyyy})\cdot h^4\right]=\frac{h^2}{4}(f_{xx}+f_{yy})+\frac{h^4}{48}(f_{xxxx}+f_{yyyy})+\cdots$$

26解. $F(r)=F(0)+\frac{1}{1!}dF(0)\cdot r+\frac{1}{2!}d^2F(0)\cdot r^2+\frac{1}{3!}d^3F(0)r^3+\frac{1}{4!}d^4F(0)r^4+\cdots$

而 $F(0)=\frac{1}{2\pi}\int_0^{2\pi}f(x,y)\cdot d\varphi=\frac{1}{2\pi}(f(x,y)\cdot\varphi)\Big|_0^{2\pi}=f(x,y)$

$$dF(0)=\frac{1}{2\pi}\int_0^{2\pi}(f_u\cdot u_r+f_v\cdot v_r)\,d\varphi=\frac{1}{2\pi}\int_0^{2\pi}\left[f_x\cdot(-\sin\varphi)+f_y\cdot\cos\varphi\right]d\varphi$$

$$=\frac{1}{2\pi}(f_x\cos\varphi+f_y\sin\varphi)\Big|_0^{2\pi}=0$$

$$d^2F(0)=\frac{1}{2\pi}\int_0^{2\pi}\left[(f_{uu}\cdot u_r+f_{uv}\cdot v_r)u_r+(f_{vu}\cdot u_r+f_{vv}\cdot v_r)v_r\right]d\varphi$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\left[f_{xx}\sin^2\varphi+2f_{xy}(-\sin\varphi\cdot\cos\varphi)+f_{yy}\cdot\cos^2\varphi\right]d\varphi$$

$$=\frac{1}{2\pi}\left[f_{xx}\left(\int_0^{2\pi}\sin^2\varphi\right)+f_{xy}(-\sin^2\varphi)\Big|_0^{2\pi}+f_{yy}\left(\int_0^{2\pi}\cos^2\varphi\right)\right]$$

$$=\frac{1}{2\pi}\cdot(f_{xx}\cdot\pi+0+f_{yy}\cdot\pi)=\frac{1}{2}(f_{xx}+f_{yy})$$

$$d^3F(0)=\frac{1}{2\pi}\int_0^{2\pi}\Big[(f_{uuu}\cdot u_r+f_{uuv}\cdot v_r)u_r^2+f_{uu}\cdot 2u_r\cdot u_{rr}+(f_{uvu}\cdot u_r+f_{uvv}\cdot v_r)u_rv_r$$

$$+f_{uv}\cdot(u_r\cdot v_{rr}+v_ru_{rr})+(f_{vuu}u_r+f_{vuv}\cdot v_r)u_rv_r+f_{vu}(u_rv_{rr}+v_ru_{rr})$$

$$+(f_{vvu}u_r+f_{vvv}\cdot v_r)\cdot v_r^2+f_{vv}\cdot 2v_r\cdot v_{rr}$$

$$=\cdots=0$$

$$d^4F(0)=\frac{1}{2\pi}\int_0^{2\pi}\Big[(f_{uuuu}\cdot u_r+f_{uuuv}\cdot v_r)\cdot u_r^3+f_{uuu}\cdot 3u_r^2\cdot u_{rr}$$

$$+(f_{uuvu}\cdot u_r+f_{uuvv}\cdot v_r)v_r\cdot u_r^2+f_{uuv}\cdot(v_r\cdot 2u_ru_{rr}+u_r^2\cdot v_{rr})$$

$$+(f_{uuu}\cdot u_r+f_{uuv}\cdot v_r)\cdot 2u_r\cdot u_{rr}+2f_{uu}\cdot(u_{rr}^2+u_ru_{rrr})$$

$$+\cdots+(f_{vvu}\cdot u_r+f_{vvv}\cdot v_r)\cdot 2v_r\cdot v_{rr}+2f_{vv}\cdot(v_{rr}^2+v_r\cdot v_{rrr})\Big]d\varphi$$

$$=\frac{1}{2\pi}\cdot\left[f_{xxxx}\cdot\int_0^{2\pi}\sin^4\varphi\,d\varphi+6f_{xxyy}\int_0^{2\pi}\sin^2\varphi\cos^2\varphi\,d\varphi+f_{yyyy}\int_0^{2\pi}\cos^4\varphi\,d\varphi\right]$$

$$=\frac{1}{2\pi}\left(\frac{3\pi}{4}f_{xxxx}+\frac{3\pi}{2}f_{xxyy}+\frac{3\pi}{4}f_{yyyy}\right).$$

$$\therefore F(r)=f(x,y)+\frac{1}{4}(f_{xx}+f_{yy})r^2+\frac{1}{64}(f_{xxxx}+f_{yyyy}+2f_{xxyy})r^4+\cdots$$

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27.解：对 $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}$ 两边关于R求偏导得：$-\frac{\frac{\partial R}{\partial R_1}}{R^2}=-\frac{1}{R_1^2}$

$\therefore \frac{\partial R}{\partial R_1}=\frac{R^2}{R_1^2}$ (*) 当 $R_1=30, R_2=45, R_3=90$ 时，$R=15$ 代入(*)得 $\frac{\partial R}{\partial R_1}=\frac{1}{4}$

28解：(1). $\frac{\partial u}{\partial x}=y^2\cdot(z^3+3z^2\cdot\frac{\partial z}{\partial x}\cdot x)$ ① 对方程 $x^2+y^2+z^2-3xyz=0$ 两边关于x求偏导得.

$2x+2z\cdot\frac{\partial z}{\partial x}-3y(z+x\cdot\frac{\partial z}{\partial x})=0$ …②

把 $(x,y,z)=(1,1,1)$ 代入①②两式解得 $\frac{\partial u}{\partial x}=-2$

(2). $\frac{\partial u}{\partial x}=z^3(y^2+2y\cdot\frac{\partial y}{\partial x}\cdot x)$ …③. 对方程 $x^2+y^2+z^2-3xyz=0$ 两边关于x求偏导得.

$2x+2y\cdot\frac{\partial y}{\partial x}-3z(y+x\cdot\frac{\partial y}{\partial x})=0$ …④

把 $(x,y,z)=(1,1,1)$ 代入③④两式解得 $\frac{\partial u}{\partial x}=-1$.

29.证明：设 $u=u(x,y,z)=x^2+y^2+z^2$. 对方程 $ax+by+cz=F(x^2+y^2+z^2)$ 两边关于x, y求偏导得：

$a+c\cdot\frac{\partial z}{\partial x}=F_u\cdot(2x+2z\cdot\frac{\partial z}{\partial x})$ …　$b+c\frac{\partial z}{\partial y}=F_u(2y+2z\cdot\frac{\partial z}{\partial y})$

$\therefore \frac{\partial z}{\partial x}=(2F_u x-a)/(c-2F_u\cdot z)$　$\frac{\partial z}{\partial y}=(2F_u y-b)/(c-2F_u\cdot z)$

$\therefore (cy-bz)\cdot\frac{\partial z}{\partial x}+(az-cx)\cdot\frac{\partial z}{\partial y}=\frac{1}{(c-2F_u\cdot z)}((cy-bz)(2F_u x-a)+(az-cx)(2F_u y-b))$

$=\frac{1}{c-2F_u z}\cdot(2F_u cxy-2F_u xbz-acy+abz+2F_u ayz-2F_u cxy-abz+bcx)$

$=\frac{1}{c-2F_u z}[-2F_u z(bx-ay)+c(bx-ay)]=\frac{1}{c-2F_u z}(bx-ay)\cdot(c-2F_u z)$

$=bx-ay$. 证毕.

30解：$\frac{\partial F}{\partial x}=2x$, $\frac{\partial F}{\partial y}=2y$. 由 $\begin{cases}\frac{\partial F}{\partial x}=0\\ \frac{\partial F}{\partial y}=0\end{cases}$ 得 $\begin{cases}x=0\\ y=0\end{cases}$. 显然点(0,0)在三角区域内 而 $f(0,0)=0$

又 $f(x,y)=x^2+y^2\geqslant 0$. $\therefore f(x,y)$ 在三角区域上的最小值为0.

易知 $f(x,y)$ 在三角区域边界点(0,2)中取得最大值4.

$\therefore f(x,y)$ 在三角区域上的最大值为4. 最小值为0

31.解. $\frac{\partial F}{\partial x}=2x+y-6$. $\frac{\partial F}{\partial y}=x+2y$　由 $\begin{cases}\partial F/\partial x=0\\ \partial F/\partial y=0\end{cases}$ 得 $\begin{cases}x=4\\ y=-2\end{cases}$. 显然(4,-2)在矩形域内.

易证 $F(4,-2)=-10$ 为 $F(x,y)$ 在矩形域内的最小值.

易知 $F(x,y)$ 在其矩形域的边界上点(0,-3)取得最大值 $f(0,-3)=X$.

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∴ $F(x,y)$在矩形域上的最大值为11. 最小值为 -10.

32解 $\frac{\partial z}{\partial x} = \cos x - \sin(x-y)$ $\frac{\partial z}{\partial y} = -\sin y + \sin(x-y)$.

由$\begin{cases}\partial z/\partial x = 0 \\ \partial z/\partial y = 0\end{cases}$ 解得 $\begin{cases} x = \pi/3 \\ y = \pi/6 \end{cases}$ ($\because x, y \in [0, \frac{\pi}{2}]$)

设 $A = \frac{\partial^2 z}{\partial x^2} = -\sin x - \cos(x-y)$. $B = \frac{\partial^2 z}{\partial x \partial y} = \cos(x-y)$. $C = \frac{\partial^2 z}{\partial y^2} = -\cos y - \cos(x-y)$

当 $x = \pi/3, y = \pi/6$时. $A = -\sqrt{3}$ $B = \frac{\sqrt{3}}{2}$. $C = -\sqrt{3}$. 此时 $B^2 < AC$且$A < 0$

∴ $z(x,y)$的极大值为 $z(\pi/3, \pi/6) = \frac{3}{2}\sqrt{3}$

33解. 易知椭圆中心为$(0,0)$. 根据椭圆定义. 只需求得 $f(x,y) = x^2 + y^2$的最大值.最小值.

则 $a = \sqrt{f_{max}(x,y)}$, $b = \sqrt{f_{min}(x,y)}$.

构造函数 $F(x,y) = x^2 + y^2 - \lambda(5x^2 + 8xy + 5y^2 - 9)$.

解方程组 $\begin{cases} F_x = 0 \\ F_y = 0 \\ F_\lambda = 0 \end{cases}$ 即 $\begin{cases} x - 5\lambda x - 4\lambda y = 0 \\ y - 5\lambda y - 4\lambda x = 0 \\ 5x^2 + 8xy + 5y^2 - 9 = 0 \end{cases}$ 解得 $\begin{cases} x = \pm\frac{3\sqrt{2}}{2} \\ y = \mp\frac{3\sqrt{2}}{2} \\ \lambda = 1 \end{cases}$ 或 $\begin{cases} x = \pm\frac{\sqrt{2}}{2} \\ y = \pm\frac{\sqrt{2}}{2} \\ \lambda = \frac{1}{9} \end{cases}$

易验证 $f(x,y)$在$(\pm\frac{3\sqrt{2}}{2}, \mp\frac{3\sqrt{2}}{2})$处取最大值9, 在$(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2})$处取最小值1

∴ 长半轴$a = 3$, 短半轴$b = 1$.

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7 场论初步

1.解：(1) $div\vec{F}|_{(6,1,3)} = (5y+\frac{x}{z}+\frac{3}{xz}\cdot x)|_{(6,1,3)} = 5+2+\frac{3}{3}=8$

(2) $div\vec{F}|_{(3,4,3)} = (\frac{x}{\sqrt{x^2+y^2}}+\frac{1}{y+\sqrt{y^2+z^2}}\cdot(1+\frac{y}{\sqrt{y^2+z^2}})+\frac{z}{\sqrt{y^2+z^2}})|_{(3,4,3)} = \frac{7}{5}$

2解：(1) 设 $\vec{a}=(a,b,c)$ 其中 a,b,c 为常数，则 $r\cdot\vec{a}=\sqrt{x^2+y^2+z^2}(a,b,c)$

$\therefore \nabla\cdot(r\cdot\vec{a}) = a\cdot\frac{x}{\sqrt{x^2+y^2+z^2}}+b\cdot\frac{y}{\sqrt{x^2+y^2+z^2}}+c\cdot\frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{1}{r}\cdot\vec{a}\cdot\vec{r}$

(2) $r^2\vec{a}=(x^2+y^2+z^2)(a,b,c)$

$\therefore \nabla\cdot(r^2\vec{a}) = 2x\cdot a+2y\cdot b+2z\cdot c = 2\vec{a}\cdot\vec{r}$

(3) $r^n\vec{a} = (x^2+y^2+z^2)^{\frac{n}{2}}\cdot(a,b,c)$

$\therefore \nabla\cdot(r^n\vec{a}) = 2x\cdot\frac{n}{2}\cdot(x^2+y^2+z^2)^{\frac{n}{2}-1}a+2y\cdot\frac{n}{2}\cdot(x^2+y^2+z^2)^{\frac{n}{2}-1}\cdot b+2z\cdot\frac{n}{2}\cdot(x^2+y^2+z^2)^{\frac{n}{2}-1}\cdot c$

$= nr^{n-2}\cdot\vec{a}\cdot\vec{r}$

(4) $\frac{\vec{r}}{r^3} = \frac{1}{\sqrt{x^2+y^2+z^2}^3}(x,y,z)$. $\therefore \nabla\cdot(\frac{\vec{r}}{r^3}) = \frac{y^2+z^2-2x^2}{r^3}+\frac{x^2+z^2-2y^2}{r^3}+\frac{x^2+y^2-2z^2}{r^3}=0$

(5) $f(r)\cdot\vec{a}=f(r)\cdot(a,b,c)$.

$\therefore \nabla\cdot[f(r)\cdot\vec{a}] = \frac{\partial f(r)}{\partial x}\cdot a+\frac{\partial f(r)}{\partial y}\cdot b+\frac{\partial f(r)}{\partial z}\cdot c = f'(r)\cdot\frac{\partial r}{\partial x}\cdot a+f'(r)\cdot\frac{\partial r}{\partial y}\cdot b+f'(r)\cdot\frac{\partial r}{\partial z}\cdot c$

$= f'(r)\cdot\frac{ax+by+cz}{\sqrt{x^2+y^2+z^2}} = \frac{1}{r}f'(r)\cdot\vec{r}\cdot\vec{a}$

(6) $grad f(r) = (\frac{\partial f(r)}{\partial x},\frac{\partial f(r)}{\partial y},\frac{\partial f(r)}{\partial z}) = f'(r)\cdot(\frac{x}{\sqrt{x^2+y^2+z^2}},\frac{y}{\sqrt{x^2+y^2+z^2}},\frac{z}{\sqrt{x^2+y^2+z^2}})$

$\therefore \nabla\cdot[grad f(r)] = f'(r)\cdot[\frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}+\frac{x^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}+\frac{x^2+y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}]+f''(r) = \frac{2}{r}f'(r)+f''(r)$

(7) $f(r)\cdot\vec{r} = f(r)\cdot(x,y,z)$

$\therefore \nabla\cdot[f(r)\cdot\vec{r}] = f'(r)\cdot\frac{\partial r}{\partial x}\cdot x+f(r)+f'(r)\cdot\frac{\partial r}{\partial y}\cdot y+f(r)+f'(r)\cdot\frac{\partial r}{\partial z}z+f(r)$

$= f'(r)\cdot\frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}}+3f(r) = r\cdot f'(r)+3f(r)$

3解：$rot\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix} = (xz^2-xy^2)i+(x^2y-yz^2)j+(y^2z-x^2z)k$

$\therefore rot\vec{F}|_{(1,3,2)} = (-5,-9,16)$

$\therefore$ 沿 $\vec{n}=(1,2,2)$ 的方向旋量为 $rot\vec{F}\cdot\vec{n}=\dfrac{\vec{n}\cdot rot\vec{F}}{|\vec{n}|}=\dfrac{(1,2,2)(-5,-9,16)}{3}=3$

4解. (1) $rot\vec{F}=\left(\dfrac{\partial R}{\partial y}-\dfrac{\partial Q}{\partial z},\ \dfrac{\partial P}{\partial z}-\dfrac{\partial R}{\partial x},\ \dfrac{\partial Q}{\partial x}-\dfrac{\partial P}{\partial y}\right)$

$=(0-2z,\ 0-2x,\ 0-2y)=-2(z,x,y)$

(2) $rot\vec{F}=\left(\dfrac{\partial R}{\partial y}-\dfrac{\partial Q}{\partial z},\ \dfrac{\partial P}{\partial z}-\dfrac{\partial R}{\partial x},\ \dfrac{\partial Q}{\partial x}-\dfrac{\partial P}{\partial y}\right)=(x-x,\ y-y,\ z-z)=\vec{0}$

5证明: 设 [redacted]. $\vec{F}=P\vec{i}+Q\vec{j}+R\vec{j}$

(1) $grad\,u=\left(\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y},\dfrac{\partial u}{\partial z}\right)$ $\therefore rot(grad\,u)=\left(\dfrac{\partial^2 u}{\partial y\partial z}-\dfrac{\partial^2 u}{\partial y\partial z},\ \dfrac{\partial^2 u}{\partial x\partial z}-\dfrac{\partial^2 u}{\partial x\partial z},\ \dfrac{\partial^2 u}{\partial x\partial y}-\dfrac{\partial^2 u}{\partial x\partial y}\right)$

即 $rot(grad\,u)=\vec{0}$

(2) $rot\vec{F}=\left(\dfrac{\partial R}{\partial y}-\dfrac{\partial Q}{\partial z},\ \dfrac{\partial P}{\partial z}-\dfrac{\partial R}{\partial x},\ \dfrac{\partial Q}{\partial x}-\dfrac{\partial P}{\partial y}\right)$

$\therefore div(rot\vec{F})=\dfrac{\partial\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)}{\partial x}+\dfrac{\partial\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)}{\partial y}+\dfrac{\partial\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)}{\partial z}$

$=\dfrac{\partial^2 R}{\partial x\partial y}-\dfrac{\partial^2 Q}{\partial x\partial z}+\dfrac{\partial^2 P}{\partial y\partial z}-\dfrac{\partial^2 R}{\partial x\partial y}+\dfrac{\partial^2 Q}{\partial x\partial z}-\dfrac{\partial^2 P}{\partial y\partial z}=0$

(3) $rot(u\vec{F})=\left(\dfrac{\partial uR}{\partial y}-\dfrac{\partial uQ}{\partial z},\ \dfrac{\partial uP}{\partial z}-\dfrac{\partial uR}{\partial x},\ \dfrac{\partial uQ}{\partial x}-\dfrac{\partial uP}{\partial y}\right)$

$=\left(\dfrac{\partial u}{\partial y}\cdot R+u\cdot\dfrac{\partial R}{\partial y}-\dfrac{\partial u}{\partial z}\cdot Q-\dfrac{\partial Q}{\partial z}\cdot u,\ \dfrac{\partial u}{\partial z}\cdot P+u\cdot\dfrac{\partial P}{\partial z}-\dfrac{\partial u}{\partial x}\cdot R-u\dfrac{\partial R}{\partial x},\right.$

$\left.\dfrac{\partial u}{\partial x}\cdot Q+\dfrac{\partial Q}{\partial x}\cdot u-\dfrac{\partial u}{\partial y}\cdot P-u\cdot\dfrac{\partial P}{\partial y}\right)$

$=\left(\dfrac{\partial u}{\partial y}R-\dfrac{\partial u}{\partial z}Q,\ \dfrac{\partial u}{\partial z}P-\dfrac{\partial u}{\partial x}R,\ \dfrac{\partial u}{\partial x}\cdot Q-\dfrac{\partial u}{\partial y}\cdot P\right)+u\left(\dfrac{\partial R}{\partial y}-\dfrac{\partial Q}{\partial z},\ \dfrac{\partial P}{\partial z}-\dfrac{\partial R}{\partial x},\ \dfrac{\partial Q}{\partial x}-\dfrac{\partial P}{\partial y}\right)$

$=\nabla u\times\vec{F}+u\,rot\vec{F}$. 证毕.

6证明: 由斯托克斯公式得: $\oint_\Gamma \vec{F}d\vec{r}=\iint_S rot\vec{F}\cdot\vec{n}\,dS$. ($\Gamma$为任意闭曲线)

而 $rot\vec{F}=\left(\dfrac{\partial R}{\partial y}-\dfrac{\partial Q}{\partial z},\ \dfrac{\partial P}{\partial z}-\dfrac{\partial R}{\partial x},\ \dfrac{\partial Q}{\partial x}-\dfrac{\partial P}{\partial y}\right)$

$=\left(e^x\cos z(-\sin y)-e^x\sin y(-\cos z),\ e^x\cos y\cos z-e^x\cos y\cos z,\ -e^x\sin y\sin z-e^x\sin z(-\sin y)\right)$

$=\vec{0}$

$\therefore\vec{F}$为保守场. 证毕. 设C为$(1,0,\frac{\pi}{2})$. AC为 $y=0,\ z=\frac{\pi}{2},\ 0\le x\le 1$. $\widehat{CB}$为 $x=1,\ z=\frac{\pi}{2},\ 0\le y\le\pi$

$\int_{\widehat{AB}}\vec{F}\cdot d\vec{r}=\int_{\widehat{AC}}\vec{F}d\vec{r}+\int_{\widehat{CB}}\vec{F}d\vec{r}=\int_{\widehat{AC}}e^x\cos y\sin z\,dx+\int_{\widehat{CB}}(-e^x\sin y\sin z)dy=\int_0^1e^xdx-\int_0^\pi e\sin y\,dy$

$=e^x\Big|_0^1+e\cos y\Big|_0^\pi=e-1$

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7解.(1) $\because \mathrm{rot}\,\vec{F}=(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z})i+(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x})j+(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})k$

$=(0-0)i+[-z\cdot x\cos(xz)-\sin(xz)-(-x\cdot z\cos(xz)-\sin(xz))]j$

$[-xy(-\sin(xy))-\cos(xy)-(0-y)x\cdot(-\sin(xy)-\cos(xy))]k=0$

∴该向量场为保守场. 从而知:该向量场为有势场.

则有 $\vec{F}=-\mathrm{grad}\,\vec{f}=(-\frac{\partial f}{\partial x},-\frac{\partial f}{\partial y},-\frac{\partial f}{\partial z})$

$$\therefore\begin{cases} f=-\int[-z\sin(xz)-y\cos(xy)]dx=-\cos(xz)+\sin(xy)+u(y,z)\\ f=-\int[-x\cos(xy)]dy=\sin(xy)+v(x,z)\\ f=-\int[x\sin(xz)]dz=-\cos(xz)+w(x,y). \end{cases}$$

比较③式知 $f=\sin(xy)-\cos(xz)+C$

(2) $\because \mathrm{rot}\,\vec{F}=(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z})i+(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x})j+(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})k$

$=\left[x[\ln(1+z^2)+\frac{2z^2}{1+z^2}]-[xz\cdot\frac{2z}{1+z^2}+x\ln(1+z^2)]\right]i$

$+\left[[yz\cdot\frac{2z}{1+z^2}+y\ln(1+z^2)]-y[\ln(1+z^2)+\frac{2z^2}{1+z^2}]\right]j$

$+[z\ln(1+z^2)-z\ln(1+z^2)]k=0$

∴该向量场为保守场. 从而知:该向量场为有势场.

则有 $\vec{F}=-\mathrm{grad}\,\vec{f}=(-\frac{\partial f}{\partial x},-\frac{\partial f}{\partial y},-\frac{\partial f}{\partial z})$

$$\therefore\begin{cases} f=-\int yz\ln(1+z^2)dx=-xyz\ln(1+z^2)+u(y,z)\\ f=-\int xz\ln(1+z^2)dy=-xyz\ln(1+z^2)+v(x,z)\\ f=-\int xy[\ln(1+z^2)+\frac{2z^2}{1+z^2}]=-xyz\ln(1+z^2)+w(x,y) \end{cases}$$

由以上3式知 $f(x,y,z)=-xyz\ln(1+z^2)+C$

8解.(1) 由2题(7)知. $\mathrm{div}[f(r)\cdot\vec{r}]=r\cdot f'(r)+3f(r)=r\cdot\frac{df(r)}{dr}+3f(r)=0$

$\therefore \frac{df(r)}{f(r)}=-3\cdot\frac{dr}{r}$ 两边求积分得 $\ln f(r)=-3\ln r+C$ $\therefore f(r)=\frac{C}{r^3}$

其中C为任意给定常数.

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(2) 由2题(6)知 $\mathrm{div}[\mathrm{grad} f(r)] = \frac{2}{r} f'(r) + f''(r) = 0$

$\therefore \frac{2}{r} f'(r) + \frac{d f'(r)}{dr} = 0$ 即 $\frac{d f'(r)}{f'(r)} = -2 \cdot \frac{dr}{r}$. 两边求积分得

$\ln f'(r) = -2 \ln r + C_1$ 即 $f'(r) = \frac{C_1}{r^2}$

$\therefore f(r) = \int f'(r) dr = -\frac{C_1}{r} + C_2$ (其中 C_1, C_2 为任意常数)

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§ 外微分形式与一般形式的斯托克斯公式

1. $d[y^2dx-(e^{y^3}+\sin x)dy]=(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dx\wedge dy=(-\cos x-2y)dx\wedge dy$

2. $d(2xydx+x^2dy)=(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dx\wedge dy=(2x-2x)dx\wedge dy=0$

3. $d(z\cdot e^{(x+y)}dx\wedge dy+y\tan(x^2z^3)dx\wedge dz)=(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z})dx\wedge dy\wedge dz$

$=(0+[-\tan(x^2z^3)]+e^{x+y})dx\wedge dy\wedge dz=[e^{x+y}-\tan(x^2z^3)]dx\wedge dy\wedge dz$

4. 证明：设 $\omega=Pdx+Qdy+Rdz$

则 $d(f\cdot\omega)=d(fPdx+fQdy+fRdz)=(\frac{\partial Rf}{\partial y}-\frac{\partial Qf}{\partial z})dy\wedge dz+(\frac{\partial Pf}{\partial z}-\frac{\partial Rf}{\partial x})dz\wedge dx$

$+(\frac{\partial Qf}{\partial x}-\frac{\partial Pf}{\partial y})dx\wedge dy$

$=(R\cdot\frac{\partial f}{\partial y}+f\cdot\frac{\partial R}{\partial y}-Q\frac{\partial f}{\partial z}-f\cdot\frac{\partial Q}{\partial z})dy\wedge dz$

$(P\cdot\frac{\partial f}{\partial z}+f\cdot\frac{\partial P}{\partial z}-R\cdot\frac{\partial f}{\partial x}-f\cdot\frac{\partial R}{\partial x})dz\wedge dx$

$+(Q\frac{\partial f}{\partial x}+f\frac{\partial Q}{\partial x}-P\frac{\partial f}{\partial y}-f\cdot\frac{\partial P}{\partial y})dx\wedge dy$

$=f\cdot[(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z})dy\wedge dz+(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x})dz\wedge dx+(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dx\wedge dy]$

$+R\frac{\partial f}{\partial y}dy\wedge dz-Q\frac{\partial f}{\partial z}dy\wedge dz+P\frac{\partial f}{\partial z}dz\wedge dx-R\frac{\partial f}{\partial x}dz\wedge dx$

$+Q\frac{\partial f}{\partial x}dx\wedge dy-P\cdot\frac{\partial f}{\partial y}dx\wedge dy$

而 $df\wedge\omega=(\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz)\wedge(Pdx+Qdy+Rdz)$

$=Q\frac{\partial f}{\partial x}dx\wedge dy-R\frac{\partial f}{\partial x}dz\wedge dx-P\frac{\partial f}{\partial y}dx\wedge dy+R\frac{\partial f}{\partial y}dy\wedge dz$

$+P\frac{\partial f}{\partial z}\cdot dz\wedge dx-Q\frac{\partial f}{\partial z}dy\wedge dz$

$\therefore d(f\cdot\omega)=df\wedge\omega+f\cdot d\omega.$ 证毕

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第八章总练习题

1. 解. $M=\int_L \frac{1}{1+t}ds=\int_0^2 \frac{1}{1+t}\cdot\sqrt{1^2+(\sqrt{2}t^{\frac{1}{2}})^2+t^2}dt=\int_0^2 dt=2$

$\int_L x\cdot\frac{1}{1+t}ds=\int_0^2 t\,dt=2$ 　　$\int_L y\cdot\frac{1}{1+t}ds=\int_0^2 \frac{2\sqrt{2}}{3}t^{3/2}dt=\frac{2\sqrt{2}}{3}\cdot\frac{2}{5}\cdot t^{\frac{5}{2}}\Big|_0^2=\frac{32}{15}$

$\int_L z\cdot\frac{1}{1+t}ds=\int_0^2\frac{t^2}{2}dt=\frac{1}{6}t^3\Big|_0^2=\frac{4}{3}$

∴其重心横坐标 $x_0=\frac{\int_L x\cdot\frac{1}{1+t}ds}{M}=1$. 纵坐标 $y_0=\frac{\int_L y\frac{1}{1+t}ds}{M}=\frac{16}{15}$. z坐标 $z_0=\frac{\int_L z\cdot\frac{1}{1+t}ds}{M}=\frac{2}{3}$

∴其重心坐标为 $(1,\frac{16}{15},\frac{2}{3})$

2 解. ∵ $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}=\frac{xy}{\sqrt{x^2+y^2}}$ ∴积分与路径无关. 取点 $C(5,4)$. 路径 $\overline{AC}$ 为 $y=4$, $3\le x\le 5$

路径 $\overline{CB}$ 为 $x=5$, $4\le y\le 12$.

则 $\int_{(3,4)}^{(5,12)}\frac{x\cdot dx+y\,dy}{\sqrt{x^2+y^2}}=\int_{\overline{AC}}\frac{x}{\sqrt{x^2+16}}dx+\int_{\overline{CB}}\frac{y}{\sqrt{25+y^2}}dy=\int_3^5\frac{d(x^2+16)}{2\sqrt{x^2+16}}+\int_4^{12}\frac{d(y^2+25)}{2\sqrt{y^2+25}}$

$=\sqrt{x^2+16}\Big|_3^5+\sqrt{y^2+25}\Big|_4^{12}=8$

3 解. ∵ $\frac{\partial g(y)}{\partial x}=\frac{\partial f(x)}{\partial y}=0$. ∴积分与路径无关. 取点 (x_1,y_2). 路径 $\overline{AC}$ 为 $x=x_1$, $y_1\le y\le y_2$

路径 $\overline{CB}$ 为 $y=y_2$, $x_1\le x\le x_2$.

则 $\int_{(x_1,y_1)}^{(x_2,y_2)}f(x)dx+g(y)dy=\int_{y_1}^{y_2}g(y)dy+\int_{x_1}^{x_2}f(x)dx$

4 解. ∵ $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}=\frac{1}{(x-y)^2}$ ∴积分与路径无关. 取点 $C(1,6)$ 路径 $\overline{AC}$ 为 $x=1$, $2\le y\le 6$

路径 $\overline{CB}$ 为 $y=6$, $1\le x\le 5$.

则 $\int_{(1,2)}^{(5,6)}\frac{x\,dy-y\,dx}{(x-y)^2}=\int_2^6\frac{1}{(1-y)^2}dy+\int_1^5\frac{-6}{(x-6)^2}dx=-\int_2^6\frac{d(1-y)}{(1-y)^2}-\int_1^5\frac{6\,d(x-6)}{(x-6)^2}$

$=(1-y)^{-1}\Big|_2^6+6(x-6)^{-1}\Big|_1^5=-4$

5 解. ∵ $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}=\frac{y^2}{x^3}\sin\frac{y}{x}-\frac{2y}{x^2}\cos\frac{y}{x}$. ∴积分与路径无关. 取点 $C(2,2\pi)$.

则路径 $\overline{AC}$ 为 $y=2\pi$ 　$1\le x\le 2$. 路径 $\overline{CB}$ 为 $x=2$. $2\pi\ge y\ge\pi$

∴ $\int_{(1,2\pi)}^{(2,\pi)}(1-\frac{y^2}{x^2}\cos\frac{y}{x})dx+(\sin\frac{y}{x}+\frac{y}{x}\cdot\cos\frac{y}{x})dy=\int_1^2(1-\frac{4\pi^2}{x^2}\cos\frac{2\pi}{x})dx+\int_{2\pi}^{\pi}(\sin\frac{y}{2}+\frac{y}{2}\cos\frac{y}{2})dy$

$=1+\int_1^2 2\pi\cdot\cos\frac{2\pi}{x}\,d\frac{2\pi}{x}+2\int_{2\pi}^{\pi}\sin\frac{y}{2}d\frac{y}{2}+\int_{2\pi}^{\pi}\frac{y}{2}\cos\frac{y}{2}dy$

$=1+0-2+\int_{\pi}^{\frac{\pi}{2}}2t\cos t\,dt=-1+2\left(t\sin t\Big|_{\pi}^{\frac{\pi}{2}}-\int_{\pi}^{\frac{\pi}{2}}\sin t\,dt\right)=-1+2(\frac{\pi}{2}+1)=\pi+1$

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6 证明: 只需证明该积分与路径无关,即证明 $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$, 其中 $Q=x\cdot f(xy)$, $P=y\cdot f(xy)$.

$$\frac{\partial Q}{\partial x}=x\cdot f'(xy)\cdot y+f(xy)=xy\cdot f'(xy)+f(xy)$$

$$\frac{\partial P}{\partial y}=y\cdot f'(xy)\cdot x+f(xy)=xy\cdot f'(xy)+f(xy)$$

$\therefore \frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$. 证毕. 即有 $\oint_L f(xy)(y\cdot dx+x dy)=0$.

7 证明: 设路径L的表达式 $y=f(x)$. 则 $L=\int_\alpha^\beta ds=\int_\alpha^\beta \sqrt{1+f'(x)}\,dx$

$$\because \left|\int_L P(x,y)dx+Q(x,y)dy\right|=\left|\int_\alpha^\beta [P(x,y)+Q(x,y)\cdot f'(x)]dx\right|$$

$$\le\left|\int_\alpha^\beta \sqrt{P^2(x,y)+Q^2(x,y)}\cdot\sqrt{1+f'(x)}\,dx\right|\le\left|\int_\alpha^\beta M\cdot\sqrt{1+f'(x)}\,dx\right|=LM.$$ 证毕

8解: 由7题可知: $|I_a|\le L\cdot M$. 易求 $L=2\pi a$, M为函数 $\frac{x^2+y^2}{(x^2+xy+y^2)^4}$ 的最大值.

设 $f(x,y)=\frac{x^2+y^2}{(x^2+xy+y^2)^4}=\frac{16(x^2+y^2)}{[(x^2+y^2)+(x+y)^2]^4}$

$$=\frac{16(x^2+y^2)}{[2(x^2+y^2)+2(xy)]^4}=\frac{16a^2}{(2a^2+2xy)^4}.$$

而 $-a^2\le 2xy\le a^2$. $\therefore \max(f(x,y))=\frac{16a^2}{(2a^2-a^2)^4}=\frac{16}{a^6}$ $\therefore M^2=\frac{16}{a^6}$ $\therefore M=\frac{4}{a^3}\ (M>0)$

故 $|I_a|\le L\cdot M=2\pi a\cdot\frac{4}{a^3}=\frac{8\pi}{a^2}$.

9解: $I=\oint_{L^+}\sqrt{x^2+y^2}\,dx+y[xy+\ln(x+\sqrt{x^2+y^2})]dy=\iint_D\left[\left[y^2+\frac{y}{x+\sqrt{x^2+y^2}}\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)\right]-\frac{y}{\sqrt{x^2+y^2}}\right]d\sigma$

$$=\iint_{D'}\left(r^2\sin^2\theta+\frac{r\sin\theta}{r}-\frac{r\sin\theta}{r}\right)r\,dr\,d\theta=\int_0^{2\pi}d\theta\int_0^1(r^3\sin^2\theta)dr=\frac{\pi}{4}$$

10 证明: 设曲线L的单位切向量为 $(\cos\alpha,\cos\beta)$. 则设 $\vec{n}=(n_x,n_y)$. 有

$$\begin{cases}\vec{n}\cdot(\cos\alpha,\cos\beta)=0\\ \vec{n}\times(\cos\alpha,\cos\beta)=k\end{cases}\quad 解得\quad \begin{cases}n_x=\cos\beta\\ n_y=-\cos\alpha\end{cases}\quad \therefore \vec{n}=(\cos\beta,-\cos\alpha)$$

$$\therefore \oint_L\frac{\partial u}{\partial n}ds=\oint_L\left(\frac{\partial u}{\partial x}\cos\beta-\frac{\partial u}{\partial y}\cos\alpha\right)ds=\oint_L\frac{\partial u}{\partial x}dy-\frac{\partial u}{\partial y}dx$$

$$=\iint_D\left(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}\right)d\sigma=\iint_D 0\,d\sigma=0.$$ 证毕.

11解: 根据对称性知. $I_1=3\iint_{S_1}x^2dS$. $I_2=3\iint_{S_2}x^2dS$

$$\iint_{S_1}x^2dS=\iint_{S_1'}x^2dS+\iint_{S_1''}x^2dS=2\iint_{D_1}x^2\sqrt{1+\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}}\,d\sigma=2\iint_{D_1}\frac{ax^2}{\sqrt{a^2-x^2-y^2}}d\sigma$$

$$=2\iint_{D_1}\frac{a\cdot r^2\cos^2\theta}{\sqrt{a^2-r^2}}\cdot r\,dr\,d\theta=2\int_0^{2\pi}\cos^2\theta\,d\theta\cdot\int_0^a\frac{a\cdot r^3}{\sqrt{a^2-r^2}}dr$$

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$$= 2\cdot 4\cdot \frac{1}{2}\cdot\frac{\pi}{2}\cdot\int_0^a(-a\cdot r^2)\,d\sqrt{a^2-r^2} = 2\pi\left(-ar^2\sqrt{a^2-r^2}\Big|_0^a + a\int_0^a\sqrt{a^2-r^2}\,dr^2\right) = \frac{4}{3}a^4\pi$$

$$\iint_{S_2} x^2 dS = 8\iint_{D_2} x^2\sqrt{1^2+(-1)^2+(-1)^2}\,d\sigma = 8\int_0^a dx\int_0^{a-x}\sqrt{3}\,x^2 dy = \frac{2\sqrt{3}}{3}a^4$$

$$\therefore I_1 - I_2 = 3\left(\iint_{S_1}x^2dS - \iint_{S_2}x^2dS\right) = 3\cdot\left(\frac{4}{3}a^4\pi - \frac{2\sqrt{3}}{3}a^4\right) = a^4(4\pi - 2\sqrt{3})$$

12解：由 $z\geq\sqrt{x^2+y^2}$ 与 $x^2+y^2+z^2=a^2$ 得 $x^2+y^2\leq\frac{a^2}{2}$，记为区域D.

$$\text{则 } F(a) = \iint_S f(x,y,z)dS = \iint_D (x^2+y^2)\frac{a}{\sqrt{a^2-(x^2+y^2)}}d\sigma = \iint_{D'}\frac{ar^3}{\sqrt{a^2-r^2}}dr\,d\theta$$

$$= \int_0^{2\pi}d\theta\int_0^{\frac{a}{\sqrt2}} -ar^2\,d\sqrt{a^2-r^2} = 2\pi a\left[-r^2\sqrt{a^2-r^2}\Big|_0^{\frac{a}{\sqrt2}} - \int_0^{\frac{a}{\sqrt2}}\sqrt{a^2-r^2}\,d(a^2-r^2)\right]$$

$$= 2\pi a\cdot\left(-\frac{\sqrt2}{4}a^3 - \frac{2}{3}\left[\left(\frac{a}{\sqrt2}\right)^3 - a^3\right]\right) = \frac{\pi(8-5\sqrt2)}{6}a^4$$

13证明：易知平面的单位法向量为 $(\frac{2}{3},\frac{2}{3},\frac{1}{3})$.

$$\text{则 } \oint_{L^+} 2ydx+3zdy-xdz = \iint_S (0-3)\,dydz + (0+1)\,dzdx + (0-2)\,dxdy$$

$$= \iint_S \left(-3\cdot\frac{2}{3} + 1\cdot\frac{2}{3} - 2\cdot\frac{1}{3}\right)dS = -2\iint_S dS$$

由上式得知 $\oint_{L^+} 2ydx+3zdy-xdz$ 只与L所围区域的面积有关与L的形状及位置无关.

14证明：(1) ~~[struck-out line]~~

$$\mathrm{rot}\,\vec F = (0-0)i + (0-0)j + \left(\left[x\cdot\left(-\frac{2x}{(x^2+y^2)^2}\right)+\frac{1}{x^2+y^2}\right] - \left[(-y)\cdot\left(-\frac{2y}{(x^2+y^2)^2}\right) - \frac{1}{x^2+y^2}\right]\right)k = \vec 0$$

证毕

(2) 易知 Oxy 平面上的圆周 $x^2+y^2=1$ 的参数方程为 $\begin{cases}x=\cos t\\ y=\sin t\\ z=0\end{cases}$　$(0\leq t\leq 2\pi)$

$$\therefore F(\cos t,\sin t,0) = -\sin t\,i + \cos t\,j + 0k$$

$$\therefore \int_C \vec F\,d\vec r = \int_0^{2\pi}(-\sin t)(-\sin t) + \cos t\cdot\cos t + 0\,dt = \int_0^{2\pi}dt = 2\pi \neq 0.\ \text{证毕}$$

以上两个结果与斯托克斯公式不矛盾. 因为L所围成的区域曲面S在(0,0)处要挖去. 所以S非单连通区域. 不满足斯托克斯公式的条件.

15解 易知平面的单位外法向量 $\vec n = (\cos\alpha,\cos\beta,\cos\gamma)$

$$\oint_{L^+}\begin{vmatrix} dx & dy & dz\\ \cos\alpha & \cos\beta & \cos\gamma\\ x & y & z\end{vmatrix} = \oint_{L^+}(z\cos\beta - y\cos\gamma)dx + (x\cos\gamma - z\cos\alpha)dy + (y\cos\alpha - x\cos\beta)dz.$$

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$$= \oiint_{S^+} 2\cos\alpha\, dydz + 2\cos\beta\, dzdx + 2\cos\gamma\, dxdy = 2\oiint_{S^+} d\vec{S} = 2S.$$ 证毕.

16.证明: $$\oiint_{S^+}\cos(\vec{r},\vec{n})\,dS = \oiint_{S^+}\frac{\vec{r}\cdot\vec{n}}{r\cdot|\vec{n}|}\,dS = \oiint_{S^+}\frac{1}{r}\vec{r}\cdot\vec{n}_0\,dS = \oiint_{S^+}\frac{\vec{r}}{r}\,d\vec{S}$$

$$= \oiint_{S^+}\left(\frac{x-x_0}{r},\ \frac{y-y_0}{r},\ \frac{z-z_0}{r}\right)d\vec{S} = \iiint_{\Omega}\left(\frac{\partial(\frac{x-x_0}{r})}{\partial x} + \frac{\partial(\frac{y-y_0}{r})}{\partial y} + \frac{\partial(\frac{z-z_0}{r})}{\partial z}\right)dV$$

而 $$\frac{\partial(\frac{x-x_0}{r})}{\partial x} = \frac{1\cdot\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} - (x-x_0)\frac{x-x_0}{r}}{r^2} = \frac{r^2-(x-x_0)^2}{r^3}$$

同理求得 $$\frac{\partial(\frac{y-y_0}{r})}{\partial y} = \frac{r^2-(y-y_0)^2}{r^3} \qquad \frac{\partial(\frac{z-z_0}{r})}{\partial z} = \frac{r^2-(z-z_0)^2}{r^3}$$

$$\therefore \oiint_{S^+}\cos(\vec{r},\vec{n})\,dS = \iiint_{\Omega}\left(\frac{3r^2-(x-x_0)^2-(y-y_0)^2-(z-z_0)^2}{r^3}\right)dV = 2\iiint_{\Omega}\frac{1}{r}\,dV$$

即 $$\iiint_{\Omega}\frac{dxdydz}{r} = \frac{1}{2}\oiint_{S^+}\cos(\vec{r},\vec{n})\,dS$$

17解: $$\mathrm{grad}\,u = \left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right) = \left(\frac{-1}{r}\cdot\frac{\partial r}{\partial x},\ \frac{-1}{r}\frac{\partial r}{\partial y},\ \frac{-1}{r}\cdot\frac{\partial r}{\partial z}\right)$$

$$\therefore |\mathrm{grad}\,u| = \sqrt{\frac{1}{r^2}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2+\left(\frac{\partial r}{\partial z}\right)^2\right]}$$

$$= \sqrt{\frac{1}{r^2}\cdot\left[\frac{(x-x_0)^2}{r^2}+\frac{(y-y_0)^2}{r^2}+\frac{(z-z_0)^2}{r^2}\right]} = \sqrt{\frac{1}{r^2}\cdot\frac{r^2}{r^2}} = \frac{1}{r} = 1$$

$\therefore$ $r=1$. 即所求点为整个球面$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=1$上的点

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第九章 常微分方程

1 基本概念

1解. (1) 一阶. (2) 二阶　(3) 三阶

2解 (1) $\because \dfrac{D(y,y')}{D(C_1,C_2)}=\begin{vmatrix}\cos ax & \sin ax\\ -a\sin ax & a\cos ax\end{vmatrix}=a(\sin^2 ax+\cos^2 ax)=a\neq 0$

$\therefore C_1, C_2$是两个独立常数.

(2) $\because \dfrac{D(y,y')}{D(C_1,C_2)}=\begin{vmatrix}1-\cos 2x & \sin^2 x\\ 2\sin 2x & \sin 2x\end{vmatrix}=2\sin^2 x\cdot\sin 2x-2\sin 2x\cdot\sin^2 x=0$

$\therefore C_1, C_2$不是独立的.

(3) $\because \dfrac{D(y,y')}{D(C_1,C_2)}=\begin{vmatrix}e^{\lambda_1 x} & e^{\lambda_2 x}\\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x}\end{vmatrix}=e^{\lambda_1 x+\lambda_2 x}(\lambda_2-\lambda_1)\neq 0$

$\therefore C_1, C_2$是两个独立常数.

(4) $\because \dfrac{D(y,y')}{D(C_1,C_2)}=\begin{vmatrix}e^{\lambda x} & x\cdot e^{\lambda x}\\ \lambda e^{\lambda x} & e^{\lambda x}+\lambda x e^{\lambda x}\end{vmatrix}=e^{2\lambda x}\neq 0$

$\therefore C_1, C_2$是两个独立常数.

(5) $\because \dfrac{D(y,y',y'')}{D(C_1,C_2,C_3)}=\begin{vmatrix}x & x^2 & x^2+x\\ 1 & 2x & 2x+1\\ 0 & 2 & 2\end{vmatrix}=x\cdot(4x-4x-2)+x^2(0-2)+(x^2+x)(2-0)=0$

$\therefore C_1, C_2, C_3$不是独立的.

3解: (1) $y''=-4\sin 2x$　$\therefore y''+4y=0$. 故 $y=\sin 2x$为微分方程的特解.

(2) $y''=a^2(-C_1\cos ax-C_2\sin ax)$　$\therefore y''+a^2y=0$ 且 $\dfrac{D(y,y')}{D(C_1,C_2)}=a\neq 0$.

故 $y=C_1\cos ax+C_2\sin ax$为微分方程的通解.

(3) 易验证 $y=C_1e^{\lambda_1 x}+C_2e^{\lambda_2 x}$ 满足微分方程 且 $\dfrac{D(y,y')}{D(C_1,C_2)}\neq 0$

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$\therefore C_1, C_2$ 是两个独立常数，故 $y=C_1e^{\lambda_1 x}+C_2e^{\lambda_2 x}$ 为微分方程的通解.

(4) 易验证 $y=C_1e^{\lambda x}+C_2xe^{\lambda x}$ 满足微分方程，且 $\frac{D(y,y')}{D(C_1,C_2)}\neq 0$

$\therefore y=C_1e^{\lambda x}+C_2xe^{\lambda x}$ 为常微分方程的通解.

(5). $\because$ 微分方程是二阶的，而 $y=Ce^{3x}$ 只包含一个任意常数

故 $y=Ce^{3x}$ 即不是微分方程的通解，也不是微分方程的特解.

(6) 易验证 $y=x\left(\int_1^x \frac{e^x}{x}dx+C\right)$ 为常微分方程的解，且 C 为某一常数

$\therefore y=x\left(\int_1^x \frac{e^x}{x}dx+C\right)$ 为常微分方程的特解.

4. 解. 易验证 $x(t)=\cos t+2\sin t-2t\cos t$ 为方程 $\frac{d^2x}{dt^2}+x=4\sin t$.

且 $x(0)=1+0-0=1 \qquad x'(0)=-0+2-2(1+0)=0$.

故函数 $x(t)=\cos t+2\sin t-2t\cos t$ 是题设初值问题的解.

5. 解: (1) $x(t)=\int \frac{dx}{dt}dt=\int \cos\omega t\,dt=\frac{1}{\omega}\sin\omega t+C$

$x(0)=\frac{1}{\omega}\sin 0+C=C=10$

$\therefore$ 初值问题的解为 $x(t)=\frac{1}{\omega}\sin\omega t+10$

(2). $\frac{dy}{dx}=\int \frac{d^2y}{dx^2}dx=\int 12x^2dx=4x^3+C_1$

$y=y(x)=\int \frac{dy}{dx}dx=\int(4x^3+C_1)dx=x^4+C_1x+C_2$

$\because y(0)=C_2=0 \qquad y'(0)=C_1=1 \qquad \therefore C_1=1,\ C_2=0$

故初值问题的解为 $y(x)=x^4+x$

(3) $y''=\int y'''dx=\frac{1}{2}x^2+C_1 \qquad y'=\int y''dx=\frac{1}{6}x^3+C_1x+C_2$

$y=\int y'dx=\frac{1}{24}x^4+\frac{1}{2}C_1x^2+C_2x+C_3$

且 $y(0)=C_3=a_0. \quad y'(0)=C_2=a_1 \quad y''(0)=C_1=a_2$

$\therefore$ 初值问题的解为 $y(x)=\frac{1}{24}x^4+\frac{1}{2}a_2x^2+a_1x+a_0$

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2初等积分法

1解 (1) $\because \frac{dy}{dx}=\frac{1+y^2}{(1+x^2)xy}$ $\therefore \frac{y}{1+y^2}dy=\frac{1}{(1+x^2)x}dx$ $\therefore \frac{1}{2}\frac{d(y^2+1)}{1+y^2}=\frac{1}{2}\cdot\frac{dx^2}{(1+x^2)x^2}$

$\therefore d[\ln(y^2+1)]=\frac{1}{x^2}dx^2-\frac{1}{1+x^2}d(x^2+1)=d(\ln x^2-\ln(x^2+1))$

$\therefore \ln(y^2+1)=\ln x^2-\ln(x^2+1)+c=\ln\frac{x^2}{x^2+1}\cdot e^c$ $\therefore y^2+1=C'\cdot\frac{x^2}{x^2+1}$ $(C'=e^c)$

即微分方程的通积分为 $(1+x^2)(1+y^2)=Cx^2$.

(2) 方程可化为. $\frac{y-a}{2ay}dy=\frac{1}{x}dx$ 即 $\frac{1}{2a}dy-\frac{1}{2y}dy=d(\ln x)$

$\therefore d(\frac{1}{2a}y-\frac{1}{2}\ln y)=d(\ln x)$ $\therefore \frac{1}{2a}y-\frac{1}{2}\ln y=\ln x+C$

即 $\frac{y}{a}=\ln x^2y+2C=\ln e^{2c}\cdot x^2y$ $\therefore x^2y=e^{y/a}\cdot\frac{1}{e^{2c}}=C'\cdot e^{y/a}$ $(C'=\frac{1}{e^{2c}})$

$\therefore$微分方程的通积分为: $x^2y=C\cdot e^{y/a}$

(3) 方程化为 $\frac{1}{\sqrt{1-y^2}}dy=\frac{1}{\sqrt{1+x^2}}dx$. 两边求积分得 $\int\frac{1}{\sqrt{1-y^2}}dy=\int\frac{1}{\sqrt{1+x^2}}dx$

而 $\int\frac{1}{\sqrt{1-y^2}}dy=\int\frac{\cos t}{\sqrt{1-\sin^2 t}}dt=t+C_1=\arcsin y+C_1$

$\int\frac{1}{\sqrt{1+x^2}}dx=\int\frac{1}{\sqrt{1+\tan^2 t}}\cdot\frac{1}{\cos^2 t}\cdot dt=\int\frac{1}{\cos t}dt=\int\frac{d(\sin t)}{1-\sin^2 t}=\int\frac{1}{2}(\frac{d\sin t}{1-\sin t}+\frac{d\sin t}{1+\sin t})$

$=\frac{1}{2}\int(-\frac{d(1-\sin t)}{1-\sin t})+\frac{1}{2}\int\frac{d(1+\sin t)}{1+\sin t}=-\frac{1}{2}\ln(1-\sin t)+\frac{1}{2}\ln(1+\sin t)+C_2$

$=-\frac{1}{2}\ln(1-\frac{x}{\sqrt{x^2+1}})+\frac{1}{2}\ln(1+\frac{x}{\sqrt{x^2+1}})+C_2$

$\therefore \arcsin y=-\frac{1}{2}\ln(1-\frac{x}{\sqrt{x^2+1}})+\frac{1}{2}\ln(1+\frac{x}{\sqrt{x^2+1}})+C=\ln C'(x+\sqrt{1+x^2})$

(4) 解一: $(x+2y)dx+(2x-3y)dy=0\Rightarrow xdx+(2ydx+2xdy)-3ydy=0$

$\Rightarrow \frac{1}{2}x^2+2xy-\frac{3}{2}y^2+C'=0\Rightarrow x^2+4xy-3y^2=C$

解二: 方程两边同时除以x得: $(1+2\cdot\frac{y}{x})dx+(2-3\frac{y}{x})dy=0$.

$\therefore \frac{dy}{dx}=\frac{1+2\frac{y}{x}}{3\frac{y}{x}-2}$ 令 $u=\frac{y}{x}$ 则 $y'=u'x+u$ $\therefore u'x+u=\frac{1+2u}{3u-2}$

即 $\frac{du}{dx}\cdot x=\frac{1+4u-3u^2}{3u-2}$ $\therefore \frac{3u-2}{1+4u-3u^2}du=\frac{1}{x}dx$

而 $\int\frac{3u-2}{1+4u-3u^2}du=3\int\frac{\frac{1}{2}d(u-\frac{2}{3})^2}{-3(u-\frac{2}{3})^2+\frac{7}{3}}=\frac{3}{2}\int\frac{dt}{-3t+\frac{7}{3}}$ $(t=(u-\frac{2}{3})^2)$ $=-\frac{1}{2}\int\frac{d(-3t+\frac{7}{3})}{-3t+\frac{7}{3}}$

$=-\frac{1}{2}\ln(-3t+\frac{7}{3})+C_1=-\frac{1}{2}\ln(1+4u-3u^2)+C_1$

而 $\int \frac{1}{x}dx = \ln x + C_2$ 故 $-\frac{1}{2}\ln(1+4u-3u^2) = \ln x + C'$

即 $\frac{1}{\sqrt{1+4u-3u^2}} = C''x$ 而 $u=\frac{y}{x}$ ∴ $\frac{1}{\sqrt{1+4\frac{y}{x}-3\frac{y^2}{x^2}}} = C''\cdot x$

即 $x^2+4xy-3y^2=C$

(5) 方程两边同时除以x得 $(3+5\cdot\frac{y}{x})dx+(4+6\cdot\frac{y}{x})dy=0$

∴ $\frac{dy}{dx} = -\frac{3+5\cdot\frac{y}{x}}{4+6\frac{y}{x}}$ 令 $u=\frac{y}{x}$ 得 $u'x+u=-\frac{3+5u}{4+6u}$ 即 $\frac{du}{dx}x=\frac{-6u^2-9u-3}{4+6u}$

∴ $\frac{4+6u}{-6u^2-9u-3}du=\frac{1}{x}dx$

而 $\int\frac{4+6u}{-6u^2-9u-3}du =$ [illegible]

$= -\frac{2}{3}\int\frac{2+3u}{2u^2+3u+1}du = -\frac{2}{3}\int\frac{(3u+2)du}{(2u+1)(u+1)} = -\frac{2}{3}\int \frac{du}{2u+1}+\frac{du}{u+1}$

$= -\frac{2}{3}\left(\frac{1}{2}\ln(2u+1)+\ln(u+1)\right)+C_1 = -\frac{1}{3}\ln(2u+1)-\frac{2}{3}\ln(u+1)+C_1$

而 $\int\frac{1}{x}dx=\ln x+C_2$

∴ $-\frac{1}{3}\ln(2u+1)-\frac{2}{3}\ln(u+1)=\ln x+C'$ 即 $\ln\frac{1}{(2u+1)(u+1)^2}=\ln x^3+C''$

即 $\frac{1}{(2\frac{y}{x}+1)(\frac{y}{x}+1)^2}=C'''x^3$ 即 $(x+y)^2(2y+x)=C$

(6) 两边同时除以xdx得 $2\frac{dz}{dx}-2\frac{z}{x}=\sqrt{1+4(\frac{z}{x})^2}$ 令 $u=\frac{z}{x}$，则：

$2(u'x+u)-2u=\sqrt{1+4u^2}$ ∴ $\frac{du}{dx}x=\frac{\sqrt{1+4u^2}}{2}$ 即 $\frac{2du}{\sqrt{1+4u^2}}=\frac{1}{x}dx$

而 $\int\frac{2du}{\sqrt{1+4u^2}} = \int\frac{2}{\sqrt{1+\tan^2 t}}\cdot\frac{1}{2}\cdot\frac{1}{\cos^2 t}dt = \int\frac{1}{\cos t}dt = \frac{1}{2}\ln\frac{1+\sin t}{1-\sin t}+C_1$

$=\frac{1}{2}\ln\frac{1+\frac{2u}{\sqrt{4u^2+1}}}{1-\frac{2u}{\sqrt{4u^2+1}}}+C_1 = \ln\frac{2u+\sqrt{4u^2+1}}{1}+C_1$

$\int\frac{1}{x}dx=\ln x+C_2$

∴ $\frac{2u+\sqrt{4u^2+1}}{1}=C'x$ 即 $2\frac{z}{x}+\sqrt{4\frac{z^2}{x^2}+1}=2C'x$ 即 $2z+\sqrt{x^2+4z^2}=Cx^2$

(7) 两边同时除以x^2dx得：$(2+\frac{y^2}{x^2})+\left[2\frac{y}{x}+3(\frac{y}{x})^2\right]\frac{dy}{dx}=0$. 令 $u=\frac{y}{x}$ 则有

$(2+u^2)+(2u+3u^2)(u'x+u)=0$ 即 $\frac{du}{dx}x+u=-\frac{2+u^2}{2u+3u^2}$

∴ $-\frac{2u+3u^2}{3u^3+3u^2+2}du=\frac{1}{x}dx$ 最终解得：$2x^3+3xy^2+3y^3=C$ （计算太繁）

另解：$(2x^2+y^2)dx+(2xy+3y^2)dy=2x^2dx+(y^2dx+2xydy)+3y^2dy=0$

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∴ $\frac{2}{3}x^3+xy^2+y^3=C'$ 即 $2x^3+3xy^2+3y^3=C$.

(8) 令 $u=x+y+2$. 则 $\frac{du}{dx}=1+\frac{dy}{dx}$. ∴ $\frac{dy}{dx}=\frac{du}{dx}-1=u^2$ 即 $\frac{du}{u^2+1}=\frac{dx}{\text{[redacted]}}$

故 $\arctan u=$ [redacted] $x+C'$ 即 $\arctan(x+y+2)=x+C$

(9) 令 $u=2x+3y$. 则 $\frac{du}{dx}=2+3\cdot\frac{dy}{dx}$ ∴ $\frac{dy}{dx}=\frac{1}{3}(\frac{du}{dx}-2)$

∴方程化为 $(u-1)+(2u-5)\cdot\frac{1}{3}(\frac{du}{dx}-2)=0$ 即 $\frac{2u-5}{u-7}du=dx$

即 $(2+\frac{9}{u-7})du=dx$ 即 $2u+9\ln|u-7|=x+C'$

即 $4x+6y+9\ln|2x+3y-7|=x+C'$ 即为 $x+2y+3\ln|2x+3y-7|=C$

(10) 解 $\begin{cases}2x-y+4=0\\x-2y+5=0\end{cases}$ 得 $\begin{cases}x_0=\text{[redacted]}-1\\y_0=\text{[redacted]}\,2\end{cases}$ ∴ $\frac{dy}{dx}=-\frac{x-2y+5}{2x-y+4}=-\frac{(x+1)-2(y-2)}{2(x+1)-(y-2)}$

令 $u=x+1$, $v=y-2$ ∴ $\frac{dv}{du}=-\frac{u-2v}{2u-v}=\frac{2\cdot\frac{v}{u}-1}{2-\frac{v}{u}}$

令 $z=\frac{v}{u}$. 则 $v'=z'\cdot u+z$ ∴ $\frac{dz}{du}\cdot u+z=\frac{2z-1}{2-z}$ 即 $\frac{2-z}{z^2-1}dz=\frac{1}{u}\cdot du$

即 $\frac{1}{2}(\frac{1}{z-1}-\frac{3}{z+1})dz=\frac{1}{u}du$

而 $\int\frac{1}{2}(\frac{1}{z-1}-\frac{3}{z+1})dz=\frac{1}{2}\ln|z-1|-\frac{3}{2}\ln(z+1)+C_1$

$\int\frac{1}{u}du=\ln u+C_2$

∴ $\frac{1}{2}\ln(z-1)-\frac{3}{2}\ln(z+1)=\ln u+C'$ 即 $\ln(\frac{y-2}{x+1}-1)-3\ln(\frac{y-2}{x+1}+1)=\ln(x+1)^2+C'$

即 $\frac{\frac{y-2}{x+1}-1}{(\frac{y-2}{x+1}+1)^3}=C''(x+1)^2$ 化为 $y-x-3=C(x+y-1)^3$

2解. (1) 方程化为 $4y\,dy=-x\,dx$ 即 $d(2y^2)=d(-\frac{1}{2}x^2)$ ∴ $2y^2=-\frac{1}{2}x^2+C$

又 $y(4)=2$ 即 $2\cdot2^2=-\frac{1}{2}\cdot4^2+C$ 解得 $C=16$ ∴ $y=\frac{1}{2}\sqrt{32-x^2}$

(2) 方程化为 $y\,dy+xe^x dx=0$, $\int y\,dy=\frac{1}{2}y^2+C_1$; $\int -xe^x dx=e^x-e^x x+C_2$

∴ $\frac{1}{2}y^2=e^x(1-x)+C_2$ 又∵ $y(0)=1$ ∴ $\frac{1}{2}=1\cdot1+C_2$ 即 $C_2=-\frac{1}{2}$

∴ $y=\sqrt{2e^x(1-x)-1}$ (书上答案不严谨)

(3) 方程化为 $\frac{1}{y^3}dy=\frac{x}{\sqrt{1+x^2}}dx$ 即 $d(-\frac{1}{2y^2})=\frac{\frac{1}{2}d(x^2+1)}{\sqrt{1+x^2}}=d\sqrt{x^2+1}$

∴ $-\frac{1}{2y^2}=\sqrt{x^2+1}+C$ 又 $y(0)=1$ ∴ $-\frac{1}{2}=1+C$ ∴ $C=-\frac{3}{2}$

∴ $y=\sqrt{\frac{1}{3-2\sqrt{x^2+1}}}$

(4) 易知 $P(x)=\frac{2}{x}$, $Q(x)=\frac{\sin x}{x}$. 故方程通解为

$y(x)=\left[\int_1^x \frac{\sin t}{t}\cdot e^{\int_1^t P(s)ds}dt+C'\right]\cdot e^{-\int_1^x \frac{2}{t}dt}$

$=\left[\int_1^x \frac{\sin t}{t}\cdot e^{2\ln t}dt+C'\right]e^{-2\ln x}=\left(\int_1^x \frac{\sin t}{t}\cdot t^2dt+C'\right)\frac{1}{x^2}$

$=\left(\int_1^x \sin t\cdot t\,dt+C'\right)\frac{1}{x^2}=(-x\cos x+\sin x+C)\frac{1}{x^2}$

又 $y(\pi)=\frac{1}{\pi}$ ∴ $\frac{1}{\pi}=(-\pi\cos\pi+\sin\pi+C)\frac{1}{\pi^2}$ ∴ $C=0$

故 $y=-\frac{\cos x}{x}+\frac{\sin x}{x^2}$

3解: (1) 方程化为 $\frac{dy}{dx}+(-\frac{1}{x})\cdot y=\frac{x-1}{x}e^x$. ∴ $P(x)=-\frac{1}{x}$, $Q(x)=\frac{x-1}{x}e^x$

则 $y(x)=\left(\int_1^x Q(t)e^{\int_1^t P(s)ds}dt+C\right)e^{\int_1^x -P(t)dt}$

$=\left(\int_1^x\left(\frac{t-1}{t}e^t\cdot e^{-\ln t}\right)dt+C\right)\cdot e^{\ln x}=\left(\int_1^x \frac{t-1}{t^2}e^tdt+C\right)\cdot x$

$=\left(\frac{e^x}{x}+C\right)x=e^x+Cx$ 即方程的通解为 $y=e^x+Cx$

(2) 易知 $P(x)=-\frac{2}{x+1}$, $Q(x)=(x+1)^{5/2}$

则 $y(x)=\left(\int_0^x Q(t)\cdot e^{\int_0^t P(s)ds}dt+C\right)e^{\int_1^x -P(t)dt}$

$=\left(\int_0^x (t+1)^{5/2}\cdot e^{\int_0^t -\frac{2}{s+1}ds}dt+C\right)e^{-\int_0^x(-\frac{2}{t+1})dt}$

$=\left(\int_0^x (t+1)^{5/2}\cdot(t+1)^{-2}dt+C\right)(x+1)^2=\left(\int_0^x (t+1)^{\frac{1}{2}}dt+C\right)(x+1)^2$

$=\left(\frac{2}{3}(x+1)^{\frac{3}{2}}+C\right)(x+1)^2=\frac{2}{3}(x+1)^{\frac{7}{2}}+C(x+1)^2$

∴ $y=\frac{2}{3}(x+1)^{\frac{7}{2}}+C(x+1)^2$ 为方程的通解.

(3) 易知 $P(x)=-\frac{n}{x+1}$ $Q(x)=e^x(x+1)^n$

∴ $y(x)=\left(\int_0^x Q(t)\cdot e^{\int_0^t P(s)ds}dt+C\right)e^{-\int_0^x P(t)dt}$

$=\left(\int_0^x e^t(t+1)^n\cdot\frac{1}{(t+1)^n}dt+C\right)(x+1)^n=\left(\int_0^x e^tdt+C\right)(x+1)^n$

$=(x+1)^ne^x+C(x+1)^n$

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故 $y=(x+1)^n e^x + C(x+1)^n$ 为方程的通解.

(4) 易知 $P(x)=2$. $Q(x)=x\cdot e^{-x}$ 故微分方程的通解为:

$$\therefore y(x)=\left(\int_0^x Q(t)\cdot e^{\int_0^t P(s)ds}dt+C\right)e^{\int_0^x -P(t)dt}=\left(\int_0^x t\cdot e^{-t}e^{2t}dt+C\right)e^{-2x}$$

$$=\left(\int_0^x t\cdot e^t dt+C\right)e^{-2x}=\left((x-1)e^x+C\right)\cdot e^{-2x}=C\cdot e^{-2x}+(x-1)e^{-x}$$

即.微分方程的通解为: $y=C\cdot e^{-2x}+(x-1)e^{-x}$

4.解:(1) 方程化为 $\frac{1}{y^3}\frac{dy}{dx}+\frac{1}{x}\cdot\frac{1}{y^2}=1$. 即 $-\frac{1}{2}\frac{d(y^{-2})}{dx}+\frac{1}{x}\cdot\frac{1}{y^2}=1$

令 $z=y^{-2}$ $\therefore -\frac{1}{2}\frac{dz}{dx}+\frac{1}{x}z=1$ 即 $\frac{dz}{dx}-\frac{2}{x}z=-2$

易知 $P(x)=-\frac{2}{x}$, $Q(x)=-2$. $\therefore z(x)=\left(\int_1^x Q(t)e^{\int_1^t P(s)ds}dt+C\right)e^{\int_1^x -P(t)dt}$

即 $z(x)=\left(\int_1^x -2\cdot t^{-2}dt+C\right)e^{\ln x^2}=\left(\frac{2}{x}+C\right)x^2=2x+Cx^2$

$\therefore \frac{1}{y^2}=2x+Cx^2$ 即 $Cx^2y^2+2xy^2-1=0$

(2) 方程化为 $\frac{1}{\sqrt{y}}\frac{dy}{dx}+\frac{\sqrt{y}}{x}=\frac{2}{\sqrt{x}}$ 令 $z=\sqrt{y}$. $\therefore dz=\frac{1}{2}\cdot\frac{1}{\sqrt{y}}\cdot dy$

$\therefore$ 方程化为 $\cancel{2}\frac{dz}{dx}+\frac{z}{x}=\frac{2}{\sqrt{x}}$ 即 $\frac{dz}{dx}+\frac{1}{2x}\cdot z=\frac{1}{\sqrt{x}}$

易知 $P(x)=\frac{1}{2x}$, $Q(x)=\frac{1}{\sqrt{x}}$ $\therefore z=\left(\int_1^x Q(t)e^{\int_1^t P(s)ds}dt+C\right)e^{\int_1^x -P(t)dt}$

即 $z=\left(\int_1^x \frac{1}{\sqrt{t}}\cdot t^{\frac{1}{2}}dt+C\right)\frac{1}{\sqrt{x}}=\left(\int_1^x dt+C\right)\frac{1}{\sqrt{x}}=\frac{x+C}{\sqrt{x}}$

$\therefore \sqrt{y}=\frac{x+C}{\sqrt{x}}$ 即 $x-\sqrt{xy}+C=0$

(3) 方程化为 $\frac{1}{y^{n+1}}\frac{dy}{dx}+\frac{2y^{-n}}{nx}=\frac{1}{n}$ 令 $z=y^{-n}$ 则 $\frac{dz}{dx}=(-n)\cdot y^{-(n+1)}\cdot\frac{dy}{dz}$

$\therefore$ 方程化为 $-\frac{1}{n}\frac{dz}{dx}+\frac{2z}{nx}=\frac{1}{n}$ 即 $\frac{dz}{dx}-\frac{2z}{x}=-1$

易知 $P(x)=-\frac{2}{x}$. $Q(x)=-1$ $\therefore z=\left(\int_1^x -1\cdot e^{\int_1^t -\frac{2}{s}ds}dt+C\right)e^{\int_1^x \frac{2}{t}dt}$

即 $z=\left(\int_1^x -t^{-2}dt+C\right)x^2=\left(\frac{1}{x}+C\right)x^2$ 即 $y^{-n}=x+Cx^2$

即 $Cx^2y^n+xy^n-1=0$

(4) 方程化为: $\frac{1}{y^4}\frac{dy}{dx}-\frac{1}{3x}\cdot\frac{1}{y^3}=\ln x$. 令 $z=\frac{1}{y^3}$ 则 $\frac{dz}{dx}=-3\frac{1}{y^4}\frac{dy}{dx}$

方程化为 $-\frac{1}{3}\frac{dz}{dx}-\frac{1}{3x}\cdot z=\ln x$ 即 $\frac{dz}{dx}+\frac{1}{x}z=\ln x^{-3}$

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易知 $p(x)=\frac{1}{x}$，$Q(x)=\ln x^{-3}$ $\therefore z=\left(\int_1^x \ln t^{-3}\cdot e^{\int_1^t \frac{1}{s}ds}dt+c\right)e^{\int_1^x -\frac{1}{t}dt}$

即 $z=\left(\int_1^x -3\ln t\cdot t\,dt+c\right)\frac{1}{x}=\left(-\frac{3}{2}\ln x\cdot x^2+\frac{3}{4}x^2+c\right)\frac{1}{x}$

即 $\frac{1}{y^3}=\left(-\frac{3}{2}\ln x\cdot x^2+\frac{3}{4}x^2+c\right)\frac{1}{x}$ 即 $\cancel{xy^{-3}+}\frac{3}{4}x^2(2\ln x-1)=C$

(5) 方程化为 $\frac{1}{y^2}\cdot y'-\frac{3x}{y}=x$ 令 $z=\frac{1}{y}$ 则 $\frac{dz}{dx}=-\frac{1}{y^2}\frac{dy}{dx}$

$\therefore$ 方程化为 $-\frac{dz}{dx}-3xz=x$ 即 $\frac{dz}{dx}+3x\cdot z=-x$

易知 $p(x)=3x$，$Q(x)=-x$. $\therefore z=\left(\int_0^x -t\cdot e^{\int_0^t 3s\,ds}dt+c\right)e^{\int_0^x -3t\,dt}$

即 $z=\left(-\frac{1}{3}\int_0^x e^{\frac{3}{2}t^2}d\,\frac{3}{2}t^2+c\right)e^{-\frac{3}{2}x^2}=\left(-\frac{1}{3}e^{\frac{3}{2}x^2}+c\right)e^{-\frac{3}{2}x^2}$

$\therefore \frac{1}{y}=\left(-\frac{1}{3}e^{\frac{3}{2}x^2}+c\right)e^{-\frac{3}{2}x^2}=-\frac{1}{3}+ce^{-\frac{3}{2}x^2}$ 即 $\frac{3}{y}+1=3ce^{-\frac{3}{2}x^2}$

即 $\left(1+\frac{3}{y}\right)e^{\frac{3}{2}x^2}=C$

5. 解：(1) 令 $z=y^2$. 则 $dz=2y\,dy$ $\therefore \frac{dy}{dx}=\frac{1}{2y}\cdot\frac{dz}{dx}$

$\therefore$ 方程化为 $\frac{1}{2y}\cdot\frac{dz}{dx}=\frac{x^2+z}{2y}$ 即 $\frac{dz}{dx}=x^2+z$. 即为所求线性微分方程.

(2) 令 $z=y^2$. 则 $\frac{dy}{dx}=\frac{1}{2y}\frac{dz}{dx}$

$\therefore$ 方程化为 $\frac{1}{2y}\frac{dz}{dx}=\frac{y}{x+y^2}$ 即 $\frac{dz}{dx}=\frac{2z}{x+z}$ $\therefore \frac{dx}{dz}=\frac{1}{2z}x+\frac{1}{2}$.

即为所求线性微分方程.

(3) 令 $z=y^3$. 则 $dz=3y^2dy$. $\therefore$ 方程化为 $x\cdot\frac{dz}{dx}+z+x^3=0$ 即 $\frac{dz}{dx}+\frac{1}{x}z=-x^2$

即为所求线性微分方程.

(4) 令 $z=\sin y$ 则 $dz=\cos y\,dy$. 方程化为 $\cos y\frac{dy}{dx}=1+x\cdot\sin y$

即 $\frac{dz}{dx}=1+xz$. 即为所求线性微分方程.

6. 解：依题意，曲线表达式满足 $\frac{dy}{dx}=\frac{2y+x+1}{x}$ 且 $y(1)=0$.

$\frac{dy}{dx}=\frac{2y+x+1}{x}$ 化为 $\frac{dy}{dx}-\frac{2}{x}y=\frac{x+1}{x}$. 易知 $p(x)=-\frac{2}{x}$. $Q(x)=\frac{x+1}{x}=1+\frac{1}{x}$

$\therefore y=\left(\int_1^x\left(1+\frac{1}{t}\right)\cdot e^{\int_1^t -\frac{2}{s}ds}dt+c\right)e^{\int_1^x \frac{2}{t}dt}=\left(\int_1^x\left(1+\frac{1}{t}\right)\frac{dt}{t^2}+c\right)x^2$

$=\left(\int_1^x \frac{1}{t^2}+\frac{1}{t^3}dt+c\right)x^2=\left(-\frac{1}{x}-\frac{1}{2x^2}+c\right)x^2$. 又 $y(1)=0$ $\therefore c=\frac{3}{2}$.

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∴ $y(x)=-x-\frac{1}{2}+\frac{3}{2}x^2$

7解. 设t时刻物体温度为u(t). 则有 $\frac{du}{dt}=-k(u-15)$. $(k>0)$ 且 $u(0)=95$, $u(10)=55$

∴ $u=Ce^{-kt}+15$. ∵ $u(0)=95$ ∴ $c=80$ ∴ $u(t)=80e^{-kt}+15$

又 $u(10)=55$ 即 $80e^{-k\cdot 10}+15=55$ ∴ $k=\frac{\ln 2}{10}$ ∴ $u(t)=80e^{-\frac{\ln 2}{10}t}+15$

即 $u(t)=80\cdot 2^{-\frac{t}{10}}+15$ 由 $u(t)=20$ 解得 $t=40$

答需要经过40分钟,该物体的温度才降至20℃.

8解. 设t年时镭的量为R(t). 则有 $\frac{dR}{dt}=-kR$ ∴ $R=R_0\cdot e^{-kt}$ (R_0为镭原始量)

依题意. $\frac{1}{2}=e^{-1600k}$ ∴ $k=\frac{\ln 2}{1600}$ ∴ $R=R_0\cdot 2^{-\frac{t}{1600}}$

当 $t=1$ 时 镭衰变了 $R_0-R(1)=R_0-R_0\cdot 2^{-\frac{1}{1600}}\approx 0.44\,(mg)$

9解. 令 $u=xt$. 则方程化为 $\int_0^x \varphi(u)\cdot\frac{1}{x}du=n\varphi(x)$. 两边求导得 $\varphi(x)\cdot$ [redacted] $=n(\varphi'(x)x+\varphi(x))$

即 $x\cdot\frac{d\varphi}{dx}\cdot n=$ [redacted] $(1-n)\varphi$ 化为 $n\cdot\frac{d\varphi}{\varphi}=(1-n)\frac{dx}{x}$. 两边求积分得 $n\ln\varphi=\ln|x|^{1-n}+C'$

∴ $\varphi^n(x)=C''|x|^{1-n}$ ∴ $\varphi(x)=C\cdot|x|^{\frac{1-n}{n}}$

10证明: ∵ $y_1(x)$为方程 $\frac{dy}{dx}+P(x)\cdot y=Q(x)$的解. ∴ $y_1'(x)+P(x)\cdot y_1(x)=Q(x)$

$y_0(x)$为方程 $\frac{dy}{dx}+P(x)\cdot y=0$的解 ∴ $y_0'(x)+P(x)\cdot y_0=0$.

∴有 $[y_1(x)+y_0(x)]'+P(x)[y_1(x)+y_0(x)]=Q(x)$

显然 $y_1(x)+y_0(x)$也为方程 $\frac{dy}{dx}+P(x)\cdot y=Q(x)$的解. 证毕

11证明: 依题意有 $y_1'(x)+P(x)\cdot y_1(x)=Q(x)$, $y_2'(x)+P(x)\cdot y_2(x)=Q(x)$. 两式相减得:

$[y_1'(x)-y_2'(x)]+P(x)[y_1(x)-y_2(x)]=0$. ∴ $y_1(x)-y_2(x)$为 $\frac{dy}{dx}+P(x)y=0$的解. 证毕.

12.证明: 由11题知. 齐次方程 $\frac{dy}{dx}+P(x)\cdot y=0$的解 $y_0(x)$可表示为 $y_1(x)-y_2(x)$.

设 y^*为方程 $\frac{dy}{dx}+P(x)y=Q(x)$的任意解. 则 $y^*-y_1(x)$为方程 $\frac{dy}{dx}+P(x)\cdot y=0$的解.

而 $\frac{dy}{dx}+P(x)\cdot y=0$的所有解为 $C_0e^{-\int_{x_0}^x P(t)dt}$ 故 $y^*=y_1(x)+C_0e^{-\int_{x_0}^x P(t)dt}$. 显然

$y^*(x)$包含在函数族 $y_1(x)+Ce^{-\int_{x_0}^x p(t)dt}$ 中. 证毕

13. 解 (1) 令 $z=y'$. 则有 $x^2 z'=z^2$ ∴ $\frac{dz}{z^2}=\frac{dx}{x^2}$ ∴ $-\frac{1}{z}=-\frac{1}{x}+C_1'$ ∴ $z=-\frac{x}{C_1x+1}$

即 $y'=\frac{x}{1+C_1x}$ [redacted]

∴ $y=\int\frac{x}{1+C_1x}dx=\int(\frac{dx}{C_1}+\frac{-1}{C_1}\frac{dx}{1+C_1x})=\frac{1}{C_1}x+(-\frac{1}{C_1^2})\int\frac{d(1+C_1x)}{1+C_1x}$

∴ $y=\frac{1}{C_1}x-\frac{1}{C_1^2}\ln(1+C_1x)+C_2'$ 即 $C_1x-C_1^2y=\ln|C_1x+1|+C_2$ ($y=C$, $y=\frac{x^2}{2}+C$)

(2) 令 $z=y'$. 则有 $z^2+2y\cancel{z'}=0$. 即 $z^2+2y\frac{dz}{dy}z=0$ ∴ $\frac{dz}{z}=\frac{-dy}{2y}$ ∴ $z=\frac{1}{\sqrt{y}}C_1'$

即 $\frac{dy}{dx}=C_1'\frac{1}{\sqrt{y}}$ ∴ $\sqrt{y}\,dy=C_1'dx$ 即 $\frac{2}{3}\sqrt{y}^3=C_1'x+C_2'$ 即 $y=C_1(x+C_2)^{2/3}$

(3) 令 $p=y'$ 则 $y''=\frac{dy'}{dx}=\frac{dp}{dx}$ [redacted]

∴方程化为 $\frac{dp}{dx}(e^x+1)+p=0$. [redacted]

即 $\frac{dp}{p}=\frac{-dx}{e^x+1}=-dx+\frac{e^x}{e^x+1}dx=-dx+\frac{d(e^x+1)}{e^x+1}$ ∴ $\ln p=-x+\ln(e^x+1)+C_1$

即 $p=C_1(1+e^{-x})$ ∴ $\frac{dy}{dx}=C_1(1+e^{-x})$ ∴ $y=C_1(x-e^{-x})+C_2$

(4) 令 $p=y''$ 则 $y'''=\frac{dy''}{dx}=\frac{dp}{dx}$

方程化为 $\frac{dp}{dx}=2(p-1)\cot x$ 即 $\frac{d(p-1)}{p-1}=2\cot x\,dx$

∴ $\ln(p-1)=2\ln\sin x+C_1'$ 即 $p-1=C_1'\sin^2x$ ∴ $p=1+C_1'\sin^2x$

∴ $y'=\int p\,dx=\int 1+C_1'\frac{1-\cos2x}{2}dx=(\frac{C_1'}{2}+1)x-\frac{C_1'}{4}\sin2x+C_2$

∴ $y=\int y'dx=\frac{C_1'}{8}\cos2x+(\frac{1}{2}+\frac{C_1'}{4})x^2+C_2x+C_3=C_1\cos2x+(\frac{1}{2}+2C_1)x^2+C_2x+C_3$

14 解 (1) $\frac{\partial M}{\partial y}=4\neq\frac{\partial N}{\partial x}=2$ ∴该方程不是全微分方程.

(2) ∵ $\frac{\partial M}{\partial y}=2=\frac{\partial N}{\partial x}$ ∴该方程是全微分方程.

方程化为 $xdx+2ydx+2xdy-ydy=0$ 两边求积分得 $\frac{1}{2}x^2+2xy-\frac{1}{2}y^2=C$

(3) ∵ $\frac{\partial M}{\partial y}=e^y=\frac{\partial N}{\partial x}$ ∴该方程是全微分方程.

方程化为 $(e^ydx+xe^ydy)-2ydy=0$ 两边求积分得 $xe^y-y^2=C$

(4) ∵ $\frac{\partial M}{\partial y}=x\cdot\frac{y}{\sqrt{x^2+y^2}}=\frac{\partial N}{\partial x}$ ∴该方程是全微分方程.

方程化为 $dx+(x\sqrt{x^2+y^2}dx+y\sqrt{x^2+y^2}dy)-ydy=0$ 两边求积分得.

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$x+\frac{1}{3}(x^2+y^2)^{\frac{3}{2}}-\frac{1}{2}y^2=C$

(5) $\frac{\partial M}{\partial y}=2=\frac{\partial N}{\partial x}$ ∴方程为全微分方程.

方程化为 $xdx+(2ydx+2xdy)+3ydy=0$ 两边求积分得 $\frac{1}{2}x^2+2xy+\frac{3}{2}y^2=C$

(6) ∵ $\frac{\partial M}{\partial y}=-b$，$\frac{\partial N}{\partial x}=b$ 而 $b\neq 0$ ∴ $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$. 故该方程不是全微分方程.

(7) ∵ $\frac{\partial M}{\partial y}=e^x+2y=\frac{\partial N}{\partial x}$ ∴方程是全微分方程.

方程化为 $2e^xdx+[(ye^x+y^2)dx+(e^x+2xy)dy]=0$ 两边求积分得 $2e^x+ye^x+xy^2=C$

(8) $\frac{\partial M}{\partial y}=2by$ $\frac{\partial N}{\partial x}=ly$ 当 $2b\neq l$ 时. 方程不是全微分方程

当 $2b=l$ 时. 方程化为 $(ax^2+by^2)dx+2bxydy=0$ 即 $ax^2dx+(by^2dx+2bxydy)=0$

两边求积分得: $\frac{1}{3}ax^3+bxy^2=C$

15解. (1) 积分因子 $\varphi(x)=\frac{1}{(x+y)^2}$. 两边同时乘以 $\varphi(x)$. 则化为 $dx-dy=\frac{1}{(x+y)^2}(dx+dy)$

两边求积分得: $x-y=-\frac{1}{x+y}+c$ 即 $x-y+\frac{1}{x+y}=c$

(2) 积分因子 $\varphi(x)=\frac{1}{x^2+y^2+1}$ 方程两边同时乘以 $\varphi(x)$. 则化为 $(1+\frac{x}{x^2+y^2+1})dx+\frac{y}{1+x^2+y^2}dy=0$

两边求积分得. $x+\frac{1}{2}\ln(x^2+y^2+1)$ [redacted] $=C$

(3) 积分因子 $\varphi(x)=e^x$. 方程两边同时乘以 $\varphi(x)$. 方程化为 $e^x\sin y\,dx+e^x\cos y\,dy=0$

两边求积分得: $e^x\sin y=C$

(4) 积分因子 $\varphi(x)=\frac{1}{x^2+y^2}$. 方程两边同时乘以 $\varphi(x)$. 方程化为 $(1+\frac{y}{x^2+y^2})dx-\frac{x}{x^2+y^2}dy=0$

即 $dx-(\frac{-y/x^2}{(\frac{y}{x})^2+1}dx+\frac{\frac{1}{x}}{(\frac{y}{x})^2+1}dy)=0$ 方程两边求积分得. $x-\arctan\frac{y}{x}=C$

(或者 $x+\arctan\frac{x}{y}=C$)

(5) 积分因子 $\varphi(x)=\frac{1}{xy\sqrt{1-y^2}}$. 方程两边同时乘以 $\mu(x)$. 方程化为:

($\frac{dy}{y}$ [redacted] + $\frac{dx}{x}$ [redacted]) $+\frac{1}{\sqrt{1-y^2}}dy=0$. 即 $(\frac{dx}{x}+\frac{dy}{y})+\frac{1}{\sqrt{1-y^2}}dy=0$

两边求积分得 $\ln|xy|+\arcsin y=C$

16解: (1) 方程化为 $(2xy^3\sqrt{1+x^2}-x)dx+3x^2y^2\sqrt{1+x^2}dy=0$.

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$\therefore M=2xy^3\sqrt{1+x^2}-x \quad N=3x^2y^2\sqrt{1+x^2} \quad \therefore \frac{\partial M}{\partial y}=6xy^2\sqrt{1+x^2},\ \frac{\partial N}{\partial x}=6xy^2\sqrt{1+x^2}+\frac{3x^3y^2}{\sqrt{1+x^2}}$

$\therefore$有$F(x)=\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\dfrac{-\frac{3x^3y^2}{\sqrt{1+x^2}}}{3x^2y^2\sqrt{1+x^2}}=\dfrac{-x}{1+x^2}$.

$\therefore$积分因子 $u=e^{\int_0^x F(t)dt}=e^{-\frac{1}{2}\ln(1+x^2)}=\dfrac{1}{\sqrt{1+x^2}}$

方程两边同时乘以u得: $\dfrac{x}{\sqrt{1+x^2}}dx=2xy^3dx+3x^2y^2dy$

两边同时求积分得 $\sqrt{1+x^2}=x^2y^3+C$. 即 $\sqrt{1+x^2}-x^2y^3=C$

(2) 易得 $M=x^2+y$. $N=-x$. $\therefore \frac{\partial M}{\partial y}=1$. $\frac{\partial N}{\partial x}=-1$.

$\therefore$有 $F(x)=\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=-\dfrac{2}{x}$. $\therefore$积分因子 $u=e^{\int_1^x F(t)dt}=\dfrac{1}{x^2}$

方程两边同时乘以u得 $(1+\frac{y}{x^2})dx-\frac{1}{x}dy=0$ 即 $dx=-\frac{y}{x^2}dx+\frac{1}{x}dy$

两边同时求积分得 $x=\frac{y}{x}+c$ 即 $x-\frac{y}{x}=c$

(3) $M=y(x+1)$. $N=x(y+1)$ $\therefore \frac{\partial M}{\partial y}=x+1$. $\frac{\partial N}{\partial x}=y+1$

方程两边同时乘以$\frac{1}{xy}$得 $(1+\frac{1}{x})dx+(1+\frac{1}{y})dy=0$. 即 $dx+dy+\frac{1}{x}dx+\frac{1}{y}dy=0$

两边同时求积分得: $x+y+\ln|xy|=C$

(4). $M=3x^2y+2xy+y^3$ $N=x^2+y^2$ $\therefore \frac{\partial M}{\partial y}=3x^2+2x+3y^2$. $\frac{\partial N}{\partial x}=2x$

$\therefore F(x)=\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=3$ $\therefore$积分因子 $u=e^{\int_0^x F(t)dt}=e^{3x}$

方程两边同时乘以u得: $e^{3x}(3x^2y+2xy+y^3)dx+e^{3x}(x^2+y^2)dy=0$

即 $[(3x^2y+y^3)e^{3x}+2xye^{3x}]dx+e^{3x}(x^2+y^2)dy=0$

两边求积分得 $e^{3x}(3x^2y+y^3)=C$.

(5) $M=2xy^3$. $N=x^2y^2-1$ $\therefore \frac{\partial M}{\partial y}=6xy^2$ $\frac{\partial N}{\partial x}=2xy^2$ $\therefore G(y)=\dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=-\dfrac{2}{y}$

$\therefore$积分因子$u=e^{\int_1^y -\frac{2}{t}dt}=\dfrac{1}{y^2}$. 方程两边同时乘以$u$得 $2xydx+x^2dy-\frac{1}{y^2}dy=0$

两边同时求积分得 $x^2y+\frac{1}{y}=c$

(6). $M=e^x$. $N=e^x\cot y+2y\cos y$ $\frac{\partial M}{\partial y}=0$. $\frac{\partial N}{\partial x}=e^x\cot y$ $\therefore G(y)=\dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=\cot y$

$\therefore$积分因子$u=e^{\int_{\pi/2}^y G(t)dt}=\sin y$ 方程两边同时乘以u再求积分得 $e^x\sin y-\frac{1}{2}y\cos 2y+\frac{1}{4}\sin 2y=C$

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3 微分方程解的存在唯一性定理

1. 证明：易得 $|f_y(x,y)| \le M$. 则由拉格朗日中值定理得 $|f(x,y_1)-f(x,y_2)| = |f_y(x,y_3)||y_1-y_2|$

其中 $y_3 \in (y_2, y_1)$ ∴ $|f_y(x,y_3)| \le M$ 故有 $|f(x,y_1)-f(x,y_2)| \le M|y_1-y_2|$.

∴ $f(x,y)$ 在R上对y满足李氏条件.

2 解：(1). $f_y(x,y)=\frac{1}{2\sqrt{y}}$ 在矩形域内是无界的. ∴该函数在相应闭区域上不满足李氏条件

(2). $f_y(x,y)=\frac{1}{2\sqrt{y}} \in [\frac{1}{2\sqrt{6}}, \frac{1}{2\sqrt{5}}]$. ∴该函数在相应闭区域上满足李氏条件.

(3) $f_y(x,y)=nxy^{n-1}$ ∴ $|f_y(x,y)| \le n|ab^{n-1}|$. ∴该函数相应闭区域上满足李氏条件

(4) $f_y(x,y)=2xy$. 在矩形域内是无界的 ∴该函数在相应闭区域上不满足李氏条件

3 解. $|f_y(x,y)|=|x^2| \le 1$ 由题知李氏常数 $L=1$.

$|f(x,y)|=|x^2y+x|=|x||xy+1| \le |x|(|xy|+1) \le 1\cdot(1+1)=2$ ∴ $M=\max\limits_R|f(x,y)|=2$

易知 $a=1$. $b=1$. ∴ $h=\min\{a,\frac{b}{M}\}=\frac{1}{2}$. ∴ $2h=1$

4. 解：与题设初值问题等价的是积分方程 $y=-2+\int_0^x (x+y)dx$

∴ $y_1=-2+\int_0^x (x-2)dx=\frac{1}{2}x^2-2x-2=\frac{1}{2}x^2-x-1-(1+x)$

同理求得 $y_2=\frac{1}{6}x^3-\frac{1}{2}x^2-x-1-(1+x)$　$y_3=\frac{1}{24}x^4-\frac{1}{6}x^3-\frac{1}{2}x^2-x-1-(1+x)$

$\cdots\cdots y_n=\frac{1}{(n+1)!}x^{n+1}-\frac{1}{n!}x^n-\frac{1}{(n-1)!}x^{n-1}\cdots\cdots-\frac{1}{2!}x^2-\frac{1}{1!}x^1-1-(1+x)$

注意到 $-e^x$ 的泰勒展开式为 $-e^x=-1-\frac{1}{1!}x^1-\frac{1}{2!}x^2-\cdots-\frac{1}{n!}x^n+o(x^{n+1})\cdots$

∴ $\lim\limits_{n\to\infty} y_n(x)=-e^x-(1+x)$　∴其极限 $\varphi(x)=-e^x-x-1$　$(x\in R)$

5 解：与题设初值问题等价的是积分方程 $y=0+\int_0^x (x-y^2)dx$

∴ $y_0(x)=y(0)=0$　$y_1(x)=0+\int_0^x (x-0)dx=\frac{1}{2}x^2$　$y_2(x)=0+\int_0^x (x-\frac{1}{4}x^4)dx=\frac{x^2}{2}-\frac{x^5}{20}$

6 解：(1). ∵ $|x-4| \le 2$, $|y-1| \le 1$. ∴ $\sqrt{x-y} \le \sqrt{6}$. 即有 $a=2$ $b=1$. $M=\sqrt{6}$.

∴ $h=\min\{a,\frac{b}{M}\}=\frac{1}{\sqrt{6}}$　∴所求区间为 $|x-4| \le \frac{1}{\sqrt{6}}$

(2). ∵ $\left|\frac{\sqrt{y}-x}{x-2}\right|=\frac{|\sqrt{y}-x|}{|x-2|} \le \left|\frac{-2-x}{x-2}\right|=\frac{x+2}{2-x} \le \frac{1+2}{2-(1)}=3$. 有 $a=1, b=1, M=3$

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$\therefore h = \min\{a, \frac{b}{M}\} = \frac{1}{3}$ $\therefore$ 所求区间为 $|x| \leq \frac{1}{3}$

(3). $|x+e^y| \leq |x| + |e^y| \leq 2+e < 5$ 取 $M=5$. 则所求区间为 $|x| \leq \min\{a, \frac{b}{M}\} = \frac{1}{5}$

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4 高阶线性微分方程.

1 证明: (1). 设有常数 k_1, k_2 使 $k_1 e^{\lambda x}+k_2 x e^{\lambda x}=0$. 令 $x=0,1$ 得 $\begin{cases}k_1+0\cdot k_2=0\\ k_1\cdot e^{\lambda}+k_2\cdot e^{\lambda}=0\end{cases}$ 解得 $k_1=k_2=0$

$\therefore e^{\lambda x}, xe^{\lambda x}$ 线性无关.

(2) 设有常数 k_1, k_2 使 $k_1\cos\beta x+k_2\sin\beta x=0$. 令 $x=0, \frac{\pi}{2\beta}$ 得 $\begin{cases}k_1+0\cdot k_2=0\\ 0\cdot k_1+k_2=0\end{cases}$ 解得 $k_1=k_2=0$

$\therefore \cos\beta x, \sin\beta x$ 线性无关.

(3) 设有常数 k_1, k_2 使 $k_1 e^{\alpha x}\cos\beta x+k_2 e^{\alpha x}\sin\beta x=0$ 化为 $k_1\cos\beta x+k_2\sin\beta x=0$ 由(2)知 $k_1=k_2=0$

$\therefore e^{\alpha x}\cos\beta x, e^{\alpha x}\sin\beta x$ 线性无关.

2. 证明: $\because \varphi_1(x), \varphi_2(x)$ 为方程 $y''+q(x)\cdot y=0$ 的解. $\therefore \begin{cases}\varphi_1''(x)+q(x)\varphi_1(x)=0\\ \varphi_2''(x)+q(x)\cdot\varphi_2(x)=0\end{cases}$

$\therefore \varphi_1(x)=-\dfrac{\varphi_1''(x)}{q(x)}\qquad \varphi_2(x)=-\dfrac{\varphi_2''(x)}{q(x)}$

$\therefore W'(x)=[\varphi_1(x)\cdot\varphi_2'(x)-\varphi_1'(x)\varphi_2(x)]'=\varphi_1(x)\cdot\varphi_2''(x)-\varphi_2(x)\cdot\varphi_1''(x)=-\dfrac{\varphi_1''(x)\cdot\varphi_2''(x)}{q(x)}+\dfrac{\varphi_1''(x)\cdot\varphi_2''(x)}{q(x)}$

即 $W'(x)=0\quad \therefore W(x)\equiv$ 常数. 证毕.

3. 证明: 假设 $\varphi'(x_0)=0$. 则考虑初值问题 $\begin{cases}y''+p(x)y'+q(x)\cdot y=0\\ y(x_0)=\varphi(x_0)=0\quad y'(x_0)=\varphi'(x_0)=0\end{cases}$

可以发现 $u(x)\equiv0$ 为上述方程的解且满足初值条件. 根据解的存在唯一性定理知. 此时

$\varphi(x)=u(x)\equiv0$ 与题设矛盾. $\therefore \varphi'(x_0)\neq0$.

4 证明: $\varphi_1(x)$ 与 $\varphi_2(x)$ 线性无关. 则 $W(x)=\begin{vmatrix}\varphi_1(x) & \varphi_2(x)\\ \varphi_1'(x) & \varphi_2'(x)\end{vmatrix}\neq0$.

若它们有公共零点. 设为 x_0 ($x_0\in(a,b)$) 则 $W(x_0)=\begin{vmatrix}0 & 0\\ \varphi_1' & \varphi_2'\end{vmatrix}=0$ 与 $W(x)\neq0$ 矛盾

$\therefore \varphi_1(x)$ 与 $\varphi_2(x)$ 没有公共的零点.

5 证明: 线性齐次微分方程 $y''+p(x)\cdot y'+q(x)\cdot y=0$ 的通解可表示为 $y=C_1\varphi_1(x)+C_2\varphi_2(x)$.

(其中 $\varphi_1(x), \varphi_2(x)$ 为方程的两个解). 则 C_1, C_2 是两个独立常数.

$\therefore \dfrac{D(y,y')}{D(C_1,C_2)}=\begin{vmatrix}\varphi_1(x) & \varphi_1'(x)\\ \varphi_2(x) & \varphi_2'(x)\end{vmatrix}\neq0$. 即 $W(x)\neq0$ (φ_1, φ_2 的朗斯基行列式)

$\therefore \varphi_1(x), \varphi_2(x)$ 线性无关. 证毕.

6 证明: 设 $y(x)=C_1\varphi_1(x)+C_2\varphi_2(x)+y^*(x)$ 为方程的通解. 再设 y^{**} 为方程的任意一个解.

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先记 $y^{**}-y(x)$ 为线性齐次方程 $y''+p(x)y'+q(x)y=0$ 的解.

$\because y^{**''}+p(x)y^{**'}+q(x)y^{**}=f(x)$. $y''+p(x)y'+q(x)y=f(x)$

$\therefore (y^{**}-y(x))''+p(x)(y^{**'}-y')+q(x)(y^{**}-y)=0$.

而 $y''+p(x)y'+q(x)y=0$ 的所有解均为形式 $C_1'\varphi_1(x)+C_2'\varphi_2(x)$

$\therefore y^{**}-y(x)=C_1'\varphi_1(x)+C_2'\varphi_2(x)$ 即 $y^{**}=(C_1'+C_1)\varphi_1(x)+(C_2'+C_2)\varphi_2(x)+y^{*}$

$\therefore y^{**}$ 也在函数族 $C_1\varphi_1(x)+C_2\varphi_2(x)+y^{*}$ 中.

故线性非齐次微分方程的任一解,都包含在它的通解中.

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5 二阶线性常系数微分方程

1解:(1) 其特征方程为 $\lambda^2-3\lambda+2=0$ 即 $(\lambda-1)(\lambda-2)=0$ $\therefore \lambda_{1,2}=1,2$.

$\therefore$方程有两个线性无关的特解 e^x, e^{2x}. $\therefore$方程通解为 $y=c_1e^x+c_2e^{2x}$ (c_1,c_2为独立常数)

(2) 其特征方程为 $4\lambda^2+5\lambda+1=0$ 即 $(4\lambda+1)(\lambda+1)=0$ $\therefore \lambda_{1,2}=-\frac{1}{4},-1$

$\therefore$方程有两个线性无关的特解 $e^{-x}, e^{-\frac{1}{4}x}$ $\therefore$方程通解为 $y=c_1e^{-x}+c_2e^{-\frac{1}{4}x}$ (c_1,c_2为独立常数)

(3) 其特征方程 $\lambda^2+6\lambda+9=(\lambda+3)^2=0$ $\therefore \lambda=-3$. $\therefore$方程有一特解 e^{-3x}

$\therefore$方程通解 $y=(C_1+C_2x)e^{-3x}$

(4) 其特征方程为 $\lambda^2+4\lambda+5=0$ 解得 $\lambda_{1,2}=-2\pm i$ $\therefore$方程有两个线性无关的特解

$e^{-2x}\cos x$, $e^{-2x}\sin x$. $\therefore$方程的通解为 $y=e^{-2x}(c_1\cos x+c_2\sin x)$

(5) 其特征方程为 $\lambda^2-\lambda+2=0$ 解得 $\lambda_{1,2}=\frac{1}{2}\pm\frac{\sqrt{7}}{2}x$ $\therefore$方程有两个线性无关的特解

$e^{\frac{1}{2}x}\cos\frac{\sqrt{7}}{2}x$, $e^{\frac{1}{2}x}\sin\frac{\sqrt{7}}{2}x$ $\therefore$方程的通解为 $y=e^{\frac{1}{2}x}(c_1\cos\frac{\sqrt{7}}{2}x+c_2\sin\frac{\sqrt{7}}{2}x)$

(6) 令 $z=y'$. 则 $z''+2z'-z=0$ 其特征方程为 $\lambda^2+2\lambda-1=0$ 解得 $\lambda_{1,2}=-1\pm\sqrt{2}$

$\therefore$方程通解为 $z=c_1'e^{(-1+\sqrt{2})x}+c_2'e^{(-1-\sqrt{2})x}$ (c_1',c_2'为两个独立常数)

$\therefore y=\int z\,dx=c_1e^{(\sqrt{2}-1)x}+c_2e^{-(1+\sqrt{2})x}+c_3$. ($c_1,c_2,c_3$为独立常数)

2.解.(1) 微分方程的特征方程为 $\lambda^2+2\lambda+4=0$ 解得 $\lambda_{1,2}=-1\pm\sqrt{3}i$

$\therefore$微分方程的通解 $y(x)=e^{-x}(c_1\cos\sqrt{3}x+c_2\sin\sqrt{3}x)$. ($c_1,c_2$为独立常数)

又 $y(0)=1$, $y'(0)=-1$ 可解得 $c_1=1$, $c_2=0$ $\therefore$方程的解为 $y=e^{-x}\cos\sqrt{3}x$

(2) 微分方程的特征方程为 $4\lambda^2+4\lambda+1=(2\lambda+1)^2=0$ 解得 $\lambda=-\frac{1}{2}$ $\therefore$方程的通解为:

$y(x)=(C_1+C_2x)e^{-\frac{1}{2}x}$ 又 $y(0)=0$ $\therefore C_1=0$ $\therefore y(x)=C_2xe^{-\frac{1}{2}x}$ $\therefore y'(x)=C_2(e^{-\frac{1}{2}x}-\frac{1}{2}xe^{-\frac{1}{2}x})$

又 $y'(0)=2$ $\therefore C_2=2$ 故初值问题的解为 $y=2xe^{-\frac{1}{2}x}$

3解:(1) 对应的齐次方程的特征根 $\lambda_{1,2}=\frac{3}{2}\pm\frac{\sqrt{11}}{2}i$. $\therefore 0$不是其特征根. 又 $f(x)=P_0(x)=6$

$\therefore$方程有一特解 $Q_0(x)=C$. 代入方程得 $0-3\cdot0+5C=6$ $\therefore C=\frac{6}{5}$ 即其特解为 $y=Q_0(x)=\frac{6}{5}$

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(2). 方程齐次的特征根为 $\lambda_1=-3$ 和 $\lambda_2=0$. 显然 $\lambda=2$ 不是齐次方程的特征根.

∴该微分方程有一特解 $y=Ae^{2x}$ 代入方程得 $4Ae^{2x}+3\cdot 2Ae^{2x}=6e^{2x}$ ∴ $A=\frac{3}{5}$

∴方程的一个特解为 $y=\frac{3}{5}e^{2x}$

(3) 对应的齐次方程的特征根为 $\lambda_1=4, \lambda_2=5$. 显然 0 不是其特征根且 $f(x)=P_1(x)=x+1$

∴该微分方程有一特解 $y=Q_1(x)=b_0x+b_1$. 代入方程得 $0-9b_0+20b_0x+20b_1=x+1$

∴ $b_0=\frac{1}{20}$ $b_1=\frac{29}{400}$ ∴该方程的一个特解为 $y=\frac{1}{20}x+\frac{29}{400}$

(4). 显然 $\pm i$ 是对应的齐次方程的特征根. ∴方程有一特解 $y=(A\cos x+B\sin x)x$

代入方程得 $[x(A\cos x+B\sin x)]''+x(A\cos x+B\sin x)=4\sin x$ 解得 $A=-2$, $B=0$

∴方程的一个特解为 $y=-2x\cos x$

(5) 显然 $\pm i$ 不是对应的齐次方程的特征根. ∴方程有一特解 $y=Q_1(x)\cos x+R_1(x)\sin x$

其中 $Q_1(x)=b_0x+b_1$ $R_1(x)=a_0x+a_1$. 代入方程得:

$$-2b_0\sin x-\cos x(b_0x+b_1)+2a_0\cos x-\sin x(a_0x+a_1)-3[b_0\cos x-\sin x(b_0x+b_1)$$

$$+a_0\sin x+\cos x(a_0x+a_1)]+2[(b_0x+b_1)\cos x+(a_0x+a_1)\sin x]=x\cos x$$

解得. $b_0=0.1$ $b_1=-0.12$ $a_0=0.3$ $a_1=-0.34$

(6). 显然. $3\pm i$ 不是对应的齐次方程的特征根. ∴方程特解有 $y=e^{3x}(c_1\cos x+c_2\sin x)$

代入方程得 $e^{3x}[(8c_1+6c_2)\cos x+(8c_2-6c_1)\sin x]-9e^{3x}(c_1\cos x+c_2\sin x)=e^{3x}\cos x$

解得 $a=-\frac{1}{37}$ $b=\frac{6}{37}$ ∴方程特解为 $y=e^{3x}(\frac{6}{37}\sin x-\frac{1}{37}\cos x)$

(7) 分别求 $y''-y=2e^x$, $y''-y=-x^2$ 的特解 $y_1(x), y_2(x)$.

①先求 $y_1(x)$. 显然 $\lambda=1$ 是 $y''-y=0$ 的特征方程的单特征根. ∴ $y_1(x)=Axe^x$

代入方程得 $Ae^x(x+2)-Axe^x=2e^x$ ∴ $A=1$ ∴ $y_1(x)=x\cdot e^x$

②求 $y_2(x)$. 显然 $\lambda=0$ 不是 $y''-y=0$ 的特征方程的单特征根. ∴ $y_2(x)=Q_2(x)$.

其中 $Q_2(x)=b_0x^2+b_1x+b_2$ 代入方程得 $(2b_0-(b_0x^2+b_1x+b_2))=-x^2$

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解得 $b_0=1$, $b_1=0$, $b_2=2$ ∴方程的特解 $y=y_1+y_2=xe^x+x^2+2$

(8) 设方程 $y''+y'=\sin 4x$, $y''+y'=-2\sin 2x$ 的特解分别 $y_1(x)$, $y_2(x)$.

① 求 $y_1(x)$. 显然 $\pm 4i$ 不是 $y''+y'=0$ 的特征方程的特征根. ∴ $y_1(x)=A\cos 4x+B\sin 4x$

代入方程得 $-16A\cos 4x-16B\sin 4x-4A\sin 4x+4B\cos 4x=\sin 4x$ 解得 $A=-\frac{1}{68}$, $B=-\frac{1}{17}$

∴ $y_1(x)=-\frac{1}{68}\cos 4x-\frac{1}{17}\sin 4x$

② 求 $y_2(x)$ 显然 $\pm 2i$ 不是 $y''+y'=0$ 的特征方程的特征根. ∴ $y_2(x)=A'\cos 2x+B'\sin 2x$

代入方程得. $-4A'\cos 2x-4B'\sin 2x-2A'\sin 2x+2B'\cos 2x=-2\sin 2x$ 解得 $A'=\frac{1}{5}$ $B'=\frac{2}{5}$

∴ $y_2(x)=\frac{1}{5}\cos 2x+\frac{2}{5}\sin 2x$

∴方程的特解为 $y=y_1+y_2=-\frac{1}{17}(\sin 4x+\frac{1}{4}\cos 4x)+\frac{1}{5}(\cos 2x+2\sin 2x)$

4解: (1) $f(x)=P_2(x)=x^2+x$. 且 $\lambda=0$ 不是 $y''+y=0$ 对应的特征方程的特征根.

∴方程的特解形式为 $y=Q_2(x)=A_0x^2+A_1x+A_2$

(2). $f(x)=P_1(x)=x-2$. 且 $\lambda=0$ 是 $y''+y'=0$ 对应的特征方程的单特征根.

∴方程的特解形式为 $y=xQ_1(x)=x(A_0x+A_1)$

(3) $f(x)=e^{\lambda x}P_1(x)$ 且 $\lambda=3$ 不是 $y''+y=0$ 对应的特征方程的根. ∴方程的特解形式为:

$y=Q_1(x)\cdot e^{3x}=(A_0x+A_1)e^{3x}$

(4) $f(x)=e^{\lambda x}P_2(x)=e^x(x^2-1)$ 且 $\lambda=1$ 为 $y''-y=0$ 对应的特征方程的单特征根. ∴方程的特解形式为. $y=xQ_2(x)\cdot e^x=x\cdot e^x(A_0x^2+A_1x+A_2)$

(5) $f(x)=e^{\lambda x}P_1(x)=e^{-x}(x-5)$. 且 $\lambda=-1$ 为方程的3重特征根. ∴方程的特解形式为:

$y=x^3\cdot e^{-x}\cdot Q_1(x)=x^3\cdot e^{-x}(A_0x+A_1)$

(6) 方程 $y''-2y'+2y=e^x$ 和方程 $y''-2y'+2y=x\cos x$ 的两个特解分别为 $y_1(x)$, $y_2(x)$.

易知 $y_1(x)=A_0e^x$　$y_2(x)=Q_1\cos x+R_1(x)\sin x=(b_0x+b_1)\cos x+(C_0+C_1)\cos x$

∴原方程的特解形式为 $y=y_1+y_2=A_0e^x+(b_0x+b_1)\cos x+(c_0+C_1)\sin x$

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5.解：依题意有．$LC\frac{d^2u_c(t)}{dt^2}+RC\frac{du_c(t)}{dt}+u_c(t)=E$．即 $LCu_c''+RCu_c'+u_c=E$

且有 $u_c(0)=0$，$u_c'(0)=0$

∴所求解初值问题为 $\begin{cases} LC\frac{d^2u_c}{dt^2}+RC\cdot\frac{du_c}{dt}+u_c=E. \\ u_c(0)=0 \quad u_c'(0)=0 \end{cases}$

6解：易知．$u_c(0)=E$．$u_c'(0)=0$．当开关K打向2时有 $LC\frac{d^2u_c(t)}{dt^2}+RC\frac{du_c(t)}{dt}+u_c(t)=0$

即所求的初值问题为 $\begin{cases} LC\frac{d^2u_c}{dt^2}+RC\cdot\frac{du_c}{dt}+u_c=0 \\ u_c(0)=E, \quad u_c'(0)=0. \end{cases}$

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6 用常数变易法求解二阶线性非齐次方程与欧拉方程的解法：

1. 解：齐次方程对应的特征方程的特征根为 $\lambda_{1,2}=-1,-2$. 则方程的通解形式为

$y=C_1(x)\cdot e^{-x}+C_2(x)\cdot e^{-2x}$ 代入方程化简可得 $\begin{cases}C_1'(x)e^{-x}+C_2'(x)e^{-2x}=0\\ -C_1'(x)\cdot e^{-x}-2C_2'(x)e^{-2x}=\dfrac{1}{e^x+1}\end{cases}$

解得 $\begin{cases}C_1'(x)=\dfrac{e^x}{e^x+1}\\ C_2'(x)=\dfrac{-e^{2x}}{e^x+1}\end{cases}$ $\therefore C_1(x)=\ln(e^x+1)+C_1'$ $\quad C_2(x)=-e^x+\ln(e^x+1)+C_2'$

$\therefore$ 方程的通解为 $y=(e^{-x}+e^{-2x})\ln(e^x+1)+C_1e^{-x}+C_2e^{-2x}$

2. 解：齐次方程对应的特征方程的特征根为 $\lambda_{1,2}=\pm i$ 则方程的通解形式为

$y=C_1(x)\cdot\cos x+C_2(x)\cdot\sin x$. 代入方程化简得 $\begin{cases}C_1'(x)\cos x+C_2'(x)\cdot\sin x=0\\ C_1'(x)\cdot(-\sin x)+C_2'(x)\cdot\cos x=\dfrac{1}{\sin x}\end{cases}$

解得 $\begin{cases}C_1'(x)=\text{[redacted]}-1\\ C_2'(x)=\dfrac{\cos x}{\sin x}\end{cases}$ $\therefore C_1(x)=-x+C_1$ $\quad C_2(x)=C_2+\ln|\sin x|$

$\therefore$ 方程的通解为 $y=(C_2+\ln|\sin x|)\sin x+(C_1-x)\cos x$

3. 解：齐次方程对应的特征方程的特征根为 $\lambda_{1,2}=\pm 2i$. 则方程的通解形式为：

$y=C_1(x)\cdot\cos 2x+C_2(x)\cdot\sin 2x$. 代入方程化简得 $\begin{cases}C_1'(x)\cos 2x+C_2'(x)\cdot\sin 2x=0\\ -2C_1'(x)\cdot\sin 2x+2C_2'(x)\cdot\cos 2x=2\tan x\end{cases}$

解得 $\begin{cases}C_1'(x)=-2\sin^2 x\\ C_2'(x)=\dfrac{2\cos^2 x-1}{\cos x}\cdot\sin x\end{cases}$ $\therefore C_1(x)=-x+\frac{1}{2}\sin 2x+C_1'$ $\quad C_2(x)=-\cos^2 x+\ln|\cos x|+C_2'$

$\therefore$ 方程的通解为：$y=\sin 2x\ln|\cos x|-x\cos 2x+C_1\sin 2x+C_2\cos 2x$

4. 解：齐次方程对应的特征方程的特征根为 $\lambda_{1,2}=\pm i$. 则方程的通解形式为：

$y=C_1(x)\cos x+C_2(x)\sin x$. 代入方程化简得 $\begin{cases}C_1'(x)\cos x+C_2'(x)\sin x=0\\ C_1'(x)\cdot(-\sin x)+C_2'(x)\cdot\cos x=2\sec^3 x\end{cases}$

解得 $\begin{cases}C_1'(x)=-2\sin x\cdot\sec^3 x\\ C_2'(x)=2\sec^2 x\end{cases}$ $\therefore C_1(x)=-\sec^2 x+C_1'$ $\quad C_2(x)=2\tan x+C_2'$

$\therefore$ 方程的通解为 $y=C_1\cos x+C_2\sin x-\dfrac{1}{\cos x}+\dfrac{2\sin^2 x}{\cos x}=C_1\cos x+C_2\sin x-\dfrac{\cos 2x}{\cos x}$

5. 解：令 $x=e^t$. $\therefore \dfrac{dy}{dx}=\dfrac{dy}{dt}\cdot e^{-t}$, $\dfrac{d^2y}{dx^2}=\left(\dfrac{d^2y}{dt^2}-\dfrac{dy}{dt}\right)e^{-2t}$ 代入方程化为：

$\dfrac{d^2y}{dt^2}-5\cdot\dfrac{dy}{dt}+6y=0$. 其对应的特征方程的特征根为 $\lambda_{1,2}=2,3$.

$\therefore y(t)=C_1e^{2t}+C_2e^{3t}$ $\quad\therefore y(x)=C_1x^2+C_2x^3$

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6解. 令$x=e^t$. $\therefore \frac{dy}{dx}=\frac{dy}{dt}e^{-t}$. $\frac{d^2y}{dx^2}=(\frac{d^2y}{dt^2}-\frac{dy}{dt})e^{-2t}$ 代入方程得.

$y''-2y'-3y=0$. ($y''=\frac{d^2y}{dt^2}$, $y'=\frac{dy}{dt}$). 其特征方程的特征根为 $\lambda_{1,2}=-1,3$

$\therefore y(t)=c_1e^{-t}+c_2\cdot e^{3t} \Rightarrow y(x)=\frac{c_1}{x}+c_2x^3$

7解. 令$x=e^t$ $\therefore \frac{dy}{dx}=\frac{dy}{dt}e^{-t}$, $\frac{d^2y}{dx^2}=(\frac{d^2y}{dt^2}-\frac{dy}{dt})e^{-2t}$, $\frac{d^3y}{dx^3}=(\frac{d^3y}{dt^3}-3\frac{d^2y}{dt^2}+2\frac{dy}{dt})e^{-3t}$

代入方程得: $\frac{d^3y}{dt^3}-3\frac{d^2y}{dt^2}+3\frac{dy}{dt}-y=0$. 其特征方程的特征根为 $\lambda=1$

$\therefore y(t)=e^t(C_1+C_2t+C_3t^2)$ $\therefore y(x)=x\cdot(c_1+c_2\ln|x|+c_3\ln^2|x|)$

8解: 令$x=e^t$ $\therefore \frac{dy}{dx}=\frac{dy}{dt}e^{-t}$, $\frac{d^2y}{dx^2}=(\frac{d^2y}{dt^2}-\frac{dy}{dt})e^{-2t}$. 代入方程得

$\frac{d^2y}{dx^2}+4y=10$. 其齐次方程的特征方程的特征根为 $\lambda_{1,2}=\pm 2i$ 则0不是其特征根

$\therefore$方程的通解形式为 $y(t)=C_1\cos 2t+C_2\sin 2t+y^*$.

y^*为方程的一个特解. 易求$y^*=\frac{5}{2}$ $\therefore y(t)=C_1\cos 2t+C_2\sin 2t+\frac{5}{2}$

$\therefore y(x)=C_1\cos(2\ln|x|)+C_2\sin(2\ln|x|)+\frac{5}{2}$

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7 常系数线性微分方程组

1. 解: (1) $\begin{cases} \dfrac{dx}{dt}=-x-5y & \text{①} \\ \dfrac{dy}{dt}=x+y & \text{②} \end{cases}$ [redacted] 由①得 $y=\frac{-1}{5}\frac{dx}{dt}-\frac{1}{5}x$ ③

对③式两边对t求导 $\frac{dy}{dt}=-\frac{1}{5}\frac{d^2x}{dt^2}-\frac{1}{5}\frac{dx}{dt}$ 代入②式得:

$-\frac{1}{5}\frac{d^2x}{dt^2}-\frac{1}{5}\frac{dx}{dt}=x+(-\frac{1}{5}\frac{dx}{dt}-\frac{1}{5}x)$ 即 $\frac{d^2x}{dt^2}+4x=0$

其对应的特征方程的根为 $\lambda_{1,2}=\pm 2i$ $\therefore x(t)=A_1\cos 2t+A_2\sin 2t$

$\therefore y=(-\frac{2}{5}A_2-\frac{1}{5}A_1)\cos 2t+(\frac{2}{5}A_1-\frac{1}{5}A_2)\cdot\sin 2t$

(2) $\begin{cases} \dfrac{dx}{dt}=x+y+2e^t & \text{①} \\ \dfrac{dy}{dt}=4x+y-e^t & \text{②} \end{cases}$ 由①得 $y=\frac{dx}{dt}-x-2e^t$ ③

对③式进行求导得 $\frac{d^2x}{dt^2}-\frac{dx}{dt}-2e^t=4x+(\frac{dx}{dt}-x-2e^t)-e^t$. 化简得

$x''-2x'-3x=-e^t$. 对应的齐次方程的特征根为 $\lambda_{1,2}=-1,3$. 且易求其一个特解为 $\frac{1}{4}e^t$

$\therefore x(t)=C_1e^{-t}+C_2e^{3t}+\frac{1}{4}e^t$ $\therefore y(t)=2C_2e^{3t}-2C_1e^{-t}-2e^t$

(3). $\begin{cases} \dfrac{dx}{dt}=2x-5y-\sin 2t & \text{①} \\ \dfrac{dy}{dt}=x-2y+t & \text{②} \\ x(0)=0,\ y(0)=1 \end{cases}$ 由②得 $x=\frac{dy}{dt}+2y-t$ $\therefore \frac{dx}{dt}=\frac{d^2y}{dt^2}+2\frac{dy}{dt}-1$

代入①得 $\frac{d^2y}{dt^2}+2\frac{dy}{dt}-1=2(\frac{dy}{dt}+2y-t)-5y-\sin 2t$. 化简为 $y''+y=1-2t-\sin 2t$

解得 $y(t)=C_1\cos t+C_2\sin t+\frac{1}{3}\sin 2t+1-2t$. $\therefore x(t)=(C_2+2C_1)\cos t+(2C_2-C_1)\sin t+\frac{2}{3}(\sin 2t+\cos 2t)-5t$

又 $x(0)=0,\ y(0)=1$ $\therefore C_1=0,\ C_2=-\frac{2}{3}$

$\therefore x(t)=-\frac{2}{3}\cos t-\frac{4}{3}\sin t+\frac{2}{3}\sin 2t+\frac{2}{3}\cos 2t-5t$ $\quad y=-\frac{2}{3}\sin t+\frac{1}{3}\sin 2t-2t+1$

(4) $\begin{cases} \dfrac{dx}{dt}=x & \text{①} \\ \dfrac{dy}{dt}=-y+\sqrt{2}z & \text{②} \\ \dfrac{dz}{dt}=\sqrt{2}y & \text{③} \end{cases}$ 由①可得 $x=C_1e^t$ 由③②得 $\begin{cases} \dfrac{dy}{dt}=\dfrac{1}{\sqrt{2}}\dfrac{d^2z}{dt^2} \\ \dfrac{dy}{dt}=-\dfrac{1}{\sqrt{2}}\dfrac{dz}{dt}+\sqrt{2}z \end{cases}$

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$\therefore \frac{1}{\sqrt{2}}\frac{d^2z}{dt^2}=-\frac{1}{\sqrt{2}}\frac{dz}{dt}+\sqrt{2}z.$ $\therefore z''=-z'+2z$ $\therefore z''+z'-2z=0$

其特征方程为 $\lambda^2+\lambda-2=0$ 解得 $\lambda_{1,2}=-2$和1. $\therefore z=C_2e^{-2t}+\sqrt{2}C_3e^t$

代入③得 $y=-\sqrt{2}C_2e^{-2t}+C_3e^t$

2解:(1) $\begin{cases}\dot{x}=x+2y & ①\\ \dot{y}=x-5\sin t & ②\end{cases}$ 由①得 $y=\frac{1}{2}\dot{x}-\frac{1}{2}x$ $\therefore \dot{y}=\frac{1}{2}\ddot{x}-\frac{1}{2}\dot{x}$ 代入②得

$\frac{1}{2}\ddot{x}-\frac{1}{2}\dot{x}=x-5\sin t$. 即 $\ddot{x}-\dot{x}-2x=-10\sin t$. 对应的齐次方程的特征根为$-1,2$

而 $\pm i$不是其特征方程的根. ∴方程的特解形式为 $x=A\cos t+B\sin t$

可解得 $A=-1, B=3$ $\therefore x=3\sin t-\cos t$ 可解得 $y=2\cos t-5\sin t$

(2) $\begin{cases}\dot{x}=2x+y+e^t & —①\\ \dot{y}=-2x+2t & —②\end{cases}$. 由②得 $x=-\frac{1}{2}\dot{y}+t$ 代入①得 $-\frac{1}{2}\ddot{y}+1=-\dot{y}+2t+y+e^t$

$\therefore \ddot{y}-2\dot{y}+2y=2-4t-2e^t$. 可分解方程 $\begin{cases}\ddot{y}-2\dot{y}+2y=2-4t\\ \ddot{y}-2\dot{y}+2y=-2e^t\end{cases}$ 对应特解为 $\begin{cases}y_1=-2t-1\\ y_2=-2e^t\end{cases}$

$\therefore y=-(2e^t+2t+1)$. 代入②解得 $x=e^t+t+1$

(3) $\begin{cases}\dot{x}=2x-y & —①\\ \dot{y}=y-2x+18t & —②\end{cases}$ 由①得 $y=-\dot{x}+2x$ 代入②得 $-\ddot{x}+2\dot{x}=-\dot{x}+2x-2x+18t$

化为 $\ddot{x}-3\dot{x}=-18t$. 有特解 $x=3t^2+2t$. 代入①得 $y=6t^2-2t-2$ ……

(4) $\begin{cases}\dot{x}=2x-y & \cdots①\\ \dot{y}=x+2e^t & \cdots②\end{cases}$ 由①得 $y=-\dot{x}+2x$ 代入②得 $-\ddot{x}+2\dot{x}=x+2e^t$

化为 $\ddot{x}-2\dot{x}+x=-2e^t$. 有特解 $x=-t^2e^t$ 代入①得 $y=(2t-t^2)e^t$

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第十章 无穷级数

1 柯西收敛原理与数项级数的概念

1. 证明:(1) 级数的部分和 $S_n=\sum_{k=1}^{n}a_k$. 下面证明序列 $\{S_n\}$ 有极限

$|a_k|=\left|\frac{\cos k}{k(k+1)}\right|\leq\left|\frac{1}{k(k+1)}\right|=\left|\frac{1}{k}-\frac{1}{k+1}\right|$

$\therefore |S_m-S_n|=\left|\sum_{k=n+1}^{m}a_k\right|\leq\left|\left(\frac{1}{n+1}-\frac{1}{n+2}\right)+\left(\frac{1}{n+2}-\frac{1}{n+3}\right)+\cdots+\frac{1}{m}-\frac{1}{m+1}\right|<\frac{1}{n+1}\leq\frac{1}{N+1}$

令 $\varepsilon=\frac{1}{N+1}$ $\therefore |S_m-S_n|<\varepsilon.(m>n\geq N)$ $\therefore \{S_n\}$ 有极限. $\therefore$ 级数 $\sum_{n=1}^{\infty}\frac{\cos n}{n(n+1)}$ 收敛.

(2). 考虑 $\sum_{k=n+1}^{n+p}\frac{1}{\sqrt{k}}=\frac{1}{\sqrt{n+1}}+\cdots+\frac{1}{\sqrt{n+p}}$ 的和当 $n\to\infty$ 时是否趋于零. 任意给定 n. 取 $p=n$

$\therefore \sum_{k=n+1}^{2n}\frac{1}{\sqrt{k}}\geq\frac{1}{n+1}+\cdots+\frac{1}{2n}\geq\frac{n}{2n}=\frac{1}{2}$. $\therefore$ 级数 $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ 发散.

(3) 级数 $\sum_{n=1}^{\infty}a_n,\sum_{n=1}^{\infty}b_n,\sum_{n=1}^{\infty}u_n$ 的部分和分别为 S_n,T_n,R_n. 当 $n\geq N$ 时. 根据题知.

$S_n.T_n$ 有极限 S 和 T. 且有 $S_n\leq R_n\leq T_n$.

$\therefore |R_m-R_n|\leq|T_m-S_n|\leq|T_m|+|S_n|<\varepsilon_1+\varepsilon_2=\varepsilon.$ ($\because T_n,S_n$ 有极限)

$\therefore$ 级数 $\sum_{n=1}^{\infty}u_n$ 收敛.

2. 由于级数 $\sum_{n=1}^{\infty}b_n$ 发散. $\therefore$ 级数 $\sum_{n=1}^{\infty}(a_n\pm b_n)$ 发散 (其中级数 $\sum_{n=1}^{\infty}a_n$ 收敛)

3. (1) 发散 (2) 收敛 (3) 收敛 (4) 发散 (5) 发散 (6) 收敛 (7) 发散 (8) 发散.

4 证明: 只需证明 $|S_m-S_n|<\varepsilon\ (m,n\geq N)$. 不妨设. m 为偶数. n 为奇数

$|S_m-S_n|=\left|\sum_{k=1}^{\frac{m}{2}}(S_{2k}+S_{2k-1})-\sum_{k=1}^{\frac{n-1}{2}}(S_{2k}+S_{2k+1})\right|<\varepsilon$. ($\because \{S_{2n}\}$ 与 $\{S_{2n+1}\}$ 收敛). 证毕.

5 证明: ① 先证 $2nu_{2n}\to0$. $\because$ 级数 $\sum_{n=1}^{\infty}u_n$ 收敛 $\therefore \left|\sum_{k=n+1}^{n+p}u_k\right|<\varepsilon.(n\geq N,p\geq1)$.

不妨取 $p=n$. 有 $\lim_{n\to\infty}(u_{n+1}+u_{n+2}+\cdots+u_{2n})=0$

$\therefore \lim_{n\to\infty}0\leq\lim_{n\to\infty}nu_{2n}\leq\lim_{n\to\infty}(u_{n+1}+\cdots+u_{2n})=0$ $\therefore nu_{2n}\to0$ $\therefore 2nu_{2n}\to0$

② 再证 $(2n+1)u_{2n+1}\to0$. 同理取 $p=n+1$ 即可获证.

综上对 $n\in N^*$ 有 $\lim_{n\to\infty}nu_n=0$. 证毕.

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2 正项级数的收敛判法.

1.解:(1) $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}2\cdot\frac{\sin\frac{\pi}{4^{n+1}}}{\sin\frac{\pi}{4^n}}\xlongequal{\theta=\pi/4^n}\lim\limits_{\theta\to0}2\cdot\frac{\sin\frac{\theta}{4}}{\sin\theta}\xlongequal{洛必达法则}\lim\limits_{\theta\to0}2\cdot\frac{1}{4}\cdot\frac{\cos\frac{\theta}{4}}{\cos\theta}=\frac{1}{2}<1$

∴ $\sum\limits_{n=1}2^n\sin\frac{\pi}{4^n}$ 是收敛的.

(2) ∵ $u_n=\frac{1}{\sqrt{2n^3+1}}<\frac{1}{\sqrt{2n^3}}=\frac{1}{n^{\frac{3}{2}}}\cdot\frac{1}{\sqrt{2}}=v_n$. 显然 v_n 是收敛的.

∴ $\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt{2n^3+1}}$ 是收敛的

(3) ∵ $\frac{1}{\sqrt[n]{n}}=\frac{1}{n^{\frac{1}{n}}}\geqslant\frac{1}{n}$. 而 $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ 发散 ∴ $\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt[n]{n}}$ 也发散.

(4) ∵ $\lim\limits_{n\to\infty}\frac{\frac{4n}{n^2+4n-3}}{\frac{1}{n}}=\lim\limits_{n\to\infty}\frac{4n^2}{n^2+4n-3}=4>0$ 又 $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ 发散 ∴ $\sum\limits_{n=1}^{\infty}\frac{4n}{n^2+4n-3}$ 发散

(5) ∵ $\frac{\frac{n^n}{(n^2+3n+1)^{\frac{n+2}{2}}}}{\frac{1}{n^2}}=\frac{n^{n+2}}{(n^2+3n+1)^{\frac{n+2}{2}}}=\frac{1}{\sqrt{(1+\frac{3}{n}+\frac{1}{n^2})}^{\,n+2}}$

而 $\lim\limits_{n\to\infty}\frac{1}{(1+\frac{3}{n}+\frac{1}{n^2})^{\frac{n+2}{2}}}=e^{\lim\limits_{n\to\infty}(-\frac{n+2}{2})\ln(1+\frac{3}{n}+\frac{1}{n^2})}=e^{-\frac{1}{2}}\cdot e^{\lim\limits_{n\to\infty}(n+2)\ln(1+\frac{3}{n}+\frac{1}{n^2})}$

又 $\lim\limits_{n\to\infty}(n+2)\ln(1+\frac{3}{n}+\frac{1}{n^2})=\lim\limits_{n\to\infty}\frac{\ln(1+\frac{3}{n}+\frac{1}{n^2})}{\frac{1}{n+2}}\xlongequal{洛必达法则}\lim\limits_{n\to\infty}\frac{(3n+2)(n+2)^2}{n(n^2+3n+1)}=3$

∴ $\lim\limits_{n\to\infty}\frac{1}{(1+\frac{3}{n}+\frac{1}{n^2})^{\frac{n+2}{2}}}=e^{-\frac{1}{2}}\cdot e^3=e^{\frac{5}{2}}$

∴ $\lim\limits_{n\to\infty}\frac{u_n}{\frac{1}{n^2}}=e^{\frac{5}{2}}>0$ 又 $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$ 收敛 ∴ $\sum\limits_{n=1}^{\infty}\frac{n^n}{(n^2+3n+1)^{\frac{n+2}{2}}}$ 也收敛.

(6) 注意到 $n>e^{e^3}$ 时,即 $\ln(\ln n)>3$ 时有 $(\ln n)^{\ln n}=e^{\ln n\cdot\ln(\ln n)}=n^{\ln(\ln n)}>n^3$.

∴ $\frac{n}{(\ln n)^{\ln n}}<\frac{n}{n^3}=\frac{1}{n^2}$ 而 $\sum\limits_{n=2}^{\infty}\frac{1}{n^2}$ 收敛 ∴ $\sum\limits_{n=2}^{\infty}\frac{n}{(\ln n)^{\ln n}}$ 也收敛

(7) ∵ $\lim\limits_{n\to\infty}\frac{n\cdot\tan\frac{1}{3^n}}{n\cdot\frac{1}{3^n}}=\lim\limits_{n\to\infty}\frac{\tan\frac{1}{3^n}}{\frac{1}{3^n}}=1>0$ 下面考虑 $\sum\limits_{n=1}n\cdot\frac{1}{3^n}$ 的敛散性. (令 $v_n=n\cdot\frac{1}{3^n}$

∵ $\lim\limits_{n\to\infty}\frac{v_{n+1}}{v_n}=\lim\limits_{n\to\infty}\frac{(n+1)\cdot\frac{1}{3^{n+1}}}{n\cdot\frac{1}{3^n}}=\frac{1}{3}\lim\limits_{n\to\infty}\frac{n+1}{n}=\frac{1}{3}<1$ ∴ $\sum\limits_{n=1}v_n$ 收敛

故 $\sum\limits_{n=1}n\tan\frac{1}{3^n}$ 收敛.

2.解:(1) $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{1}{n+1}\cdot\frac{(n+1)^5}{n^5}=\lim\limits_{n\to\infty}\frac{(n+1)^4}{n^5}=0<1$ ∴ $\sum\limits_{n=1}^{\infty}\frac{n^5}{n!}$ 收敛

(2) $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{n^2}{(n+1)^2}\to\infty$. ∴ $\sum\limits_{n=1}^{\infty}\frac{n!}{3n^2}$ 发散.

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(3). $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}3\cdot\left(\frac{n}{n+1}\right)^n=3\cdot\lim\limits_{n\to\infty}\left(\frac{n}{n+1}\right)^n=3\cdot e^{\lim\limits_{n\to\infty}n\cdot\ln\frac{n}{n+1}}\xlongequal{\text{洛比达}}3\cdot e^{-1}=\frac{3}{e}>1$

$\therefore\sum\limits_{n=1}^{\infty}\frac{3^n\cdot n!}{n^n}$发散

(4). 注意到 $\frac{1}{n^{1+1/n}}\sim\frac{1}{n}(n\to\infty)$ 而 $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ 发散. $\therefore\sum\limits_{n=1}^{\infty}\frac{1}{n^{1+1/n}}$ 发散

(5) 注意到 $\lim\limits_{n\to\infty}n^{\frac{2}{n}}=e^{\lim\limits_{n\to\infty}\frac{2}{n}\ln n}=e^{2\cdot\lim\limits_{n\to\infty}\frac{\ln n}{n}}\xlongequal{\text{洛比达}}e^{2\cdot\lim\limits_{n\to\infty}\frac{\frac{1}{n}}{1}}=e^0=1$

$\therefore\lim\limits_{n\to\infty}\sqrt[n]{u_n}=\lim\limits_{n\to\infty}\frac{n^{\frac{2}{n}}}{3-\frac{1}{n}}=\frac{1}{3}<1\quad\therefore\sum\limits_{n=1}^{\infty}\frac{n^2}{(3-\frac{1}{n})^n}$ 收敛.

(6). $\lim\limits_{n\to\infty}\sqrt[n]{u_n}=\lim\limits_{n\to\infty}n^{\frac{1}{n}}\cdot\lim\limits_{n\to\infty}\frac{1}{\ln n}=1\cdot 0=0<1.\quad\therefore\sum\limits_{n=2}^{\infty}\frac{n}{(\ln n)^n}$ 收敛.

(7). $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{1000}{n+1}=0<1\quad\therefore\sum\limits_{n=1}^{\infty}\frac{1000^n}{n!}$ 收敛.

(8). $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n+2)}=\lim\limits_{n\to\infty}\frac{n+1}{2(2n+1)}=\frac{1}{4}<1\quad\therefore\sum\limits_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$ 收敛.

(9). $\lim\limits_{n\to\infty}\sqrt[n]{u_n}=\lim\limits_{n\to\infty}\frac{1}{3}\left(\frac{n+1}{n}\right)^n=\frac{1}{3}e^{\lim\limits_{n\to\infty}n\ln\frac{n+1}{n}}=\frac{1}{3}e^{\lim\limits_{n\to\infty}\frac{\ln\frac{n+1}{n}}{\frac{1}{n}}}\xlongequal{\text{洛比达}}\frac{1}{3}e<1$

$\therefore\sum\limits_{n=1}^{\infty}\frac{1}{3^n}\left(\frac{n+1}{n}\right)^{n^2}$ 收敛.

(10). $f(n)=u_n=\frac{1}{n(\ln n)^p}$ 为单调递减的非负函数.

而 $\int_2^{+\infty}f(x)dx=\int_2^{+\infty}\frac{d\ln x}{(\ln x)^p}$

当 $p=1$ 时 $\int_2^{+\infty}f(x)dx=\lim\limits_{A\to\infty}\ln(\ln x)-\ln(\ln 2)\to\infty$. 此时 $\sum\limits_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$ 发散.

当 $0<p<1$ 时 $\int_2^{+\infty}f(x)dx=\lim\limits_{x\to\infty}\frac{1}{1-p}(\ln x)^{1-p}-\frac{1}{1-p}(\ln 2)^{1-p}\to\infty$. 此时 $\sum\limits_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$ 发散

当 $p>1$ 时 $\int_2^{+\infty}f(x)dx=\lim\limits_{x\to\infty}\frac{1}{1-p}\cdot\frac{1}{(\ln x)^{p-1}}-\frac{1}{1-p}\frac{1}{(\ln 2)^{p-1}}\to-\frac{1}{1-p}\frac{1}{(\ln 2)^{p-1}}$

$\therefore\sum\limits_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ 收敛.

综上述. 当 $p>1$ 时 $\sum\limits_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ 收敛. 当 $0<p\le 1$ 时 $\sum\limits_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ 发散

(11). 注意到 x 比 $\ln x$ 高阶. 故当 n 足够大时. $(\ln(\ln n))^q<(\ln n)^{1/2}$

$\therefore u_n>\frac{1}{n\cdot(\ln n)^{1/2}}=V_n$. $f(n)=V_n=\frac{1}{n(\ln n)^{1/2}}$

$\therefore\int_3^{+\infty}f(x)dx=\int_3^{+\infty}\frac{d(\ln n)}{(\ln n)^{1/2}}=2\lim\limits_{x\to\infty}\sqrt{\ln n}-2\sqrt{\ln 3}\to\infty$. 发散. 即 $\sum\limits_{n=3}^{\infty}V_n$ 发散

$\therefore\sum\limits_{n=3}^{\infty}u_n$ 也发散.

(12). 分别讨论以下4种情况:

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① $p>1$时. $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=\lim\limits_{n\to\infty}\frac{u_n}{\frac{1}{n(\ln n)^p}}=\lim\limits_{n\to\infty}\frac{1}{(\ln\ln n)^q}=0.$

由第(10)题知. $p>1$时 $\sum\limits_{n=3}^{\infty}v_n=\sum\limits_{n=3}^{\infty}\frac{1}{n(\ln n)^p}$ 收敛. ∴该级数收敛.

② $p<1$时. $\lim\limits_{n\to\infty}\frac{v_n}{u_n}=\lim\limits_{n\to\infty}(\ln\ln n)^q\to\infty$. 由 $\sum\limits_{n=1}^{\infty}v_n$发散知 $\sum\limits_{n=1}^{\infty}u_n$发散.

③. $p=1, q>1$时. 设 $f(n)=\frac{1}{n(\ln n)(\ln\ln n)^q}$ ∴ $\int_3^A f(x)dx=\int_3^A\frac{d(\ln\ln x)}{(\ln\ln x)^q}=\frac{1}{1-q}\frac{1}{(\ln\ln x)^{q-1}}\Big|_3^A\to 0$

∴ $\sum\limits_{n=1}^{\infty}u_n$收敛

④ $p=1$, $q<1$时. $\int_3^A f(x)dx=\frac{1}{1-q}\cdot(\ln\ln x)^{1-q}\Big|_3^A\to\infty$. ∴ $\sum\limits_{n=1}^{\infty}u_n$发散

3. 证明: ∵ $\sum\limits_{n=1}^{\infty}u_n$收敛. ∴ $\lim\limits_{n\to\infty}u_n=0$

∴ $\lim\limits_{n\to\infty}\frac{u_n^2}{u_n}=\lim\limits_{n\to\infty}u_n=0$ 故 $\sum u_n^2$收敛

反之不一定成立. 如 $u_n=\frac{1}{n}$时 $\sum\limits_{n=1}u_n^2$收敛 而 $\sum\limits_{n=1}u_n$发散.

4. 证明: ∵ $|a_nb_n|\le\frac{a_n^2}{2}+\frac{b_n^2}{2}$ ∴ $\sum\limits_{n=1}^{\infty}|a_nb_n|$收敛

∵ $(a_n+b_n)^2\le 2a_n^2+2b_n^2$ ∴ $\sum\limits_{n=1}^{\infty}(a_n+b_n)^2$收敛

令 $b_n=\frac{1}{n}$. 显然符合题意. ∴ $\sum\limits_{n=1}^{\infty}|a_nb_n|=\sum\limits_{n=1}^{\infty}\frac{|a_n|}{n}$ 收敛. 证毕

5. 解: (1) 一定发散.

(2) 不一定发散. 令 $u_n=v_n=\frac{1}{n}$. 则 $\sum\limits_{n=1}^{\infty}(u_n-v_n)$收敛

(3) 不一定发散. 令 $u_n=v_n=\frac{1}{n}$. 则 $\sum\limits_{n=1}u_n\cdot v_n$ 收敛.

6. 证明: ∵ $\lim\limits_{n\to\infty}\frac{\frac{u_n}{n}}{\frac{1}{n^2}}=\lim\limits_{n\to\infty}n\cdot u_n=l$. 而 $\sum\limits_{n=1}\frac{1}{n^2}$收敛 ∴ $\sum\limits_{n=1}^{\infty}\frac{u_n}{n}$ 收敛.

又 $\lim\limits_{n\to\infty}\frac{u_n^2}{\frac{u_n}{n}}=\lim\limits_{n\to\infty}nu_n=l$ ∴ $\sum\limits_{n=1}^{\infty}\frac{u_n}{n}$收敛

而 $\lim\limits_{n\to\infty}\frac{u_n}{\frac{1}{n}}=\lim\limits_{n\to\infty}nu_n=l$ 且 $\sum\limits_{n=1}^{\infty}\frac{1}{n}$发散 ∴ $\sum\limits_{n=1}u_n$发散.

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5任意项级数

1.解: (1) $\sum_{n=1}^{\infty}|u_n|=\sum_{n=1}^{\infty}\frac{1}{(2n)^2}=\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}$ 收敛. ∴该级数绝对收敛.

(2) $\sum_{n=1}^{\infty}|u_n|=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^p}$.

①当 $P>1$ 时. $\sum_{n=1}^{\infty}|u_n|<\sum_{n=2}^{\infty}\frac{1}{(2n-2)^p}+1=\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^p}+1$. 收敛. ∴该级数绝对收敛.

②当 $0<p\le 1$ 时. $\sum_{n=1}^{\infty}|u_n|\geq\sum_{n=1}^{\infty}\frac{1}{(2n)^p}=\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^p}$ 发散. 而又.

$\sum_{n=1}^{\infty}u_n$ 收敛. ($\because \lim_{n\to\infty}u_n=0$ 且 $\frac{1}{(2n-1)^p}>\frac{1}{(2n+1)^p}$

∴该级数条件收敛

(3).设 $u_n=\frac{(-1)^{n-1}}{(n+1)\ln(n+1)}$ 则正项级数 $\sum_{n=1}^{\infty}|u_n|=\sum_{n=1}^{\infty}\frac{1}{(n+1)\ln(n+1)}=\sum_{n=1}^{\infty}\frac{1}{n\ln n}$ 发散.

而 $|u_n|\geq|u_{n+1}|$ 且 $\lim_{n\to\infty}|u_n|=0$ ∴ $\sum_{n=1}^{\infty}u_n$ 收敛

故该级数条件收敛.

(4). 正项级数 $\sum_{n=1}^{\infty}\frac{\sqrt{n}-1}{n}=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}-\sum_{n=1}^{\infty}\frac{1}{n}$ 发散.

而 $u_n\geq u_{n+1}$ 且 $\lim_{n\to\infty}|u_n|=0$ ∴ $\sum_{n=1}^{\infty}u_n$ 收敛. 故该级数条件收敛.

(5). 正项级数 $\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}=\sum_{n=1}^{\infty}\frac{1}{n+1}+\sum_{n=1}^{\infty}\frac{1}{n}$ 发散.

而 $u_n>u_{n+1}$ 且 $\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{2n+1}{n(n+1)}=\lim_{n\to\infty}(\frac{1}{n}+\frac{1}{n+1})=0$. ∴ $\sum_{n=1}^{\infty}u_n$ 收敛.

故该级数条件收敛

(6) 正项级数 $\sum_{n=1}^{\infty}|u_n|=\sum_{n=1}^{\infty}\frac{n!}{3^{n^2}}$. $\because \lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\lim_{n\to\infty}\frac{n+1}{3^{2n+1}}=0<1$. ∴ $\sum_{n=1}^{\infty}|u_n|$ 收敛

故该级数绝对收敛

(7) $\sum_{n=2}^{\infty}|u_n|=\sum_{n=2}^{\infty}\frac{1}{n}\cdot\sin\frac{\pi}{n}$. 令 $a_n=\frac{1}{n}$. $b_n=\sin\frac{\pi}{n}$. 显然 a_n 单调且 $\lim_{n\to\infty}a_n=0$

而 b_n 的部分和 B_n 有界. ∴ $\sum_{n=2}^{\infty}\frac{1}{n}\cdot\sin\frac{\pi}{n}$ 收敛 即有 该级数绝对收敛 (或: $|u_n|<\frac{\pi}{n^2}$

(8) 正项级数. $\sum_{n=1}^{\infty}|\tan\frac{\varphi}{n}|>\sum_{n=1}^{\infty}|\varphi|\cdot\frac{1}{n}$ (发散) ∴ $\sum_{n=1}^{\infty}|\tan\frac{\varphi}{n}|$ 发散.

而 $u_n=|\tan\frac{\varphi}{n}|>|\tan\frac{\varphi}{n+1}|=u_{n+1}$. 且 $\lim_{n\to\infty}u_n=0$ ∴ $\sum_{n=1}^{\infty}u_n$ 收敛

故该级数条件收敛.

(9) 分以下四种情况讨论.

① $t>1$ 时. 注意到 $\lim\limits_{n\to\infty}\frac{|u_n|}{\frac{1}{n^t}} = \lim\limits_{n\to\infty}\frac{1}{(\ln n)^t}=0$ 而 $\sum\limits_{n=1}^{\infty}\frac{1}{n^t}$ 此时收敛.

∴ $\sum\limits_{n=1}^{\infty}|u_n|$ 也收敛. 即该级数绝对收敛.

② $0<t<1$ 时. $\lim\limits_{n\to\infty}\frac{\frac{1}{n^t}}{|u_n|} = \lim\limits_{n\to\infty}(\ln n)^t\to\infty$. 故 $\sum\limits_{n=1}^{\infty}|u_n|$ 发散.

而 $u_n\geq u_{n+1}$. $\lim\limits_{n\to\infty}u_n=0$ ∴ $\sum\limits_{n=1}^{\infty}u_n$ 收敛 ∴该级数条件收敛

③ $t=1$ 且 $0<s\leq1$ 时. $u_n=\frac{1}{n(\ln n)^s}$ ∵ $\int_1^A\frac{1}{x(\ln x)^s}dx=\int_1^A\frac{d\ln x}{(\ln x)^s}=\frac{1}{1-s}(\ln x)^{1-s}\Big|_1^A\to\infty\ (A\to\infty)$

∴ $\sum\limits_{n=1}^{\infty}|u_n|$ 发散. 而 $u_n\geq u_{n+1}$ 且 $\lim\limits_{n\to\infty}u_n=0$ ∴ $\sum\limits_{n=1}^{\infty}u_n$ 收敛 ∴该级数条件收敛

④ $t=1$ 且 $s>1$ 时. $u_n=\frac{1}{n(\ln n)^s}$ ∵ $\int_1^A\frac{1}{x(\ln x)^s}dx=\frac{1}{1-s}\frac{1}{(\ln x)^{s-1}}\Big|_1^A\to 0\ (A\to\infty)$

∴ $\sum\limits_{n=1}^{\infty}|u_n|$ 收敛. 故该级数绝对收敛.

(10) $\sum\limits_{n=1}^{\infty}\sin(\pi\sqrt{n^2+1})=\sum\limits_{n=1}^{\infty}\sin[\pi(\sqrt{n^2+1}+n)-n\pi]=\sum\limits_{n=1}^{\infty}(-1)^n\sin[\pi(\sqrt{n^2+1}+n)]$

∴ $\sum\limits_{n=1}^{\infty}|u_n|=\sum\limits_{n=1}^{\infty}|\sin\pi(\sqrt{n^2+1}+n)|=\sum\limits_{n=1}^{\infty}\sin[\pi\cdot\frac{1}{\sqrt{n^2+1}-n}]=\sum\limits_{n=1}^{\infty}v_n$.

∵ $\lim\limits_{n\to\infty}\frac{v_n}{\frac{\pi}{\sqrt{n^2+1}-n}}=1$. 且 $\sum\limits_{n\to\infty}\frac{\pi}{\sqrt{n^2+1}-n}$ 发散. ∴ $\sum\limits_{n=1}^{\infty}|u_n|$ 发散.

而 $u_n\geq u_{n+1}$ 且 $\lim\limits_{n\to\infty}u_n=0$ ∴ $\sum\limits_{n=1}^{\infty}u_n$ 收敛 故该级数条件收敛

(11) 令 $u_n=\frac{1\cdot3\cdots(2n-1)}{2\cdot4\cdots(2n)}$. 由 $\frac{n+1}{n}<\frac{n}{n-1}$ 得 $u_n<\frac{1}{u_n}\cdot\frac{1}{2n}$ 即 $u_n<\frac{1}{\sqrt{2n}}$. ∴ $u_n\to0\ (n\to\infty)$

又 $\frac{n}{n+1}>\frac{n-1}{n}$ ∴ $u_n>\frac{1}{2u_n}\cdot\frac{1}{2n}$ ∴ $u_n>\frac{1}{2}\cdot\frac{1}{\sqrt{n}}$ 而 $\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ 发散 ∴ $\sum\limits_{n=1}^{\infty}u_n$ 发散

显然 $u_n\geq u_{n+1}$ 且 $\lim\limits_{n\to\infty}u_n=0$ ∴ 该级数条件收敛.

2. 证一: 显然 $\frac{1}{n^p}$ 和 $\frac{n}{n+1}=1-\frac{1}{n+1}$ 单调有界. 而 $\sum\limits_{n=1}^{\infty}u_n$ 收敛. 命题得证.

证二: ∵ $\lim\limits_{n\to\infty}\frac{\frac{u_n}{n^p}}{u_n}=\lim\limits_{n\to\infty}\frac{1}{n^p}=0$ 而 $\sum\limits_{n=1}^{\infty}u_n$ 收敛. ∴ $\sum\limits_{n=1}^{\infty}\frac{u_n}{n^p}$ 收敛.

同理证得 $\sum\limits_{n=1}^{\infty}\frac{n}{n+1}u_n$ 收敛.

3. 证明: ① $p>1$ 时. $\sum\limits_{n=1}^{\infty}|u_n|\leq\sum\limits_{n=1}^{\infty}\frac{1}{n^p}$ 而 $p>1$ 时 $\sum\limits_{n=1}^{\infty}\frac{1}{n^p}$ 收敛 ∴该级数绝对收敛.

② $0<p\leq1$ 时. $\left|\frac{\cos n\varphi}{n^p}\right|\geq\frac{\cos^2 n\varphi}{n}=\frac{1}{2n}-\frac{\cos 2n\varphi}{2n}\geq0$ 而 $\sum\limits_{n=1}^{\infty}\frac{1}{2n}$ 发散. $\sum\limits_{n=1}^{\infty}\frac{\cos 2n\varphi}{2n}$ 收敛

∴ $\sum\limits_{n=1}^{\infty}\left|\frac{\cos n\varphi}{n^p}\right|$ 发散.

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而 $\left|\sum_{n=1}^{\infty}\cos n\varphi\right| = \left|\dfrac{\sin\frac{2n+1}{2}\varphi - \sin\frac{\varphi}{2}}{2\sin\frac{\varphi}{2}}\right| \leq \left|\dfrac{1}{2\sin\frac{\varphi}{2}}\right|$ （即 $\left|\sum_{n=1}^{\infty} b_n\right| \leq M$）.

而 $a_n = \frac{1}{n^p}$ 单调且 $\lim\limits_{n\to\infty} a_n = 0$ $\therefore \sum_{n=1}^{\infty} \frac{\cos n\varphi}{n^p}$ 收敛. 故该级数条件收敛

4. 由3题知. $\sum_{n=1}^{\infty} \frac{\cos n\varphi}{n^p}$ 收敛. 由阿贝尔判别法知. 如果该级数收敛. 只需证 $(1+\frac{1}{n})^n$ 单调有界

$\because (1+\frac{1}{n+1})^{n+1} > (1+\frac{1}{n})^{n+1} > (1+\frac{1}{n})^n$ $\therefore (1+\frac{1}{n})^n$ 单调.

$\therefore (1+\frac{1}{n})^n < \cdots < \lim\limits_{n\to\infty}(1+\frac{1}{n})^n = e$. 即 $(1+\frac{1}{n})^n$ 有界.

故 $\sum_{n=1}^{\infty} \frac{\cos n\varphi}{n^p}(1+\frac{1}{n})^n$ 收敛.

5 证明: 先记 $b_n = \frac{1}{n^{x-x_0}}$ 单调有界. 显然 b_n 递减 $(x>x_0)$

且 $\lim\limits_{n\to\infty} b_n = 0$ ($x>x_0$时) $\therefore$ 当 $x>x_0$ 时. $\sum_{n=1}^{\infty} \frac{a_n}{n^{x_0}} \cdot \frac{1}{n^{x-x_0}} = \sum_{n=1}^{\infty} \frac{a_n}{n^x}$ 收敛.

当 $x<x_0$ 时. $\lim\limits_{n\to\infty} \dfrac{\frac{a_n}{n^x}}{\frac{a_n}{n^{x_0}}} = \lim\limits_{x\to\infty} n^{x_0-x} \to \infty$. 而 $\sum_{n=1}^{\infty} \frac{a_n}{n^{x_0}}$ 发散. $\therefore \sum_{n=1}^{\infty} \frac{a_n}{n^x}$ 发散. 证毕

6. 证明: 只需证明 $\left|\frac{2n-1}{n}\right|$ 单调有界 $\because a_n = \left|\frac{2n-1}{n}\right| = \left|2-\frac{1}{n}\right| = 2-\frac{1}{n}$

显然 $a_{n+1} > a_n$ 且 $a_n < 2$. $\therefore a_n$ 单调有界. 依题意 $\sum_{n=1}^{\infty}|u_n|$ 收敛.

$\therefore \sum_{n=1}^{\infty} a_n \cdot |u_n|$ 也收敛 即 $\sum_{n=1}^{\infty}\left|\frac{2n-1}{n}u_n\right|$ 收敛. 证毕.

7. 证明: 令 $V_k = \frac{1}{\sqrt{4k-3}} + \frac{1}{\sqrt{4k-1}} - \frac{1}{\sqrt{2k}}$ $\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} = \sum_{k=1}^{\infty} V_k$.

而 $V_k \geq \frac{1}{\sqrt{4k}} + \frac{1}{\sqrt{4k}} - \frac{1}{\sqrt{2k}} = (1-\frac{1}{\sqrt{2}})\frac{1}{\sqrt{k}}$ 且 $\sum_{k=1}^{\infty}(1-\frac{1}{\sqrt{2}})\frac{1}{\sqrt{k}}$ 发散.

$\therefore \sum_{k=1}^{\infty} V_k$ 发散. 命题得证.

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第十章总练习题

1.证明:(1) 当n充分大时有 $\left|\ln\frac{n-1}{n+1}\right|=\left|\ln\left(1-\frac{2}{n+1}\right)\right|=\left|-\frac{2}{n+1}\ln\left(1+\frac{1}{-\frac{n+1}{2}}\right)^{-\frac{(n+1)}{2}}\right|<\left|-\frac{2}{n+1}\cdot1\right|=\frac{2}{n+1}$

$\therefore |a_n|\leq\frac{1}{\sqrt{n+1}+\sqrt{n}}\cdot\frac{2}{n+1}\leq\frac{1}{2\sqrt{n}}\cdot\frac{2}{n}=\left|\frac{1}{n^{1+\frac{1}{2}}}\right|=1\cdot|b_n|$. 得证.

(2) 当n充分大时有. $\left|\ln(\sec\frac{\pi}{n})\right|=\left|\frac{1}{2}\ln(1+\tan^2\frac{\pi}{n})\right|\leq\left|\frac{1}{2}\ln(1+\frac{\pi^2}{n^2})\right|\leq\frac{\pi^2}{2}\cdot\frac{1}{n^2}$.

$\therefore |a_n|\leq\frac{\pi^2}{2}\cdot\frac{1}{n^2}=\frac{\pi^2}{2}\cdot|b_n|$ 证毕

(3). 令 $\ln n=t$. 则n充分大时. $\left|n^{1+\frac{k}{\ln n}}\right|=\left|(e^t)^{(1+\frac{k}{t})}\right|=|e^{t+k}|\leq|e^k\cdot e^t|=e^k\cdot|n|=e^k\cdot|b_n|$

证毕

(4). 显然. $[\sqrt{2}+(-1)^n]^n\leq(\sqrt{2}+1)^n$.

$\therefore |a_n|=\left|\frac{n^3[\sqrt{2}+(-1)^n]^n}{3^n}\right|\leq\left|\frac{n^3(\sqrt{2}+1)^n}{3^n}\right|=|b_n|$. 证毕.

2.解(1). 由第1题(1)知. $|a_n|\leq\frac{1}{(2\sqrt{n})^p}\cdot\frac{2}{n}=\left|\frac{2}{2^p}\right|\cdot\left|\frac{1}{n^{1+\frac{p}{2}}}\right|=\left|\frac{2}{2^p}\right||b_n|$ $(b_n=\frac{1}{n^{1+\frac{p}{2}}})$

显然 $\sum_{n=1}^{\infty}|b_n|$ 收敛. 故级数 $\sum_{n=1}^{\infty}a_n$ 收敛 $(a_n=|a_n|)$

(2). 由第1题(2)知. $|a_n|\leq(\frac{\pi^2}{2}\cdot\frac{1}{n^2})^p=\frac{\pi^{2p}}{2^p}\cdot\left|\frac{1}{n^{2p}}\right|=\frac{\pi^{2p}}{2^p}\cdot|b_n|$

当 $p>\frac{1}{2}$ 时 $\sum_{n=1}^{\infty}|b_n|$ 收敛. 当 $0<p\leq\frac{1}{2}$ 时. $\sum_{n=1}^{\infty}|b_n|$ 发散.

$\therefore$ 当 $p>\frac{1}{2}$ 时 $\sum_{n=1}^{\infty}a_n$ 收敛. 当 $0<p\leq\frac{1}{2}$ 时. $\sum_{n=1}^{\infty}a_n$ 发散.

(3) 由第1题(3)可知. $|a_n|\leq e^k\cdot\left|\frac{1}{n^p}\right|=e^k|b_n|$

显然 $p>1$ 时 $\sum_{n=2}^{\infty}|b_n|$ 收敛. $\therefore\sum_{n=2}^{\infty}a_n$ 收敛.

(4). 由第1题(4)知. $|a_n|\leq\left|\frac{(\sqrt{2}+1)^n}{3^n}\cdot n^3\right|=|b_n|$

$\lim_{n\to\infty}\frac{b_{n+1}}{b_n}=\frac{\sqrt{2}+1}{3}<1$. $\therefore\sum_{n=1}^{\infty}b_n$ 收敛 $\therefore\sum_{n=1}^{\infty}a_n$ 收敛.

3.证明:显然有 $\frac{1}{a_n}\geq n^{1+\alpha}$. $\therefore a_n\leq\frac{1}{n^{1+\alpha}}$. (n充分大).

当 $\alpha>0$ 时 $\sum_{n=1}^{\infty}\frac{1}{n^{1+\alpha}}$ 收敛. 故 $\sum_{n=1}^{\infty}a_n$ 收敛.

4.证明:由 $\sum_{n=1}^{\infty}\frac{a_n}{n^{x_1}}$ 收敛知: $\left|\frac{a_n}{n^{x_1}}\right|\leq M\frac{1}{n^2}$(M为常数) $\therefore\left|\frac{a_n}{n^x}\right|=\left|\frac{a_n}{n^{x_1}}\cdot n^{x_1-x}\right|\leq\left|\frac{M}{n^2}\cdot\frac{1}{n^{x-x_1}}\right|$ $(x>x_1)$

$\therefore \sum_{n=1}^{\infty}\frac{a_n}{n^x}$ 对任意一点 $x>x_0$ 也收敛，令 $x_0=\alpha$.

对 $x\in[\alpha,\beta]$. $|u_n(x)|=|\frac{a_n}{n^x}|\le|\frac{M}{n^2}\cdot\frac{1}{n^{x-\alpha}}|\le|\frac{M}{n^2}\cdot\frac{1}{n^{\beta-\alpha}}|=|a_n|$

显然 $\sum_{n=1}^{\infty}|a_n|$ 收敛. 故 狄氏级数在其收敛区间 $(\alpha,+\infty)$ 中的任意一个闭区间 $[a,b]$ 上一致收敛.

5. 证明: 令 $u_n(x)=\frac{a_n}{n^x}$. 则 $u_n'(x)=a_n\cdot\ln\frac{1}{n}\cdot(\frac{1}{n})^x$. 仍在收敛区间上连续.

下面只需证明 $\sum_{n=1}^{\infty}u_n'(x)$ 在其收敛区间上一致收敛即可.

显然 $|u_n'(x)|\le|\frac{M}{n^2}\cdot\ln\frac{1}{n}\cdot\frac{1}{n^{\beta-\alpha}}|\le|\frac{M}{n^2}\cdot n\cdot\frac{1}{n^{\beta-\alpha}}|=|\frac{M}{n^{1+\beta-\alpha}}|=|b_n|$

而 $\sum_{n=1}^{\infty}|b_n|$ 收敛. 故 $u_n'(x)$ 在 $[\alpha,\beta]$ 上一致收敛. 故命题得证.

6. 证明: $|z_n-z_0|<\varepsilon\Leftrightarrow\sqrt{(x_n-x_0)^2+(y_n-y_0)^2}<\varepsilon$.

根据连续性知 存在 δ_1,δ_2 有 $|x_n-x_0|<\delta_1$, $|y_n-y_0|<\delta_2$ 成立. (只需 $n>N$)

不妨令 $\delta_1=\delta_2=\frac{\varepsilon}{\sqrt{2}}$ 则有对任意 $\varepsilon>0$, $|x_n-x_0|<\frac{\varepsilon}{\sqrt{2}}=\varepsilon_1$, $|y_n-y_0|<\frac{\varepsilon}{\sqrt{2}}=\varepsilon_2$ (只需 $n>N$)

故有 $x_0\to x_0$, $y_0\to y_0$ $(n\to\infty)$. 证毕

7. 证明: 由6题知, $\sum_{n=0}^{\infty}z^n$ 收敛于 $\sum_{n=0}^{\infty}|z_0|^n$. 而 $a_n=\frac{1}{n!}$ 单调有界. $\therefore \sum_{n=0}^{\infty}\frac{1}{n!}z^n$ 对任意复数z收敛

8 证明: $e^{ix}=e^z=\sum_{n=0}^{\infty}\frac{1}{n!}\cdot(ix)^n=(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4\cdots)+(x-\frac{x^3}{3}+\frac{x^5}{5}+\cdots)i=\cos x+i\sin x$ 证毕.

9. 证明: $f(x)=\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+R_n(x)$. 其中 $R_n(x)=\frac{f^{(n+1)}[x_0+\theta(x-x_0)]}{n!}(1-\theta)^n(x-x_0)^{n+1}$ $(0<\theta<1)$

下面证明 $R_n(x)\to0$ $(n\to\infty)$

$\because |R_n(x)|=|\frac{f^{(n+1)}[x_0+\theta(x-x_0)]}{(n+1)!}\cdot(x-x_0)^{n+1}|\le|\frac{M}{(n+1)!}(a)^{n+1}|\to0$ $(n\to\infty)$

故 $R_n(x)\to0$ $(n\to\infty)$

$\therefore f(x)=\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n$.

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第十一章 广义积分与含参变量的积分

1. 广义积分

1.解: (1) $\int_0^{+\infty} x\cdot e^{-x}dx=\lim\limits_{A\to+\infty}\int_0^A xe^{-x}dx=\lim\limits_{A\to+\infty}\int_0^A(-x)de^{-x}=\lim\limits_{A\to+\infty}(-x\cdot e^{-x}-e^{-x})\Big|_0^A=\lim\limits_{A\to+\infty}(-\frac{A}{e^A}-\frac{1}{e^A}+1)$

$=\lim\limits_{A\to+\infty}(-\frac{A}{e^A})+1\xlongequal{\text{洛必达}}-\lim\limits_{A\to+\infty}\frac{1}{e^A}+1=1.$

∴该广义积分收敛. 其值为1

(2). $\int_0^{+\infty}\frac{dx}{(x+1)(x+2)}=\lim\limits_{A\to+\infty}\int_0^A(\frac{1}{x+1}-\frac{1}{x+2})dx=\lim\limits_{A\to+\infty}[\ln(x+1)-\ln(x+2)]\Big|_0^A=\lim\limits_{A\to+\infty}\ln\frac{A+1}{A+2}+\ln 2$

$=\lim\limits_{A\to+\infty}\ln(\frac{1+\frac{1}{A}}{1+\frac{2}{A}})+\ln 2=\ln 2.$ ∴该广义积分收敛,其值为$\ln 2$.

(3) 令$t=(x-a)/\sqrt{2}\sigma$. ∴$dt=\frac{1}{\sqrt{2}\sigma}dx$.

∴$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-\frac{(x-a)^2}{2\sigma^2}}dx=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-t^2}dt=\frac{1}{\sqrt{\pi}}\left(\int_0^{+\infty}e^{-t^2}dt+\int_{-\infty}^0e^{-t^2}dt\right)=\frac{2}{\sqrt{\pi}}\int_0^{+\infty}e^{-t^2}dt$

$=\frac{2}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2}=1$. 故该广义积分收敛,其值为1.

(4) 令$u=x-\frac{1}{x}$. 则$x\to+\infty$时, $u\to+\infty$, $x\to 0^+$时 $u\to-\infty$.

∵$du=1+\frac{1}{x^2}$　　$u^2=x^2+\frac{1}{x^2}-2$.

∴$\int_0^{+\infty}\frac{1+x^2}{1+x^4}dx=\int_0^{+\infty}\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx=\int_{-\infty}^{+\infty}\frac{du}{u^2+2}=2\int_0^{+\infty}\frac{du}{u^2+2}=\lim\limits_{A\to+\infty}\sqrt{2}\int_0^A\frac{d(\frac{u}{\sqrt{2}})}{(\frac{u}{\sqrt{2}})^2+1}=\lim\limits_{A\to+\infty}\sqrt{2}\arctan$

$=\lim\limits_{A\to+\infty}\sqrt{2}\arctan\frac{A}{\sqrt{2}}=\sqrt{2}\cdot\frac{\pi}{2}=\frac{\pi}{\sqrt{2}}$. ∴该广义积分收敛.其值为$\frac{\pi}{\sqrt{2}}$.

(5). $\int_0^{+\infty}x\sin x dx=\lim\limits_{A\to+\infty}\int_0^A x\sin x dx=\lim\limits_{A\to+\infty}(\sin x-x\cos x)\Big|_0^A=\lim\limits_{A\to+\infty}(\sin A-A\cdot\cos A)$.

由于$\lim\limits_{A\to+\infty}\sin A$. $\lim\limits_{A\to+\infty}\cos A$不存在. 故广义积分发散.

(6). 令$t=\sqrt{x-1}$. 则 $x=t^2+1$ (∵$x\geq 4>0$) ∴x对应$4,+\infty$时. t对应$\sqrt{3},+\infty$,且$2t\,dt=dx$

∴$\int_4^{+\infty}\frac{dx}{x\sqrt{x-1}}=\int_{\sqrt{3}}^{+\infty}\frac{2t\,dt}{(t^2+1)\cdot t}=2\int_{\sqrt{3}}^{+\infty}\frac{dt}{t^2+1}=2\cdot\lim\limits_{A\to+\infty}(\arctan t\Big|_{\sqrt{3}}^A)=2(\frac{\pi}{2}-\frac{\pi}{3})=\frac{\pi}{3}$

∴该广义积分发散.

(7) 令$x=\tan\theta$. 则θ对应上下限为 $0,\frac{\pi}{2}$. ∴$\arctan x=\theta$. $\sqrt{1+x^2}=\frac{1}{\cos\theta}$.　$dx=\frac{1}{\cos^2\theta}d\theta$

∴$\int_0^{+\infty}\frac{\arctan x}{(1+x^2)^{3/2}}dx=\int_0^{\frac{\pi}{2}}\cos^3\theta\cdot\theta\cdot\frac{1}{\cos^2\theta}d\theta=\int_0^{\frac{\pi}{2}}\cos\theta\,\theta\,d\theta=\theta\sin\theta+\cos\theta\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{2}-1$

(8) $\int_{-\infty}^{+\infty}\frac{dx}{x^2+2x+2}=\int_{-\infty}^{+\infty}\frac{d(x+1)}{(x+1)^2+1}\xlongequal{t=x+1}\int_{-\infty}^{+\infty}\frac{dt}{t^2+1}=\int_{0}^{+\infty}\frac{dt}{t^2+1}+\int_{-\infty}^{0}\frac{dt}{t^2+1}$

$=\lim_{A\to+\infty}\arctan t\Big|_0^A+\lim_{B\to-\infty}\arctan t\Big|_B^0=\frac{\pi}{2}+[-(-\frac{\pi}{2})]=\pi$

∴该广义积分收敛.其值为π.

(9). ∵$\int_0^{+\infty}e^{-x}\cos x\,dx=-\int_0^{+\infty}\cos x\,de^{-x}=-\cos x\cdot e^{-x}\Big|_0^A(A\to+\infty)+\int_0^{+\infty}e^{-x}d\cos x$ …⊛

而 $\int_0^{+\infty}e^{-x}d\cos x=-\int_0^{+\infty}e^{-x}\sin x\,dx=\int_0^{+\infty}\sin x\,de^{-x}=\lim_{B\to+\infty}e^{-x}\sin x\Big|_0^B-\int_0^{+\infty}e^{-x}\cos x\,dx$

代入⊛式得 $\int_0^{+\infty}e^{-x}\cos x\,dx=\frac{1}{2}\lim_{A\to+\infty}(e^{-x}\sin x-e^{-x}\cos x)\Big|_0^A=\frac{1}{2}[\lim_{A\to+\infty}\frac{1}{e^A}(\sin A-\cos A)+1]$

$=\frac{1}{2}$. ∴该广义积分收敛.其值为$\frac{1}{2}$.

(10)令$x=\sin t$. 则x为$-1,1$时, t对应值为$-\frac{\pi}{2},\frac{\pi}{2}$. ∴$\sqrt{1-x^2}=\cos t$. $dx=\cos t\,dt$

∴$\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}}=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos t\,dt}{\cos t}=t\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\pi$. 故广义积分收敛,其值为$\pi$.

(11).令$t=\ln x$. 则x为0, $1/2$时. t对应$-\infty$, $\ln\frac{1}{2}$ $dt=\frac{1}{x}dx$

∴$\int_0^{1/2}\frac{dx}{x\ln x}=\int_{-\infty}^{\ln\frac{1}{2}}\frac{dt}{t}=\lim_{A\to-\infty}(\ln|t|)\Big|_A^{\ln\frac{1}{2}}=\ln(\ln\frac{1}{2})-\lim_{A\to-\infty}\ln|A|$.不存在.

∴该广义积分发散.

(12).令$t=\ln x$. 则$\int_0^{1/2}\frac{dx}{x\ln^2x}=\int_{-\infty}^{\ln\frac{1}{2}}\frac{dt}{t^2}=\lim_{A\to-\infty}(-\frac{1}{t})\Big|_A^{-\ln 2}=\frac{1}{\ln 2}-\lim_{A\to-\infty}(-\frac{1}{A})=\frac{1}{\ln 2}$

∴该广义积分收敛.其值为$\frac{1}{\ln 2}$

(13).令$x=\frac{1}{t}$. 则 $\int_0^{+\infty}\frac{dx}{1+x^4}=\int_{+\infty}^{0}(-\frac{t^2}{t^4+1})dt=\int_0^{+\infty}\frac{t^2}{t^4+1}dt=\int_0^{+\infty}\frac{x^2}{1+x^4}dx$

∴$\int_0^{+\infty}\frac{dx}{1+x^4}=\frac{1}{2}(\int_0^{+\infty}\frac{dx}{1+x^4}+\int_0^{+\infty}\frac{x^2}{1+x^4}dx)=\frac{1}{2}\int_0^{+\infty}\frac{1+x^2}{1+x^4}dx=\frac{1}{2}\cdot\frac{\pi}{\sqrt{2}}=\frac{\pi}{2\sqrt{2}}$

∴该广义积分收敛.其值为$\frac{\pi}{2\sqrt{2}}$

2.证明: (1)令$t=\frac{x-a}{\sqrt{2}\sigma}$. 则原式$=\frac{2\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{+\infty}t^2e^{-t^2}dt=\frac{\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{+\infty}-t\,de^{-t^2}$

$=\frac{\sigma^2}{\sqrt{\pi}}(\lim_{A\to+\infty,B\to-\infty}-t\cdot e^{-t^2}\Big|_B^A+\int_{-\infty}^{+\infty}e^{-t^2}dt)=\frac{\sigma^2}{\sqrt{\pi}}\cdot2\int_0^{+\infty}e^{-t^2}dt=\frac{\sigma^2}{\sqrt{\pi}}\cdot2\cdot\frac{\sqrt{\pi}}{2}=\sigma^2$

证毕.

(2).令$t=\frac{x-a}{\sqrt{2}\sigma}$. 则. $x=\sqrt{2}\sigma t+a$. $dx=\sqrt{2}\sigma dt$

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原式 $= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty}(\sqrt{2}\sigma t+a)e^{-t^2}\sqrt{2}\sigma dt = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}(\sqrt{2}\sigma\cdot t\cdot e^{-t^2}+ae^{-t^2})dt$

$= \frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-t^2}dt^2+\frac{a}{\sqrt{\pi}}\cdot 2\int_0^{+\infty}e^{-t^2}dt = 0+\frac{a}{\sqrt{\pi}}\cdot 2\cdot\frac{\sqrt{\pi}}{2}=a.$ 证毕

3.解 (1) 当 $x\ge 1$ 时. $\sqrt[3]{x^4+3}>x$. $\therefore 0<\frac{1}{x^2+\sqrt[3]{x^4+3}}<\frac{1}{x^2+x}=\frac{1}{x}-\frac{1}{x+1}$.

下面证明 $\int_1^{+\infty}(\frac{1}{x}-\frac{1}{x+1})dx$ 收敛. 从而原积分收敛

$\because \int_1^{+\infty}(\frac{1}{x}-\frac{1}{1+x})dx=\lim_{A\to+\infty}(\ln\frac{x}{1+x})\Big|_1^A=\lim_{A\to+\infty}\ln\frac{1}{1+\frac{1}{A}}-\ln\frac{1}{2}=-\ln\frac{1}{2}$.

$\therefore \int_1^{+\infty}(\frac{1}{x}-\frac{1}{x+1})dx$ 收敛. 故原积分收敛.

(2). 当 $x\ge 2$ 时 $\sqrt[3]{x^2+1}\le x$. $\therefore \frac{1}{2x+\sqrt[3]{x^2+1}+6}\ge\frac{1}{3x+6}=\frac{1}{3}\frac{1}{x+2}>0$

而 $\int_1^{+\infty}\frac{1}{3}\frac{1}{x+2}dx=\frac{1}{3}\lim_{A\to+\infty}\ln(x+2)\Big|_1^A$ 发散. 故原积分发散.

(3). [redacted]

令 $t=\sqrt[12]{x}$. 则 x 对应 $0, 2$ 时 t 对应 $0, \sqrt[12]{2}$. $\therefore dx=12t^{11}dt$. $\sqrt[3]{x}=t^4$, $\sqrt[4]{x}=t^3$, $x^3=t^{36}$

$\therefore \int_0^2\frac{dx}{\sqrt[3]{x}+3\sqrt[4]{x}+x^3}=\int_0^{\sqrt[12]{2}}\frac{12t^{11}}{t^4+3t^3+t^{36}}dt=\int_0^{\sqrt[12]{2}}\frac{12t^8}{t+3+t^{33}}dt<\int_0^{\sqrt[12]{2}}\frac{12t^8}{t}dt=\frac{12}{8}t^8\Big|_0^{\sqrt[12]{2}}=\frac{3}{2}\cdot 2^{\frac{2}{3}}$

故原积分收敛.

(4). $\lim_{x\to 1^-}\frac{1}{\sqrt[3]{1-x^4}}/\frac{1}{\sqrt[3]{1-x}}=\lim_{x\to 1^-}\frac{1}{\sqrt[3]{(1+x^2)(1+x)}}=\frac{1}{\sqrt[3]{4}}>0$.

且瑕积分 $\int_0^1\frac{1}{\sqrt[3]{1-x}}dx=-\int_0^1\frac{1}{\sqrt[3]{1-x}}d(1-x)=-\frac{3}{2}(1-x)^{\frac{2}{3}}\Big|_0^1=\frac{3}{2}$. 收敛.

故瑕积分 $\int_0^1\frac{dx}{\sqrt[3]{1-x^4}}$ 也收敛.

(5). $\because \left|\frac{\sin x}{x^{3/2}}\right|=\left|\frac{\sin x}{x}\cdot\frac{1}{x^{1/2}}\right|<M\cdot\frac{1}{x^{1/2}}$. ($\because \frac{\sin x}{x}$ 有界)

而又 $\int_0^1\frac{dx}{x^{1/2}}$ 收敛. 故阿贝尔判别法知 $\int_0^1\frac{\sin x}{x^{3/2}}dx$ 收敛.

(6). 对任意 $A\ge 0$, 有 $\left|\int_1^A\sin x dx\right|=|\cos 1-\cos A|\le 2$. 即 $\int_1^A\sin x dx$ 有界.

而 $\frac{1}{x}$ 在 $[1,+\infty)$ 上单调下降且趋于0 ($x\to+\infty$ 时). 由狄利克雷判别法即知 $\int_1^{+\infty}\frac{\sin x}{x}dx$ 收敛

下面证明 $\int_0^1\frac{\sin x}{x}dx$ 收敛. $\because \frac{\sin x}{x}<1$ ($0<x<1$ 时) $\therefore \int_0^1\frac{\sin x}{x}dx<\int_0^1 dx=1$. 收敛.

故 $\int_0^1\frac{\sin x}{x}dx$ 收敛. 综上知 $\int_0^{+\infty}\frac{\sin x}{x}dx=\int_0^1\frac{\sin x}{x}dx+\int_1^{+\infty}\frac{\sin x}{x}dx$ 收敛.

(7) 令 $t=x^2$. 则 $\int_1^{+\infty} x^{\alpha}\cdot e^{-x^2}dx=\int_1^{+\infty}\frac{1}{2}t^{\frac{\alpha-1}{2}}\cdot e^{-t}dt$

当 $0<\alpha<3$ 时. $\frac{\alpha-1}{2}<1$ $\therefore$ $t^{\frac{\alpha-1}{2}}\le t^{\frac{[\alpha]-1}{2}}$ 注意到 $[\alpha]$ 取 0, 1, 2.

且 $\int_1^{+\infty}\frac{1}{2}t^{-\frac{1}{2}}e^{-t}dt$, $\int_1^{+\infty}\frac{1}{2}t^{0}e^{-t}dt$, $\int_1^{+\infty}\frac{1}{2}t^{\frac{1}{2}}e^{-t}dt$ 均收敛.

$\therefore$ 此时 $\int_1^{+\infty}\frac{1}{2}t^{\frac{\alpha-1}{2}}e^{-t}dt$ 收敛. 即原积分收敛.

当 $\alpha=3$ 时. 容易证明 原积分收敛.

当 $\alpha>3$ 时. $\frac{\alpha-1}{2}>1$ $\therefore$ $t^{\frac{\alpha-1}{2}}<t^{\frac{[\alpha]+1-1}{2}}=t^{\frac{[\alpha]}{2}}<t^{n}$. ($n$ 为大于 $[\alpha]$ 的某个正整数).

$\therefore \int_1^{+\infty}\frac{1}{2}t^{\frac{\alpha-1}{2}}\cdot e^{-t}dt<\int_1^{+\infty}\frac{1}{2}t^{n}\cdot e^{-t}dt=-\int_1^{+\infty}\frac{1}{2}t^{n}de^{-t}=-\frac{1}{2}t^{n}\cdot e^{-t}\Big|_1^{+\infty}+\int_1^{+\infty}\frac{1}{2}e^{-t}n\cdot t^{n-1}dt$

容易证明. $-\frac{1}{2}t^{n}e^{-t}\Big|_1^{+\infty}$ 是存在的.

而 $\int_1^{+\infty}\frac{1}{2}ne^{-t}t^{n-1}dt=-\int_1^{+\infty}\frac{1}{2}nt^{n-1}de^{-t}=-\frac{1}{2}nt^{n-1}\cdot e^{-t}\Big|_1^{+\infty}+\int_1^{+\infty}\frac{1}{2}n(n-1)t^{n-2}\cdot e^{-t}dt$

$=\cdots\cdots=-\frac{1}{2}nt^{n-1}\cdot e^{-t}\Big|_1^{+\infty}-\frac{1}{2}n(n-1)t^{n-2}e^{-t}\Big|_1^{+\infty}-\cdots\cdots-\frac{1}{2}(n)!\cdot te^{-t}\Big|_1^{+\infty}+\int_1^{+\infty}\frac{1}{2}n!e^{-t}dt$

收敛. 故原积分收敛

综上述. 积分 $\int_1^{+\infty}x^{\alpha}e^{-x^2}dx$ ($\alpha>0$) 收敛

(8) $\int_0^1\frac{\ln x}{1-x}dx=\lim_{\varepsilon_1\to 0+0}\int_{\varepsilon_1}^{1/2}\frac{\ln x}{1-x}dx+\lim_{\varepsilon_2\to 1-0}\int_{1/2}^{\varepsilon_2}\frac{\ln x}{1-x}dx$

① 先证 $\lim_{\varepsilon_1\to 0+0}\int_{\varepsilon_1}^{1/2}\frac{\ln x}{1-x}dx$ 收敛. $\because$ $\lim_{x\to 0}\frac{\ln x}{1-x}\Big/-\frac{1}{1-x}=\lim_{x\to 0}(-\ln x)\to+\infty$.

而 $\int_0^{1/2}-\frac{1}{1-x}dx=\int_0^{1/2}\frac{1}{x-1}dx=\ln|x-1|\Big|_0^{1/2}=\ln\frac{1}{2}$. 收敛.

$\therefore \int_0^{1/2}\frac{\ln x}{1-x}dx$ 收敛

② 再证 $\int_{1/2}^{1}\frac{\ln x}{1-x}dx$ 收敛.

$\lim_{x\to 1-0}\frac{\ln x}{1-x}\Big/\frac{\ln x}{x}=\lim_{x\to 1-0}\frac{x}{1-x}\to+\infty$. 而 $\int_{1/2}^{1}\frac{\ln x}{x}dx$ 收敛.

故 $\int_{1/2}^{1}\frac{\ln x}{1-x}dx$ 收敛.

综上所述得 $\int_0^1\frac{\ln x}{1-x}dx$ 收敛.

(9). $\int_0^{\pi/2}\frac{dx}{\sin^2x\cdot\cos^2x}=2\int_0^{\pi/2}\frac{dx}{\sin^2x}$ ($\because \int_0^{\pi/2}\frac{dx}{\sin^2x}=\int_0^{\pi/2}\frac{dx}{\cos^2x}$)

而 $\lim_{x\to0}\frac{\frac{1}{\sin^2x}}{\frac{1}{x^2}}=\lim_{x\to0}\left(\frac{x}{\sin x}\right)^2=1$. 且 $\int_0^{\pi/2}\frac{1}{x^2}dx=\lim_{\varepsilon\to0+0}\left(-\frac{1}{x}\right)\Big|_\varepsilon^{\pi/2}\to+\infty$. 发散.

∴ $\int_0^{\pi/2}\frac{dx}{\sin^2x}$ 也发散. 故原积分发散.

(10) 不妨设 $p\ge q\ge0$. $\int_0^{+\infty}\frac{dx}{x^p+x^q}=\int_0^A\frac{dx}{x^p+x^q}+\int_A^{+\infty}\frac{dx}{x^p+x^q}\le\int_0^A\frac{dx}{x^q}+\int_A^{+\infty}\frac{dx}{x^p}$ …… ⊛

只有当 $q<1$ 且 $p>1$ 时. 上式收敛. 下面证明:

假设 $q\ge1$ 或 $p\le1$ 时. 上式收敛. 则当 $q\ge1$ 时 $\int_0^{+\infty}\frac{dx}{x^p+x^q}\ge\int_0^{+\infty}\frac{dx}{2x^q}=\frac12\int_0^A\frac{dx}{x^q}+\frac12\int_A^{+\infty}\frac{dx}{x^q}$ 发散

当 $p\le1$ 时. $\int_0^{+\infty}\frac{dx}{x^p+x^q}\ge\int_0^1\frac{dx}{x^p+x^q}+\int_1^{+\infty}\frac{dx}{x^p+x^q}\ge\int_0^1\frac{dx}{2x^q}+\int_1^{+\infty}\frac{dx}{2x^p}$ 发散.

∴假设不成立. 容易证明当 $q<1$ 且 $p>1$ 时. 原式收敛.

综上述 当 $\max\{p,q\}>1$ 且 $\min\{p,q\}<1$ 时. 原积分收敛. 其余情况原积分均发散.

4解:(1). $\int_0^{+\infty}\frac{\sqrt x\cos x}{x+3}dx=\int_0^1\frac{\sqrt x\cos x}{x+3}dx+\int_1^{+\infty}\frac{\sqrt x\cos x}{x+3}dx\le\int_0^1\sqrt x\cos x\,dx+\int_1^{+\infty}\frac{\sqrt x}{x}|\cos x|\,dx$

$=\int_0^1\sqrt x\cos x\,dx+\int_1^{+\infty}\frac{|\cos x|}{\sqrt x}dx$.

∵ $\int_\varepsilon^1\cos x\,dx=|\sin1-\sin\varepsilon|<2$. 且 $\sqrt x\to0$ ($x\to0^+$时). ∴ $\int_0^1\sqrt x\cos x\,dx$ 收敛.

$\int_1^A|\cos x|\,dx=|\sin A-\sin1|<2$ 且 $\frac{1}{\sqrt x}\to0$ ($x\to+\infty$时) ∴ $\int_1^{+\infty}\frac{|\cos x|}{\sqrt x}dx$ 收敛

故有 $\int_0^{+\infty}\frac{\sqrt x\cos x}{x+3}dx$ 收敛.

而 $\int_0^{+\infty}\left|\frac{\sqrt x\cos x}{x+3}\right|dx\ge\int_0^{+\infty}\frac{\sqrt x}{x+3}dx\overset{t=\sqrt x}{=}\int_0^{+\infty}\frac{t}{t^2+3}\cdot2t\,dt=\int_0^{+\infty}\left(2+\frac{6}{t^2+3}\right)dt$ 发散

综上知. $\int_0^{+\infty}\frac{\sqrt x\cos x}{x+3}dx$ 条件收敛.

(2). ∵ $\left|\int_0^A\cos(3x+2)\,dx\right|=\left|\frac13\sin(3A+2)-\frac13\sin2\right|<2$ 且 $\left|\frac{1}{\sqrt{x^2+1}\sqrt[3]{x+1}}\right|$ 单调并且趋于0 ($x\to+\infty$)

∴ $\int_1^{+\infty}\frac{\cos(3x+2)}{\sqrt{x^2+1}\sqrt[3]{x+1}}$ 绝对收敛.

5. (狄利克雷判别法). 设 $f(x)$ 及 $g(x)$ 在 $(a,b]$ 上有定义, 并考虑瑕积分 $\int_a^b f(x)g(x)dx$ 设对一切 $a<A\le b$. 积分 $\int_A^b f(x)dx$ 有界. 即存在常数 $M>0$ 使 $\left|\int_A^b f(x)dx\right|\le M$, 又设 $g(x)$ 在 $(a,b]$ 上单调趋于零 (当 $x\to a$ 时). 则上述瑕积分收敛.

(阿贝尔判别法). 设 $f(x)$ 与 $g(x)$ 在 $(a,b]$ 上有定义, 并考虑瑕积分 $\int_a^b f(x)\cdot g(x)dx$, 若瑕积

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分$\int_a^b f(x)dx$收敛，且函数$g(x)$在$(a,b]$上单调有界，则瑕积分$\int_a^b f(x)g(x)dx$收敛.

6. 证明，令$t=\frac{1}{x-a}$ 则 $x=\frac{1}{t}+a$. 则x对应a,b时 t对应$+\infty$, $\frac{1}{b-a}$.

$\therefore \int_a^b f(x)dx=\int_{+\infty}^{\frac{1}{b-a}} f(\frac{1}{t}+a)\cdot(-\frac{1}{t^2})dt=\int_{\frac{1}{b-a}}^{+\infty} f(a+\frac{1}{t})\cdot\frac{1}{t^2}dt$.

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2.含参变量的正常积分

1.解: (1). $\lim_{k\to0}\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}=\int_0^{\pi/2}\lim_{k\to0}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}=\int_0^{\pi/2}d\varphi=\frac{\pi}{2}$

(2). $\lim_{k\to1-0}\int_0^{\pi/2}\sqrt{1-k^2\sin^2\varphi}\,d\varphi=\int_0^{\pi/2}\lim_{k\to1-0}\sqrt{1-k^2\sin^2\varphi}\,d\varphi=\int_0^{\pi/2}\sqrt{1-\sin^2\varphi}\,d\varphi=\int_0^{\pi/2}\cos\varphi\,d\varphi=1$

(3) 令 $t=-\alpha+x$. 则 $x=t+\alpha$. ∴ x对应 α, $\alpha+\frac{\pi}{2}$时, t对应 0, $\frac{\pi}{2}$

∴原式 $=\lim_{\alpha\to0}\int_0^{\pi/2}\frac{\sin^2(t+\alpha)}{4+\alpha(t+\alpha)^2}dt=\int_0^{\pi/2}\lim_{\alpha\to0}\frac{\sin^2(t+\alpha)}{4+\alpha(t+\alpha)^2}dt=\int_0^{\pi/2}\frac{\sin^2 t}{4}dt=\frac{\pi}{16}$

(4) 原式 $=\lim_{\alpha\to0}\int_{0+\alpha}^{1+\alpha}\frac{dx}{1+x^2+\alpha^2}=\lim_{\alpha\to0}\int_0^1\frac{dt}{1+(t+\alpha)^2+\alpha^2}=\int_0^1\lim_{\alpha\to0}\frac{dt}{1+\alpha^2+(t+\alpha)^2}=\int_0^1\frac{dt}{1+t^2}=\arctan t\Big|_0^1=$

(5) 原式 $=\int_0^1\lim_{y\to0}\frac{e^x\sin xy}{y+1}dx=\int_0^1\frac{e^x\sin0}{1}dx=\int_0^1 0\,dx=0$

2.解: (1). $g'(y)=\int_{a-ky}^{a+ky}f_y(x)dx+f(a+ky)\cdot(a+ky)'-f(a-ky)\cdot(a-ky)'=0+kf(a+ky)+kf(a-ky)$

$=k(f(a+ky)+f(a-ky))$

(2). $g'(y)=\int_{\sin y}^{\cos y}f_y(x,y)dx+f(\cos y,y)\cdot(\cos y)'-f(\sin y,y)\cdot(\sin y)'$

$=\int_{\sin y}^{\cos y}\sqrt{1-x^2}\cdot e^{y\sqrt{1-x^2}}dx+e^{y\cdot\sin y}\cdot(-\sin y)-e^{y\cos y}\cdot\cos y$

$=\int_{\sin y}^{\cos y}\sqrt{1-x^2}\cdot e^{y\sqrt{1-x^2}}dx-\sin y\cdot e^{y\cdot\sin y}-\cos y\cdot e^{y\cos y}$

(3) $g'(y)=\int_0^y f_y(x,y)dx+f(y,y)\cdot1-f(0,y)\cdot0$.

$=\int_0^y\left(\frac{x}{1+xy}\cdot\frac{1}{x}\right)dx+\frac{\ln(1+y^2)}{y}=\int_0^y\frac{1}{1+xy}dx+\frac{\ln(1+y^2)}{y}$

$=\frac{1}{y}\ln(1+y^2)+\frac{\ln(1+y^2)}{y}=\frac{2}{y}\cdot\ln(1+y^2)$

(4) $g'(y)=\int_0^{y^2}f_y(x,y)dx+f(y^2,y)\cdot2y-f(0,y)\cdot0$

$=\int_0^{y^2}2y\cdot\cos(x^2+y^2)dx+\sin(y^4+y^2)\cdot2y=2y\cdot\sin(y^4+y^2)+2y\int_0^{y^2}\cos(x^2+y^2)dx$

3.解 (1).

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