第七章 带电粒子和电磁场的相互作用

（1）已知粒子运动求电磁场

（2）粒子激发电磁场对粒子的反作用

（3）粒子与外电磁场的相互作用

§1 运动带电粒子的势和辐射电磁场

1. 任意运动带电粒子的势
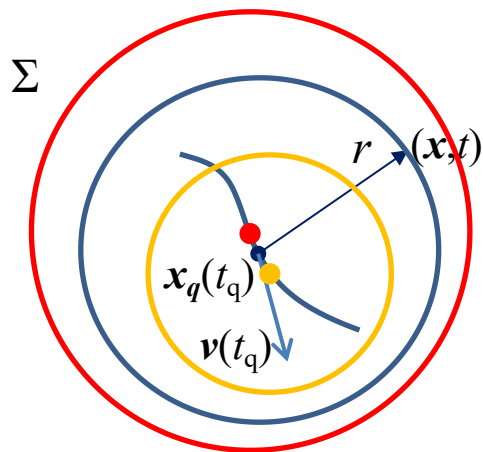

图1. 电荷运动轨迹x_q和场点(x,t).
红圈、蓝圈、黄圈分别表示电荷在红点、
蓝点、黄点激发的电磁场在t时刻所到达的
球面. 三个球面不可能有交点。

电荷在t_q激发的电磁场局限在以$x_q(t_q)$为圆心、以光速扩张的无穷薄球面上

$$r \equiv | x - x_q(t_q) | = c(t - t_q)$$

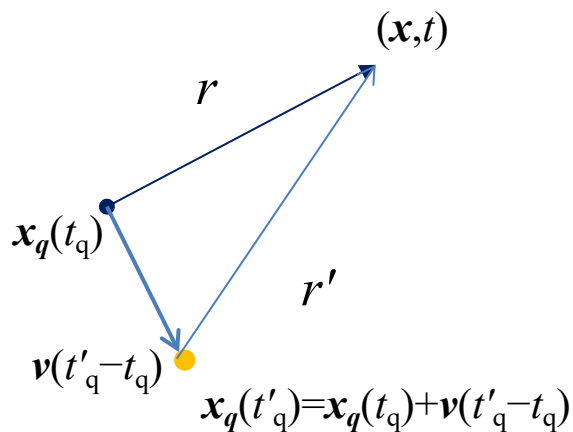因为光速与发光电荷的运动无关，而电荷的速度小于光速，所以三个球面不可能有交点，即对给定的$\boldsymbol{x}$和t只有唯一的t_q满足下式

$$|\boldsymbol{x} - \boldsymbol{x}_q(t_q)| = c(t - t_q) \qquad (1)$$

证明：设 $r = c(t - t_q)$ ，$r' = |\boldsymbol{x} - \boldsymbol{x}_q(t'_q)|$ ， $t'_q > t_q$.

因为三角形两边和大于第三边，

$$r' + v(t'_q - t_q) > r = c(t - t_q)$$

所以

$$r' > c(t - t_q) - v(t'_q - t_q)$$
$$= c(t - t'_q) + (c - v)(t'_q - t_q)$$
$$> c(t - t'_q)$$

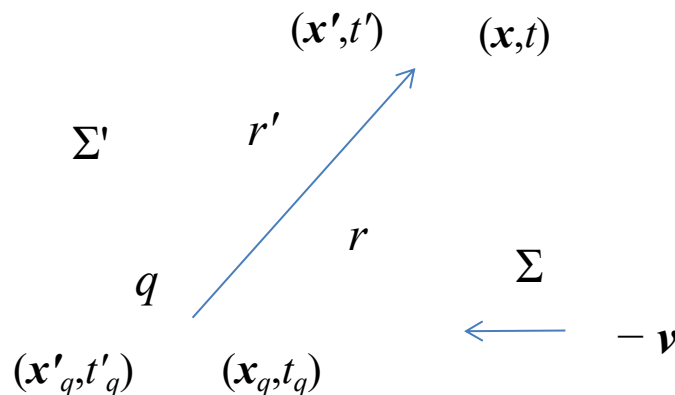所以，$\boldsymbol{x}_q(t'_q)$发出的电磁波不可能在t 时刻达到$\boldsymbol{x}$点。

场方程

$$(\partial_\nu \partial^\nu) A^\mu - \partial^\mu (\partial_\nu A^\nu) = -\mu_0 j^\mu$$

带电物质仅通过四维电流密度与电磁势发生相互作用，所以带电物质激发的电磁势与电荷加速度无关。从而在电荷瞬时静止的惯性系Σ′中，电磁势等于静电荷激发的静电势（已取无穷远处为零的边界条件）

$$A'_a(r') = \left(0,0,0, \frac{i}{c} \frac{q}{4\pi\varepsilon_0 r'} \right) \tag{2}$$

q可以有加速度

(x',t') (x,t)

Σ′ r'

注意：$r' \neq r$

r

Σ

q

$-v$

(x'_q, t'_q) (x_q, t_q)

设在实验室惯性系Σ的t_q时刻电荷具有瞬时速度$\boldsymbol{v}(t_q)$，则Σ相对Σ′以速度$-\boldsymbol{v}(t_q)$运动. 不妨设运动沿X轴方向. 由典型洛伦兹变换得Σ系中的电磁势

$$\begin{pmatrix} A^1 \\ A^2 \\ A^3 \\ \dfrac{i}{c}\varphi \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\dfrac{iv}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \dfrac{iv}{c}\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dfrac{i}{c}\dfrac{q}{4\pi\varepsilon_0 r'} \end{pmatrix} = \gamma(v) \begin{pmatrix} \dfrac{qv}{4\pi\varepsilon_0 c^2 r'} \\ 0 \\ 0 \\ \dfrac{i}{c}\dfrac{q}{4\pi\varepsilon_0 r'} \end{pmatrix} \tag{3}$$

其中 $v = v(t_{\mathrm{q}})$，和 $\quad \gamma(v) = \dfrac{1}{\sqrt{1-v^2(t_q)/c^2}}$

$$t' - t'_q = \frac{t - t_q - \dfrac{v}{c^2}(x^1 - x_q^1)}{\sqrt{1-(v/c)^2}}$$

根据光速不变原理，在Σ′系源点与场点的距离

$$r' = c(t' - t_q') = \frac{c(t - t_q) - \frac{1}{c}v(x^1 - x_q^1)}{\sqrt{1 - v^2/c^2}} = \frac{r - \frac{1}{c}v(x^1 - x_q^1)}{\sqrt{1 - v^2/c^2}}$$

通过空间转动，可以从电荷沿X轴运动的结果得到电荷任意运动的结果。

存在空间转动三维正交矩阵 $\hat{O}$， $\hat{O}\hat{O}^T = \hat{O}^T\hat{O} = I$，使得

$$\begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} \xrightarrow{\hat{O}} \begin{pmatrix} v''^1 \\ 0 \\ 0 \end{pmatrix} = \hat{O}\begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix}, \qquad \begin{pmatrix} x^1 - x_q^1 \\ x^2 - x_q^2 \\ x^3 - x_q^3 \end{pmatrix} \xrightarrow{\hat{O}} \begin{pmatrix} x''^1 - x_q''^1 \\ 0 \\ 0 \end{pmatrix} = \hat{O}\begin{pmatrix} x^1 - x_q^1 \\ x^2 - x_q^2 \\ x^3 - x_q^3 \end{pmatrix}$$

从而

$$r' = \frac{r - \frac{1}{c}v''^1(x''^1 - x_q''^1)}{\sqrt{1 - v^2/c^2}} = \frac{r - \frac{1}{c}\boldsymbol{v}^T\hat{O}^T\hat{O}\boldsymbol{r}}{\sqrt{1 - v^2/c^2}} = \frac{r - \frac{1}{c}\boldsymbol{v}^T\boldsymbol{r}}{\sqrt{1 - v^2/c^2}} = \frac{r - \frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{r}}{\sqrt{1 - v^2/c^2}} \qquad (4)$$

其中 $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}_q$和$r = \sqrt{(\boldsymbol{r}\cdot\boldsymbol{r})} = c(t - t_q)$. 右边表达式在空间坐标转动下不变，因此它对任意选取的笛卡尔坐标架普遍地成立.

同理可以通过空间转动，从电荷瞬时沿X轴运动的结果(3)式和(4)得到任意运动电荷辐射的电磁势。

李纳-维谢尔势

$$A = \frac{q\boldsymbol{v}}{4\pi\varepsilon_0 c^2 (r - \frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{r})} \tag{5a}$$

$$\varphi = \frac{q}{4\pi\varepsilon_0 (r - \frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{r})} \tag{5b}$$

复杂性包含在$\boldsymbol{v}$和$\boldsymbol{r}$中，他们均依赖于t_q，而后者是$\boldsymbol{x}$和t 的隐函数。所以求$\boldsymbol{E}$和$\boldsymbol{B}$时涉及隐函数求导。

2. 连续分布带电体的势(推迟势)

考虑带电粒子集合 $\{q_n | n=1,2,3...\}$，q_n 的轨迹为 $\boldsymbol{x}_n(t')$. 记 $\boldsymbol{r}_n(t')=\boldsymbol{x} - \boldsymbol{x}_n(t')$.

$$r_n \frac{\partial r_n}{\partial t'} = -\frac{d\boldsymbol{x}_n}{dt'} \cdot \boldsymbol{r}_n = -\boldsymbol{v}_n(t') \cdot \boldsymbol{r}_n, \qquad (给定\boldsymbol{x}和t)$$

$$r_n - \tfrac{1}{c}\boldsymbol{v}_n \cdot \boldsymbol{r}_n = r_n + r_n \frac{\partial r_n}{\partial t'} = r_n \frac{\partial}{\partial t'}\left(t' - t + \tfrac{1}{c}r_n\right) \qquad (6)$$

设 q_n 在 t_n 时刻发射的电磁波在 t 时刻达到场点 $\boldsymbol{x}$ 即 $r_n(t_n) = c(t-t_n)$.

可利用(6)式把 q_n 激发的李纳-维谢尔矢势写成

$$A_n(\boldsymbol{x},t) = \frac{q_n \boldsymbol{v}_n(t_n)}{4\pi\varepsilon_0 c^2(r_n - \tfrac{1}{c}\boldsymbol{v}_n(t_n)\cdot \boldsymbol{r}_n)} = \frac{q_n \boldsymbol{v}_n(t_n)}{4\pi\varepsilon_0 c^2 r_n \frac{\partial}{\partial t_n}(t_n - t + \tfrac{1}{c}r_n(t_n))} \qquad (7)$$

为了方便对 n 求和，改写(7)式（目的是消去分母对 n 的依赖），

$$A_n(\boldsymbol{x},t) = \int dV' \int dt' \frac{\delta(t'-t_n)}{\frac{\partial}{\partial t_n}(t_n - t + \tfrac{1}{c}r_n(t_n))} \frac{\delta(\boldsymbol{x}'-\boldsymbol{x}_n(t'))q_n \boldsymbol{v}_n(t')}{4\pi\varepsilon_0 c^2 |\boldsymbol{x} - \boldsymbol{x}'|} \qquad (8)$$

记：$f(t') = t' - t + r_n(t')/c$

因为 $r_n(t_n) = c(t - t_n)$，所以$f(t_n) = 0$，即t_n的是方程$f(t') = 0$的根，而且前面已经分析过，这个根是唯一的。从而

$$\delta(f(t')) = \delta\left(\left. \frac{\partial f(t')}{\partial t'} \right|_{t'=t_n} (t' - t_n) \right) = \frac{1}{\left. \frac{\partial f(t')}{\partial t'} \right|_{t'=t_n}} \delta(t' - t_n) \tag{9}$$

利用(9)式并记$r = |\boldsymbol{x} - \boldsymbol{x}'|$，(8)式写成

$$\begin{aligned}
\boldsymbol{A}_n &= \int dV' \int dt' \delta(t' - t + \tfrac{1}{c} r_n(t')) \frac{\delta(\boldsymbol{x}' - \boldsymbol{x}_n(t')) q_n \boldsymbol{v}_n(t')}{4\pi\varepsilon_0 c^2 |\boldsymbol{x} - \boldsymbol{x}'|} \\
&= \int dV' \int dt' \delta(t' - t + \tfrac{1}{c} |\boldsymbol{x} - \boldsymbol{x}'|) \frac{\delta(\boldsymbol{x}' - \boldsymbol{x}_n(t')) q_n \boldsymbol{v}_n(t')}{4\pi\varepsilon_0 c^2 |\boldsymbol{x} - \boldsymbol{x}'|} \\
&= \int dV' \frac{\delta(\boldsymbol{x}' - \boldsymbol{x}_n(t - \tfrac{1}{c} r)) q_n \boldsymbol{v}_n(t - \tfrac{1}{c} r)}{4\pi\varepsilon_0 c^2 r}
\end{aligned} \tag{10}$$

回忆电流密度

$$J(x,t) = \sum_n q_n \delta(x - x_n(t)) v_n(t) \qquad (11)$$

从而总矢势

$$A(x,t) = \sum_n A_n = \int dV' \frac{J(x', t - \frac{1}{c}r)}{4\pi\varepsilon_0 c^2 r} \qquad (12)$$

类似可得标势

$$\varphi(x,t) = \int dV' \frac{\rho(x', t - \frac{1}{c}r)}{4\pi\varepsilon_0 r} \qquad (13)$$

公式(12)和(13)式正是推迟势公式，它们也可以在罗伦芝规范条件下直接从麦克斯韦方程方程组推导出来. 推迟势反映了真空中光信号传播速度恒等于c.

数学准备。记

$$s \equiv r - \frac{1}{c}\boldsymbol{v} \cdot \boldsymbol{r} \tag{14}$$

则(4)式给出

$$r' = \gamma(v)s \tag{15}$$

对给定的$\boldsymbol{x}$和t，t_q 唯一，故存在单值函数$t_q = t_q(\boldsymbol{x},t)$. 可证下面偏导数(附录1），

$$\frac{\partial t_q}{\partial t} = \frac{r}{s}, \qquad \left.\frac{\partial \boldsymbol{r}}{\partial t}\right|_x = -\frac{r}{s}\boldsymbol{v}, \qquad \left.\frac{\partial r}{\partial t}\right|_x = -\frac{\boldsymbol{r} \cdot \boldsymbol{v}}{s}, \tag{16a}$$

$$\frac{\partial t_q}{\partial x^j} = -\frac{r^j}{cs}, \qquad \left.\frac{\partial \boldsymbol{r}}{\partial x^j}\right|_t = \hat{\boldsymbol{e}}_j + \frac{r^j}{cs}\boldsymbol{v}, \qquad \left.\frac{\partial r}{\partial x^j}\right|_t = \frac{r^j}{s} \tag{16b}$$

利用上述数学关系式，计算李纳-维谢尔势对应的场强，可得(附录2，3)

$$E = -\nabla\varphi - \frac{\partial A}{\partial t} = E_c + E_{rad}, \qquad B = \nabla \times A = B_c + B_{rad} \tag{17}$$

非辐射场（最右表达式作了低速近似）

$$E_c = \frac{q}{4\pi\varepsilon_0 r^2} \frac{\hat{e}_r - \frac{1}{c}v}{(1 - \frac{1}{c}v \cdot \hat{e}_r)^3 \gamma^2} \sim \frac{q}{4\pi\varepsilon_0 r^2}\hat{e}_r \tag{18}$$

$$B_c = \frac{q}{4\pi\varepsilon_0 c^2 r^2} \frac{(1 - \frac{v^2}{c^2})v \times \hat{e}_r}{(1 - \frac{1}{c}v \cdot \hat{e}_r)^3} \sim \frac{q}{4\pi\varepsilon_0 c^2 r^2}v \times \hat{e}_r \tag{19}$$

辐射场（最右表达式作了低速近似）

$$E_{rad} = \frac{q}{4\pi\varepsilon_0 c^2 r} \frac{\hat{e}_r \times [(\hat{e}_r - \frac{1}{c}v) \times \dot{v}]}{(1 - \frac{1}{c}v \cdot \hat{e}_r)^3} \sim \frac{q}{4\pi\varepsilon_0 c^2 r}\hat{e}_r \times (\hat{e}_r \times \dot{v}) \tag{20}$$

$$B_{rad} = \frac{q}{4\pi\varepsilon_0 c^3 r} \frac{\dot{v} \times \hat{e}_r + \frac{1}{c}[(\dot{v} \cdot \hat{e}_r)(v \times \hat{e}_r) - (v \cdot \hat{e}_r)(\dot{v} \times \hat{e}_r)]}{(1 - \frac{1}{c}v \cdot \hat{e}_r)^3} = \frac{1}{c}\hat{e}_r \times E_{rad} \sim \frac{q}{4\pi\varepsilon_0 c^3 r}\dot{v} \times \hat{e}_r \tag{21}$$

辐射场的特点：

辐射场强以1/r衰减，而非辐射场强以1/r^2衰减.

与 $\left| \dot{\boldsymbol{v}} \right|$ 成正比

$\boldsymbol{E}_{rad}$ 和 $\boldsymbol{B}_{rad}$ 互相正交并分别与 $\boldsymbol{r}$ 正交（横场。在自由空间的情形）

可直接验证

$$\boldsymbol{B}_{rad} = \frac{1}{c}\hat{\boldsymbol{e}}_r \times \boldsymbol{E}_{rad} \tag{22}$$

§2 高速运动带电粒子的辐射

1. 高速运动带电粒子的辐射功率和角分布

辐射场的能量密度

$$S = E \times H = \frac{1}{\mu_0} E \times B = \frac{1}{\mu_0 c} E \times (\hat{e}_r \times E) = \varepsilon_0 c E^2 \hat{e}_r \qquad (23)$$

辐射场

$$E = \frac{q}{4\pi\varepsilon_0 c^2 r} \frac{\hat{e}_r \times [(\hat{e}_r - \frac{1}{c}v) \times \dot{v}]}{(1 - \frac{1}{c} v \cdot \hat{e}_r)^3} \qquad (24)$$

故 t 时刻 x 处（场点）

$$S = \frac{q^2}{16\pi^2 \varepsilon_0 c^3 r^2} \frac{\left| \hat{e}_r \times [(\hat{e}_r - \frac{1}{c}v) \times \dot{v}] \right|^2}{(1 - \frac{1}{c} v \cdot \hat{e}_r)^6} \hat{e}_r \qquad (25)$$

在源点附近**S**很复杂，所以一般不在源点附近求辐射功率.
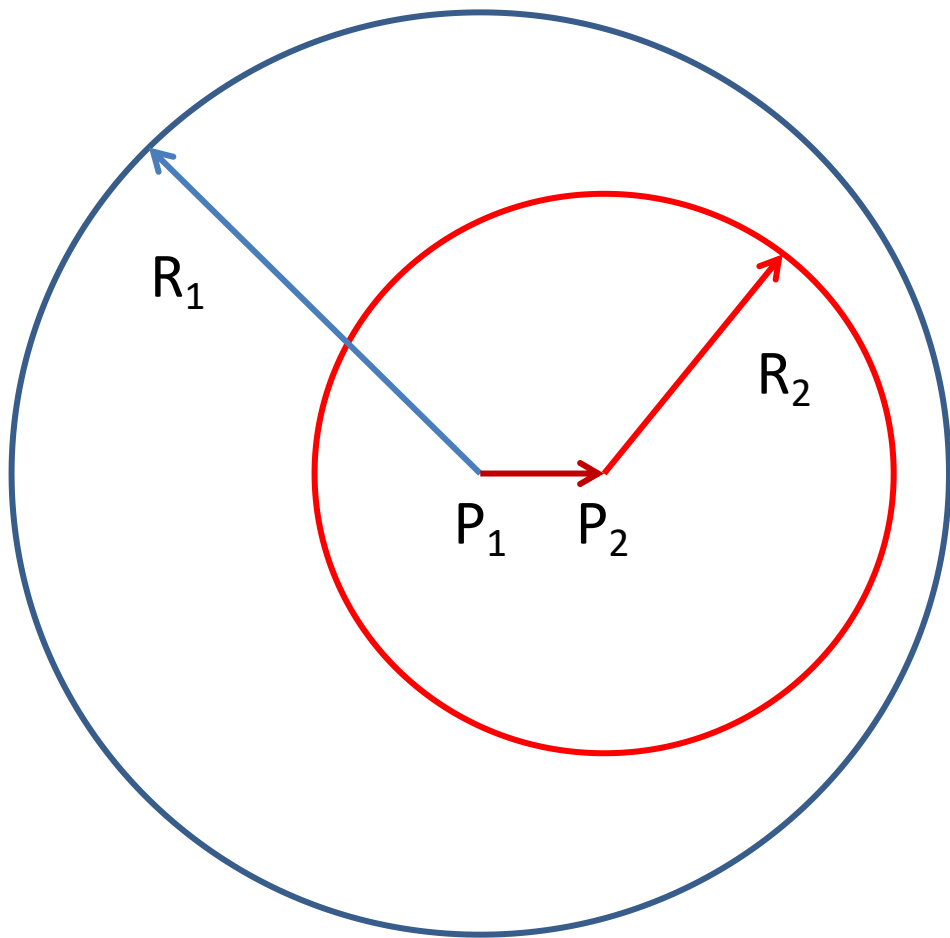
P_1: 电荷在t'时刻所在位置
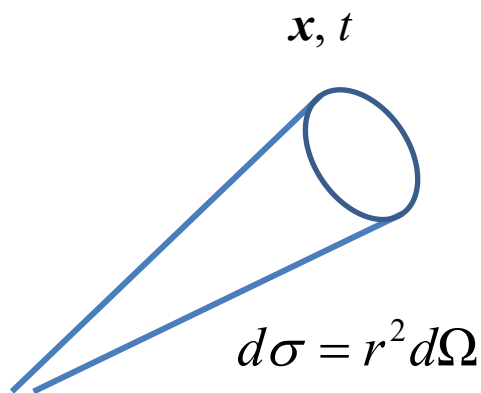
P_2: 电荷在t'+dt'时刻所在位置

在时刻t，P_1处发出的辐射达到半径为R_1的球面

在时刻t，P_2处发出的辐射达到半径为R_2的球面

$$R_1 = c(t - t') \qquad R_2 = c(t - t' - dt')$$

$$\overline{P_1 P_2} = v(t')dt'$$

电荷在dt'时间辐射的能量等于两球面之间的辐射场能量

辐射功率角分布 [方法一]

$\boldsymbol{x}, t$

$d\boldsymbol{\sigma} = r^2 d\Omega$

因为$\boldsymbol{x}_q(t')$已知，从 $|\boldsymbol{x} - \boldsymbol{x}_q(t')| = c(t-t')$ 可解出

$$t = t' + \frac{1}{c}|\boldsymbol{x} - \boldsymbol{x}_q(t')| \equiv t(\boldsymbol{x}, t')$$

电荷在dt'时间产生的辐射经过$\boldsymbol{x}$处面元$d\sigma$的时间为

$$dt = \left.\frac{\partial t}{\partial t'}\right|_{\boldsymbol{x}} dt' = \left(\left.\frac{\partial t'}{\partial t}\right|_{\boldsymbol{x}}\right)^{-1} dt' = (1 - \frac{1}{c}\boldsymbol{v} \cdot \hat{\boldsymbol{e}}_r)dt' \qquad (26)$$

记电荷辐射功率角分布为$P(t', \Omega)$，即在dt'时间内辐射到$\mathrm{d}\Omega$立体角方向的能量为 $P(t', \Omega)dt'd\Omega$ （课本省略了变量Ω). 电荷在dt'辐射到$\mathrm{d}\Omega$的能量在$t\sim t+dt$通过$\boldsymbol{x}$处的$\mathrm{d}\sigma$面元，

$$P(t', \Omega)dt'd\Omega = S dt d\sigma$$

$$= S r^2 (1 - \frac{1}{c}\boldsymbol{v} \cdot \hat{\boldsymbol{e}}_r)dt'd\Omega = \frac{q^2}{16\pi^2 \varepsilon_0 c^3} \frac{\left|\hat{\boldsymbol{e}}_r \times [(\hat{\boldsymbol{e}}_r - \frac{1}{c}\boldsymbol{v}) \times \dot{\boldsymbol{v}}]\right|^2}{(1 - \frac{1}{c}\boldsymbol{v} \cdot \hat{\boldsymbol{e}}_r)^5} dt'd\Omega \qquad (27)$$

辐射功率角分布[方法二]
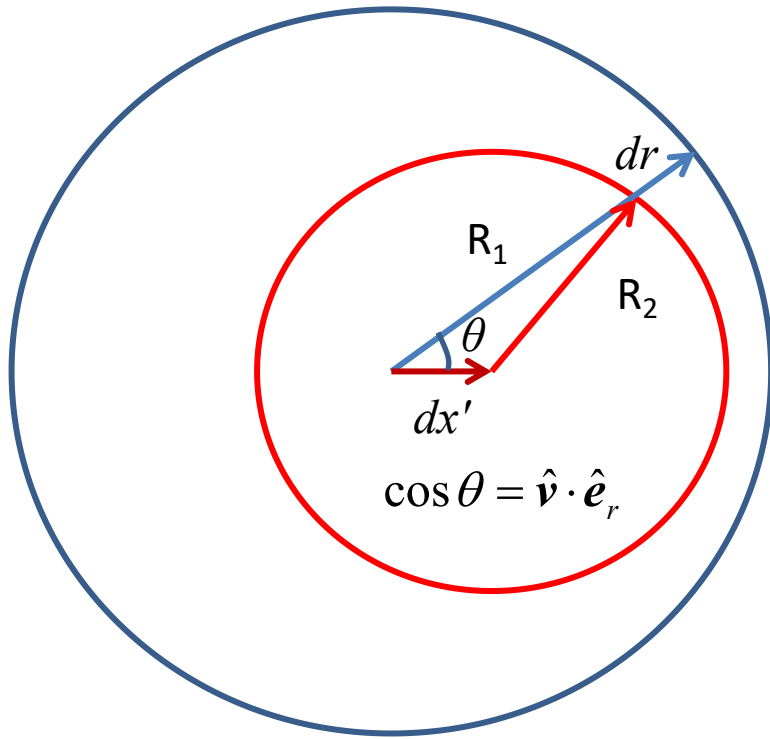
当 $dt' \to 0$ 时，$\quad dx' = vdt'$

设 dt' 辐射到 θ 方向的场到达 $\boldsymbol{x}$ 处厚为 dr 的壳层.

$$R_1 = c(t - t') = r,$$

$$R_2 = c(t - t' - dt') = r - cdt'$$

$$R_2^2 = (R_1 - dr)^2 + dx'^2 - 2(R_1 - dr)dx'\cos\theta$$

从而（只需比较一阶无穷小量）

$$dr = (c - v\cos\theta)dt' = c(1 - \tfrac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)dt'$$

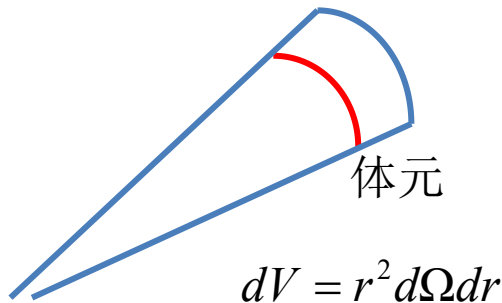在 t 时刻 $\boldsymbol{x}$ 处的辐射能量密度 $w = \varepsilon_0 E^2$

立体角 $d\Omega$ 内 dr 厚体元 dV 的辐射能量即

$$P(t',\Omega)dt'd\Omega = wdV = \varepsilon_0 E^2(\boldsymbol{x},t)r^2 d\Omega dr$$

$$= \varepsilon_0 c E^2(\boldsymbol{x},t)r^2(1 - \tfrac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)dt'd\Omega$$

它等于 dt' 辐射到 $d\Omega$ 的能量.

与(27)一致

体元

$$dV = r^2 d\Omega dr$$

$\cos\theta = \hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{e}}_r$

辐射功率

记电荷辐射功率为$P(t')$.

$$P(t') = \int P(t', \Omega) d\Omega \tag{28}$$

辐射功率角分布为

$$\frac{dP(t')}{d\Omega} \equiv P(t', \Omega) = \frac{q^2}{16\pi^2 \varepsilon_0 c^3} \frac{\left| \hat{\boldsymbol{e}}_r \times [(\hat{\boldsymbol{e}}_r - \frac{1}{c}\boldsymbol{v}) \times \dot{\boldsymbol{v}}] \right|^2}{(1 - \frac{1}{c}\boldsymbol{v} \cdot \hat{\boldsymbol{e}}_r)^5} \tag{29}$$

故辐射功率为

$$P(t') = \frac{q^2}{16\pi^2 \varepsilon_0 c^3} \int \frac{\left| \hat{\boldsymbol{e}}_r \times [(\hat{\boldsymbol{e}}_r - \frac{1}{c}\boldsymbol{v}) \times \dot{\boldsymbol{v}}] \right|^2}{(1 - \frac{1}{c}\boldsymbol{v} \cdot \hat{\boldsymbol{e}}_r)^5} d\Omega \tag{30}$$

高速运动: $v/c \sim 1$

$$1 - \tfrac{1}{c} \boldsymbol{v} \cdot \hat{\boldsymbol{e}}_r = 1 - \tfrac{v}{c} \cos\theta$$

当 $\theta \sim 0$ 时 $1 - \tfrac{1}{c} \boldsymbol{v} \cdot \hat{\boldsymbol{e}}_r \approx \tfrac{1}{2}(\gamma^{-2} + \theta^2)$
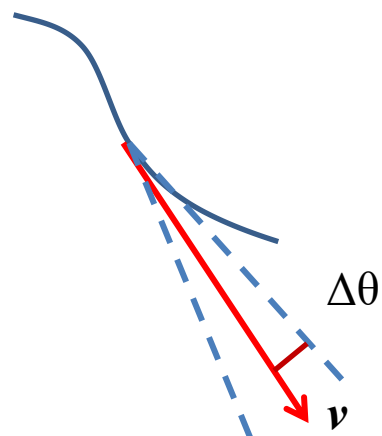
$$\frac{dP(t')}{d\Omega} = \frac{q^2}{16\pi^2 \varepsilon_0 c^3} \frac{\left| \hat{\boldsymbol{e}}_r \times [(\hat{\boldsymbol{e}}_r - \tfrac{1}{c}\boldsymbol{v}) \times \dot{\boldsymbol{v}}] \right|^2}{(1 - \tfrac{1}{c} \boldsymbol{v} \cdot \hat{\boldsymbol{e}}_r)^5}$$
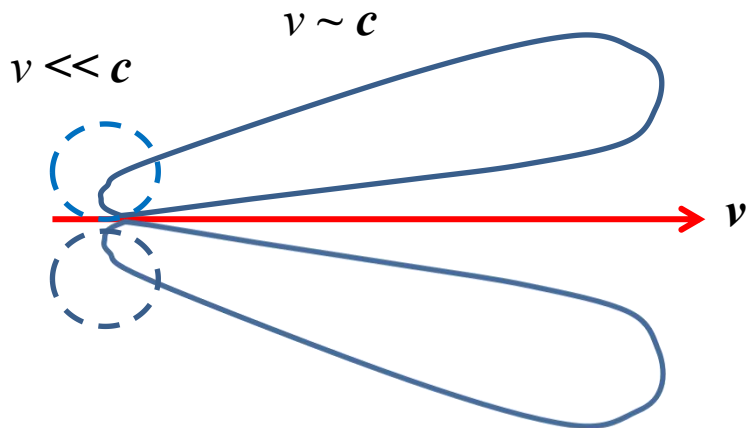
上式分母很小

$$\frac{1}{(1 - \tfrac{1}{c} \boldsymbol{v} \cdot \hat{\boldsymbol{e}}_r)^5} \approx \frac{32}{(\gamma^{-2} + \theta^2)^5} \tag{31}$$

当 $\theta > \gamma^{-1}$，辐射功率迅速减少，从而射功率集中在粒子运动方向（此因子不涉及加速度方向）

$$\Delta\theta \sim \frac{1}{\gamma} \tag{32}$$

2. 加速度平行速度情形

例如直线加速器

辐射功率角分布

$$\frac{dP(t')}{d\Omega} = \frac{q^2 \dot{v}^2}{16\pi^2 \varepsilon_0 c^3} \frac{\sin^2 \theta}{\left(1 - \frac{v}{c}\cos\theta\right)^5} \quad (33)$$

辐射功率

$$P(t') = \frac{q^2 \dot{v}^2}{16\pi^2 \varepsilon_0 c^3} \int \frac{\sin^2 \theta}{\left(1 - \frac{v}{c}\cos\theta\right)^5} \, d\Omega$$

$$= \frac{q^2 \dot{v}^2}{6\pi \varepsilon_0 c^3} \gamma^6 \quad (34)$$

电荷受力（总力）

$$\boldsymbol{F}(t') = \frac{d(\gamma m \boldsymbol{v})}{dt'}$$

$$= m \frac{\dot{\boldsymbol{v}}(1-(\frac{v}{c})^2) + \frac{1}{c^2}(\boldsymbol{v} \cdot \dot{\boldsymbol{v}})\boldsymbol{v}}{(1-(\frac{v}{c})^2)^{3/2}} = \gamma^3 m \dot{\boldsymbol{v}} \tag{35}$$
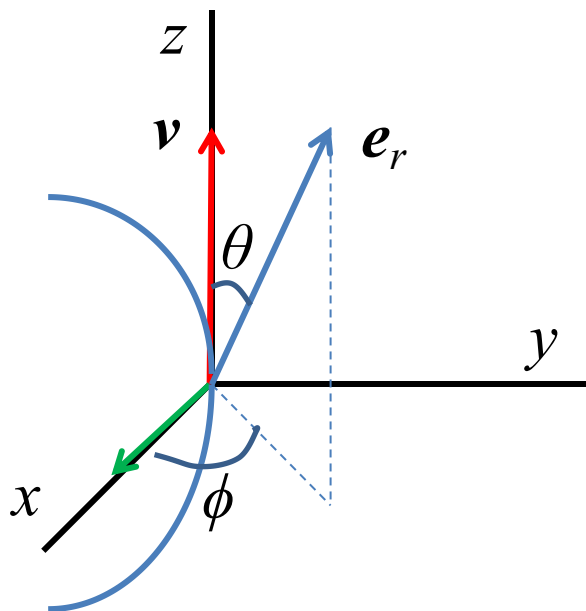
从而

$$P(t') = \frac{q^2}{6\pi\varepsilon_0 m^2 c^3} F^2 \tag{36}$$

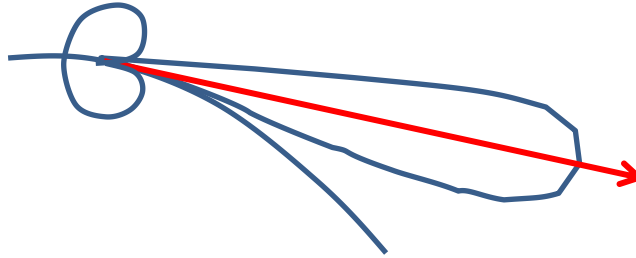　　直线加速运动电荷辐射功率只与其受力有关，而与粒子能量无关，由此可产生高能粒子.

3. 加速度垂直速度情形

例如回旋加速器的同步辐射

$$\boldsymbol{v} \cdot \hat{\boldsymbol{e}}_r = v \cos\theta$$

$$\dot{\boldsymbol{v}} \cdot \hat{\boldsymbol{e}}_r = \dot{v} \sin\theta \cos\phi$$

$$\dot{\boldsymbol{v}} \cdot \boldsymbol{v} = 0$$

$$\hat{\boldsymbol{e}}_r \times [(\hat{\boldsymbol{e}}_r - \tfrac{1}{c}\boldsymbol{v}) \times \dot{\boldsymbol{v}}]$$
$$= \dot{v} \sin\theta \cos\phi (\hat{\boldsymbol{e}}_r - \tfrac{1}{c}\boldsymbol{v}) - (1 - \tfrac{v}{c}\cos\theta)\dot{\boldsymbol{v}}$$

$$\left| \hat{\boldsymbol{e}}_r \times [(\hat{\boldsymbol{e}}_r - \tfrac{1}{c}\boldsymbol{v}) \times \dot{\boldsymbol{v}}] \right|^2$$
$$= \dot{v}^2 \left[(1 - \tfrac{v}{c}\cos\theta)^2 - (1 - \tfrac{v^2}{c^2})\sin^2\theta\cos^2\phi \right]$$

辐射功率角分布

$$\frac{dP(t')}{d\Omega} = \frac{q^2 \dot{v}^2}{16\pi^2 \varepsilon_0 c^3} \frac{(1 - \frac{v}{c}\cos\theta)^2 - (1 - \frac{v^2}{c^2})\sin^2\theta\cos^2\phi}{(1 - \frac{v}{c}\cos\theta)^5} \qquad (37)$$

辐射功率

$$P(t') = \frac{q^2}{16\pi^2 \varepsilon_0 c^3} \int \frac{dP(t')}{d\Omega} d\Omega = \frac{q^2 \dot{v}^2}{6\pi \varepsilon_0 c^3} \gamma^4 \qquad (38)$$

电荷受力（总力）

$$\boldsymbol{F}(t') = \frac{d(\gamma m \boldsymbol{v})}{dt'} = m \frac{\dot{\boldsymbol{v}}(1-(\frac{v}{c})^2) + \frac{1}{c^2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\boldsymbol{v}}{(1-(\frac{v}{c})^2)^{3/2}} = \gamma m \dot{\boldsymbol{v}} \tag{39}$$

辐射功率写成

$$P(t') = \frac{q^2}{6\pi\varepsilon_0 m^2 c^3} \gamma^2 F^2$$

$$= \frac{q^2}{6\pi\varepsilon_0 m^4 c^7} E_q^2 F^2, \tag{40}$$

电荷能量

$$E_q = \gamma m c^2. \tag{41}$$

当F给定后，辐射功率与电荷能量平方成正比.当辐射功率等于输入功率时，粒子能量不能继续增加.但由此产生的"同步辐射"是很有用的.

§3 辐射的频谱分析

1. 频谱分析的一般公式

傅里叶变换

$$f(t) = \int_{-\infty}^{\infty} f_\omega e^{-i\omega t} d\omega \tag{42}$$

$$f_\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt \tag{43}$$

若$f(t)$为实函数，

$$f_\omega^* = f_{-\omega} \tag{44}$$

$$\int_{-\infty}^{\infty} f^2(t) dt = 4\pi \int_0^{\infty} \left| f_\omega \right|^2 d\omega \tag{45}$$

例

（1）电流密度

$$\boldsymbol{J}(\boldsymbol{x},t) = \int_{-\infty}^{\infty} \boldsymbol{J}_\omega(\boldsymbol{x}) e^{-i\omega t} d\omega, \qquad \boldsymbol{J}_\omega(\boldsymbol{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \boldsymbol{J}(\boldsymbol{x},t) e^{i\omega t} dt \qquad (46)$$

（2）推迟势

$$\begin{aligned}
\boldsymbol{A}(\boldsymbol{x},t) &= \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{x}', t-\frac{r}{c})}{r} dV' \\
&= \frac{\mu_0}{4\pi} \int \frac{dV'}{r} \int_{-\infty}^{\infty} \boldsymbol{J}_\omega(\boldsymbol{x}') e^{-i\omega t + i\omega \frac{r}{c}} d\omega \\
&= \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} e^{-i\omega t} d\omega \left[\int \frac{\boldsymbol{J}_\omega(\boldsymbol{x}')}{r} e^{i\omega \frac{r}{c}} dV' \right]
\end{aligned} \qquad (47)$$

可见

$$\boldsymbol{A}_\omega(\boldsymbol{x}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}_\omega(\boldsymbol{x}')}{r} e^{i\omega \frac{r}{c}} dV' = \frac{\mu_0}{8\pi^2} \int e^{i\omega(t'+\frac{r}{c})} dt' \int \frac{\boldsymbol{J}(\boldsymbol{x}',t')}{r} dV' \qquad (48)$$

类似可得

$$\varphi_\omega(\boldsymbol{x}) = \frac{1}{8\pi^2 \varepsilon_0} \int e^{i\omega(t'+\frac{r}{c})} dt' \int \frac{\rho(\boldsymbol{x}',t')}{r} dV' \qquad (49)$$

直接用电流密度计算势的频谱分量更方便.

一般推迟势的频谱分量公式

$$A_\omega(\boldsymbol{x}) = \frac{\mu_0}{8\pi^2}\int e^{i\omega(t'+\frac{r}{c})}dt'\int \frac{\boldsymbol{J}(\boldsymbol{x}',t')}{r}dV', \tag{48}$$

$$\varphi_\omega(\boldsymbol{x}) = \frac{1}{8\pi^2\varepsilon_0}\int e^{i\omega(t'+\frac{r}{c})}dt'\int \frac{\rho(\boldsymbol{x}',t')}{r}dV' \tag{49}$$

电流密度和电荷密度分别为

$$\boldsymbol{J}(\boldsymbol{x}',t') = \rho(\boldsymbol{x}',t')\boldsymbol{v}(t'), \tag{50}$$

$$\rho(\boldsymbol{x}',t') = q\delta(\boldsymbol{x}' - \boldsymbol{x}_q(t')) \tag{51}$$

故

$$A_\omega(\boldsymbol{x}) = \frac{\mu_0}{8\pi^2}\int_{-\infty}^{\infty} \frac{q\boldsymbol{v}(t')}{r} e^{i\omega(t'+\frac{r}{c})}dt' \tag{52}$$
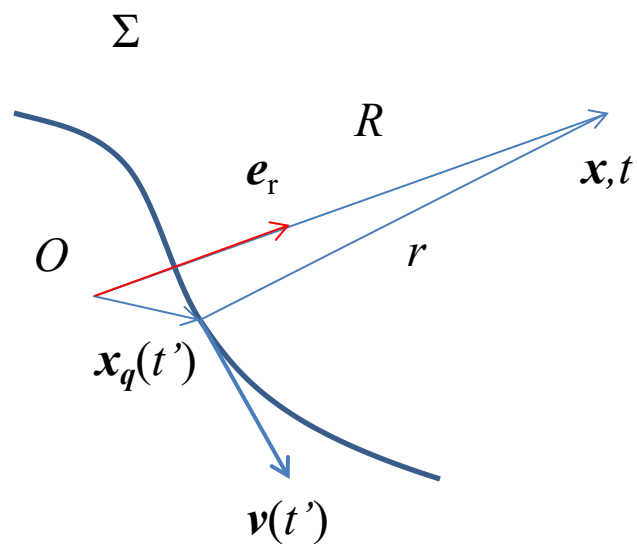
远场近似(如同多极矩展开)

$$r \approx R - \hat{\boldsymbol{e}}_r \cdot \boldsymbol{x}_q(t') \tag{53}$$

$$\boldsymbol{A}_\omega(\boldsymbol{x}) = \frac{\mu_0 q}{8\pi^2} \frac{e^{ikR}}{R} \int_{-\infty}^{\infty} \boldsymbol{v}(t') e^{i\omega(t' - \frac{1}{c}\hat{\boldsymbol{e}}_r \cdot \boldsymbol{x}_q)} dt' \tag{54}$$

其中 $k=\omega/c$ ， $R=|\boldsymbol{x}|$.

由 (54)：
$$\boldsymbol{A}_\omega(\boldsymbol{x}) = \frac{\mu_0 q}{8\pi^2} \frac{e^{ikR}}{R} \int_{-\infty}^{\infty} \boldsymbol{v}(t') e^{i\omega(t' - \frac{1}{c}\hat{\boldsymbol{e}}_r \cdot \boldsymbol{x}_q)} dt' \qquad 可得$$

磁场（不用对分母求导）

$$\boldsymbol{B}_\omega = \nabla \times \boldsymbol{A}_\omega = ik\hat{\boldsymbol{e}}_r \times \boldsymbol{A}_\omega$$

$$= \frac{iq\omega}{8\pi^2 \varepsilon_0 c^3} \frac{e^{ikR}}{R} \int_{-\infty}^{\infty} \hat{\boldsymbol{e}}_r \times \boldsymbol{v}(t') e^{i\omega(t' - \frac{1}{c}\hat{\boldsymbol{e}}_r \cdot \boldsymbol{x}_q)} dt' \tag{55}$$

电场

$$\boldsymbol{E}_\omega = -c\hat{\boldsymbol{e}}_r \times \boldsymbol{B}_\omega$$

$$= -\frac{iq\omega}{8\pi^2 \varepsilon_0 c^2} \frac{e^{ikR}}{R} \int_{-\infty}^{\infty} \hat{\boldsymbol{e}}_r \times (\hat{\boldsymbol{e}}_r \times \boldsymbol{v}) e^{i\omega(t' - \frac{1}{c}\hat{\boldsymbol{e}}_r \cdot \boldsymbol{x}_q)} dt' \tag{56}$$

这两个式子和直接对辐射场$\boldsymbol{E}(t)$和$\boldsymbol{B}(t)$进行傅里叶变换的结果一致.

直接对辐射场 $\boldsymbol{E}(t)$ 进行傅里叶变换:

$$\boldsymbol{E}_\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \boldsymbol{E}_{rad} e^{i\omega t} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \boldsymbol{E}_{rad} e^{i\omega t} \partial_{t'} t dt'$$

由 $\partial_{t'} t = 1 - \frac{1}{c} \boldsymbol{v} \cdot \hat{\boldsymbol{e}}_r$ 和 $\boldsymbol{E}_{rad} = \frac{q}{4\pi\varepsilon_0 c^2 r} \frac{\hat{\boldsymbol{e}}_r \times [(\hat{\boldsymbol{e}}_r - \frac{1}{c}\boldsymbol{v}) \times \dot{\boldsymbol{v}}]}{(1 - \frac{1}{c}\boldsymbol{v} \cdot \hat{\boldsymbol{e}}_r)^3}$ 得

$$\boldsymbol{E}_\omega = \frac{q}{8\pi^2 \varepsilon_0 c^2} \int \frac{\hat{\boldsymbol{e}}_r \times [(\hat{\boldsymbol{e}}_r - \frac{1}{c}\boldsymbol{v}) \times \dot{\boldsymbol{v}}]}{r(1 - \frac{1}{c}\boldsymbol{v} \cdot \hat{\boldsymbol{e}}_r)^2} e^{i\omega t} dt'$$

利用 $\frac{d}{dt'}\left[\frac{\hat{\boldsymbol{e}}_r \times (\hat{\boldsymbol{e}}_r \times \boldsymbol{v})}{1 - \frac{1}{c}\boldsymbol{v} \cdot \hat{\boldsymbol{e}}_r}\right] = \frac{\hat{\boldsymbol{e}}_r \times [(\hat{\boldsymbol{e}}_r - \frac{1}{c}\boldsymbol{v}) \times \dot{\boldsymbol{v}}]}{(1 - \frac{1}{c}\boldsymbol{v} \cdot \hat{\boldsymbol{e}}_r)^2}$ 得

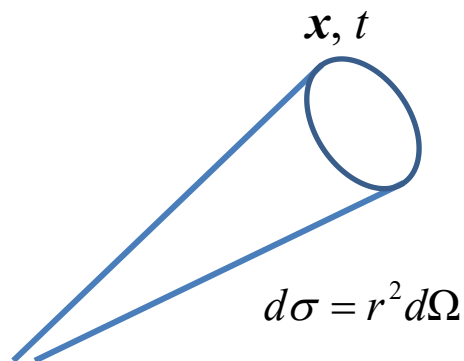$$\boldsymbol{E}_\omega = \frac{q}{8\pi^2 \varepsilon_0 c^2} \int \frac{e^{i\omega t}}{r} \frac{d}{dt'}\left[\frac{\hat{\boldsymbol{e}}_r \times (\hat{\boldsymbol{e}}_r \times \boldsymbol{v})}{1 - \frac{1}{c}\boldsymbol{v} \cdot \hat{\boldsymbol{e}}_r}\right] dt' = -\frac{q}{8\pi^2 \varepsilon_0 c^2} \int \frac{\hat{\boldsymbol{e}}_r \times (\hat{\boldsymbol{e}}_r \times \boldsymbol{v})}{1 - \frac{1}{c}\boldsymbol{v} \cdot \hat{\boldsymbol{e}}_r} \frac{d}{dt'}\left[\frac{e^{i\omega t}}{r}\right] dt'$$

略去 $1/r^2$ 项,

$$\boldsymbol{E}_\omega = -\frac{iq\omega}{8\pi^2 \varepsilon_0 c^2} \int \frac{\hat{\boldsymbol{e}}_r \times (\hat{\boldsymbol{e}}_r \times \boldsymbol{v})}{r} e^{i\omega(t' - \frac{r}{c})} dt'$$

与(56)一致

辐射能量的角分布

$$x, t$$

$$d\sigma = r^2 d\Omega$$

$$\frac{dW}{d\Omega} = \int \frac{dP}{d\Omega} dt = \int_{-\infty}^{\infty} \boldsymbol{S} \cdot \hat{\boldsymbol{e}}_r R^2 dt \tag{57}$$

代入 $\boldsymbol{S} = \varepsilon_0 c E^2 \hat{\boldsymbol{e}}_r$，利用 $\int_{-\infty}^{\infty} f^2(t) dt = 4\pi \int_0^{\infty} |f_\omega|^2 d\omega$

$$\frac{dW}{d\Omega} = \varepsilon_0 c R^2 \int_{-\infty}^{\infty} E^2 dt = 4\pi \varepsilon_0 c R^2 \int_0^{\infty} |E_\omega|^2 d\omega \equiv \int_0^{\infty} \frac{dW_\omega}{d\Omega} d\omega \tag{58}$$

单位频率间隔辐射能量角分布　（角频率在ω）

$$\frac{dW_\omega}{d\Omega} = 4\pi\varepsilon_0 cR^2 \mid E_\omega \mid^2 .$$

(59)

$$\left(\frac{dW_\omega}{d\Omega} = \frac{d^2W}{d\Omega d\omega} \right)$$

对Ω积分得**单位频率间隔辐射能量**

$$W_\omega = 4\pi\varepsilon_0 cR^2 \oint \mid E_\omega \mid^2 d\Omega$$

(60)

先求E_ω，然后通过上面两式进行辐射能量的频谱分析.

电场傅里叶分量（复习）

$$E_\omega = \frac{q}{8\pi^2\varepsilon_0 c^2} \int \frac{\hat{e}_r \times [(\hat{e}_r - \frac{1}{c}\boldsymbol{v}) \times \dot{\boldsymbol{v}}]}{r(1 - \frac{1}{c}\boldsymbol{v}\cdot\hat{e}_r)^2} e^{i\omega t} dt'$$

匀速运动，$\boldsymbol{v}$ 为常矢量，故 $E_\omega = 0$.

远场近似 $r \approx R - \hat{e}_R \cdot \boldsymbol{x}_q(t')$,

$$E_\omega = -\frac{iq\omega}{8\pi^2\varepsilon_0 c^2} \frac{e^{ikR}}{R} \hat{e}_R \times \left[\hat{e}_R \times \int_{-\infty}^{\infty} \boldsymbol{v} e^{i\omega(t' - \frac{1}{c}\hat{e}_R \cdot \boldsymbol{x}_q)} dt' \right] \tag{61}$$

周期直线运动 $\boldsymbol{x}_q(t') = \boldsymbol{x}_{q0}\cos(\omega_0 t')$

$$E_\omega \approx -\frac{q\omega\omega_0}{16\pi^2\varepsilon_0 c^2} \frac{e^{ikR}}{R} \hat{e}_R \times \left[\hat{e}_R \times \boldsymbol{x}_{q0} \int_{-\infty}^{\infty} e^{i(\omega-\omega_0)t' - i\frac{\omega}{c}x_{q0R}\cos(\omega_0 t')} dt' \right] \tag{62}$$

低速， $\omega_0 x_{q0R}/c \ll 1$

$$E_\omega \approx -\frac{q\omega\omega_0}{8\pi\varepsilon_0 c^2} \frac{e^{ikR}}{R} \hat{e}_R \times \left(\hat{e}_R \times \boldsymbol{x}_{q0} \right) \delta(\omega - \omega_0) \tag{63}$$

可见，辐射频率集中在粒子运动的频率附近。

2. 低速运动带电粒子在碰撞过程中的辐射频谱

光子

$\boldsymbol{v}_1$

电子
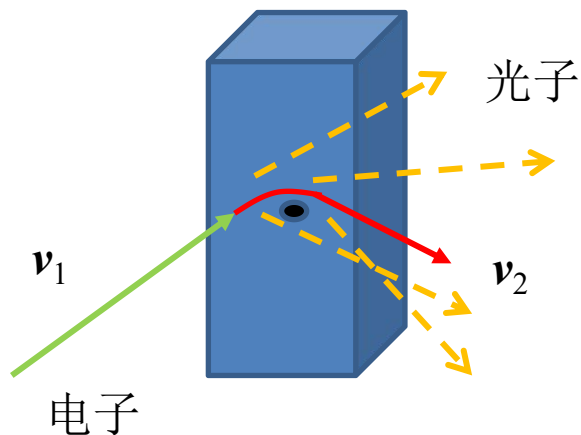
$$E_\omega = \frac{q}{8\pi^2\varepsilon_0 c^2} \int \frac{\hat{\boldsymbol{e}}_r \times [(\hat{\boldsymbol{e}}_r - \frac{1}{c}\boldsymbol{v}) \times \dot{\boldsymbol{v}}]}{r(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^2} e^{i\omega t} dt'$$

设 $v << c$。采用偶极近似

$$r \approx R - \boldsymbol{e}_r \cdot \boldsymbol{x}_q \approx R, \quad t \approx t' + \frac{1}{c}(R - \boldsymbol{e}_r \cdot \boldsymbol{x}_q) \approx t' + \frac{1}{c}R$$

$$E_\omega = \frac{q}{8\pi^2\varepsilon_0 c^2} \frac{e^{ikR}}{R} \int \hat{\boldsymbol{e}}_r \times (\hat{\boldsymbol{e}}_r \times \dot{\boldsymbol{v}}) e^{i\omega t'} dt' \tag{64}$$

轫致辐射, $\Delta\boldsymbol{v} = \boldsymbol{v}_2 - \boldsymbol{v}_1$

物理过程特点： $\dot{\boldsymbol{v}}$ 只在很短的时间 τ 内非零，从而

(1)若 $\omega\tau << 1$，因为 $e^{i\omega t'} \sim 1$, 所以

$$E_\omega \approx \frac{q}{8\pi^2\varepsilon_0 c^2} \frac{e^{ikR}}{R} \hat{\boldsymbol{e}}_r \times (\hat{\boldsymbol{e}}_r \times \Delta\boldsymbol{v}) \tag{65}$$

(2)若 $\omega\tau >> 1$，因为 $e^{i\omega t'}$ 快速振荡，所以

$$E_\omega \approx 0 \tag{67}$$
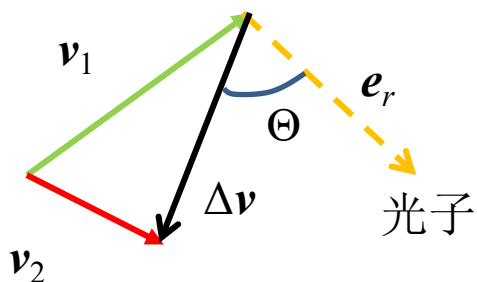
对(1): $\omega\tau \ll 1$ 的情形,

$$|\boldsymbol{E}_\omega| \approx \frac{q}{8\pi^2\varepsilon_0 c^2}\frac{1}{R}|\hat{\boldsymbol{e}}_r\times(\hat{\boldsymbol{e}}_r\times\Delta\boldsymbol{v})|$$

$$=\frac{q}{8\pi^2\varepsilon_0 c^2}\frac{1}{R}|\Delta v\sin\Theta| \qquad (68)$$

单位频率间隔辐射能量角分布为

$$\frac{dW_\omega}{d\Omega}=4\pi\varepsilon_0 cR^2|E_\omega|^2=\frac{q^2}{16\pi^3\varepsilon_0 c^3}|\Delta v|^2\sin^2\Theta \qquad (69)$$

单位频率间隔辐射能量

$$W_\omega=\int\frac{dW_\omega}{d\Omega}d\Omega=\frac{q^2}{6\pi^2\varepsilon_0 c^3}|\Delta v|^2 \qquad (70)$$

与ω无关！

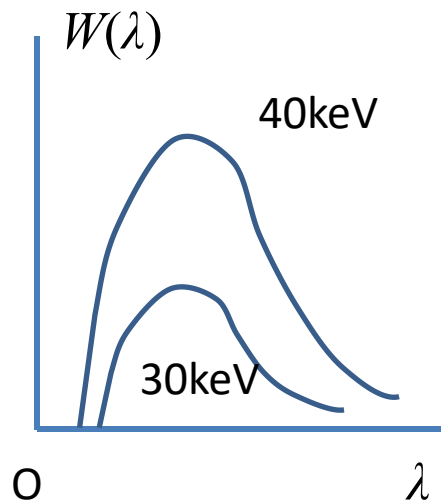# 辐射能量按波长的分布

对(1): $\omega\tau \ll 1$, 有 $c\tau/\lambda \ll 1$.

$$W(\lambda) = W_\omega \left|\frac{d\omega}{d\lambda}\right| = \frac{q^2}{3\pi\varepsilon_0}\left|\frac{\Delta v}{c}\right|^2 \frac{1}{\lambda^2} \qquad (71)$$

对(2): $\omega\tau \gg 1$, 有 $c\tau/\lambda \gg 1$

$$W(\lambda) = 0 \qquad (72)$$
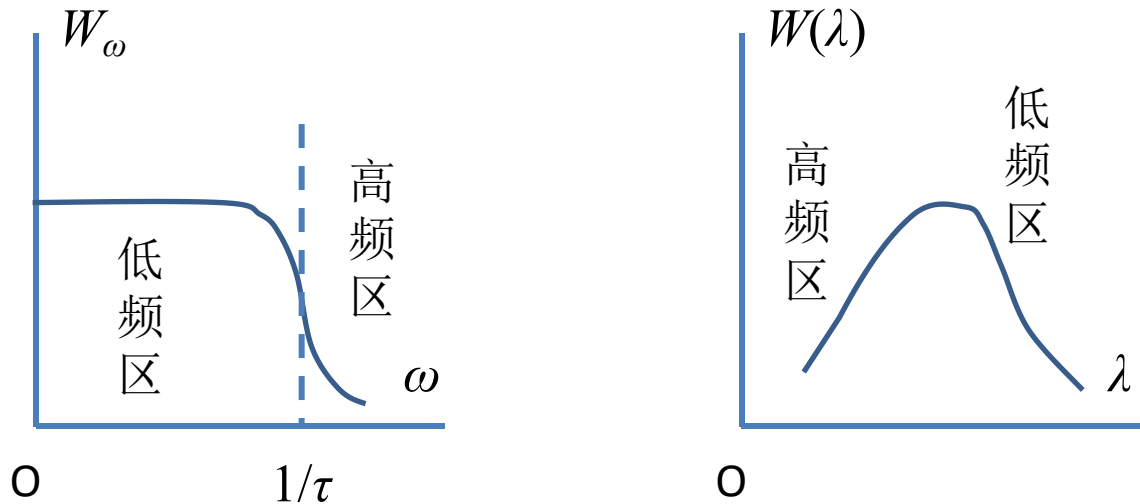
实验

短波区的截止波长是量子效应

$$E_e = \frac{1}{2} m v_1^2 = \hbar \omega_c \tag{73}$$
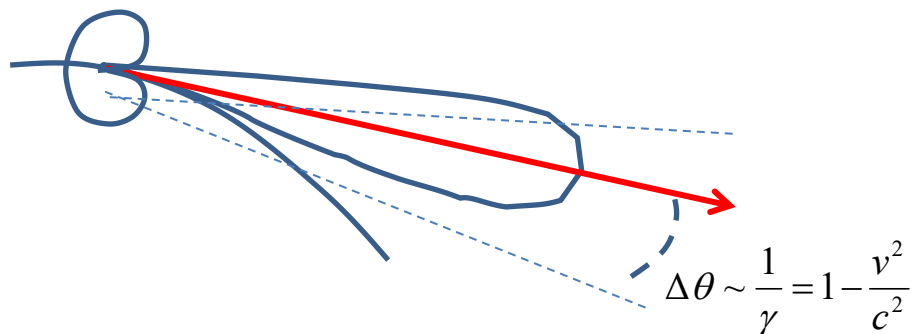
记光子数分布为N(ω)。对 $\omega \tau << 1,$

$$N(\omega)d\omega = \frac{W_\omega}{\hbar \omega} d\omega = \frac{2\alpha}{3\pi} \left(\frac{\Delta v}{c} \right)^2 \frac{d\omega}{\omega}, \tag{74}$$

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = \frac{1}{137.0359895(61)} \tag{75}$$

3. 高速圆周运动带电粒子的辐射频谱

加速度沿垂直速度方向

$$\frac{dP(t')}{d\Omega} = \frac{q^2 \dot{v}^2}{16\pi^2 \varepsilon_0 c^3} \frac{(1 - \frac{v}{c}\cos\theta)^2 - (1 - \frac{v^2}{c^2})\sin^2\theta\cos^2\phi}{(1 - \frac{v}{c}\cos\theta)^5} \tag{76}$$
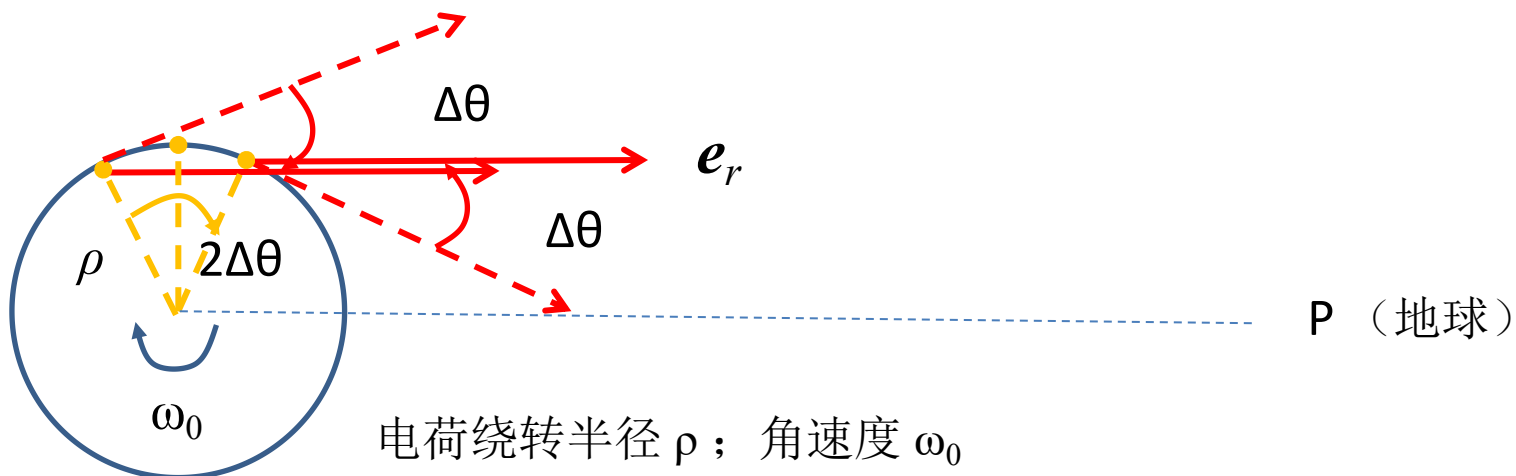
$$\frac{1}{(1 - \frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^5} \approx \frac{32}{(\gamma^{-2} + \theta^2)^5} \tag{77}$$

$$\Delta\theta \sim \frac{1}{\gamma} = 1 - \frac{v^2}{c^2}$$

当 $v \sim c$ 时，$\Delta\theta$ 非常小

高速圆周运动的带电粒子产生的辐射像一个旋转的探照灯一样

中子星

脉冲星转过角度$\Delta\theta$的时间 $\Delta t' = 2\Delta\theta/\omega_0$

收到脉冲星辐射的持续时间$\mathrm{d}t = (\mathrm{d}t/\mathrm{d}t')\,\mathrm{d}t'$, $\qquad \dfrac{dt}{dt'} = 1 - \dfrac{1}{c}\boldsymbol{v}\cdot\boldsymbol{e}_r$

$$\Delta t = \int_0^{\Delta t'} \frac{dt}{dt'}\,dt' = \int_0^{\Delta t'}\left(1 - \frac{v}{c}\cos(\Delta\theta - \omega_0 t')\right)dt'$$

$$= \Delta t' - \frac{2v}{c\omega_0}\sin\frac{\omega_0\Delta t'}{2} \approx \left(1 - \frac{v}{c}\right)\Delta t' \approx \frac{\Delta t'}{2\gamma^2} = \frac{\Delta\theta}{\omega_0\gamma^2} = \frac{1}{\omega_0\gamma^3} \approx \frac{\rho}{c\gamma^3}$$

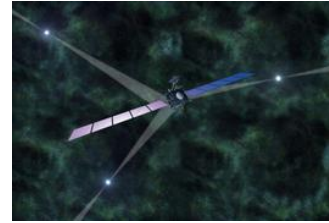$$\Delta t = \frac{1}{\omega_0 \gamma^3} \approx \frac{\rho}{c\gamma^3} \qquad (78)$$

$1/(\omega_0\gamma^3)$

$2\pi/\omega_0$

t
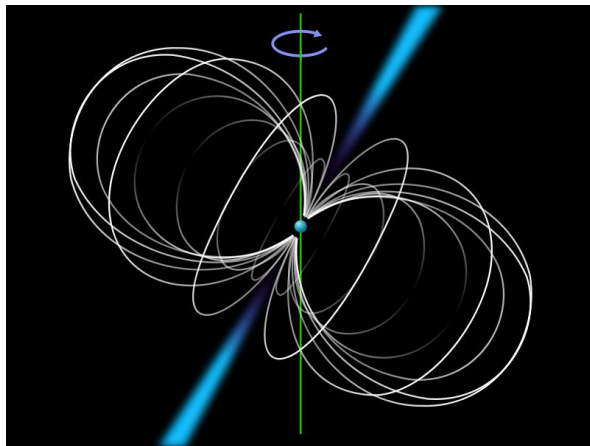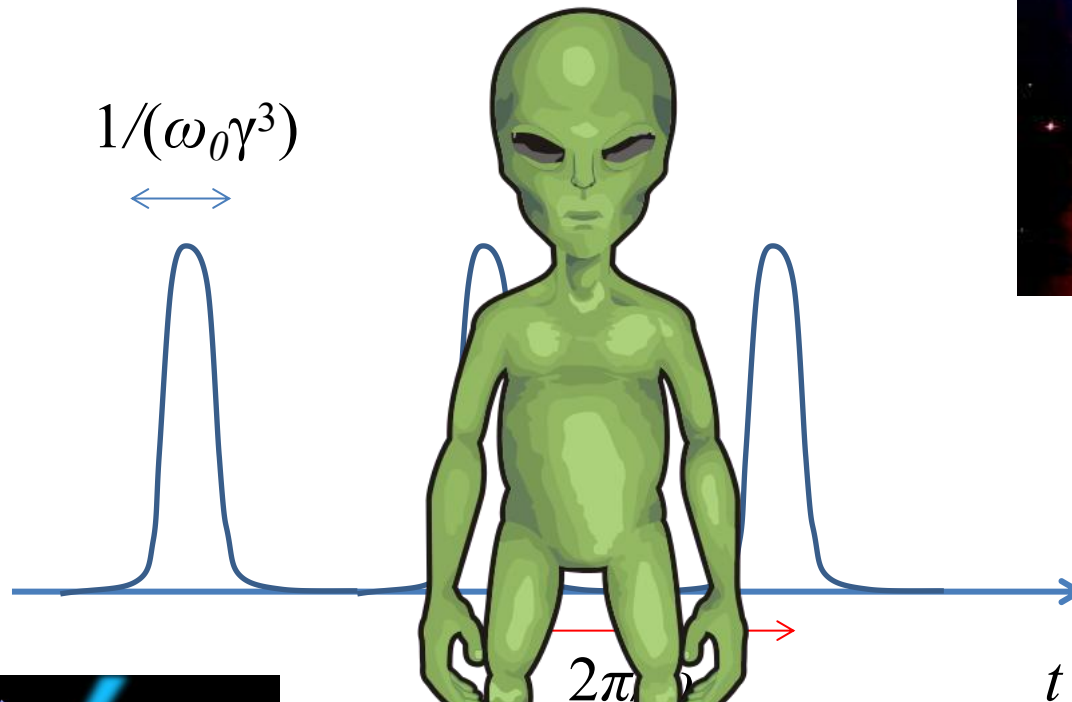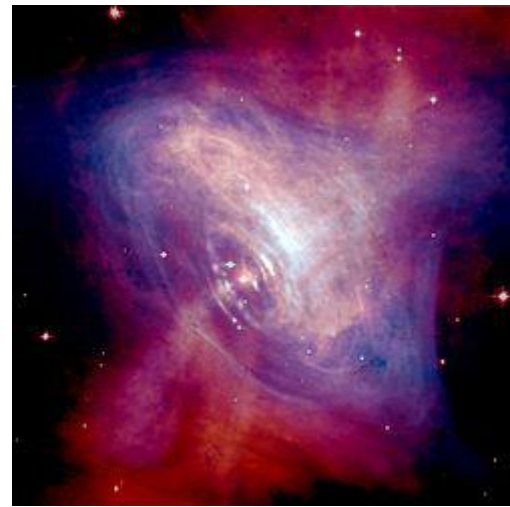
"小绿人"

1967.8 Susan Jocelyn Bell Burnell

中国于2016年11月发射首颗脉冲星
导航试验卫星（XPNAV－1）

"中国天眼" 500米口径球面射电望
远镜（FAST）

上海光源是一台高性能的中能第三代同步辐射光源，
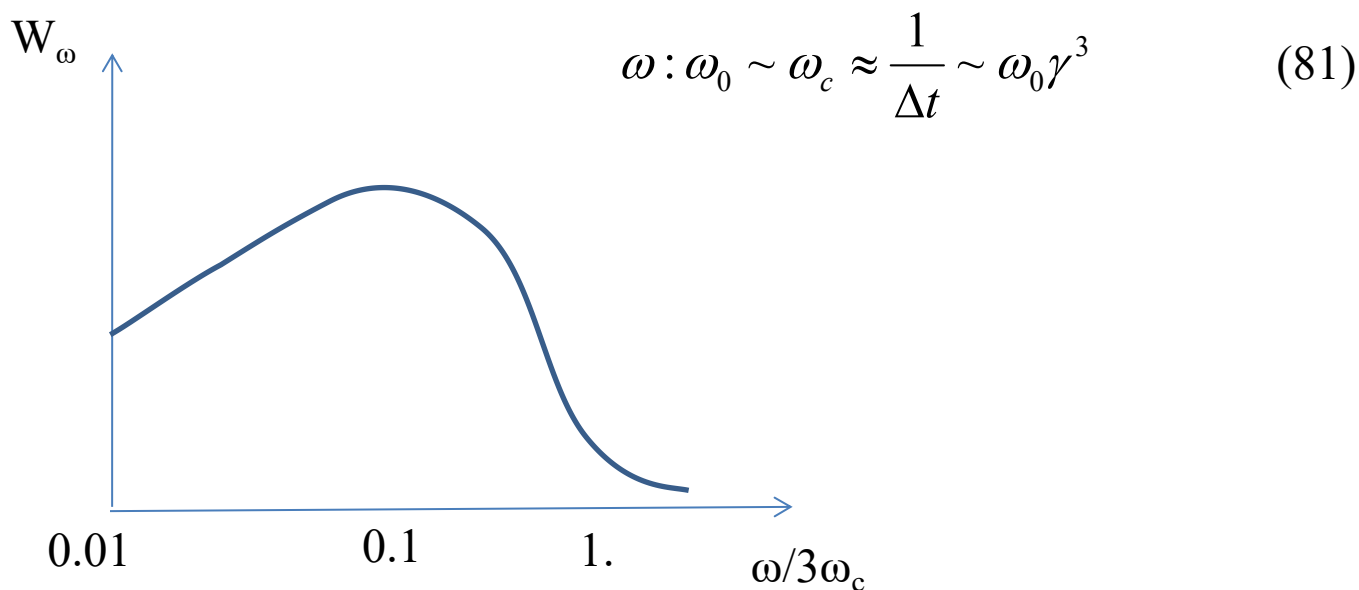它的英文全名为Shanghai Synchrotron Radiation
facility，简称SSRF。

单位立体角内辐射能量

$$\frac{dW}{d\Omega} = \varepsilon_0 c R^2 \int_{-\infty}^{\infty} E^2 dt = 4\pi\varepsilon_0 c R^2 \int_0^{\infty} |E_\omega|^2 d\omega \equiv \int_0^{\infty} \frac{dW_\omega}{d\Omega} d\omega \qquad (79)$$

单位频率间隔辐射能量

$$W_\omega = 4\pi\varepsilon_0 c R^2 \oint |\boldsymbol{E}_\omega|^2 d\Omega \qquad (80)$$

同步辐射：$E_e = 100\text{MeV}$, 轨道半径 $\rho = 0.4\text{m}$， $\omega_0 \sim 8 \times 10^8 \text{s}^{-1}$，$\gamma \sim 200$

$$\omega : \omega_0 \sim \omega_c \approx \frac{1}{\Delta t} \sim \omega_0 \gamma^3 \qquad (81)$$

作业

1. 证明(45)式
$$\int_{-\infty}^{\infty} f^2(t)dt = 4\pi \int_0^{\infty} \left| f_\omega \right|^2 d\omega$$

2. 从

$$\frac{dP(t')}{d\Omega} = \frac{q^2 \dot{v}^2}{16\pi^2 \varepsilon_0 c^3} \frac{(1 - \frac{v}{c}\cos\theta)^2 - (1 - \frac{v^2}{c^2})\sin^2\theta\cos^2\phi}{(1 - \frac{v}{c}\cos\theta)^5}$$

推导$P(t')$.

3. p272第2题

附录1 推导偏导数关系

对给定的 x 和 t，t_q 唯一，因此存在函数关系 $t_q = t_q(x,t)$. 对 $r = x - x_q$ 求 t 的偏导数，

$$\left.\frac{\partial \boldsymbol{r}}{\partial t}\right|_x = -\left.\frac{\partial \boldsymbol{x}_q(t_q)}{\partial t}\right|_x = -\left.\frac{\partial t_q}{\partial t}\right|_x \frac{d\boldsymbol{x}_q}{dt_q} = -\left.\frac{\partial t_q}{\partial t}\right|_x \boldsymbol{v} \tag{附1}$$

对 $r = \sqrt{(\boldsymbol{r} \cdot \boldsymbol{r})}$ 求 t 的偏导数，利用(附1)，

$$\left.\frac{\partial r}{\partial t}\right|_x = \frac{\boldsymbol{r}}{r} \cdot \left.\frac{\partial \boldsymbol{r}}{\partial t}\right|_x = -\frac{\boldsymbol{r} \cdot \boldsymbol{v}}{r} \left.\frac{\partial t_q}{\partial t}\right|_x \tag{附2}$$

此外，由 $r = c(t - t_q)$，

$$\left.\frac{\partial r}{\partial t}\right|_x = c\left(1 - \left.\frac{\partial t_q}{\partial t}\right|_x\right) \tag{附3}$$

结合(附2)和(附3)，

$$\frac{\partial t_q}{\partial t} = \frac{r}{r - \frac{1}{c}\boldsymbol{v} \cdot \boldsymbol{r}} = \frac{r}{s} \tag{附4}$$

代回(附1)和(附2)，

$$\left.\frac{\partial \boldsymbol{r}}{\partial t}\right|_x = -\frac{r}{s}\boldsymbol{v}, \qquad\qquad \left.\frac{\partial r}{\partial t}\right|_x = -\frac{\boldsymbol{r} \cdot \boldsymbol{v}}{s} \tag{附5}$$

对 $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}_q$ 求 x^j 的偏导数，考虑到 $\boldsymbol{x}_q$ 是 t_q 的函数，

$$\frac{\partial \boldsymbol{r}}{\partial x^j}\bigg|_t = \hat{\boldsymbol{e}}_j - \frac{\partial \boldsymbol{x}_q}{\partial x^j}\bigg|_t = \hat{\boldsymbol{e}}_j - \frac{\partial t_q}{\partial x^j}\bigg|_t \dot{\boldsymbol{x}}_q = \hat{\boldsymbol{e}}_j - \frac{\partial t_q}{\partial x^j}\bigg|_t \boldsymbol{v} \tag{附6}$$

对 $r = \sqrt{(\boldsymbol{r} \cdot \boldsymbol{r})}$ 求 x^j 的偏导数，利用(附6)，

$$\frac{\partial r}{\partial x^j}\bigg|_t = \frac{\boldsymbol{r}}{r} \cdot \frac{\partial \boldsymbol{r}}{\partial x^j}\bigg|_t = \frac{r^j}{r} - \frac{\partial t_q}{\partial x^j}\bigg|_t \frac{\boldsymbol{r} \cdot \boldsymbol{v}}{r} \tag{附7}$$

此外，由 $r = c(t - t_q)$，

$$\frac{\partial r}{\partial x^j}\bigg|_t = -c\frac{\partial t_q}{\partial x^j}\bigg|_t \tag{附8}$$

结合(附7)和(附8)，

$$\frac{\partial t_q}{\partial x^j}\bigg|_t = -\frac{r^j}{c(r - \frac{1}{c}\boldsymbol{v} \cdot \boldsymbol{r})} = -\frac{r^j}{cs} \tag{附9}$$

代回(附6)和(附8)，

$$\frac{\partial \boldsymbol{r}}{\partial x^j}\bigg|_t = \hat{\boldsymbol{e}}_j + \frac{r^j}{cs}\boldsymbol{v}, \qquad \frac{\partial r}{\partial x^j}\bigg|_t = \frac{r^j}{s} \tag{附10}$$

$$E = -\nabla\varphi - \frac{\partial A}{\partial t} \tag{附11}$$

以李纳-维谢尔势代入

$$E = -\nabla\varphi - \frac{\partial A}{\partial t} = -\nabla\frac{q}{4\pi\varepsilon_0(r - \frac{1}{c}v\cdot r)} - \frac{\partial}{\partial t}\frac{qv}{4\pi\varepsilon_0 c^2(r - \frac{1}{c}v\cdot r)}$$

$$= \frac{q}{4\pi\varepsilon_0(r - \frac{1}{c}v\cdot r)^2}\left[\nabla r - \frac{1}{c}\nabla(v\cdot r) + \frac{v}{c^2}\left(\partial_t r - \frac{1}{c}\partial_t(v\cdot r)\right) - (r - \frac{1}{c}v\cdot r)\frac{\partial_t v}{c^2}\right] \tag{附12}$$

应用(附4,5)式和(附9,10)式，

$$\nabla r = \frac{\hat{e}_r}{1 - \frac{1}{c}v\cdot\hat{e}_r}, \qquad \nabla(v\cdot r) = v + \frac{\frac{1}{c}v^2\hat{e}_r}{1 - \frac{1}{c}v\cdot\hat{e}_r} - \frac{\frac{1}{c}(\hat{e}_r\cdot\dot{v})r}{1 - \frac{1}{c}v\cdot\hat{e}_r},$$

$$\partial_t r = -\frac{v\cdot\hat{e}_r}{1 - \frac{1}{c}v\cdot\hat{e}_r}, \qquad \partial_t(v\cdot r) = \frac{r\cdot\dot{v} - v^2}{1 - \frac{1}{c}v\cdot\hat{e}_r}, \qquad \partial_t v = \frac{\dot{v}}{1 - \frac{1}{c}v\cdot\hat{e}_r} \tag{附13}$$

代回(附12)式，得到E为非辐射电场E_c和辐射电场E_{rad}之和

$$E_c = \frac{q}{4\pi\varepsilon_0 r^2}\frac{\hat{e}_r - \frac{1}{c}v}{(1 - \frac{1}{c}v\cdot\hat{e}_r)^3\gamma^2} \sim \frac{q}{4\pi\varepsilon_0 r^2}\hat{e}_r \tag{附14}$$

$$E_{rad} = \frac{q}{4\pi\varepsilon_0 c^2 r}\frac{\hat{e}_r\times[(\hat{e}_r - \frac{1}{c}v)\times\dot{v}]}{(1 - \frac{1}{c}v\cdot\hat{e}_r)^3} \sim \frac{q}{4\pi\varepsilon_0 c^2 r}\hat{e}_r\times(\hat{e}_r\times\dot{v}) \tag{附15}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{附16a}$$

在笛卡尔坐标中的分量形式(重复空间指标，j和k，隐含从1到3求和)

$$B^i = \varepsilon^{ijk} \partial_j A_k \tag{附16b}$$

以李纳-维谢尔势代入

$$B^i = \varepsilon^{ijk} \partial_j \frac{q v_k}{4\pi\varepsilon_0 c^2 s} = \frac{q\varepsilon^{ijk}}{4\pi\varepsilon_0 c^2 s} \partial_j v_k - \frac{q\varepsilon^{ijk} v_k}{4\pi\varepsilon_0 c^2 s^2} \partial_j s$$

$$= \frac{q\varepsilon^{ijk}}{4\pi\varepsilon_0 c^2 s} \dot{v}_k \frac{\partial t_q}{\partial x^j} - \frac{q\varepsilon^{ijk} v_k}{4\pi\varepsilon_0 c^2 s^2} \left[\frac{\partial r}{\partial x^j} - \frac{\boldsymbol{r}}{c} \cdot \frac{\partial \boldsymbol{v}}{\partial x^j} - \frac{\boldsymbol{v}}{c} \cdot \frac{\partial \boldsymbol{r}}{\partial x^j} \right]$$

$$= \frac{q\varepsilon^{ijk}}{4\pi\varepsilon_0 c^2 s} \dot{v}_k \frac{\partial t_q}{\partial x^j} - \frac{q\varepsilon^{ijk} v_k}{4\pi\varepsilon_0 c^2 s^2} \left[\frac{\partial r}{\partial x^j} - \frac{\boldsymbol{r}}{c} \cdot \dot{\boldsymbol{v}} \frac{\partial t_q}{\partial x^j} - \frac{\boldsymbol{v}}{c} \cdot \frac{\partial \boldsymbol{r}}{\partial x^j} \right] \tag{附17}$$

其中 $s = r - \frac{1}{c} \boldsymbol{v} \cdot \boldsymbol{r}$. 应用(附9、10)式，

$$\boldsymbol{B} = \frac{q\dot{\boldsymbol{v}} \times \boldsymbol{r}}{4\pi\varepsilon_0 c^3 s^2} - \frac{q}{4\pi\varepsilon_0 c^2 s^2} \left[\frac{\boldsymbol{r} \times \boldsymbol{v}}{s} + \frac{(\dot{\boldsymbol{v}} \cdot \boldsymbol{r})(\boldsymbol{r} \times \boldsymbol{v})}{c^2 s} - \frac{v^2 (\boldsymbol{r} \times \boldsymbol{v})}{c^2 s} \right]$$

$$= \frac{q}{4\pi\varepsilon_0 c^2 r^2} \frac{(1 - \frac{v^2}{c^2})\boldsymbol{v} \times \hat{\boldsymbol{e}}_r}{(1 - \frac{1}{c}\boldsymbol{v} \cdot \hat{\boldsymbol{e}}_r)^3} + \frac{q}{4\pi\varepsilon_0 c^3 r} \frac{\dot{\boldsymbol{v}} \times \hat{\boldsymbol{e}}_r + \frac{1}{c}[(\dot{\boldsymbol{v}} \cdot \hat{\boldsymbol{e}}_r)(\boldsymbol{v} \times \hat{\boldsymbol{e}}_r) - (\boldsymbol{v} \cdot \hat{\boldsymbol{e}}_r)(\dot{\boldsymbol{v}} \times \hat{\boldsymbol{e}}_r)]}{(1 - \frac{1}{c}\boldsymbol{v} \cdot \hat{\boldsymbol{e}}_r)^3} \tag{附18}$$

B的第一项是非辐射项

$$B_c = \frac{q}{4\pi\varepsilon_0 c^2 r^2} \frac{(1-\frac{v^2}{c^2})\boldsymbol{v}\times\hat{\boldsymbol{e}}_r}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^3} \tag{附19}$$

第二项是辐射项

$$\begin{aligned}
\boldsymbol{B}_{rad} &= \frac{q}{4\pi\varepsilon_0 c^3 r} \frac{\dot{\boldsymbol{v}}\times\hat{\boldsymbol{e}}_r + \frac{1}{c}[(\dot{\boldsymbol{v}}\cdot\hat{\boldsymbol{e}}_r)(\boldsymbol{v}\times\hat{\boldsymbol{e}}_r) - (\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)(\dot{\boldsymbol{v}}\times\hat{\boldsymbol{e}}_r)]}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^3} \\
&= \frac{q}{4\pi\varepsilon_0 c^3 r} \frac{\hat{\boldsymbol{e}}_r\times\hat{\boldsymbol{e}}_r\times(\hat{\boldsymbol{e}}_r\times\dot{\boldsymbol{v}}) + \frac{1}{c}[(\dot{\boldsymbol{v}}\cdot\hat{\boldsymbol{e}}_r)\boldsymbol{v} - (\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)\dot{\boldsymbol{v}}]\times\hat{\boldsymbol{e}}_r}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^3} \\
&= \frac{q}{4\pi\varepsilon_0 c^3 r} \frac{\hat{\boldsymbol{e}}_r\times\hat{\boldsymbol{e}}_r\times(\hat{\boldsymbol{e}}_r\times\dot{\boldsymbol{v}}) - \frac{1}{c}\hat{\boldsymbol{e}}_r\times[\hat{\boldsymbol{e}}_r\times(\boldsymbol{v}\times\dot{\boldsymbol{v}})]}{(1-\frac{1}{c}\boldsymbol{v}\cdot\hat{\boldsymbol{e}}_r)^3}
\end{aligned} \tag{附20}$$

与(附15)比较，

$$\boldsymbol{B}_{rad} = \frac{1}{c}\hat{\boldsymbol{e}}_r \times \boldsymbol{E}_{rad} \tag{附21}$$

低速，$v/c \ll 1$，

$$\boldsymbol{B}_c \sim \frac{q}{4\pi\varepsilon_0 c^2 r^2}\boldsymbol{v}\times\hat{\boldsymbol{e}}_r, \qquad \boldsymbol{B}_{rad} \sim \frac{q}{4\pi\varepsilon_0 c^3 r}\dot{\boldsymbol{v}}\times\hat{\boldsymbol{e}}_r \qquad \frac{|B_c|}{|B_{rad}|} \sim \frac{\lambda}{r} \tag{附22}$$