

习题 9.2

$$(1) \frac{y}{1+y^2} dy = \frac{dx}{x(1+x^2)} \quad \frac{1}{2} \ln(1+y^2) = \int \left(\frac{1}{x} - \frac{x}{1+x^2} \right) dx = \ln|x| - \frac{1}{2} \ln(1+x^2) + C$$

$$\ln(1+y^2) = \ln x^2 - \ln(1+x^2) + C \Rightarrow (1+x^2)(1+y^2) = e^C x^2 = C^* x^2, \quad C^* > 0.$$

$$(3) \frac{1}{\sqrt{1-y^2}} dy = \frac{1}{\sqrt{1+x^2}} dx, \quad \arcsin y = \ln(x + \sqrt{1+x^2}) + C = \ln e^C (x + \sqrt{1+x^2}) = \ln C^* (x + \sqrt{1+x^2}), \quad C^* > 0.$$

$$(7) \frac{dy}{dx} = -\frac{2x^2+y^2}{2xy+3y^2} = -\frac{2+u^2}{2u+3u^2}, \quad u = \frac{y}{x}$$

$$u' = \frac{y'x - y}{x^2} = \frac{1}{x} \left(-\frac{2+3u^2+3u^3}{2u+3u^2} \right), \quad \frac{2u+3u^2}{2+3u^2+3u^3} du = -\frac{1}{x} dx$$

$$\frac{1}{3} \ln|2+3u^2+3u^3| = -\ln|x| + C, \quad |2+3u^2+3u^3| = C|x|^{-3}, \quad C > 0, \quad 2+3u^2+3u^3 = Cx^{-3}, \quad C \neq \frac{2}{3}$$

$$(8) y' = (x+y+2)^2, \quad z = x+y+2, \quad z' = 1+y' = 1+z^2, \quad \frac{dz}{1+z^2} = dx, \quad \arctan z = x + C$$

$$(10) \frac{dy}{dx} = \frac{-(x-2y+5)}{2x-y+4}, \quad \Delta = \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} \neq 0, \quad (x_0, y_0) = (-1, 2), \quad u = x+1, \quad v = y-2$$

$$\frac{dv}{du} = \frac{-u+2v}{2u-v} = \frac{-1+2\frac{v}{u}}{2-\frac{v}{u}}, \quad \text{令 } \frac{v}{u} = z, \quad z' = \frac{1}{u} (h(z) - z) = \frac{1}{u} \cdot \frac{-1+2z-2z+3z^2}{2-z} = \frac{1}{u} \cdot \frac{z^2-1}{2-z}$$

$$z = \pm 1 \text{ 是特解. } \frac{(z^2-1)dz}{z^2-1} = \frac{1}{u} du \Rightarrow \left[-\frac{1}{1+z} - \frac{1}{1-z} - \frac{2z}{2(z^2-1)} \right] dz = \frac{1}{u} du$$

$$-\ln|1+z| + \ln|1-z| - \frac{1}{2} \ln|z^2-1| = \ln|u| + C$$

$$-\ln|(1+z)^2| + \ln|(1-z)^2| - \ln|z^2-1| = \ln u^2 + C$$

$$|z^2-1| = C \cdot \frac{(1-z)^2}{(1+z)^2} \cdot \frac{1}{u^2}, \quad C > 0, \Rightarrow z^2-1 = C \cdot \frac{(1-z)^2}{(1+z)^2} \cdot \frac{1}{u^2}, \quad C \neq 0.$$

$$x \text{ 或 } z = \pm 1 \text{ 是特解. } z^2-1 = C \cdot \frac{(1-z)^2}{(1+z)^2} \cdot \frac{1}{u^2}, \quad C \in \mathbb{R}, \quad \frac{(z+1)^3}{z-1} = C \frac{1}{u^2}, \quad (u+v)^3 = C(u-v)$$

$$8. \frac{dR}{dt} = -kR, \quad k > 0, \quad R = Ce^{-kt}$$

$$\text{在 } t=0, R_0 = C, \quad t=1600, R_{1600} = R_0 e^{-1600k} = \frac{1}{2} R_0, \quad e^{-1600k} = \frac{1}{2}, \quad k = \frac{\ln 2}{1600}$$

$$R_0 = 1(g), \quad t=1, \quad R_1 = e^{-k}, \quad R_0 R_1 = 1 \cdot e^{-k} = 1 \cdot e^{-\frac{\ln 2}{1600}} = 1 - 2^{-\frac{1}{1600}} \approx 0.00043(g)$$

$$9. u = tx, \quad \int_0^x g(u) \cdot \frac{1}{x} du = n g(x), \quad \int_0^x g(u) u du = nx g(x), \quad \text{两边求导: } g(x) = n g(x) + nx g'(x)$$

$$\frac{dg}{g} = \frac{tn}{nx} dx, \quad \ln|g| = \frac{tn}{n} \ln|x| + C, \quad |g| = |x|^{\frac{tn}{n}} \cdot C_1, \quad C_1 > 0, \Rightarrow g = |x|^{\frac{tn}{n}} \cdot C_1, \quad C_1 \in \mathbb{R}$$

$$13. (1) \text{ 令 } z = y', \quad x^2 z' = z^2, \quad \frac{dz}{z^2} = \frac{dx}{x^2}, \quad -\frac{1}{z} = -\frac{1}{x} + C_1, \quad z = \frac{x}{1+C_1 x}$$

$$\text{当 } C_1 = 0, \quad z = x \Rightarrow y' = x, \quad y = \frac{x^2}{2} + C_2, \quad \text{当 } C_1 \neq 0, \quad y' = \frac{x}{1+C_1 x} = \frac{1}{C_1} - \frac{1}{1+C_1 x}$$

$$y = \frac{x}{C_1} - \frac{1}{C_1^2} \ln|1+C_1 x| + C_2 \Rightarrow C_1 x - C_1^2 y = \ln|1+C_1 x| + C_2, \quad \text{另还有特解 } z=0 \Rightarrow y=C$$

$$(2) \text{ 令 } p = y', \quad p^2 + 2y \cdot p \cdot p' = 0, \quad p \neq 0 \Rightarrow y = C.$$

$$\frac{dp}{p} = -\frac{dy}{2y}, \quad \ln|p| = -\frac{1}{2} \ln|y| + C, \quad |p| = |y|^{-\frac{1}{2}} \cdot C, \quad C > 0, \quad p = C|y|^{-\frac{1}{2}}, \quad C \neq \frac{2}{3}$$

$$y' = C|y|^{-\frac{1}{2}}, \quad |y|^{\frac{1}{2}} dy = C dx, \quad \frac{2}{3} |y|^{\frac{3}{2}} \cdot \text{sgn}(y) = Cx + C_1, \quad \frac{2}{3} |y|^{\frac{3}{2}} = Cx + C_1$$

$$|y|^{\frac{3}{2}} = Cx + C_1 = C(x + C_1), \quad |y| = C^{\frac{2}{3}} (x + C_1)^{\frac{2}{3}} = C_2 (x + C_1)^{\frac{2}{3}}, \quad C_2 \geq 0.$$

$$y = C_2 (x + C_1)^{\frac{2}{3}}, \quad C_1, C_2 \in \mathbb{R}$$

$$14. (3) \cdot \frac{\partial P}{\partial y} = e^y \cdot \frac{\partial Q}{\partial x} = e^y. \quad u(x, y) = x e^y + \varphi(y). \quad \frac{\partial u}{\partial y} = x e^y + \varphi'(y)$$

$$\varphi'(y) = -2y \quad \varphi(y) = -y^2 + C. \quad u = x e^y - y^2 + C = C', \quad x e^y - y^2 \equiv C.$$

$$(6) \quad \frac{\partial P}{\partial y} = -6 \cdot \frac{\partial Q}{\partial x} = 6 \quad \text{不成立}$$

$$(7) \cdot \frac{\partial P}{\partial y} = e^x + 2y. \quad \frac{\partial Q}{\partial x} = e^x + 2y. \quad u = y e^x + 2e^x + x y^2 + \varphi(y), \quad \frac{\partial u}{\partial y} = e^x + 2x y + \varphi'(y)$$

$$\varphi(y) \equiv 0. \quad y e^x + 2e^x + x y^2 \equiv C$$

$$15. (1) \quad dx - dy = \frac{1}{(x+y)^2} (dx + dy) \Rightarrow d(x-y) = -d \frac{1}{x+y}. \quad x-y = -\frac{1}{x+y} + C$$

$$(4) (x^2 + y^2) dx + y dx - x dy = 0. (x^2 + y^2) dx + d\left(\frac{x}{y}\right) \cdot y^2 = 0$$

$$dx + d\left(\frac{x}{y}\right) \cdot \frac{1}{1 + \left(\frac{x}{y}\right)^2} = 0. \quad dx + d\left(\frac{x}{y}\right) \cdot d \arctan\left(\frac{x}{y}\right) = 0.$$

$$x + \arctan \frac{x}{y} \equiv C$$

$$16. (3) \cdot y(x+1) dx + x(y+1) dy = 0, \quad xy \neq 0.$$

$$\frac{\partial P}{\partial y} = x+1. \quad \frac{\partial Q}{\partial x} = y+1$$

$$\text{两边同除 } xy. \quad (1 + \frac{1}{x}) dx + (1 + \frac{1}{y}) dy = 0. \quad x + \ln|x| + y + \ln|y| = C$$

$$(6) \quad e^x dx + (e^x \cos y + 2y \cos y) dy = 0.$$

$$\frac{\partial M}{\partial y} = 0. \quad \frac{\partial N}{\partial x} = e^x \cos y. \quad \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \cos y. \quad \mu = e^{\int \cos y dy} = \sin y$$

$$\sin y \cdot e^x dx + \sin y (e^x \cos y + 2y \cos y) dy = 0.$$

$$u = \sin y \cdot e^x + \varphi(y). \quad \frac{\partial u}{\partial y} = e^x \cos y + \varphi'(y) = e^x \cos y + y \sin 2y$$

$$\varphi'(y) = y \sin 2y. \quad \varphi(y) = -\frac{1}{2} y \cos 2y + \frac{1}{4} \sin 2y$$

或更简单法：两边乘 $\sin y$.

$$e^x \sin y dx + e^x \cos y dy + 2y \sin y \cos y dy = 0$$

$$d(\sin y e^x) + y \sin 2y dy = 0$$

$$d(\sin y e^x) + d(-\frac{1}{2} y \cos 2y + \frac{1}{4} \sin 2y)$$

习题 9.4

2. 设 y_1, y_2 是 $y'' + p(x)y = 0$ 的两个解.

$$y_1'' = -p(x)y_1, \quad y_2'' = -p(x)y_2.$$

$$\begin{aligned} W(x) &= y_1 y_2' - y_1' y_2, \quad W'(x) = y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2' \\ &= y_1 y_2'' - y_1'' y_2 = -p(x)y_1 y_2 + p(x)y_1 y_2 = 0. \end{aligned}$$

$$W(x) \equiv C.$$

3. 设 y_1, y_2 是齐次方程 $y'' + p(x)y' + q(x)y = 0$ 的线性无关解.

则 $y(x)$ 可表示为 $C_1 y_1(x) + C_2 y_2(x)$, C_1, C_2 为某固定常数. (通解包含一切解)

$$\text{若 } y(x_0) = 0, \Rightarrow C_1 y_1(x_0) + C_2 y_2(x_0) = 0.$$

$$\text{若 } y'(x_0) = 0 \Rightarrow C_1 y_1'(x_0) + C_2 y_2'(x_0) = 0$$

由于 $y(x)$ 是非零解, C_1, C_2 一定不同吋为零. 则上述方程组有非零解.

$$\Rightarrow \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0, \text{ 总存在 } x_0, W(x_0) = 0 \text{ 与 } y_1, y_2 \text{ 线性无关矛盾.}$$

4. 若存在 x_0 是 y_1, y_2 的公共零点, 即 $y_1(x_0) = 0, y_2(x_0) = 0$.

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0. \text{ 也即存在 } x_0, W(x_0) = 0$$

与 y_1, y_2 线性无关矛盾.

习题 9.6

1. $y'' + 3y' + 2y = \frac{1}{e^x + 1}$

$y'' + 3y' + 2y = 0 \Rightarrow \lambda^2 + 3\lambda + 2 = 0, \lambda = -1, -2 \quad C_1 e^{-x} + C_2 e^{-2x}$

$\begin{cases} C_1' e^{-x} + C_2' e^{-2x} = 0 \\ -C_1' e^{-x} - 2C_2' e^{-2x} = \frac{1}{e^x + 1} \end{cases} \Rightarrow \begin{cases} C_1'(x) = \frac{e^x}{e^x + 1} \\ C_2'(x) = -\frac{e^{2x}}{e^x + 1} \end{cases} \Rightarrow \begin{cases} C_1(x) = \ln(1 + e^x) + C_1 \\ C_2(x) = -e^x + \ln(1 + e^x) + C_2 \end{cases}$

$C_1(x) = e^x + \frac{e^x}{1 + e^x}, \quad C_2(x) = -e^x + \ln(1 + e^x) + C_2$

$y = e^{-x} \ln(1 + e^x) + C_1 e^{-x} - e^{-x} + e^{-2x} \ln(1 + e^x) + C_2 e^{-2x}$

$= (e^{-x} + e^{-2x}) \ln(1 + e^x) + (C_1 - 1) e^{-x} + C_2 e^{-2x}$

$\checkmark C_1 \left(1 \neq \frac{1}{e}\right)$

2. $y' + y = \frac{1}{\sin x}$

$\lambda^2 + 1 = 0, \lambda = \pm i, \quad C_1 \cos x + C_2 \sin x$

$\begin{cases} C_1' \cos x + C_2' \sin x = 0 \\ -C_1' \sin x + C_2' \cos x = \frac{1}{\sin x} \end{cases} \Rightarrow \begin{cases} C_1' = -1 \\ C_2' = \cot x \end{cases} \Rightarrow \begin{cases} C_1(x) = -x + C_1 \\ C_2(x) = \ln|\sin x| + C_2 \end{cases}$

$y = (-x + C_1) \cos x + (\ln|\sin x| + C_2) \sin x = -x \cos x + \sin x \ln|\sin x| + C_1 \cos x + C_2 \sin x$

3. $y'' + 4y = 2 \tan x$

$\lambda^2 + 4 = 0, \lambda = \pm 2i, \quad C_1 \cos 2x + C_2 \sin 2x$

$\begin{cases} C_1' \cos 2x + C_2' \sin 2x = 0 \\ -2C_1' \sin 2x + 2C_2' \cos 2x = 2 \tan x \end{cases} \Rightarrow \begin{cases} C_1' = -\frac{\sin^2 x}{\cos^2 x} = -1 + \tan^2 x \\ C_2' = \tan x \cos 2x = \sin 2x - \tan x \end{cases} \Rightarrow \begin{cases} C_1(x) = -x + \frac{1}{2} \sin 2x + C_1 \\ C_2(x) = -\frac{1}{2} \cos 2x + \ln|\cos x| + C_2 \end{cases}$

$y = (-x + \frac{1}{2} \sin 2x + C_1) \cos 2x + (-\frac{1}{2} \cos 2x + \ln|\cos x| + C_2) \sin 2x$

$= -x \cos 2x + \sin 2x \ln|\cos x| + C_1 \cos 2x + C_2 \sin 2x$

4. $y'' + y = 2 \sec^3 x, \quad C_1 \cos x + C_2 \sin x$

$\begin{cases} C_1' \cos x + C_2' \sin x = 0 \\ -C_1' \sin x + C_2' \cos x = 2 \sec^3 x \end{cases} \Rightarrow \begin{cases} C_1' = -2 \sec^3 x \cdot \sin x = -\frac{2 \sin x}{\cos^3 x} \\ C_2' = 2 \sec^3 x \cdot \cos x = 2 \sec^2 x \end{cases} \Rightarrow \begin{cases} C_1(x) = -\sec^2 x + C_1 \\ C_2(x) = 2 \tan x + C_2 \end{cases}$

$y = (-\sec^2 x + C_1) \cos x + (2 \tan x + C_2) \sin x = -\sec x + 2 \tan x \sin x + C_1 \cos x + C_2 \sin x$

$= C_1 \cos x + C_2 \sin x - \frac{1 - 2 \sin^2 x}{\cos x} = C_1 \cos x + C_2 \sin x - \frac{\cos 2x}{\cos x}$

5. $x^2 y'' - 4x y' + 6y = 0$

$x = e^t, \quad y_x' = y_t' \cdot \frac{1}{x}, \quad y_x'' = (y_t'' - y_t') \frac{1}{x^2}$

$\left(t = \ln|x| \right) \frac{dt}{dx} = \frac{1}{x}$

$$y_t'' - y_t' - 4y_t' + 6y_t = 0 \quad \lambda^2 - 5\lambda + 6 = 0 \quad \lambda = 2, \lambda = 3$$

$$e^{2t} \cdot e^{3t} \Rightarrow C_1 x^2 + C_2 x^3$$

$$6. x^2 y'' - x y' - 3y = 0 \quad y_t'' - y_t' - y_t' - 3y = 0 \quad \lambda^2 - 2\lambda - 3 = 0 \quad \lambda = -1, \lambda = 3$$

$$C_1 \frac{1}{x} + C_2 x^3$$

$$7. x^3 y''' + x y' - y = 0 \quad y_t''' = \left[(y_t''' - y_t'') \frac{1}{x^2} - 2(y_t'' - y_t') \frac{1}{x^3} x \right] \cdot \frac{1}{x} = (y_t''' - 3y_t'' + 2y_t') \frac{1}{x^3}$$

$$y_t''' - 3y_t'' + 2y_t' + y_t' - y = 0 \Rightarrow y_t''' - 3y_t'' + 3y_t' - y = 0 \quad \lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0 \quad (\lambda - 1)^3 = 0$$

$$C_1 e^t + C_2 t e^t + C_3 t^2 e^t \Rightarrow x(C_1 + C_2 \ln|x| + C_3 (\ln|x|)^2)$$

$$8. x^2 y' + x y' + 4y = 10$$

$$\frac{5}{x^2} \cdot y = \frac{5}{2}$$

$$y_t'' - y_t' + y_t' + 4y = 10 \quad y_t'' + 4y = 10 \quad \lambda^2 + 4 = 0 \quad \lambda = \pm 2i$$

$$C_1 \cos 2x + C_2 \sin 2x + \frac{5}{2} \Rightarrow C_1 \cos(\ln x^2) + C_2 \sin(\ln x^2) + \frac{5}{2}$$

7.5

$$1. (1) y' - 3y' + 2y = 0 \quad \lambda^2 - 3\lambda + 2 = 0 \quad \lambda = 1, \lambda = 2 \quad C_1 e^x + C_2 e^{2x}$$

$$(3) y'' + 6y' + 9y = 0 \quad \lambda^2 + 6\lambda + 9 = 0 \quad \lambda = -3 \quad C_1 e^{-3x} + C_2 x e^{-3x}$$

$$(5) y'' - y' + 2y = 0 \quad \lambda^2 - \lambda + 2 = 0 \quad \lambda = \frac{1 \pm \sqrt{7}i}{2} \quad C_1 e^{\frac{x}{2}} \sin \frac{\sqrt{7}}{2} x + C_2 e^{\frac{x}{2}} \cos \frac{\sqrt{7}}{2} x$$

$$3. (7) y' - y = 2e^x - x^2 \quad \lambda^2 - 1 = 0 \quad \lambda = \pm 1 \quad C_1 e^x + C_2 e^{-x}$$

$$y = A x e^x: y'' = 2A e^x + A x e^x \quad 2A e^x = 2e^x \Rightarrow A = 1 \quad y_1 = x e^x$$

$$y = a x^2 + b x + c \quad y'' = 2a \quad 2a - a x^2 - b x - c = -x^2 \quad y_2 = x^2 + 2$$

$$y = C_1 e^x + C_2 e^{-x} + x e^x + x^2 + 2$$

$$(8) y'' + y' = 84x - 2 \sin 2x \quad \lambda^2 + \lambda = 0 \quad \lambda = 0, -1 \quad C_1 + C_2 e^{-x}$$

$$y = A \sin 4x + B \cos 4x \quad y' = 4A \cos 4x - 4B \sin 4x \quad y'' = -16A \sin 4x - 16B \cos 4x$$

$$\begin{cases} -16A - 4B = 1 \\ 4A - 16B = 0 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{17} \\ B = -\frac{1}{68} \end{cases} \Rightarrow y_1 = -\frac{1}{17} \sin 4x - \frac{1}{68} \cos 4x$$

$$y = A \sin 2x + B \cos 2x \quad y' = 2A \cos 2x - 2B \sin 2x \quad y'' = -4A \sin 2x - 4B \cos 2x$$

$$\begin{cases} -4A - 2B = -2 \\ 2A - 4B = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{2}{5} \\ B = \frac{1}{5} \end{cases} \Rightarrow y_2 = \frac{2}{5} \sin 2x + \frac{1}{5} \cos 2x$$

$$y = C_1 + C_2 e^{-x} - \frac{1}{17} \sin 4x - \frac{1}{68} \cos 4x + \frac{2}{5} \sin 2x + \frac{1}{5} \cos 2x$$

$$4. (2) y'' + y' = x - 2 \quad \lambda^2 + \lambda = 0 \quad \lambda = 0, -1. \quad x(ax + b)$$

$$(4) y'' - y = e^x(x^2 - 1), \quad \lambda^2 - 1 = 0, \quad \lambda = \pm 1. \quad x(ax^2 + bx + c)e^x$$

$$(5) y''' + 3y'' + 3y' + y = e^{-x}(x - 5)$$

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0, (\lambda + 1)^3 \quad \lambda = -1. \quad x^3(ax + b)e^{-x}$$