09 级一期 B 卷参考解答

一.(每小题 6 分,共 12 分)求极限:

(1) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{2x^3}$;

解　原式 $=\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{2x^3(\sqrt{1+\tan x}+\sqrt{1+\sin x})}=\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{4x^3}$

$$=\lim_{x\to 0}\frac{\sec^2 x-\cos x}{12x^2}=\lim_{x\to 0}\frac{1-\cos^3 x}{12x^2\cos^2 x}=\lim_{x\to 0}\frac{1-\cos^3 x}{12x^2}=\lim_{x\to 0}\frac{3\cos^2 x\sin x}{24x}=\frac{1}{8}.$$

(2) $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{\sin^2 x}}$.

解　$\lim\limits_{x\to 0}(\cos x)^{\frac{1}{\sin^2 x}}=\lim\limits_{x\to 0}e^{\frac{1}{\sin^2 x}\ln\cos x}$.而

$$\lim_{x\to 0}\frac{\ln\cos x}{\sin^2 x}=\lim_{x\to 0}\frac{-\sin x/\cos x}{2\sin x\cos x}=-\frac{1}{2}\lim_{x\to 0}\frac{1}{\cos^2 x}=-\frac{1}{2}.$$

故　$\lim\limits_{x\to 0}(\cos x)^{\frac{1}{\sin^2 x}}=e^{-\frac{1}{2}}=\dfrac{1}{\sqrt{e}}$.

二. (每小题 6 分,共 24 分)求下列积分:

(1) $\displaystyle\int\frac{dx}{x\left(2+x^{10}\right)}$;

解　$\displaystyle\int\frac{dx}{x\left(2+x^{10}\right)}=\int\frac{x^9dx}{x^{10}\left(2+x^{10}\right)}=\frac{1}{10}\int\frac{dx^{10}}{x^{10}\left(2+x^{10}\right)}$;

$$=\frac{1}{20}\left(\int\frac{dx^{10}}{x^{10}}-\int\frac{dx^{10}}{2+x^{10}}\right)=\frac{1}{20}\ln\left|\frac{x^{10}}{2+x^{10}}\right|+C.$$

(2) $\displaystyle\int\cos(\ln x)dx$;

解　$\displaystyle\int\cos(\ln x)dx\overset{u=\ln x}{=}\int e^u\cos udu=\int e^u d\sin u=e^u\sin u-\int\sin ude^u$

$$=e^u\sin u+\int e^u d\cos u=e^u\sin u+e^u\cos u-\int e^u\cos udu$$

所以 $\displaystyle\int\cos(\ln x)dx=\frac{e^u}{2}[\sin u+\cos u]+C=\frac{x}{2}[\sin\ln x+\cos\ln x]+C.$

(3) $\int_1^e \frac{dx}{x\left(2+\ln^2 x\right)}$;

解 $$\int_1^e \frac{dx}{x\left(2+\ln^2 x\right)} \overset{u=\ln x}{=} \int_0^1 \frac{du}{2+u^2} = \frac{1}{\sqrt{2}}\int_0^1 \frac{d(u/\sqrt{2})}{1+(u/\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}}\arctan\frac{u}{\sqrt{2}}\bigg|_0^1 = \frac{1}{\sqrt{2}}\arctan\frac{1}{\sqrt{2}}.$$

(4) $\int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t+\cos t}dt$

解 $$\int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t+\cos t}dt \overset{s=\frac{\pi}{2}-t}{=} \int_{\frac{\pi}{2}}^0 \frac{\cos\left(\frac{\pi}{2}-s\right)}{\sin\left(\frac{\pi}{2}-t\right)+\cos\left(\frac{\pi}{2}-t\right)}d\left(\frac{\pi}{2}-t\right)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin s}{\sin s+\cos s}ds = \int_0^{\frac{\pi}{2}} \frac{\sin t}{\sin t+\cos t}dt$$

而 $$\int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t+\cos t}dt + \int_0^{\frac{\pi}{2}} \frac{\sin t}{\sin t+\cos t}dt = \int_0^{\frac{\pi}{2}} dt = \frac{\pi}{2},$$

于是 $$\int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t+\cos t}dt = \frac{\pi}{4}.$$

三. (每小题 7 分, 共 21 分)

(1)设 $z(x,y)=\frac{x}{\sqrt{x^2+y^2}}$, 求 $dz\big|_{(0,1)}$;

解 $$\frac{\partial z}{\partial x} = \frac{\sqrt{x^2+y^2}-x\frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{y^2}{(x^2+y^2)^{3/2}}, \qquad \frac{\partial z}{\partial y} = \frac{-x\frac{y}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{-xy}{(x^2+y^2)^{3/2}},$$

于是 $\frac{\partial z}{\partial x}\Big|_{(0,1)}=1,\ \frac{\partial z}{\partial y}\Big|_{(0,1)}=0,$ 故 $dz\big|_{(0,1)} = \frac{\partial z}{\partial x}\Big|_{(0,1)}dx + \frac{\partial z}{\partial y}\Big|_{(0,1)}dy = dx.$

(2)已知 $f(x,y,z)=\ln\left(x+\sqrt{y^2+z^2}\right)$ 及点 $A(1,0,1), B(3,-2,-2)$, 求函数 $f(x,y,z)$ 在点 A 处沿由 A 到 B 的方向导数, 并求此函数在点 A 处方向导数的最大值.

解　$\overrightarrow{AB}=\boldsymbol{l}=(2,-2,-3)$, 故 $(\cos\alpha,\cos\beta,\cos\gamma)=\left(\frac{2}{\sqrt{17}},\frac{-2}{\sqrt{17}},\frac{-3}{\sqrt{17}}\right)$.,　$\frac{\partial f}{\partial x}=\frac{1}{x+\sqrt{y^2+z^2}}$,

$$\frac{\partial f}{\partial y}=\frac{\frac{y}{\sqrt{y^2+z^2}}}{x+\sqrt{y^2+z^2}}=\frac{y}{x\sqrt{y^2+z^2}+y^2+z^2},\quad \frac{\partial f}{\partial y}=\frac{\frac{z}{\sqrt{y^2+z^2}}}{x+\sqrt{y^2+z^2}}=\frac{z}{x\sqrt{y^2+z^2}+y^2+z^2}$$

于是　$\left.\frac{\partial f}{\partial x}\right|_{(1,0,1)}=\frac{1}{2},\left.\frac{\partial f}{\partial y}\right|_{(1,0,1)}=0,\left.\frac{\partial f}{\partial z}\right|_{(1,0,1)}=\frac{1}{2}$,

$$\left.\frac{\partial f}{\partial \boldsymbol{l}}\right|_{(1,0,1)}=\frac{1}{2}\times\frac{2}{\sqrt{17}}+0\times\frac{(-2)}{\sqrt{17}}+\frac{1}{2}\frac{(-3)}{\sqrt{17}}=-\frac{1}{2\sqrt{17}}.$$

在 A 点的方向导数最大值为 $\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2}=\sqrt{\left(\frac{1}{2}\right)^2+0+\left(\frac{1}{2}\right)^2}=\frac{\sqrt{2}}{2}$.

(3)设函数 $z=z(x,y)$ 由方程 $x+y+z=e^z$ 给出,求 $\frac{\partial z}{\partial x}.\frac{\partial z}{\partial y}$ 及 $\frac{\partial^2 z}{\partial x^2}$.

解　令 $F(x,y,z)=e^z-x-y-z$,　则　$\frac{\partial F}{\partial x}=-1,\frac{\partial F}{\partial y}=-1,\frac{\partial F}{\partial z}=e^z-1$. 于是

$$\frac{\partial z}{\partial x}=-\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial z}=\frac{1}{e^z-1},\frac{\partial z}{\partial y}=-\frac{\partial F}{\partial y}\Big/\frac{\partial F}{\partial z}=\frac{1}{e^z-1},$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{1}{e^z-1}\right)=\frac{-e^z}{(e^z-1)^2}\frac{\partial z}{\partial x}=\frac{-e^z}{(e^z-1)^3}.$$

四.(第一小题 4 分, 第二小题 6 分, 共 10 分)

(1)给定空间三点: $A(1,2,0),B(-1,3,1),C(2,-1,2)$, 求 ΔABC 的面积 S.

解　根据向量积的定义，可知三角形 ABC 的面积

$$2S_{\Delta ABC}=|AB||AC|\sin\angle A=\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$$

由于 AB=(-2，1，1)，AC=(1，-3，1)，故

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\-2&1&1\\1&-3&2\end{vmatrix}=5\boldsymbol{i}+5\boldsymbol{j}+5\boldsymbol{k},\qquad \left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\sqrt{5^2+5^2+5^2}=5\sqrt{3}$$

$$S_{\Delta ABC}=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{5\sqrt{3}}{2}.$$

(2)求经过直线 $L_1: \dfrac{x-1}{2}=\dfrac{y+2}{3}=\dfrac{z+3}{4}$ 且平行于直线 $L_2: x=y=\dfrac{z}{2}$ 的平面方程.

解　平面的法向量为 $\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 3 & 4 \\ 1 & 1 & 2 \end{vmatrix} = 2\boldsymbol{i}+0\boldsymbol{j}-\boldsymbol{k}$, 又显然所求平面过点(1, −2, −3)

故所求平面方程为　$2(x-1)+0(y+2)-(z+3)=0$, 即 $2x-z-5=0$.

五. (7 分) 求函数 $f(x)=x^{\frac{1}{x}}, x>0$ 的极值.

解　　$f'(x)=\dfrac{d}{dx}(e^{\frac{\ln x}{x}})=e^{\frac{\ln x}{x}}\cdot\dfrac{1-\ln x}{x^2}=x^{\frac{1}{x}}\cdot\dfrac{1-\ln x}{x^2}$,

令　$f'(x)=0$, 得驻点 $x=e$. 又 当 $0<x<e$ 时, $f'(x)>0$, 当 $x>e$ 时, $f'(x)<0$, 故此点为极大值点, 极大值为 $f(e)=e^{\frac{1}{e}}$.

六. (12 分) 设函数 $f(x)=\dfrac{(x-1)^3}{(x+1)^2}$, 求(1)此函数的单调区间与极值点;(2)此函数的凹凸区间与拐点;(3)此函数的渐近线.

解　$f'(x)=\dfrac{3(x-1)^2(x+1)^2-2(x-1)^3(x+1)}{(x+1)^4}$

$$=\frac{(x-1)^2[3(x+1)-2(x-1)]}{(x+1)^3}=\frac{(x-1)^2(x+5)}{(x+1)^3},$$

$$f''(x)=\frac{[2(x-1)(x+5)+(x-1)^2](x+1)^3-3(x-1)^2(x+5)(x+1)^2}{(x+1)^6}$$

$$=\frac{(x-1)[(3x+9)(x+1)-3(x-1)(x+5)]}{(x+1)^4}=\frac{24(x-1)}{(x+1)^4}.$$

x	$(-\infty,-5)$	-5	$(-5,-1)$	$(-1,1)$	1	$(1,+\infty)$
$f'(x)$	+	0	−	+	0	+
$f''(x)$	−	−	−	−	0	+
$f(x)$	凸, ↗	极大值-27/2	凸, ↘	凸, ↗	拐点(1, 0)	凹, ↗

单调增加区间为$(-\infty,-5)$, $(-1,1)$和$(1,+\infty)$, 单调减少区间为$(-5,-1)$ 函数在点

x=−5 处取到极大值, 极大值为 $f(-5) = -27/2$.

曲线的凸区间为$(-\infty, -1)$和$(-1, 1)$, 凹区间为$(1, +\infty)$, 曲线拐点为$(1, 0)$.

显然曲线有垂直渐近线 x=−1, 曲线无水平渐近线. 由于

$$k = \lim_{x\to\infty}\frac{f(x)}{x} = \lim_{x\to\infty}\frac{(x-1)^3}{x(x+1)^2} = 1,$$

$$b = \lim_{x\to\infty}[f(x) - kx] = \lim_{x\to\infty}\left[\frac{(x-1)^3}{(x+1)^2} - x\right] = \lim_{x\to\infty}\frac{-5x^2+2x-1}{(x+1)^2} = -5,$$

因此曲线有斜渐近线 $y = x-5$.

(七. (每小题 7 分, 共 14 分)

1.求证不等式　$\sin x + \tan x > 2x,\quad 0 < x < \dfrac{\pi}{2}$;

证　令 $f(x) = \sin x + \tan x - 2x,\quad 0 < x < \dfrac{\pi}{2}$, 则 $f'(x) = \cos x + \sec^2 x - 2$,

$$f''(x) = -\sin x + 2\sec x\cdot\sec x\tan x = \frac{\sin x(2-\cos^3 x)}{\cos^3 x} \ge 0,$$

故 $f'(x) = \cos x + \sec^2 x - 2$, 在区间 $0 < x < \dfrac{\pi}{2}$ 内单调增加, 而 $f'(0) = 0$, 于是

在区间 $0 < x < \dfrac{\pi}{2}$ 内 $f'(x) > 0$, 从而 $f(x) = \sin x + \tan x - 2x$在区间$0 < x < \dfrac{\pi}{2}$内单调增加,

于是 $f(x) = \sin x + \tan x - 2x > f(0) = 0$. 证毕.

2.设函数 $f(x)$ 在闭区间$[a, b]$上二阶可导, 且 $f(a) = f(b) = 0$, $f''(x) \ne 0, x \in (a,b)$.

求证: $f(x) \ne 0, x \in (a,b)$.

证　用反证法. 设存在 $c\in(a, b)$, 使得 $f(c)$=0. 于是由罗尔定理,

$$\exists\xi \in (a,c), \exists\eta \in (c,b), \text{使得} f'(\xi) = f'(\eta) = 0.$$

同样由罗尔定理, $\exists\zeta \in (\xi,\eta) \subset (a,b)$, 使得$f''(\zeta) = 0$. 矛盾.