电动力学 第10课

静电场与电势

Maxwell equations and Lorentz force

微分形式

$$\begin{cases} \nabla\cdot\vec{E}=\dfrac{\rho}{\varepsilon_0} \\ \nabla\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t} \\ \nabla\cdot\vec{B}=0 \\ \nabla\times\vec{B}=\mu_0\vec{J}+\varepsilon_0\mu_0\dfrac{\partial\vec{E}}{\partial t} \end{cases}$$

Gauss’ s law

Faraday’ s law

No monopole

Modified Ampera’ s law

Lorentz force

$$\vec{F}=q(\vec{E}+\vec{V}\times\vec{B})+\vec{R}$$

积分形式

$$\begin{cases} \oiint_S \vec{E}\cdot d\vec{\Sigma}=\dfrac{Q}{\varepsilon_0} \\ \oint\vec{E}\cdot d\vec{l}=-\iint\dfrac{\partial\vec{B}}{\partial t}\cdot d\vec{\Sigma} \\ \oiint_S \vec{B}\cdot d\vec{\Sigma}=0 \\ \oint\vec{B}\cdot d\vec{l}=\mu_0\iint\vec{J}\cdot d\vec{\Sigma}+\mu_0\varepsilon_0\iint\dfrac{\partial\vec{E}}{\partial t}\cdot d\vec{\Sigma} \end{cases}$$

1. 电势、电偶极矩

Faraday定律 $\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$

若系统的行为与时间无关， $\frac{\partial\vec{B}}{\partial t}=0$ 则 $\nabla\times\vec{E}=0$ 静电场是无旋的

另一方面，由数学上，对任意标量场 φ ， $\nabla\times\nabla\varphi=0$ （梯度的旋度恒为零）

可引入标量场 φ ，使得 $\vec{E}=-\nabla\varphi$ 静电场可写成另一标量函数的负梯度

电势 φ

$$
\begin{aligned}
\vec{E}\cdot d\vec{l} &= -(\frac{\partial\varphi}{\partial x}\vec{e}_x+\frac{\partial\varphi}{\partial y}\vec{e}_y+\frac{\partial\varphi}{\partial z}\vec{e}_z)\cdot(dx\cdot\vec{e}_x+dy\cdot\vec{e}_y+dz\cdot\vec{e}_z) \\
&= -(\frac{\partial\varphi}{\partial x}dx+\frac{\partial\varphi}{\partial y}dy+\frac{\partial\varphi}{\partial z}dz) \\
&= -d\varphi
\end{aligned}
$$

对任意两点 P_1 和 P_2 ，积分

$$\varphi(P_1)-\varphi(P_2)=-\int_{P_2}^{P_1}\vec{E}\cdot d\vec{l}$$

积分与路径无关

对于有限分布的电荷体系，其中一种比较通用的选择是取无穷远处的电势为零

$$\varphi(\infty)=0$$

于是任一点的电势可表示为

$$\varphi(r)=\int_{r}^{\infty}\vec{E}\cdot d\vec{l}$$

$$\left.\begin{aligned}&\nabla\cdot\vec{E}=\frac{\rho}{\varepsilon_0}\\&\vec{E}=-\nabla\varphi\end{aligned}\right\}\Longrightarrow\nabla\cdot\vec{E}=-\nabla\cdot\nabla\varphi=-\nabla^2\varphi=\frac{\rho}{\varepsilon_0}$$

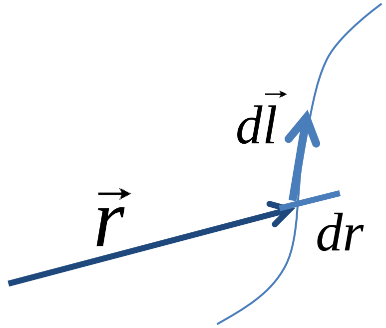
$$\nabla^2\varphi=-\frac{\rho}{\varepsilon_0}$$

Poission方程

几种典型带电体系的电势

（i）单个点电荷 q 的电势

$$\varphi(r)=\int_r^\infty \vec{E}\cdot d\vec{l}=\int_r^\infty \frac{q\vec{e}_r}{4\pi\varepsilon_0 r^2}\cdot d\vec{l}=\int_r^\infty \frac{q}{4\pi\varepsilon_0 r^2}dr=\frac{q}{4\pi\varepsilon_0 r}$$

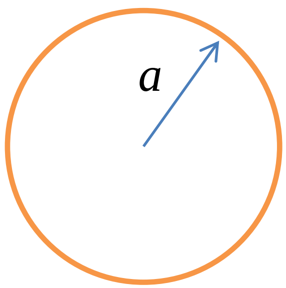


推广（一堆点电荷形成的）带电体系的电势：$\varphi=\sum_i \frac{q_i}{4\pi\varepsilon_0 r_i}$

$$\vec{e}_r\cdot d\vec{l}=dr$$

（ii）半径为 a 、电量Q均匀分布的球状体的球内外的电势

球外的电势 $\varphi(r)=\int_r^\infty \frac{Q}{4\pi\varepsilon_0 r^2}dr=\frac{Q}{4\pi\varepsilon_0 r}$

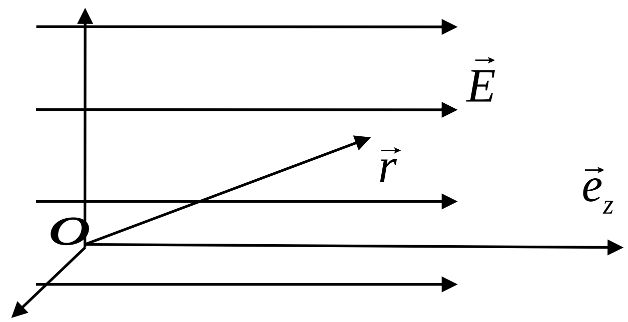


球内的电势 $\varphi(r)=\int_a^\infty \vec{E}_1\cdot d\vec{l}+\int_r^a \vec{E}_2\cdot d\vec{l}$

$$=\int_a^\infty \frac{Q\vec{e}_r}{4\pi\varepsilon_0 r^2}\cdot d\vec{l}+\int_r^a \frac{Q\vec{r}}{4\pi\varepsilon_0 a^3}\cdot d\vec{l}$$

$$=\frac{Q}{4\pi\varepsilon_0 a}+\frac{Q}{4\pi\varepsilon_0 a^3}\int_r^a rdr=\frac{3Q}{8\pi\varepsilon_0 a}-\frac{Qr^2}{8\pi\varepsilon_0 a^3}$$

（iii）均匀电场 的电势



$$\vec{E} = E\vec{e}_z$$

$$\varphi(r) = \varphi(0) - \int_0^r \vec{E}\cdot d\vec{l} = \varphi(0) - Ez = -\vec{E}\cdot\vec{r}$$

（iv）多个点电荷体系的电势

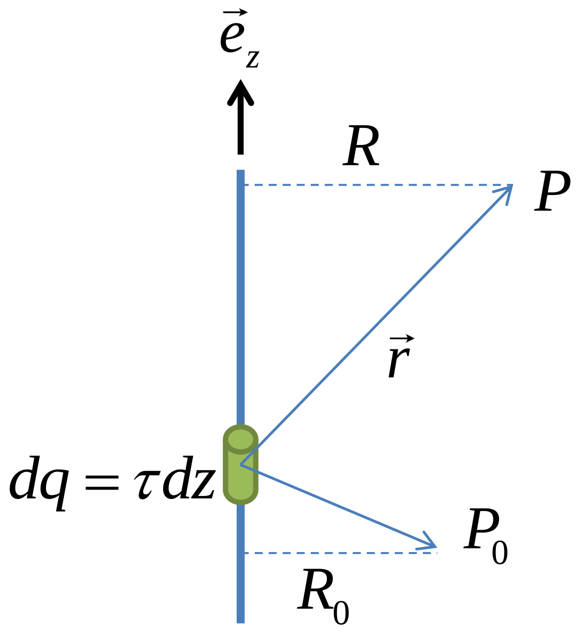
$$\varphi = \int_P^\infty \vec{E}\cdot d\vec{l} = \int_P^\infty \sum_i \frac{q_i \vec{r}_i}{4\pi\varepsilon_0 r_i^3}\cdot d\vec{l} = \sum_i \int_r^\infty \frac{q_i}{4\pi\varepsilon_0 r_i^2} dr_i$$

$$\varphi = \sum_i \frac{q_i}{4\pi\varepsilon_0 r_i}$$

多个点电荷体系的电势等于各个点电荷电势贡献的代数叠加

（v）均匀带电的无限长直线（电荷线密度 τ ）的电势

$$d\varphi = \frac{dq}{4\pi\varepsilon_0 r} = \frac{\tau dz}{4\pi\varepsilon_0\sqrt{R^2+z^2}}$$



$$\varphi(P) = \int_{-\infty}^{\infty} \frac{\tau}{4\pi\varepsilon_0\sqrt{R^2+z^2}} dz = \frac{\tau}{4\pi\varepsilon_0} \ln\left(z+\sqrt{R^2+z^2}\right)\Big|_{-\infty}^{\infty} \longrightarrow \text{发散}$$

原因：电荷不是分布在有限区域

对策：选 P_0 处的电势为零

$$\varphi(P)-\varphi(P_0)=\frac{\tau}{4\pi\varepsilon_0}\left[\ln\left(z+\sqrt{R^2+z^2}\right)\Big|_{-\infty}^{\infty}-\ln\left(z+\sqrt{R_0^2+z^2}\right)\Big|_{-\infty}^{\infty}\right]$$

$$=\frac{\tau}{4\pi\varepsilon_0}\lim_{z\to\infty}\ln\frac{z+\sqrt{R^2+z^2}}{-z+\sqrt{R^2+z^2}}\cdot\frac{-z+\sqrt{R_0^2+z^2}}{z+\sqrt{R_0^2+z^2}}$$

$$=\frac{\tau}{4\pi\varepsilon_0}\lim_{z\to\infty}\ln\frac{1+\sqrt{(R/z)^2+1^2}}{1+\sqrt{(R_0/z)^2+1^2}}\cdot\frac{-1+\sqrt{(R/z)^2+1^2}}{-1+\sqrt{(R_0/z)^2+1^2}}$$

利用Taylor展开： $\sqrt{1+x^2}\approx 1+\dfrac{x^2}{2}$ $\quad (x\ll 1)$

$$\frac{1+\sqrt{(R/z)^2+1^2}}{1+\sqrt{(R_0/z)^2+1^2}}\cdot\frac{-1+\sqrt{(R/z)^2+1^2}}{-1+\sqrt{(R_0/z)^2+1^2}}\approx\frac{2+0.5(R/z)^2}{2+0.5(R_0/z)^2}\cdot\left(\frac{R_0/z}{R/z}\right)^2 \Rightarrow \left(\frac{R_0}{R}\right)^2$$

$$\varphi(P)=\frac{\tau}{2\pi\varepsilon_0}\ln\frac{R_0}{R}$$

在柱坐标下 $$\nabla\varphi=\frac{\partial\varphi}{\partial R}\vec{e}_r+\frac{1}{R}\cdot\frac{\partial\varphi}{\partial\phi}\vec{e}_\phi+\frac{\partial\varphi}{\partial z}\vec{e}_z$$

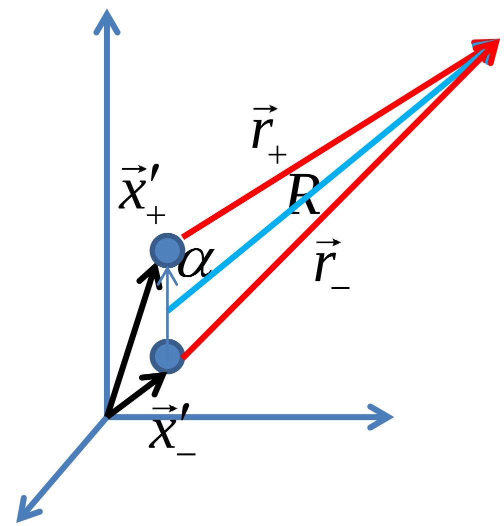
$$E_R=-\frac{\partial\varphi}{\partial R}=\frac{\tau}{2\pi\varepsilon_0 R}$$

$$\vec{E}=-\nabla\varphi$$

$$E_\phi=E_z=0$$

（vi）电偶极子在远处的电势

两个等量异号的点电荷 q 和 $-q$ 就构成了一个电偶极子



定义其电偶极矩 $\vec{p}=\sum_{i=1}^{2} q_i \vec{x}_i\,'$

$$\vec{l}=\vec{x}_+\,'-\vec{x}_-\,'$$

$$\vec{p}=\sum_i q_i\vec{x}_i=q\vec{x}_+\,'+(-q)\vec{x}_-\,'=q(\vec{x}_+\,'-\vec{x}_-\,')=q\vec{l}$$

$$\varphi=\frac{q}{4\pi\varepsilon_0}\frac{1}{r_+}+\frac{-q}{4\pi\varepsilon_0}\frac{1}{r_-}=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_+}-\frac{1}{r_-}\right)$$

$$\frac{1}{r_+}-\frac{1}{r_-}=\frac{r_- - r_+}{r_+ r_-}\approx\frac{r_- - r_+}{R^2}$$

$$r_- - r_+\approx l\cos\alpha$$

$$\varphi=\frac{q}{4\pi\varepsilon_0}\frac{l\cos\alpha}{R^2}=\frac{1}{4\pi\varepsilon_0}\frac{pR\cos\alpha}{R^3}=\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{R}}{R^3}$$

$$\vec{E}=-\nabla\varphi=-\frac{1}{4\pi\varepsilon_0}\nabla\left(\vec{p}\cdot\frac{\vec{R}}{R^3}\right)$$

由数学公式 $$\nabla\left(\vec{A}\cdot\vec{B}\right)=\vec{A}\times\left(\nabla\times\vec{B}\right)+\left(\vec{A}\cdot\nabla\right)\vec{B}+\vec{B}\times\left(\nabla\times\vec{A}\right)+\left(\vec{B}\cdot\nabla\right)\vec{A}$$

$$\vec{E}=-\frac{1}{4\pi\varepsilon_0}\left[\vec{p}\times\left(\underbrace{\nabla\times\frac{\vec{R}}{R^3}}_{=0}\right)+\left(\vec{p}\cdot\nabla\right)\frac{\vec{R}}{R^3}+\frac{\vec{R}}{R^3}\times\underbrace{\left(\nabla\times\vec{p}\right)}_{=0}+\underbrace{\left(\frac{\vec{R}}{R^3}\cdot\nabla\right)\vec{p}}_{=0}\right]$$

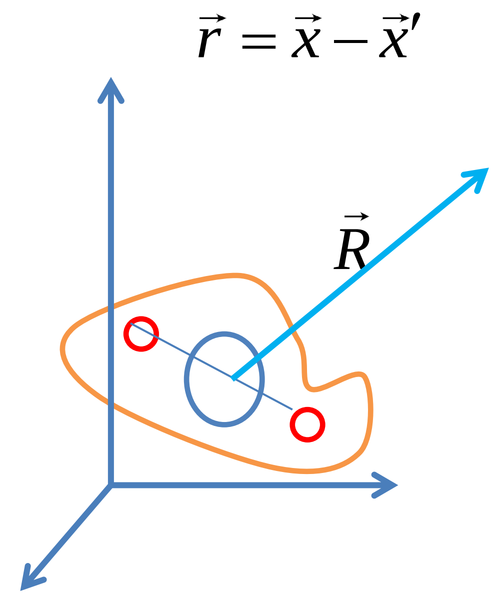
$$=-\frac{\vec{p}}{4\pi\varepsilon_0}\cdot\nabla\frac{\vec{R}}{R^3}=-\frac{\vec{p}}{4\pi\varepsilon_0}\cdot\left(R^{-3}\nabla\vec{R}+\left(\nabla R^{-3}\right)\vec{R}\right)$$

$$=-\frac{\vec{p}}{4\pi\varepsilon_0}\cdot\left[-3\frac{\vec{R}\vec{R}}{R^5}+\frac{\overleftrightarrow{I}}{R^3}\right]=\frac{1}{4\pi\varepsilon_0}\left[\frac{3\left(\vec{p}\cdot\vec{R}\right)}{R^5}\vec{R}-\frac{\vec{p}}{R^3}\right]\cdot$$

其中 $$\nabla R^{-3}=\frac{dR^{-3}}{dR}\nabla R=-\frac{3}{R^4}\frac{\vec{R}}{R}$$

$$\nabla\vec{R}=\overleftrightarrow{I}$$

任意形状带电体系的电势，电势的多极展开



在无界空间中， Poisson方程的解就是

$$\varphi=\sum_i \frac{q_i}{4\pi\varepsilon_0 r_i}=\int \frac{\rho(\vec{x}')}{4\pi\varepsilon_0 r}dV'$$

一般情况下，系统的形状不一定会具有对称性，积分实际上是很困难的

最简单粗糙的近似，看成是点电荷

$$\varphi^{(0)}=\frac{Q}{4\pi\varepsilon_0 R}$$

$$Q=\sum_i q_i$$ 分立电荷分布系统

$$Q=\int \rho(\vec{x}')dV'$$ 连续电荷分布系统

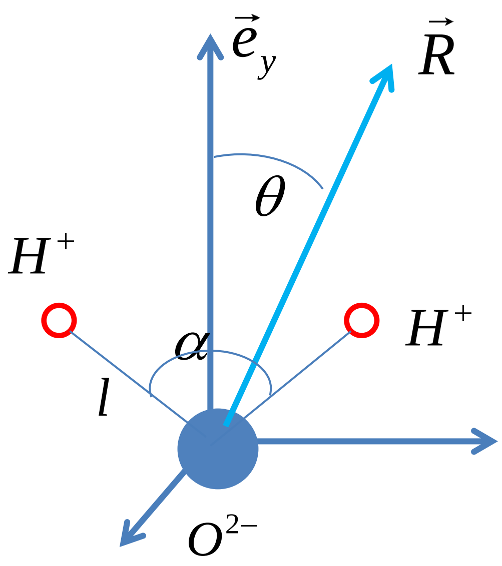
最基本的修正就是把带电体看成是一个电偶极子

$$\varphi^{(1)}=\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{R}}{R^3}$$

$$\vec{p}=\sum_{i=1}^{n} q_i \vec{x}_i'$$ 分立电荷分布系统

$$\vec{p}=\int \rho(\vec{x}')\vec{x}'dV'$$ 连续电荷分布系统

更高级的修正是把带电体看成是一个电四极矩



$$\varphi = \varphi^{(0)} + \varphi^{(1)} + \varphi^{(2)} + \ldots$$

$$\varphi^{(0)} \sim \frac{1}{R} \qquad \varphi^{(1)} \sim \frac{1}{R^2} \qquad \varphi^{(2)} \sim \frac{1}{R^3}$$

例：水分子的远场电势

$$Q = -2e + 2 \times e = 0$$

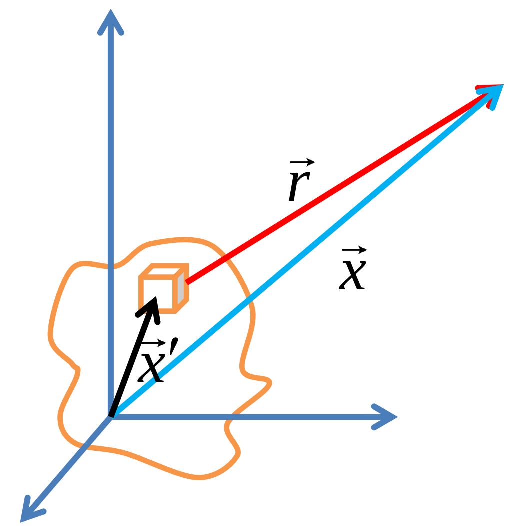
$$\varphi^{(0)} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{R} = 0$$

$$\vec{p} = \sum_i q_i \vec{r}_i' = -2e \times 0 + e\vec{r}_1 + e\vec{r}_2 = e(\vec{r}_1 + \vec{r}_2) = 2el \cdot \cos\frac{\alpha}{2} \cdot \vec{e}_y$$

$$\varphi \approx \varphi^{(1)} = \frac{1}{4\pi\varepsilon_0} \frac{\vec{P} \cdot \vec{R}}{R^3} = \frac{el \cos\frac{\alpha}{2} \cos\theta}{2\pi\varepsilon_0 R^2}$$

电势的多极展开

$$\vec{r}=\vec{x}-\vec{x}'$$



考虑单变量函数 $f\left(x-x'\right)$

在x附近 $\left(x'=0\right)$ 作Taylor展开：

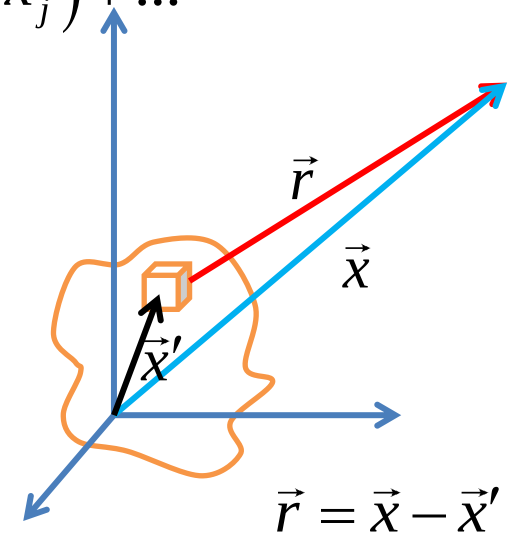
$$f\left(x-x'\right)=f\left(x\right)+\left.\frac{\partial f}{\partial x}\right|_{x'=0}\left(x-x'-x\right)+\frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x'=0}\left(x'x'\right)+\ldots$$

$$=f\left(x\right)-\left.\frac{\partial f}{\partial x}\right|_{x'=0}\cdot x'+\frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x'=0}\left(x'x'\right)+\ldots$$

推广到三维空间

$$f\left(\vec{x}-\vec{x}'\right)=f\left(\vec{x}\right)-\sum_{i=1}^{3}\left.\frac{\partial f}{\partial x_i}\right|_{x'=0}\cdot x_i'+\frac{1}{2}\sum_{ij}\left.\frac{\partial^2 f}{\partial x_i\partial x_j}\right|_{x'=0}\cdot\left(x_i'x_j'\right)+\ldots$$

$$f\left(\vec{x}-\vec{x}'\right)=f\left(\vec{x}\right)-x_i'\cdot\nabla f\Big|_{x'=0}+\frac{1}{2}\sum_{ij}\frac{\partial^2 f}{\partial x_i\partial x_j}\Bigg|_{x'=0}\cdot\left(x_i'x_j'\right)+\ldots$$



令 $f\left(\vec{x}-\vec{x}'\right)=\frac{1}{r}$

$$\frac{1}{r}=\frac{1}{R}-\vec{x}'\cdot\nabla\frac{1}{R}+\frac{1}{2}\sum_{ij}x_i'x_j'\frac{\partial^2\ {}^1\!/_R}{\partial x_i\partial x_j}+\ldots$$

$$=\frac{1}{R}+\frac{\vec{x}'\cdot\vec{R}}{R^3}+\frac{1}{2}\sum_{ij}x_i'x_j'\frac{\partial^2\ {}^1\!/_R}{\partial x_i\partial x_j}+\ldots$$

$$\varphi=\int\frac{\rho(\vec{x}')}{4\pi\varepsilon_0}\left[\frac{1}{R}+\frac{\vec{x}'\cdot\vec{R}}{R^3}+\frac{1}{2}\sum_{ij}x_i'x_j'\frac{\partial^2\ {}^1\!/_R}{\partial x_i\partial x_j}+\ldots\right]dV'$$

$$=\int\frac{\rho(\vec{x}')}{4\pi\varepsilon_0 R}dV'+\int\frac{\rho}{4\pi\varepsilon_0}\frac{\vec{x}'\cdot\vec{R}}{R^3}dV'+\int\frac{1}{8\pi\varepsilon_0}\left[\sum_{ij}x_i'x_j'\rho(\vec{x}')\right]dV'\frac{\partial^2\ {}^1\!/_R}{\partial x_i\partial x_j}+\ldots$$

$$=\frac{Q}{4\pi\varepsilon_0 R}+\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{R}}{R^3}+\frac{1}{24\pi\varepsilon_0}\overleftrightarrow{D}:\nabla\nabla\frac{1}{R}+\ldots$$

双点乘

$$=\varphi^{(0)}+\varphi^{(1)}+\varphi^{(2)}+\ldots$$

电偶极矩 $\vec{p}$ 是衡量系统电荷**偏离中心对称分布**的一个物理量，对于具有球对称的带电系统，电偶极矩为零

思考：均匀电荷分布的椭球的电偶极矩是多少？

电四极矩 $\overleftrightarrow{D}$ 是衡量系统电荷**偏离球对称分布**的一个物理量

$\nabla\nabla\frac{1}{R}$ 是张量

$\overleftrightarrow{D}:\nabla\nabla\frac{1}{R}$ 是标量

关于电荷系统的电四极矩

$$\overleftrightarrow{D} = \int 3\rho(\vec{x}')\underline{\vec{x}'\vec{x}'}dV'$$

张量 并矢

分量形式： $D_{ij} = \int 3\rho(\vec{x}')x_i'x_j'dV'$

对于分立电荷系统 $\overleftrightarrow{D} = \sum_k 3q_k\vec{x}_k'\vec{x}_k'$ $D_{ij} = \sum_k 3q_k\left(x_i'x_j'\right)_k$

可见，它是对称张量， $\left(D_{ij} = D_{ji}\right)$ ，因此有6个分量独立：

$$D_{xx}, D_{yy}, D_{zz}, D_{xy}, D_{xz}, D_{yz}$$

注意到 $0 = \nabla^2\frac{1}{R} = \nabla\cdot\nabla\frac{1}{R} = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\frac{1}{R} = \sum_i\frac{\partial^2}{\partial x_i^2}\frac{1}{R}$

即 $0 = r'^2\nabla^2\frac{1}{R} = r'^2\sum_i\frac{\partial^2}{\partial x_i^2}\frac{1}{R} = r'^2\sum_{ij}\delta_{ij}\frac{\partial^2}{\partial x_i\partial x_j}\frac{1}{R}$

$$\varphi^{(2)}=\frac{1}{24\pi\varepsilon_0}\int\left[\sum_{ij}3x_i'x_j'\rho(\vec{x}')\frac{\partial^2\,{}^1\!/\!_R}{\partial x_i\partial x_j}\right]dV'$$

改写成 ➡ $$\varphi^{(2)}=\frac{1}{24\pi\varepsilon_0}\sum_{ij}\left[\int\left(3x_i'x_j'-r'^2\delta_{ij}\right)\rho(\vec{x}')dV'\right]\frac{\partial^2\,{}^1\!/\!_R}{\partial x_i\partial x_j}$$

重新定义电四极矩

$$D_{ij}=\int\left(3x_i'x_j'-r'^2\delta_{ij}\right)\rho\left(\vec{x}'\right)dV'$$

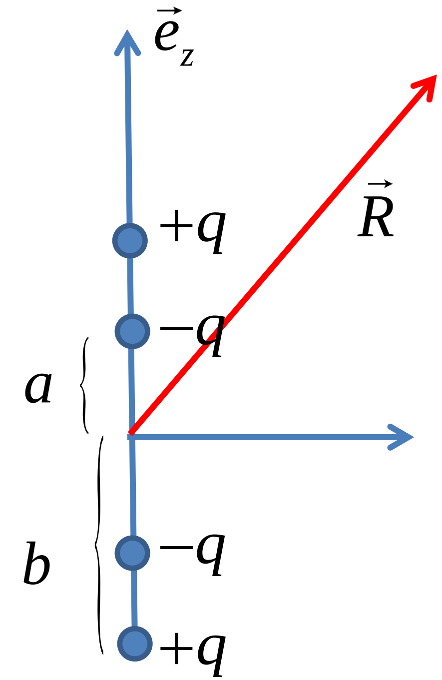
其中 $$r'^2=x'^2+y'^2+z'^2$$

此时它仍为对称张量，但满足 $D_{xx}+D_{yy}+D_{zz}=0$ 因此独立的分量只有5个

$$\varphi^{(2)}=\frac{1}{24\pi\varepsilon_0}\sum_{ij}D_{ij}\frac{\partial^2\,{}^1\!/\!_R}{\partial x_i\partial x_j}=\frac{1}{24\pi\varepsilon_0R^5}\left[\left(3x^2-R^2\right)D_{xx}+\left(3y^2-R^2\right)D_{yy}\right.$$

$$\left.+\left(3z^2-R^2\right)D_{zz}+6xyD_{xy}+6xzD_{xz}+6yzD_{yz}\right]$$

例：z轴上的线电四极子



$$Q=\sum_i q_i=+q-q-q+q=0 \qquad \varphi^{(0)}=\frac{1}{4\pi\varepsilon_0}\frac{Q}{R}=0$$

$$\vec{p}=\sum_i q_i\vec{x}_i'=\left(qb-qa-q(-a)+q(-b)\right)\vec{e}_z=0$$

$$\varphi^{(1)}=\frac{1}{4\pi\varepsilon_0}\frac{\vec{P}\cdot\vec{R}}{R^3}=0$$

$$D_{ij}=\sum_k 3q_k\left(x_i'x_j'\right)_k$$

电四极矩只有一个分量不为零

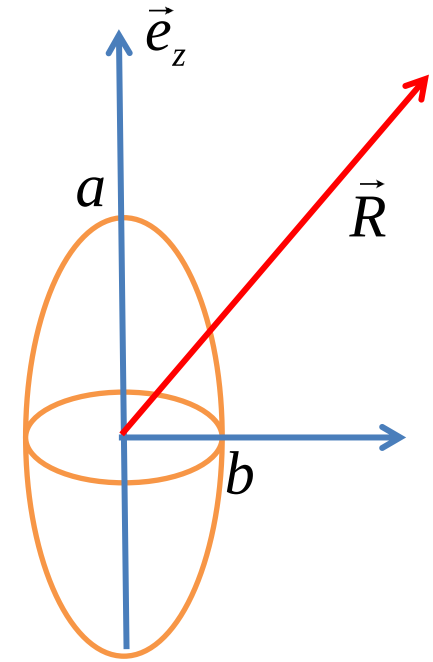
$$D_{zz}=\sum_{k=1}^{4}3q_k\left(x_z'x_z'\right)_k=3qb^2+3(-q)a^2+3(-q)(-a)(-a)+3q(-b)(-b)$$

$$=6q\left(b^2-a^2\right)$$

$$\varphi^{(2)}=\frac{1}{24\pi\varepsilon_0}D_{zz}\left(\frac{3z^2-R^2}{R^5}\right)=\frac{q\left(b^2-a^2\right)}{4\pi\varepsilon_0R^3}\left(3\cos^2\theta-1\right)$$

例：均匀带电的旋转椭圆球体——原子核的经典模型

$$\rho = \frac{Q}{V} = \frac{3Q}{4\pi ab^2} \qquad \frac{x'^2}{b^2} + \frac{y'^2}{b^2} + \frac{z'^2}{a^2} = 1$$



$$dV' = dx'dy'dz' = ab^2 r^2 \sin\theta dr d\theta d\phi$$

用新定义 $D_{ij} = \int \left(3x_i' x_j' - r'^2 \delta_{ij}\right) \rho dV'$

由于体系关于z轴对称

$$D_{xy} = \rho \int 3x_i' x_j' dV' = 3\rho \int (br\sin\theta\cos\phi)(br\sin\theta\sin\phi) ab^2 r^2 \sin\theta dr d\theta d\phi$$

$$= 3\rho ab^4 \int_0^1 r^4 dr \int_0^\pi \sin^3\theta d\theta \int_0^{2\pi} \sin\phi\cos\phi d\phi = 0$$

同理 $D_{yz} = D_{xz} = 0 \qquad D_{xx} = D_{yy}$

而 $D_{xx} + D_{yy} + D_{zz} = 0$ 即 $D_{xx} = D_{yy} = -\frac{1}{2} D_{zz}$

$$D_{zz} = \rho\int\left(3x_i'x_j' - r'^2\right)dV' = \rho\int\left(2z'^2 - x'^2 - y'^2\right)dV'$$

$$\int z'^2 dV' = \int\left(ra\cos\theta\right)^2 ab^2 r^2 \sin\theta dr d\theta d\phi$$

$$= a^3 b^2 \int_0^1 r^4 dr \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\cos^2\theta d\theta = \frac{4\pi}{15}a^3 b^2$$

$$D_{zz} = \rho\int\left(2z'^2 - 2x'^2\right)dV' = \frac{2Q}{5}\left(a^2 - b^2\right)$$

$$D_{xx} = D_{yy} = -\frac{1}{2}D_{zz} = -\frac{Q}{5}\left(a^2 - b^2\right)$$

$$\varphi^{(2)} = \frac{1}{24\pi\varepsilon_0 R^5}\left[D_{xx}\left(3x^2 - R^2\right) + D_{yy}\left(3y^2 - R^2\right) + D_{zz}\left(3z^2 - R^2\right)\right]$$

$$= \frac{D_{zz}}{24\pi\varepsilon_0 R^5}\left[-\frac{1}{2}\left(3x^2 - R^2\right) - \frac{1}{2}\left(3y^2 - R^2\right) + \left(3z^2 - R^2\right)\right]$$

$$= \frac{Q\left(a^2 - b^2\right)}{40\pi\varepsilon_0 R^5}\left(3z^2 - R^2\right) = \frac{Q\left(a^2 - b^2\right)}{40\pi\varepsilon_0 R^3}\left(3\cos^2\theta - 1\right)$$

作业

1。边长为 a 的等边三角形的三个顶点分别放置等量电荷 q ，求下列情况下远处的电势（精确到电偶极矩）。

（i）三个都是正电荷；

（ii）两个正电荷和一个负电荷。

2。如图，边长为 a 的正方形的四个顶角分别放置电荷，求下列情况下远处的电势（精确到电四极矩）。

（i）这些电荷的电量分别为 $2q$、$-q$、0、q；

（ii）这些电荷的电量分别为 $-q$、q、$-q$、q 。