Lecture 6
第六讲

Application of Derivatives
导数的应用

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目录

一、Lagrange公式

罗尔定理

$$\left\{\begin{array}{c} f(x)\text{在}[a,b]\text{上连续} \\ f(x)\text{在}(a,b)\text{内可导} \\ f(a)=f(b) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} \exists\xi\in(a,b),\ni \\ f'(\xi)=0 \end{array}\right\}$$

拉格朗日中值定理

$$\left\{\begin{array}{c} f(x)\text{在}[a,b]\text{上连续} \\ f(x)\text{在}(a,b)\text{内可导} \end{array}\right\} \Rightarrow \left\{\begin{array}{c} \exists\xi\in(a,b),\ni \\ f'(\xi)=\dfrac{f(a)-f(b)}{a-b} \end{array}\right\}$$

二、Cauchy公式

柯西中值定理

$$\boxed{\begin{gathered} f(x)\text{和}g(x)\text{在}[a,b]\text{上连续} \\ f(x)\text{和}g(x)\text{在}(a,b)\text{内可导} \\ g'(x)\neq 0 \end{gathered}}$$

$$\Downarrow$$

$$\boxed{\begin{gathered} \exists\xi\in(a,b),\ni \\ \frac{f'(\xi)}{g'(\xi)}=\frac{f(a)-f(b)}{g(a)-g(b)} \end{gathered}}$$

三、L'Hospital法则

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0(\text{或}\infty)$$

$f(x)$和$g(x)$在掏空点a的某邻域内可导

且$g'(x)\neq 0$

$$\lim_{x\to a}\frac{f'(x)}{g'(x)} = A(\text{或}\infty)$$

$$\Downarrow$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = A(\text{或}\infty)$$

其中a为有限数或∞

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$$

$f(x)$和$g(x)$在掏空点a的某邻域内可导

且$g'(x) \neq 0$

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = A(\text{或}\infty)$$

$$\Downarrow$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = A(\text{或}\infty)$$

其中a为有限数

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$$

$f(x)$和$g(x)$在无穷远处可导

且$g'(x) \neq 0$

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = A(\text{或}\infty)$$

$$\Downarrow$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = A(\text{或}\infty)$$

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty$$

$f(x)$和$g(x)$在掏空点a的某邻域内可导

且$g'(x) \neq 0$

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = A(\text{或}\infty)$$

$\Downarrow$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = A(\text{或}\infty)$$

其中a为有限数或∞

四、Taylor展开

Taylor(泰勒)展开

若函数$f(x)$在点x_0的某一邻域内具有n阶导数，则在该邻域内$f(x)$的n阶泰勒公式为

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2$$
$$+\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f^{(4)}(x_0)}{4!}(x-x_0)^4$$
$$+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+o((x-x_0)^n)$$

其中$o((x-x_0)^n)$称为Peano余项.

$o((x-x_0)^n)$是关于$(x-x_0)^n$的高阶无穷小量.

麦克劳林展开

若函数$f(x)$在点0的某一邻域内具有n阶导数，则在该邻域内$f(x)$的n阶泰勒公式为

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots+\frac{f^{(n)}(0)}{n!}x^n+o(x^n)$$

若函数$f(x)$在点x_0的某一邻域内具有$n+1$阶导数，则在该邻域内$f(x)$的n阶泰勒公式为

$$
\begin{aligned}
f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 \\
+ \frac{f'''(x_0)}{3!}(x-x_0)^3 + \frac{f^{(4)}(x_0)}{4!}(x-x_0)^4 \\
+ \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x)
\end{aligned}
$$

其中

$R_n(x)$称为Lagrange余项.

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1},\ \xi\text{在}x\text{与}x_0\text{之间}.$$

泰勒级数

$$f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2$$
$$+\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f^{(4)}(x_0)}{4!}(x-x_0)^4$$
$$+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\cdots$$

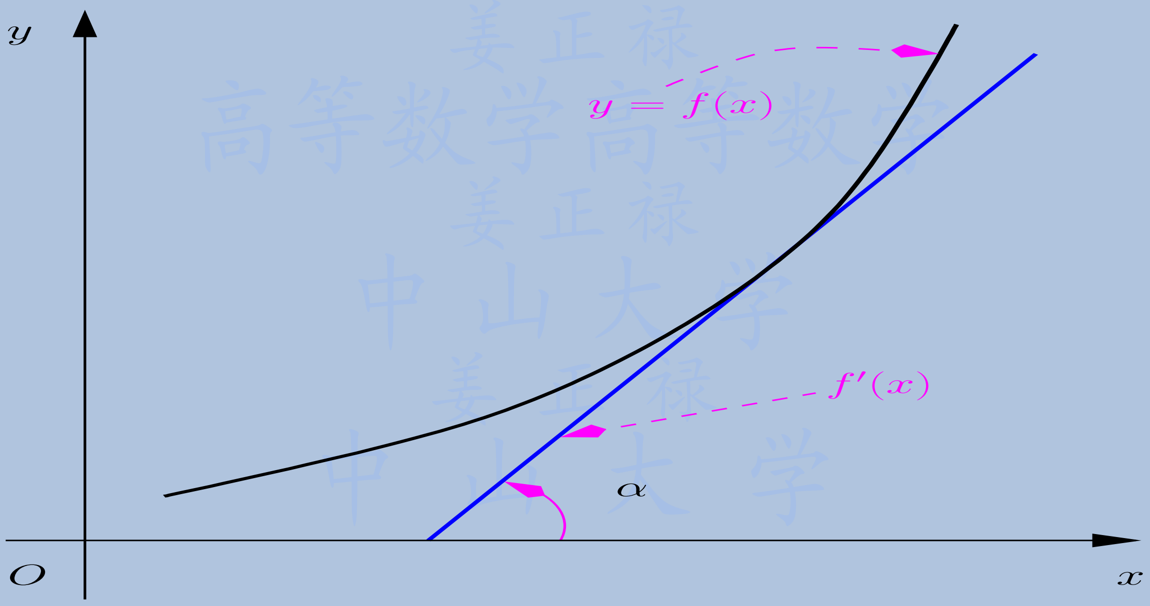
麦克劳林级数

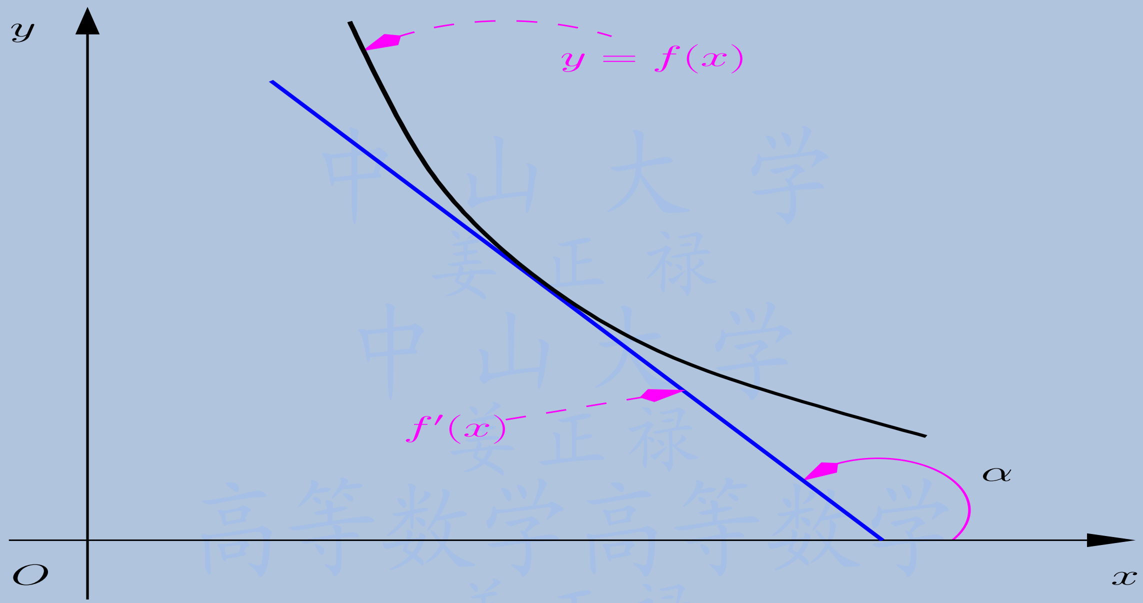
$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3$$
$$+\frac{f^{(4)}(0)}{4!}x^4+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots$$

五、函数的局部性质

增减性、凹凸性、极值、渐近性、曲率

1. *增减性*





设$y=f(x)$是一个可导函数.

如$f'(x)>0$, 则函数$y=f(x)$严格单调增加;

如$f'(x)<0$, 则函数$y=f(x)$严格单调减少.

2. 凹凸性

凸函数(convex function)

In mathematics, a real-valued function $f(x)$ defined on an interval (or on any convex subset of some vector space) is called convex, concave upwards, concave up or convex cup, if for any two points x_1 and x_2 in its domain X and any $t \in [0, 1]$,

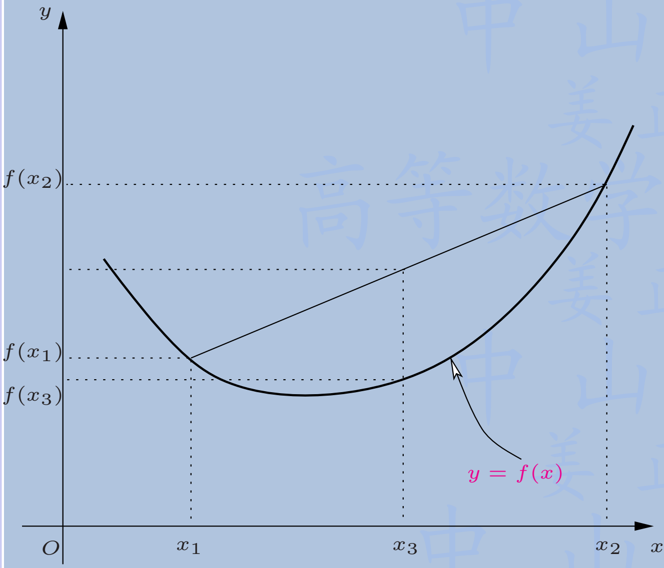
$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2).$$

A function is called strictly convex if

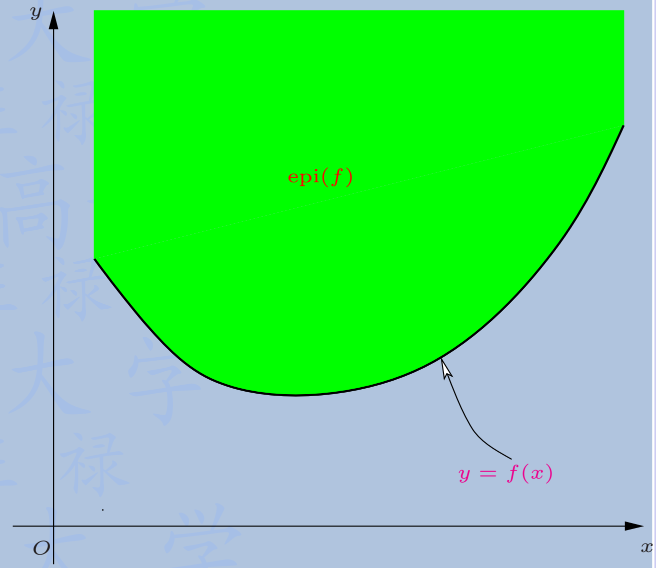
$$f(tx_1 + (1-t)x_2) < tf(x_1) + (1-t)f(x_2)$$

for any t in $(0, 1)$ and $x_1 \ne x_2$.

Sometimes an alternative definition is used:
A function is convex if its epigraph (the set of points lying on or above the graph) is a convex set. These two definitions are equivalent, i.e., one holds if and only if the other one is true.



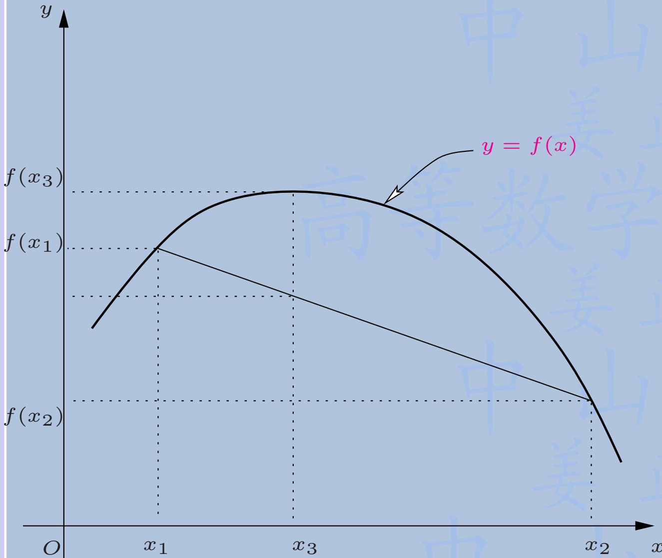
$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$



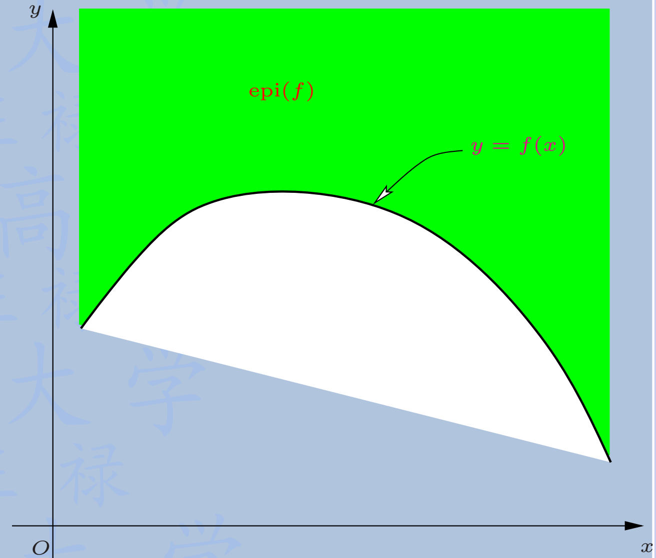
$$\mathrm{epi}(f) = \{(x, \mu) : \mu \geq f(x), x \in X\}$$

凹函数(concave function)

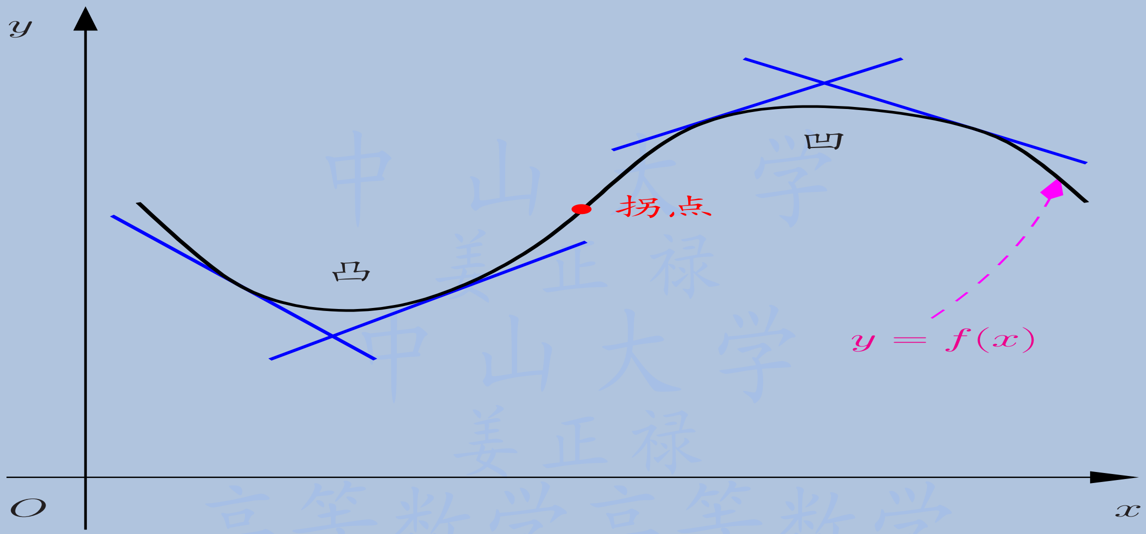
In mathematics, a concave function is the negative of a convex function. A concave function is also synonymously called concave downwards, concave down, convex cap or upper convex. A strictly concave function is the negative of a strictly convex function.



$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2)$



$\mathrm{epi}(f) = \{(x, \mu) : \mu \geq f(x), x \in X\}$凹



凹弧与凸弧的分界点 — 拐点

设$y = f(x)$是一个二阶可导函数.

如$f''(x) > 0$, 则$y = f(x)$是凸函数;

如$f''(x) < 0$, 则$y = f(x)$是凹函数.

3. 极值

$\boxed{\text{驻点} \Leftrightarrow f'(x)=0}$

设$U(x_0)$是x_0的某邻域, x_0是$f(x)$的极值点.

极值点

$\Downarrow$

极大值点

$\left(\begin{array}{c}\exists U(x_0), \forall x \in U(x_0), \\ f(x) \le f(x_0)\end{array}\right)$

$\Uparrow$

$f'(x)(x-x_0)<0$

或

$f''(x_0)<0$

极小值点

$\left(\begin{array}{c}\exists U(x_0), \forall x \in U(x_0), \\ f(x) \ge f(x_0)\end{array}\right)$

$\Uparrow$

$f'(x)(x-x_0)>0$

或

$f''(x_0)>0$

4. 渐近性

水平渐近线　垂直渐近线　斜渐近线

水平渐近线　　$y = A$

$$\lim_{x\to\infty} f(x) = A \text{ 或 } \lim_{x\to\pm\infty} f(x) = A$$

垂直渐近线　　$x = x_0$

$$\lim_{x\to x_0} f(x) = \infty \text{ 或 } \lim_{x\to x_0} f(x) = \pm\infty$$

斜渐近线

$$y = ax + b$$

$$\lim_{x\to\infty}[f(x)-(ax+b)]=0$$

或

$$\lim_{x\to\pm\infty}[f(x)-(ax+b)]=0$$

$$\Updownarrow$$

$$\lim_{x\to\infty}\frac{f(x)}{x}=a,\quad \lim_{x\to\infty}(f(x)-ax)=b$$

或

$$\lim_{x\to\pm\infty}\frac{f(x)}{x}=a,\quad \lim_{x\to\pm\infty}(f(x)-ax)=b$$

5. 曲率

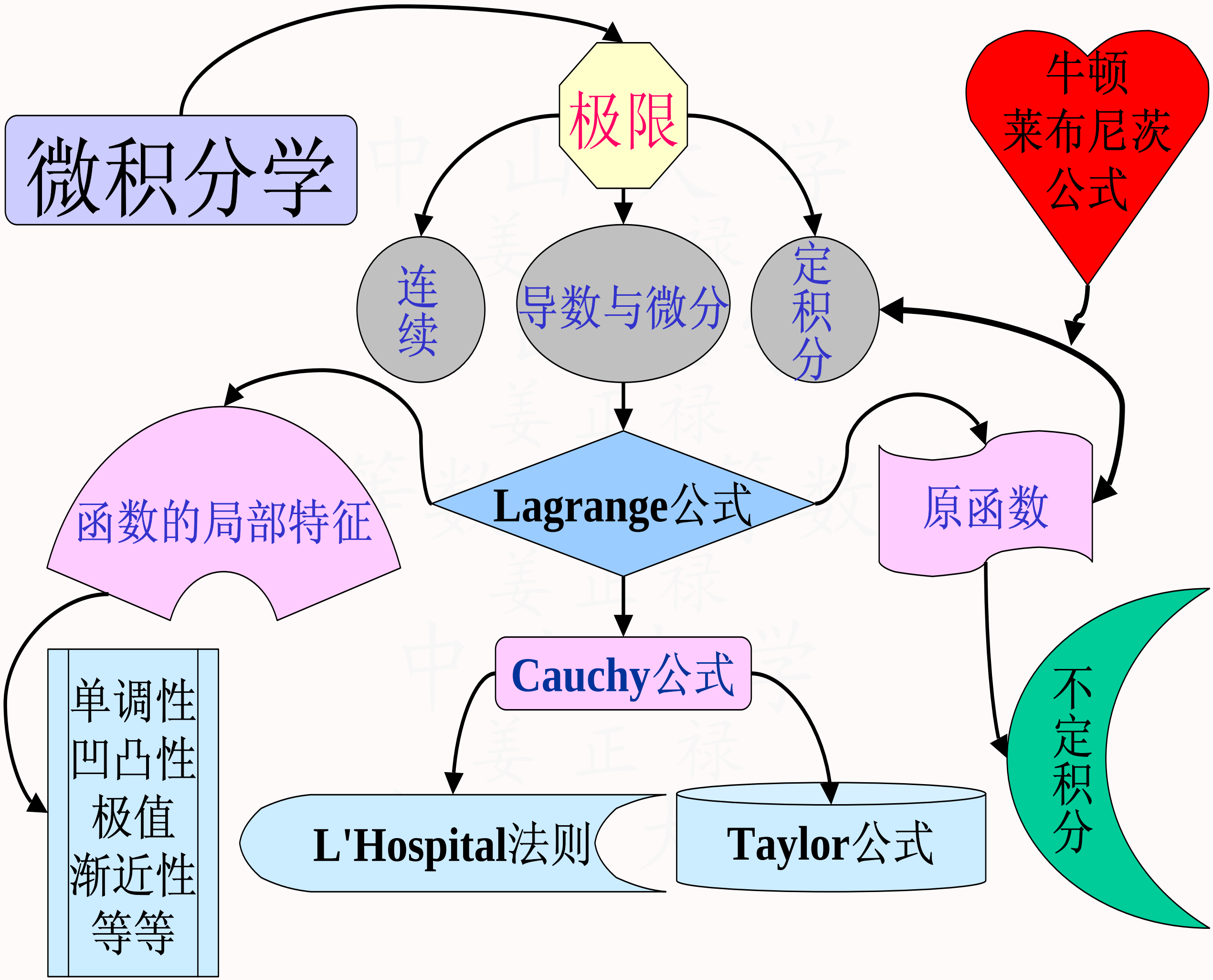
$$K = |\frac{d\alpha}{ds}| = \frac{|y''|}{(1+(y')^2)^{3/2}}$$

曲率半径

$$R = \frac{1}{K}$$

小结

- Lagrange公式、Cauchy公式
- L'Hospital法则
- Taylor展开
- 函数的局部性质（单调性、极值、凹凸性、渐近性）



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These slides are designed by Zhenglu Jiang. Please do not hesitate to contact him by email (mcsjzl@mail.sysu.edu.cn) if you have any questions!

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