學策云

中山大学地理科学与规划学院团委学生会学术策划部荣誉出品

珠海校区2010学年度1期10级高等数学2期末试题参考解答

一. 1. 用积分中值定理证明 $\lim\limits_{x\to 0}\int_{-x^3}^{0} e^{t^2}\,\mathrm{d}t = 0;$

证: 由积分中值定理, 存在$\xi\in(-x^3,0)$, 使得

$$\left|\int_{-x^3}^{0} e^{t^2}\,\mathrm{d}t\right| = \left|e^{\xi^2}[0-(-x^3)]\right| = e^{\xi^2}\left|x^3\right| \le e^{x^6}\left|x^3\right|$$

于是

$$\lim_{x\to 0}\left|\int_{-x^3}^{0} e^{t^2}\,\mathrm{d}t\right| \le \lim_{x\to 0} e^{x^6}\left|x^3\right| = \lim_{x\to 0} e^{x^6}\cdot\lim_{x\to 0}\left|x^3\right| = 0,$$

此即

$$\lim_{x\to 0}\int_{-x^3}^{0} e^{t^2}\,\mathrm{d}t = 0.$$

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2.求 $\lim\limits_{x\to 0}\dfrac{\int_{-x^3}^{0} e^{t^2}\,\mathrm{d}t}{x\tan x^2}$.

解：此为 $\dfrac{0}{0}$ 型不定式，由洛必达法则及

$$\tan x^2 \sim x^2 \quad (x\to 0)$$

$$\lim_{x\to 0}\frac{\int_{-x^3}^{0} e^{t^2}\,\mathrm{d}t}{x\tan x^2}=\lim_{x\to 0}\frac{\left(\int_{-x^3}^{0} e^{t^2}\,\mathrm{d}t\right)'}{(x\cdot x^2)'}$$

$$=\lim_{x\to 0}\frac{-e^{(-x^3)^2}(-3x^2)}{3x^2}=1.$$

二. 求函数 $f(x)=(x-1)\cos x-\sin x$ 在区间 $\left[0,\dfrac{\pi}{2}\right]$ 上的最大值和最小值.

解: $f'(x)=\cos x-(x-1)\sin x-\cos x=-(x-1)\sin x$

令 $f'(x)=0,$ 得驻点 $x_1=0, x_2=1;$ 而

$$f(0)=-1,\quad f(1)=-\sin 1,\quad f(\frac{\pi}{2})=-1,$$

因此

$$\min_{x\in\left[0,\frac{\pi}{2}\right]} f(x)=f(0)=f(\frac{\pi}{2})=-1.$$

$$\max_{x\in\left[0,\frac{\pi}{2}\right]} f(x)=f(1)=-\sin 1,$$

三. 设 $\begin{cases} x = \arccos\sqrt{1-t^2}, \\ y = \dfrac{(\ln t)^2}{2}, \end{cases} (0 < t < 1)$, 求 $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

解:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \ln t \cdot \frac{1}{t} = \frac{\ln t}{t},$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{\sqrt{1-(\sqrt{1-t^2})^2}} \cdot \frac{1}{2\sqrt{1-t^2}} \cdot (-2t)$$

$$= \frac{t}{|t|\sqrt{1-t^2}} = \frac{1}{\sqrt{1-t^2}}$$

因此

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \Big/ \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\sqrt{1-t^2}\,\ln t}{t}.$$

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四. 求下列不定积分:

$$1.\int\frac{6}{x^2-9}dx;\quad 2.\int\tan^3 x dx;\quad 3.\int\frac{x^2}{\sqrt{4-x^2}}dx;\quad 4.\int\frac{2xdx}{x^2+2x+2}.$$

解: $$1.\int\frac{6}{x^2-9}dx=\int\frac{1}{x-3}dx-\int\frac{1}{x+3}dx$$

$$=\ln|x-3|-\ln|x+3|+C=\ln\left|\frac{x-3}{x+3}\right|+C;$$

$$2.\int\tan^3 x dx=\int\tan x(\sec^2 x-1)dx$$

$$=\int\tan x d\tan x-\int\tan x dx$$

$$=\frac{1}{2}\tan^2 x-\int\frac{\sin x}{\cos x}dx$$

$$=\frac{1}{2}\tan^2 x+\ln|\cos x|+C;$$

$$3.\int \frac{x^2}{\sqrt{4-x^2}}\mathrm{d}x \xlongequal[\mathrm{d}x=2\cos t]{x=2\sin t} \int \frac{4\sin^2 t}{2\cos t}\cdot 2\cos t\mathrm{d}x$$

$$=\int 4\sin^2 t\mathrm{d}t \quad =\int (1-\cos 2t)\mathrm{d}(2t)$$

$$=2t-\sin 2t+C \quad =2t-2\sin t\cos t+C$$

$$=2\arcsin\frac{x}{2}-\frac{x\sqrt{4-x^2}}{2}+C$$

$$4.\int \frac{2x\mathrm{d}x}{x^2+2x+2}=\int \frac{2(x+1)\mathrm{d}x}{(x+1)^2+1}-\int \frac{2\mathrm{d}x}{(x+1)^2+1}$$

$$=\int \frac{\mathrm{d}[(x+1)^2+1]}{(x+1)^2+1}-\int \frac{2\mathrm{d}(x+1)}{(x+1)^2+1}$$

$$=\ln(x^2+2x+2) \quad -2\arctan(x+1)+C$$

五. 求下列定积分和反常积分:

$1.\int_0^4 \frac{3x}{\sqrt{2x+1}}\mathrm{d}x;$　　$2.\int_{-1}^{1} x(\cos^6 x+\arctan x)\mathrm{d}x$

$3.\int_{\frac{1}{e}}^{e}|\ln x|\mathrm{d}x$　　$4.\int_e^{+\infty}\frac{\mathrm{d}x}{x(\ln x)^3}$

解: 1. 令 $t=\sqrt{2x+1},$ $x=\frac{t^2-1}{2},\mathrm{d}x=t\,\mathrm{d}t,$ 则

$$\int_0^4 \frac{3x}{\sqrt{2x+1}}\mathrm{d}x=\int_1^3\frac{3(t^2-1)t\mathrm{d}t}{2t}$$

$$=\frac{3}{2}\int_1^3(t^2-1)\mathrm{d}t=\frac{1}{2}(t^3-3t)\Big|_1^3=10.$$

$2.\int_{-1}^{1} x(\cos^6 x+\arctan x)\mathrm{d}x$

由对称性, $\int_{-1}^{1} x\cos^6 x\mathrm{d}x=0$, 而

$$\int_{-1}^{1} x\arctan x\mathrm{d}x=2\int_{0}^{1} x\arctan x\mathrm{d}x$$

$$=2\left[\frac{x^2}{2}\arctan x\Big|_0^1-\frac{1}{2}\int_0^1\frac{x^2}{1+x^2}\mathrm{d}x\right]$$

$$=2\left[\frac{\pi}{8}-0-\left(\frac{x}{2}-\frac{1}{2}\arctan x\right)\Bigg|_0^1\right]=\frac{\pi}{2}-1.$$

即

$$\int_{-1}^{1} x(\cos^6 x+\arctan x)\mathrm{d}x=\frac{\pi}{2}-1.$$

$$3.\int_{\frac{1}{e}}^{e}|\ln x|\mathrm{d}x \qquad 4.\int_{e}^{+\infty}\frac{\mathrm{d}x}{x(\ln x)^3}$$

$$\int_{\frac{1}{e}}^{e}|\ln x|\mathrm{d}x=\int_{\frac{1}{e}}^{1}|\ln x|\mathrm{d}x+\int_{1}^{e}|\ln x|\mathrm{d}x$$

$$\int_{1}^{e}|\ln x|\mathrm{d}x=\int_{1}^{e}\ln x\mathrm{d}x=(x\ln x-x)\Big|_{1}^{e}=1$$

4.

$$\int_{\frac{1}{e}}^{1}|\ln x|\mathrm{d}x=-\int_{\frac{1}{e}}^{1}\ln x\mathrm{d}x=-(x\ln x-x)\Big|_{1/e}^{1}=1-2e^{-1}$$

$$\int_{\frac{1}{e}}^{e}|\ln x|\mathrm{d}x=2(1-e^{-1}).$$

$$4.\int_{e}^{+\infty}\frac{\mathrm{d}x}{x(\ln x)^3}=\int_{e}^{+\infty}\frac{\mathrm{d}(\ln x)}{(\ln x)^3}=-\frac{1}{2(\ln x)^2}\Bigg|_{e}^{+\infty}=\frac{1}{2}.$$

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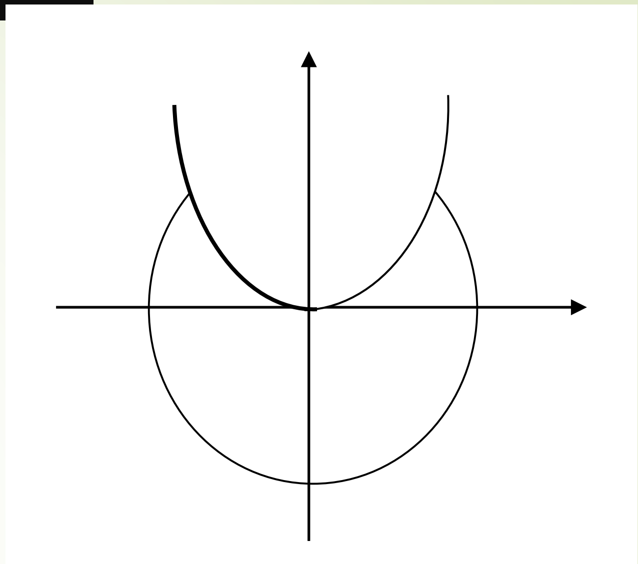
六. 求由曲线 $y=\frac{1}{2}x^2$ 与 $x^2+y^2=8$ 的上半圆周所围成图形的面积, 以及该图形分别绕 x 和 y 轴旋转所得两个旋转体的体积.



解: 解方程组 $\begin{cases} 2y=x^2, \\ x^2+y^2=8, \end{cases}$

得曲线交点(－2,2)和(2,2),从而所求图形的面积为

$$S=2\int_0^2(\sqrt{8-x^2}-\frac{1}{2}x^2)\mathrm{d}x.$$

而

$$\int_0^2\sqrt{8-x^2}\mathrm{d}x=\left(\frac{x}{2}\sqrt{8-x^2}+4\arcsin\frac{x}{2\sqrt{2}}\right)\Bigg|_0^2=2+\pi$$

$$=2\pi\int_0^2\left(8-x^2-\frac{1}{4}x^4\right)\mathrm{d}x$$

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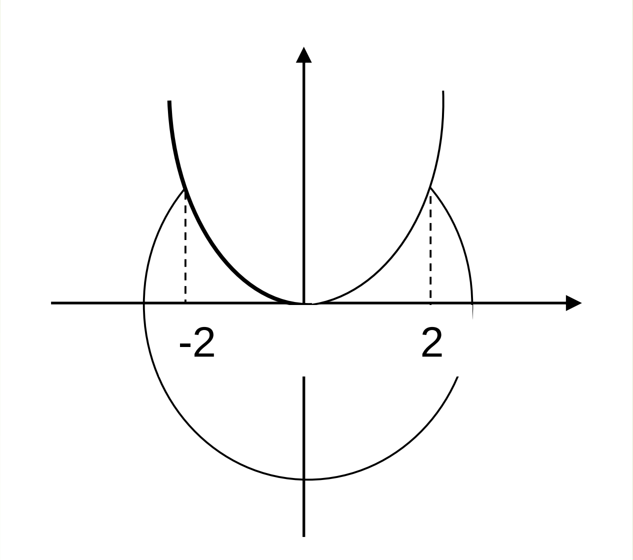
$$\int_0^2 \frac{1}{2}x^2\mathrm{d}x = \frac{x^3}{6}\bigg|_0^2 = \frac{4}{3},$$

因此

$$S = 2\left(\pi + 2 - \frac{4}{3}\right) = 2\pi + \frac{4}{3}.$$



$$\begin{cases} y = \dfrac{1}{2}x^2, \\ y = \pm\sqrt{8 - x^2}, \end{cases}$$

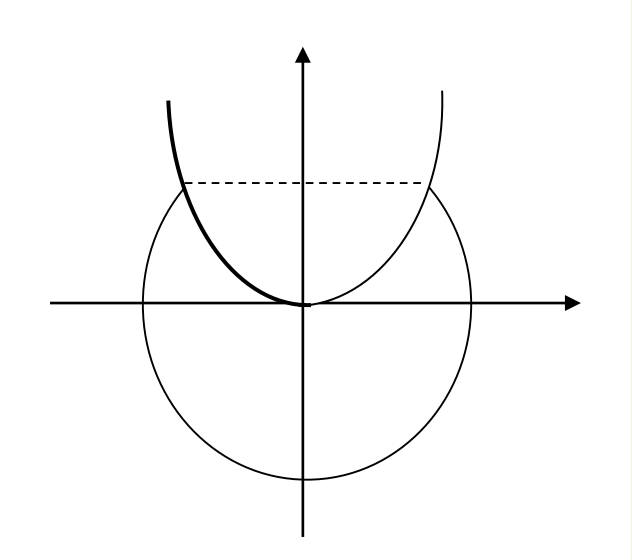
显然绕 x 轴旋转所得立体的体积为

$$V_x = \pi\left[\int_{-2}^{2}(\sqrt{8-x^2})^2\mathrm{d}x - \int_{-2}^{2}\left(\frac{1}{2}x^2\right)^2\mathrm{d}x\right]$$

$$= 2\pi\int_0^2\left(8 - x^2 - \frac{1}{4}x^4\right)\mathrm{d}x$$

$$= 2\pi\left(8x - \frac{1}{3}x^3 - \frac{1}{20}x^5\right)\bigg|_0^2 = 23\frac{7}{15}\pi.$$

$$\begin{cases} x=\sqrt{2y}, \\ x=\sqrt{8-y^2}, \end{cases}$$



绕 y 轴旋转所得立体的体积为

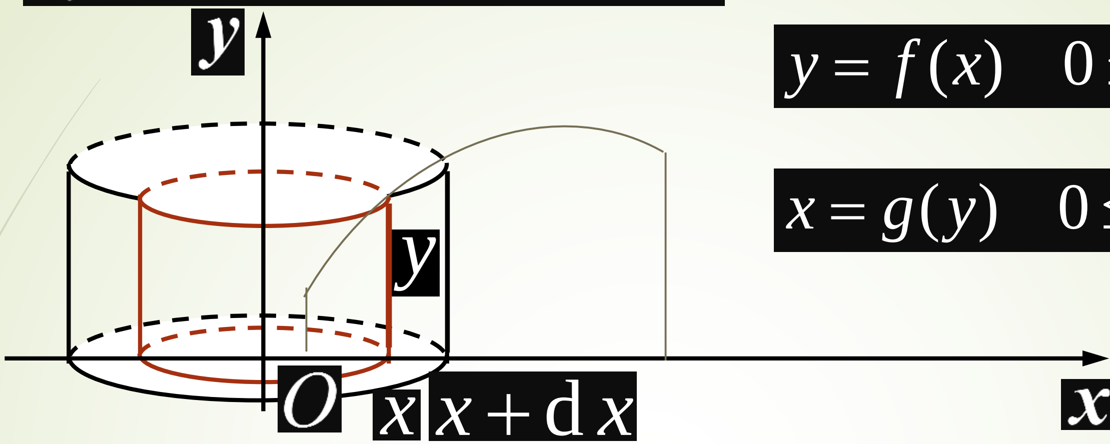
$$V_y = V_{\text{球冠}} + V_{\text{抛物线绕 } y \text{ 轴旋转}}$$

$$=\pi\left[\int_2^{2\sqrt{2}}(\sqrt{8-y^2})^2\,\mathrm{d}y\right]+\pi\int_0^2\left(\sqrt{2y}\right)^2\mathrm{d}y$$

$$=\pi\int_2^{2\sqrt{2}}(8-y^2)\mathrm{d}y+\pi\int_0^2 2y\mathrm{d}y$$

$$=\pi\left(8y-\frac{1}{3}y^3\right)\Bigg|_2^{2\sqrt{2}}+\pi(y^2)\Big|_0^2 \quad =\frac{32\sqrt{2}-28}{3}\pi.$$

说明: V_y 也可按柱壳法求出



$y=f(x)\quad 0\le a\le x\le b$

$x=g(y)\quad 0\le c\le y\le d$

柱面面积 $2\pi x\cdot y$　　　柱壳体积 $2\pi xy\cdot \mathrm{d}x$

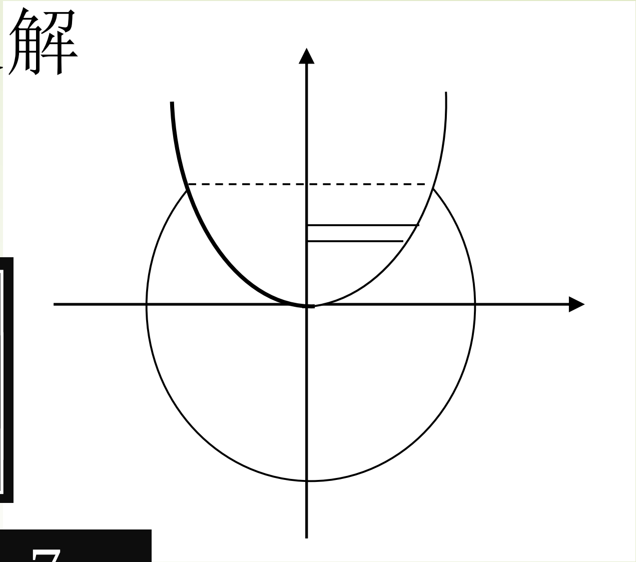
因此 $y=f(x)\quad a\le x\le b$ 绕x轴旋转一周所得立体体积

$$V_y=2\pi\int_a^b xy\,\mathrm{d}x=2\pi\int_a^b xf(x)\,\mathrm{d}x$$

类似可得 $x=g(y)\quad c\le y\le d$ 绕x轴旋转一周所得立体体积

$$V_x=2\pi\int_c^d xy\,\mathrm{d}y=2\pi\int_c^d g(y)y\,\mathrm{d}y$$

绕 x,y 轴旋转所得立体的体积又解



$$V_x = 2\left[2\pi\int_0^2 y\sqrt{2y}\,\mathrm{d}y + 2\pi\int_2^{2\sqrt{2}} y\sqrt{8-y^2}\,\mathrm{d}y\right]$$

$$= 4\pi\left[\sqrt{2}\int_0^2 y^{\frac{3}{2}}\mathrm{d}y - \frac{1}{2}\int_2^{2\sqrt{2}}(8-y^2)^{\frac{1}{2}}\mathrm{d}(8-y^2)\right]$$

$$= 4\pi\left[\sqrt{2}\cdot\frac{2}{5}y^{\frac{5}{2}}\Big|_0^2 - \frac{1}{2}\cdot\frac{2}{3}(8-y^2)^{\frac{3}{2}}\Big|_2^{2\sqrt{2}}\right] = 23\frac{7}{15}\pi.$$

$$V_y = 2\pi\int_0^2 x\left(\sqrt{8-x^2} - \frac{1}{2}x^2\right)\mathrm{d}x$$

$$= -\pi\int_0^2 (8-x^2)^{\frac{1}{2}}\mathrm{d}(8-x^2) - \pi\int_0^2 x^3\mathrm{d}x$$

$$= -\frac{2\pi}{3}(8-x^2)^{\frac{3}{2}}\Big|_0^2 - \pi\left(\frac{x^4}{4}\right)\Big|_0^2 = \frac{32\sqrt{2}-28}{3}\pi.$$

七. 1. 求微分方程 $x\frac{dy}{dx} - y\ln y = 0$ 的通解.

解: 分离变量, 得 $\frac{dy}{y\ln y} = \frac{dx}{x} \quad (\ln y \neq 0)$ ，于是

$$\int \frac{dy}{y\ln y} = \int \frac{dx}{x}$$

于是所求方程通解为 $\ln|\ln y| = \ln|x| + \ln C$ ，即

$$y = e^{Cx}.$$

而奇解 $y \equiv 1$ 对应通解中C=0的情形, 因此此通解

$$y = e^{Cx}$$

包含了方程的一切解.

2.求曲线方程, 该曲线通过原点, 且在点(x,y)处的斜率为$x+y$

解: 由题设, 得微分方程 $y' = x + y,$ 初始条件为 $y(0) = 0.$

用常数变易法来解. 其对应的齐次方程为 $y' = y,$

通解为 $y = Ce^{x}.$ 因此, 设原方程的解为 $y = C(x)e^{x},$则

$$y' = C'(x)e^{x} + C'(x)e^{x} = x + y,$$

得

$$C'(x)e^{x} = x,$$

于是 $C'(x) = xe^{-x},$

$$C(x) = \int xe^{-x}\mathrm{d}x = -\int x\mathrm{d}(e^{-x}) = -xe^{-x} + \int e^{-x}\mathrm{d}x = -xe^{-x} - e^{-x} + C,$$

从而原方程的通解为 $y = C(x)e^{x} = Ce^{x} - x - 1$

由$y(0)=0$,得$C=1$,于是所求曲线方程为 $y = e^{x} - x - 1.$

八. 求 $\lim\limits_{n\to+\infty}\int_0^1 \frac{x^n}{1+x^2}\mathrm{d}x$

解: $\forall \varepsilon > 0,$ 取 $0<\delta<1,$ 使得

$$\int_\delta^1 \frac{x^n}{1+x^2}\mathrm{d}x < \int_\delta^1 \frac{1}{1+x^2}\mathrm{d}x < (1-\delta) < \varepsilon,$$

$$\lim_{n\to+\infty}\int_0^\delta \frac{x^n}{1+x^2}\mathrm{d}x \le \lim_{n\to+\infty}\int_0^\delta \frac{\delta^n}{1+x^2}\mathrm{d}x$$

$$= \int_0^\delta \frac{1}{1+x^2}\mathrm{d}x \cdot \lim_{n\to+\infty}\delta^n = 0.$$

$$\lim_{n\to+\infty}\int_0^1 \frac{x^n}{1+x^2}\mathrm{d}x = \lim_{n\to+\infty}\int_0^\delta \frac{x^n}{1+x^2}\mathrm{d}x + \lim_{n\to+\infty}\int_\delta^1 \frac{x^n}{1+x^2}\mathrm{d}x < \varepsilon$$

因此, $\lim\limits_{n\to+\infty}\int_0^1 \frac{x^n}{1+x^2}\mathrm{d}x = 0.$