(12) Scattering, Part 2

1 Last Time

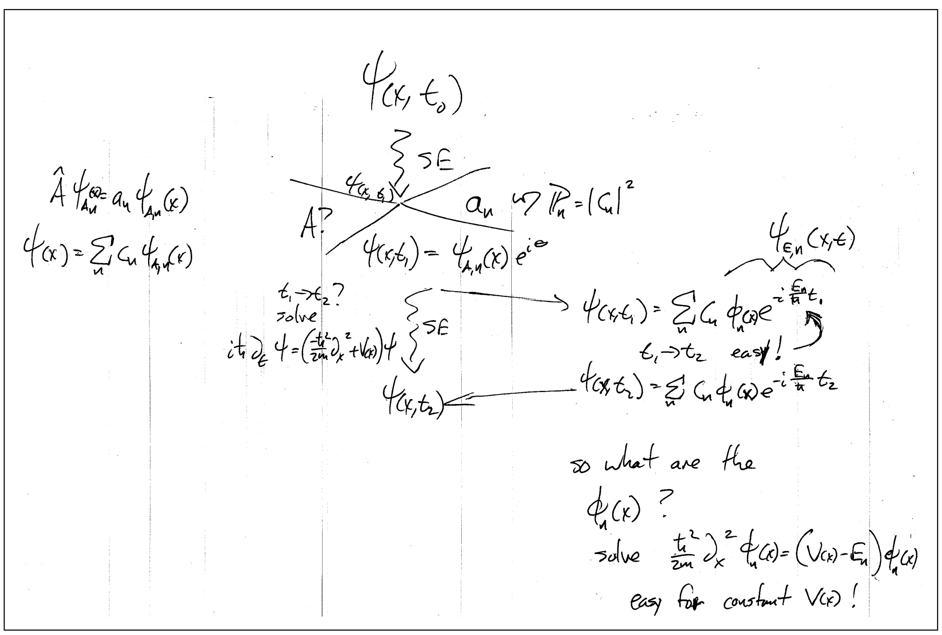
- We bounced off of a step potential,
- got past it with enough energy
- and discovered a way to quantify motion in QM: Probability Current

2 This Time

- Scattering from a barrier potential
- All there is to know about a 1D potential in a simple 2x2 matrix

3 Quick Review

Since we're all just back from Spring Break, I'll take a moment and remind everyone of the big picture.

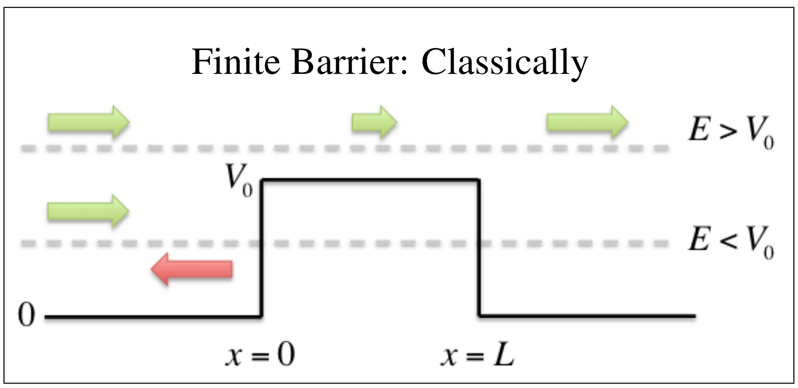


4 The Barrier Potential

Last time we worked through the transmission and reflection of a particle from a finite step potential. This time we will do a more intricate example which reveals a general approach to scattering.

Today's potential of choice is a barrier potential of height V_0.

Finite Barrier: $V(x) = \begin{cases} V_0 & 0 < x < L \\ 0 & else \end{cases}$



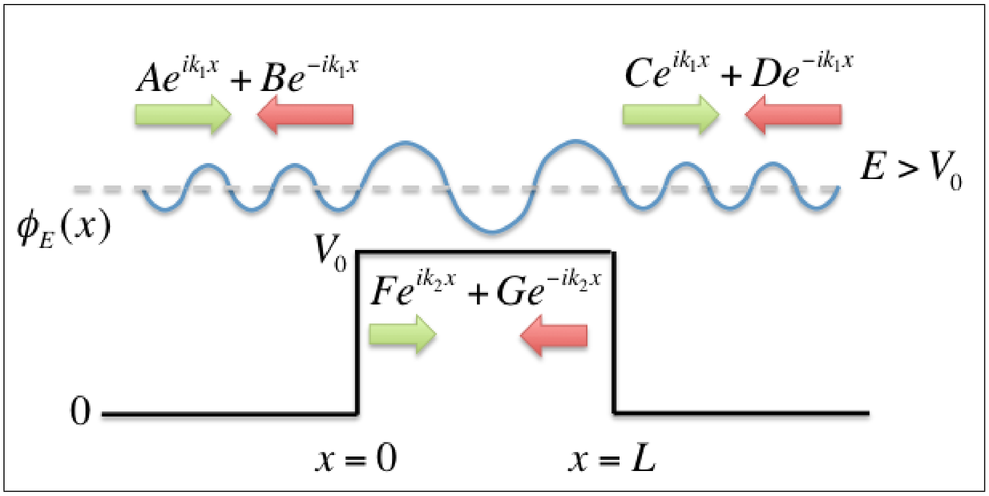
As usual, the classical expectations are very simple to imagine; for energies $E < V_0$, the particle bounces off while for energies $E > V_0$, the particle slows while crossing the barrier, but then returns to its previous velocity and departs. Classically, the finial state of the particle depend only on the height of the barrier relative to the energy of the particle. The *thickness* of the barrier is unimportant.

We will start this time with high energy case, $E > V_0$.

For $E > V_0$

$$\phi_E(x) = \begin{cases} Ae^{ik_1 x} + Be^{-ik_1 x} & x \leq 0 \\ Fe^{ik_2 x} + Ge^{-ik_2 x} & 0 < x < L \\ Ce^{ik_1 x} + De^{-ik_1 x} & x \geq L \end{cases}$$

with $k_1^2 = \frac{2m}{\hbar^2}E$ and $k_2^2 = \frac{2m}{\hbar^2}(E - V_0)$.



Show matlab animation!

As before, we will consider the case of a wave coming in from the left, so we can take $D = 0$, and we will save A for normalization. The 2 changes in $V(x)$ give us 4 matching conditions, 2 at $x = 0$ and 2 at $x = L$. Note that we *cannot* assume that $G = 0$, since some of the right moving wave will be reflected by the step at $x = L$.

Continuity with $E > V_0$

$$\begin{aligned}
\phi(x) \ @\ x = 0 \quad &\Rightarrow \quad A + B = F + G \\
\partial_x\, \phi(x) \ @\ x = 0 \quad &\Rightarrow \quad ik_1(A - B) = ik_2(F - G) \\
\\
\phi(x) \ @\ x = L \quad &\Rightarrow \quad Fe^{ik_2L} + Ge^{-ik_2L} = Ce^{ik_1L} \\
\partial_x\, \phi(x) \ @\ x = L \quad &\Rightarrow \quad ik_2(Fe^{ik_2L} - Ge^{-ik_2L}) = ik_1Ce^{ik_1L}
\end{aligned}$$

at this point a bunch of algebra happens, and we arrive at

Continuity implies...

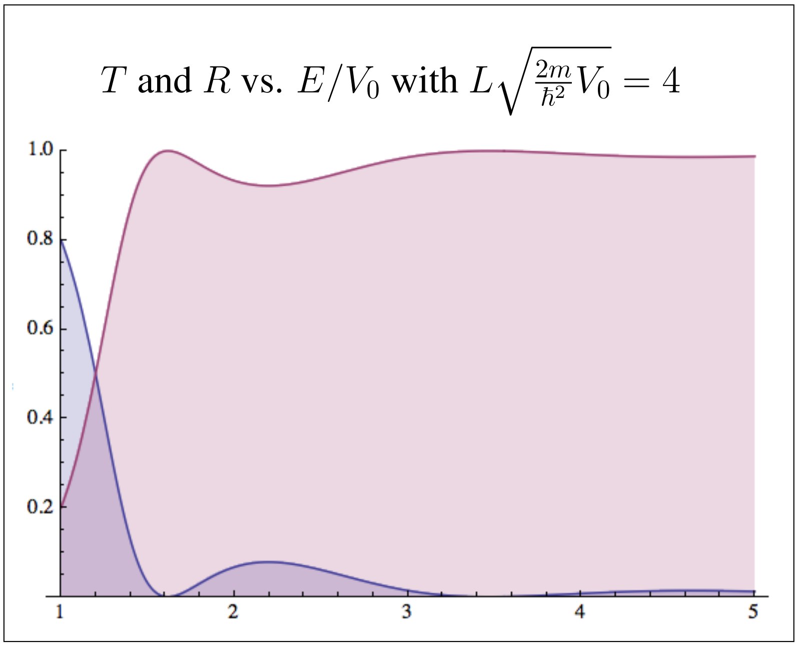
$$
\begin{aligned}
B &= \frac{(k_1^2 - k_2^2)\sin(k_2 L)}{(k_1^2 + k_2^2)\sin(k_2 L) + 2ik_1k_2\cos(k_2 L)} A \\
C &= \frac{2ik_1k_2 e^{-ik_1 L}}{(k_1^2 + k_2^2)\sin(k_2 L) + 2ik_1k_2\cos(k_2 L)} A
\end{aligned}
$$

and some similarly length expressions for F and G. Fortunately, we only need B and C to get the reflection and transmission coefficients of the barrier.

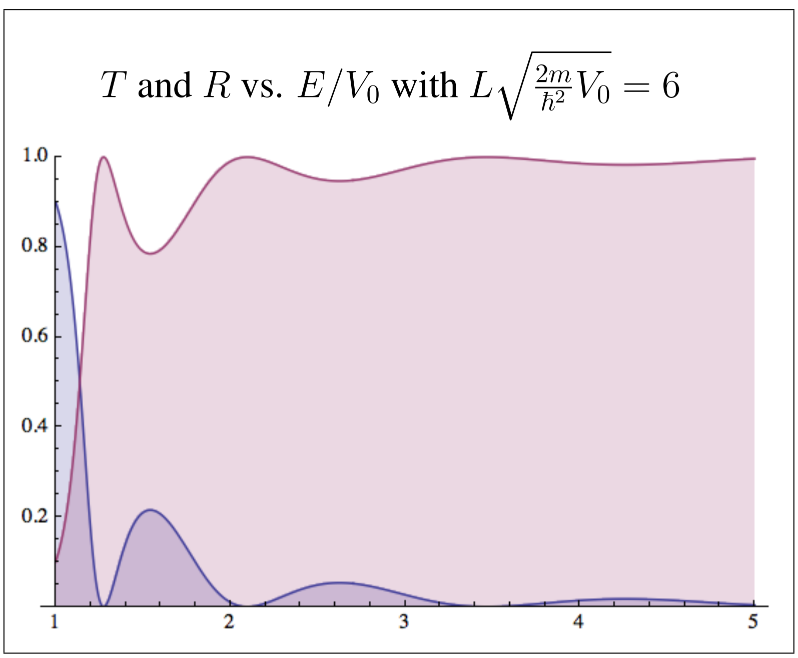
You might note that this potential has the nice feature of limited spatial extent. That is, the potential on either side of the barrier is constant and zero, unlike the finite step we studied last time. This means that we don't have to compensate for the change in energy when computing the transmission and reflection coefficients, so

$$
\begin{aligned}
T &= \left|\frac{C}{A}\right|^2 \quad \text{and} \quad R = \left|\frac{B}{A}\right|^2 = 1 - T \\
\Rightarrow T &= \frac{1}{1 + \left(\frac{k_1^2 - k_2^2}{2k_1k_2}\sin(k_2 L)\right)^2} \\
&= \frac{1}{1 + \frac{V_0^2}{4E(E-V_0)}\sin^2(k_2 L)} \\
&= \frac{1}{1 + \frac{V_0^2}{4E(E-V_0)}\sin^2\left(L\sqrt{\frac{2m}{\hbar^2}(E - V_0)}\right)}
\end{aligned}
$$

Ok, enough algebra. Let's plot this and see what we get. To do this, we need to pick some values...



and if we pick different values...



Notice that in order to make these plots I had to define some dimensionless quantities which characterize the strength of our potential barrier and the energy of the particle with respect to the barrier height.

Dimensionless Parameters

$$\text{Barrier Strength} \qquad g_0 = L\sqrt{k_1^2 - k_2^2} = L\sqrt{\frac{2m}{\hbar^2}V_0}$$

$$\text{Particle Energy} \qquad \epsilon = \frac{E}{V_0} \quad \Rightarrow \quad k_2 L = g_0\sqrt{\epsilon - 1}$$

$$\frac{k_1 k_2}{k_1^2 - k_2^2} = \sqrt{\epsilon(\epsilon - 1)} \qquad \text{and} \qquad \frac{k_1^2 + k_2^2}{k_1^2 - k_2^2} = 2\epsilon - 1$$

$$\Rightarrow T = \frac{1}{1 + \frac{1}{4\epsilon(\epsilon-1)}\sin^2\left(g_0\sqrt{\epsilon - 1}\right)}$$

You might notice that there are some interesting features in the transmission plot of our barrier. In particular, there are several points at which $T = 1$ well before $E \to \infty$. These are called "resonances" and they happen when $k_2 L = n\pi$ since these values give $\sin k_2 L = 0$ and thus $T = 1$ and $R = 0$.

Resonances: $k_2 L = n\pi$ with $n \in \{1, 2, 3, \cdots\} \quad \Rightarrow T = 1$

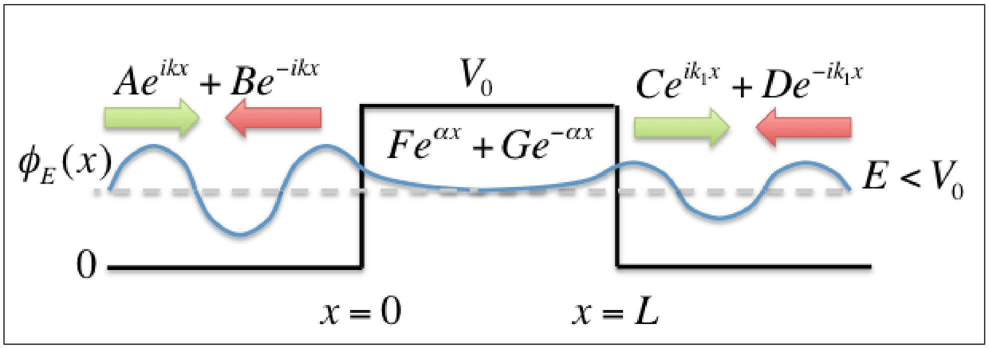
5 Tunneling

Another interesting feature of the barrier transmission is that it does not go to zero as $E \to V_0$.

$$\begin{aligned}
& \lim_{E \to V_0} \epsilon \to 1 \\
\Rightarrow \quad & \sin^2\left(g_0\sqrt{\epsilon - 1}\right) \to g_0^2(\epsilon - 1) \\
\Rightarrow \quad & T \to \frac{1}{1 + \frac{1}{4}g_0^2} \\
\Rightarrow \quad & T(E = V_0) = 0.2 \;\text{ for }\; g_0 = 4 \\
\text{and} \quad & T(E = V_0) = 0.1 \;\text{ for }\; g_0 = 6
\end{aligned}$$

Which gives values that match the plots in the previous section.

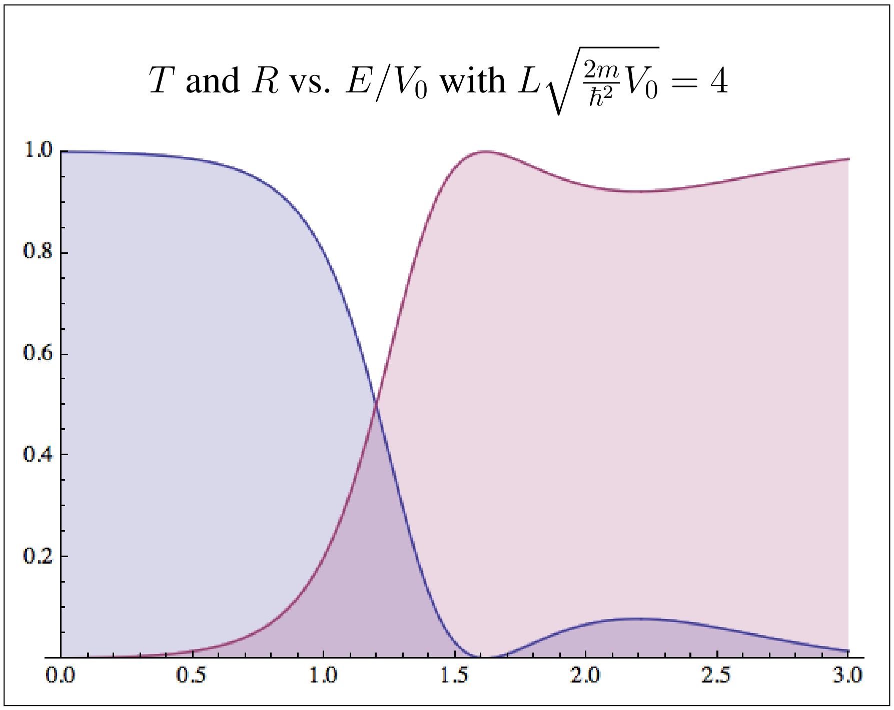
So what happens for $E < V_0$? If T is continuous, it can't go suddenly to zero as it did in the case of the finite step, and as you would expect classically. We know that the solutions should look like this:



I won't go through the derivation of T in the case of particle energies too low to pass over the potential barrier, because it is mathematically equivalent to what we have already done given a simple substitution.

For $E < V_0 \Rightarrow \epsilon - 1 < 0$

$$\begin{aligned} \text{use} \quad \sqrt{\epsilon - 1} &= i\sqrt{1-\epsilon} \\ \text{and} \quad \sin^2\left(g_0\sqrt{\epsilon-1}\right) &= -\sinh^2\left(g_0\sqrt{1-\epsilon}\right) \\ \Rightarrow T &= \frac{1}{1 + \frac{1}{4\epsilon(1-\epsilon)}\sinh^2\left(g_0\sqrt{1-\epsilon}\right)} \end{aligned}$$



The fact that our particle has a finite probability of passing through the barrier even though it doesn't have enough energy to do so (at least classically), is a feature of QM known as "tunneling". You will show on the pset that the probability of tunneling decreases exponentially with barrier thickness for $E \ll V_0$.

Show matlab animations!

Tunneling is used in a number of technologies: tunnel junction semiconductors, and scanning tunneling microscopy, just to name the first 2 that come to mind.

6 The S-Matrix

It turns out that the rabbit hole of scattering goes much deeper than just computing the transmission of piecewise constant potentials.

If we stick to localized potentials, like the finite barrier seen in today's lecture, but also more generally any potential which goes to zero far from the origin, we can write the reflection and transmission *amplitudes* of our potential as

r and t Amplitudes:

$$r = \frac{B}{A} \quad \text{and} \quad t = \frac{C}{A}$$

and thus avoid the loss of the phase information which happens when taking the absolute values associated with R and T.

We can also consider the possibility of sending in particles form the right, such that $D \neq 0$, and explore any asymmetries in the potential.

$$\tilde{r} = \frac{C}{D} \quad \text{and} \quad \tilde{t} = \frac{B}{D}$$

The combination of these coefficients into a matrix describes the scattering properties of the potential, as explored from the left or right.

$$\begin{bmatrix} B \\ C \end{bmatrix} = \begin{bmatrix} r(E) & \tilde{t}(E) \\ t(E) & \tilde{r}(E) \end{bmatrix} \begin{bmatrix} A \\ D \end{bmatrix}$$

The dependence of r and t on E is included to make explicit the fact that this is a $2x2$ matrix of *functions*.

This matrix is known as the scattering matrix or “S-matrix” , and it is at the heart of Feynman diagrams and much of high-energy physics. Everything there is to know about a localized potential can be found in its S-matrix.

For instance, you will show on the pset that for a finite well of depth V_0,

For Well of Depth V_0

$$T = \frac{1}{1 + \frac{1}{4\epsilon(\epsilon+1)} \sin^2\left(g_0\sqrt{\epsilon+1}\right)}$$

Of course, that transmission is for $E > 0$ (e.g., unbound states or “scattering states”).

But what does the equation tell us if we put in $E < 0$? We can’t send a wave in with that energy, so A and D must be zero. Thus, B and C can only be non-zero if $T \to \infty$. Can that happen? Well, yes, if the denominator goes to zero.

Pole of S-matrix elements

$$T \to \infty \quad \text{iff} \quad \sin^2\left(g_0\sqrt{\epsilon+1}\right) = -4\epsilon(\epsilon+1)$$

which sufficient algebraic manipulation and variable substitutions can eventually turn into our transcendental equations of the finite square well. For instance, the ground state in a delta well can be easily reproduced:

Ground State with $g_0 \ll 1$

$$\begin{aligned} \sin^2\left(g_0\sqrt{\epsilon+1}\right) &\simeq g_0^2(\epsilon+1) \\ \Rightarrow g_0^2 \simeq -2\epsilon \;\; \text{or} \;\; \frac{2m}{\hbar^2}V_0L^2 &\simeq \frac{-4E}{V_0} \\ \Rightarrow E &\simeq -\frac{m}{2\hbar^2}(LV_0)^2 \end{aligned}$$

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7 Next Time

- The Quantum version of our old friend, the Harmonic Oscillator.