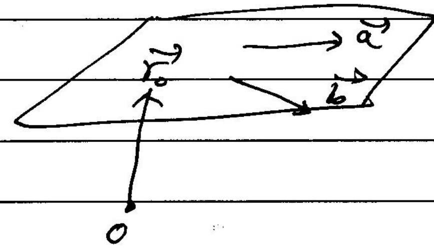
四、平面与直线的方程 —— 向量应用

1. 平面方程

(1) 点位式方程 $(\vec{r_0}, \vec{a}, \vec{b})$

① 向量式参数方程



$$\vec{r} = \vec{r_0} + u\vec{a} + v\vec{b}$$

② 坐标式参数方程

$$\begin{cases} x = x_0 + a_1 u + b_1 v \\ y = y_0 + a_2 u + b_2 v \\ z = z_0 + a_3 u + b_3 v \end{cases}$$

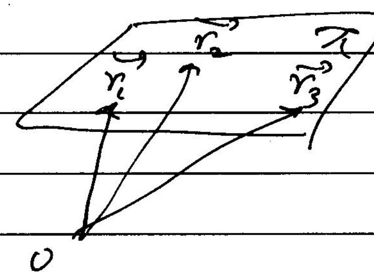
③ 混合积方程

$(\vec{r} - \vec{r_0}, \vec{a}, \vec{b}) = 0$ (共面)

④ 行列式方程

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$$

(2) 三点式方程

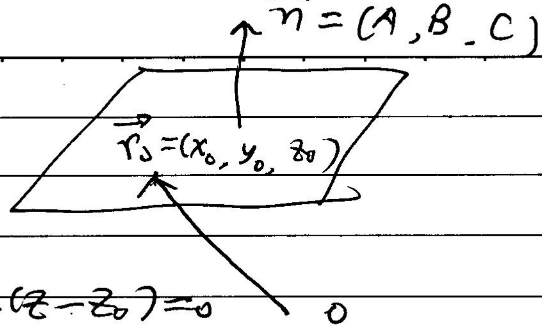


$$(\vec{r} - \vec{r_1}, \vec{r_2} - \vec{r_1}, \vec{r_3} - \vec{r_1}) = 0$$

或 $$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

(3) 点法式方程

$\vec{n}(\vec{r}-\vec{r_0})=0$



或 $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$

(4) 一般方程

$Ax+By+Cz+D=0$ （P_{22}, 定理1.7）

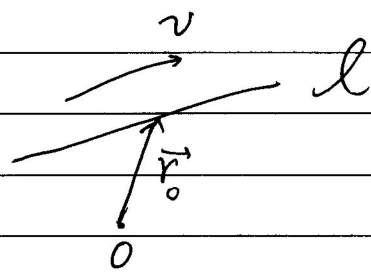
注. 每个平面均可由上述方程来表示 [由(1)、(2)、(3)知]. 下至要说明这个一般方程表示一个平面. 不妨令 $A\neq 0$. 过点 $(-\frac{D}{A},0,0)$, 作向量 $\vec{a}=(-\frac{B}{A},1,0)$, $\vec{b}=(-\frac{C}{A},0,1)$, 则由(1)知 点位式方程 $(\vec{r}-\vec{r_0},\vec{a},\vec{b})=0$, 即 $Ax+By+Cz+D=0$, 即它表示一个平面.

2. 直线方程

(1) 点向式方程

① 向量式参数方程

$\vec{r}=\vec{r_0}+t\vec{v}$



② 坐标式参数方程

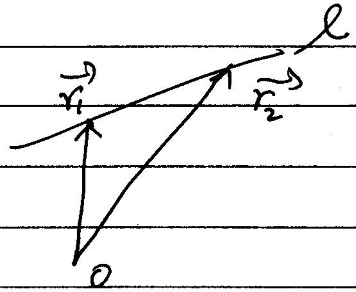
$$\begin{cases} x=x_0+v_1t \\ y=y_0+v_2t \\ z=z_0+v_3t \end{cases}$$

③ 标准方程（或对称式）方程

$$\frac{x-x_0}{v_1}=\frac{y-y_0}{v_2}=\frac{z-z_0}{v_3}$$

(2) 两点式方程

① $\vec{r} = \vec{r_1} + t(\vec{r_2} - \vec{r_1})$



② $\begin{cases} x = x_1 + t(x_2 - x_1) \\ y = y_1 + t(y_2 - y_1) \\ z = z_1 + t(z_2 - z_1) \end{cases}$

③ $\dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1} = \dfrac{z-z_1}{z_2-z_1}$.

(3) 一般方程

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

注. (i) $\vec{a_i} = (A_i, B_i, C_i)$ 互不平行, 即两平面相交于一直线;

(ii) 标准方程为

$$\frac{x-x_0}{\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}} = \frac{y-y_0}{\begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}} = \frac{z-z_0}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}$$

3. 位置关系

(1) 直线与平面 (P_{25})

直线 $\vec{v} = (l, m, n)$

平面 π : $ax + by + cz = d$

$\vec{v} \parallel \pi \iff al + bm + cn = 0$ (命题 1.8, P_{25})

1°. 相交 $\iff al + bm + cn \neq 0$

2°. 平行 $\iff al + bm + cn = 0$ $\begin{cases} \cancel{l \subset \pi} \\ l \not\subset \pi. \end{cases}$

3°. 直线 $l \in \pi$.

(2) 平面 π_1 与平面 π_2 (P26)

$\pi_i: a_i x + b_i y + c_i z = d_i \quad (i=1,2)$

1° 相交 $\Longleftrightarrow (a_1, b_1, c_1) \nparallel (a_2, b_2, c_2)$

$\Longleftrightarrow \dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$

2° 平行 $\Longleftrightarrow \left\{ \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} \neq \dfrac{d_1}{d_2} \right.$

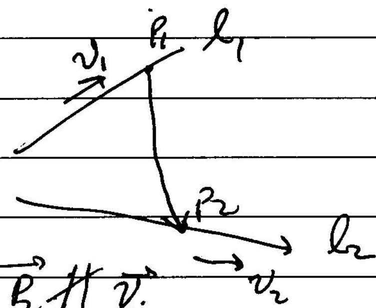
3° 重合 $\Longleftrightarrow \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} = \dfrac{d_1}{d_2}$

(3) 直线 l_1 与直线 l_2 (P26)

1° 异面

$(\overrightarrow{P_1P_2}, \vec{v}_1, \vec{v}_2) \neq 0$

(不共面)



2° 平行. $\vec{v}_1 \parallel \vec{v}_2$ 且 $\overrightarrow{P_1P_2} \nparallel \vec{v}_1$

$\left\{ \begin{array}{l} (\overrightarrow{P_1P_2}, \vec{v}_1, \vec{v}_2) = 0 \\ \vec{v}_1 \parallel \vec{v}_2 \end{array} \right.$

3° 重合. $\overrightarrow{P_1P_2}, \vec{v}_1, \vec{v}_2$ 共线 $(\vec{v}_1 \parallel \vec{v}_2)$

例1. (P26 例1.10)

过 $P_0(0,0,-2)$ $l \parallel \pi: 3x-y+2z-1=0$

l 与 $l_1: \dfrac{x-1}{4} = \dfrac{y-3}{-2} = \dfrac{z}{1}$ 相交, 求该直线的方程

解.

§4 平面束与直线束（P_{27}）

(1) 平面束

① 有轴平面束

— 通过同一直线的所有平面的集合.

该直线称为平面束的轴.

$\pi_i: a_i x + b_i y + c_i z + d_i = 0$ $(i=1,2)$. 交于一直线

$L \Rightarrow \begin{cases} \text{以 } L \text{ 为轴的平面束方程为} \\ l(a_1x+b_1y+c_1z+d_1)+m(a_2x+b_2y+c_2z+d_2)=0 \\ \text{其中 } l \text{ 与 } m \text{ 是不全为零的任意实数.} \end{cases}$

② 平行平面束

— 平行于同一个平面的所有平面的集合

$ax+by+cz=d$

$\pi': ax+by+cz=d_1$

(2) 直线束

① 中心直线束　中心 (x_0, y_0, z_0)

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

② 平行直线束 $\vec{l}$　$\vec{l} \parallel \vec{v} = (l, m, n)$

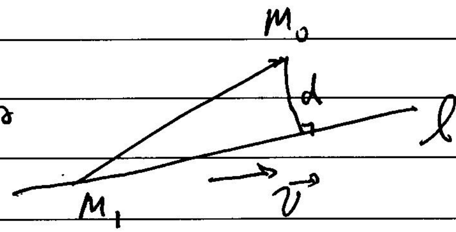
$$\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}$$

$\begin{bmatrix} (x_0, y_0, z_0) \text{ 任一点} \\ (l, m, n) \text{ 确定} \end{bmatrix}$

5. 度量性质.

(1) 距离 (P28)

① 点到直线的距离

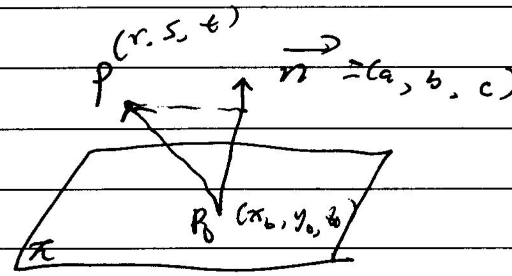


S = 以 $\vec{v}$ 和 $\overrightarrow{M_1M_0}$ 为两边的平行四边形的面积

$$\Rightarrow S = d \cdot |\vec{v}| = |\vec{v} \times \overrightarrow{M_1M_0}|$$

$$\Rightarrow d = \frac{|\vec{v} \times \overrightarrow{M_1M_0}|}{|\vec{v}|}$$

② 点到平面的距离



$\forall P_0 \in \pi: ax+by+cz-d=0$

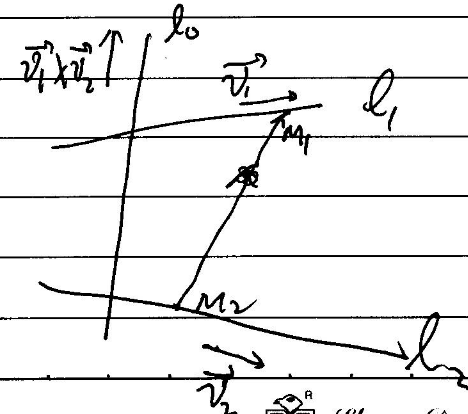
$|Pr_{\vec{n}}(\overrightarrow{P_0P})|$ = 点到平面的距离

$$= \frac{|\overrightarrow{P_0P} \cdot \vec{n}|}{|\vec{n}|} = \frac{|ar+bs+ct-d|}{\sqrt{a^2+b^2+c^2}}$$

$\overrightarrow{P_0P} = (r-x_0, s-y_0, t-z_0)$, $\vec{n} = (a, b, c)$, $P_0 \in \pi$

$$\Rightarrow \overrightarrow{P_0P} \cdot \vec{n} = ar+bs+ct-(ax_0+by_0+cz_0)$$
$$= ar+bs+ct-d$$

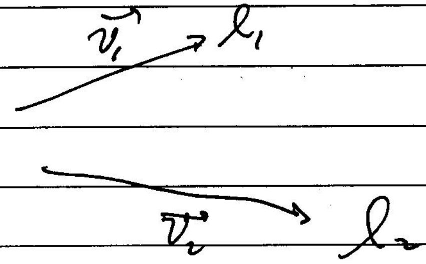
③ 异面直线的距离



$$d = \frac{|\overrightarrow{M_2M_1} \cdot (\vec{v_1} \times \vec{v_2})|}{|\vec{v_1} \times \vec{v_2}|}$$

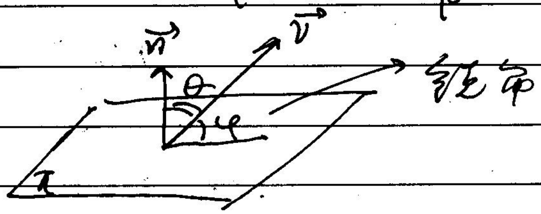
(2) 夹角 (P29)

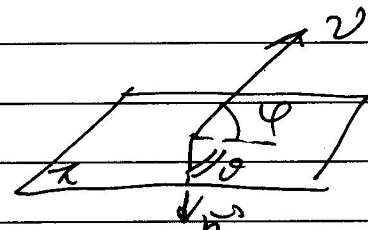
① 直线与直线的夹角



$$\cos\angle(l_1, l_2) = \pm\frac{\vec{v_1}\cdot\vec{v_2}}{|\vec{v_1}|\cdot|\vec{v_2}|}$$

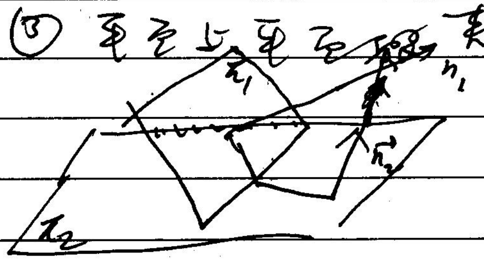
② 直线与平面的夹角





$$\varphi = \left|\frac{\pi}{2}-\theta\right| \Rightarrow \sin\varphi = |\cos\theta| = \frac{|\vec{n}\cdot\vec{v}|}{|\vec{n}|\cdot|\vec{v}|}.$$

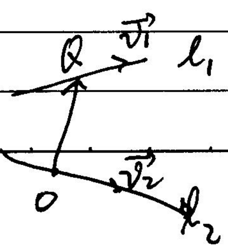
③ 平面与平面的夹角



$$\cos\angle(\pi_1, \pi_2) = \pm\cos\theta = \pm\frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\cdot|\vec{n_2}|}$$

例2. (P29. 练习1.11)

$$\left.\begin{aligned} l_1: \frac{x-1}{1} = \frac{y}{-3} = \frac{z}{3} \\ l_2: \frac{x}{2} = \frac{y}{1} = \frac{z}{-2} \end{aligned}\right\} \Rightarrow l_1 与 l_2 异面?$$



$$(\vec{OQ}, \vec{v_1}, \vec{v_2}) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -3 & 3 \\ 2 & 1 & -2 \end{vmatrix} \neq 0$$

公垂线方向 $\vec{v} = \vec{v_1}\times\vec{v_2} = (3, 8, 7)$

$$\pi_1: \begin{vmatrix} x-1 & y & z \\ 1 & -3 & 3 \\ 3 & 8 & 7 \end{vmatrix} = 0. \quad \pi_2: \begin{vmatrix} x & y & z \\ 2 & 1 & -2 \\ 3 & 8 & 7 \end{vmatrix} = 0$$