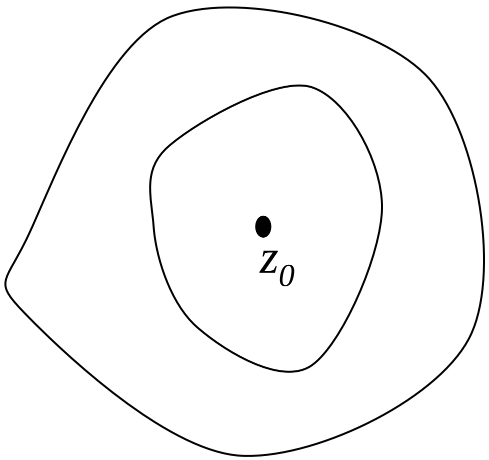
第四章　留数定理

§4.1 留数定理

$$\oint_{l} f(z)\mathrm{d}z = \oint_{l_0} f(z)\mathrm{d}z = \sum_{k=-\infty}^{+\infty} a_k \oint_{l_0} (z-z_0)^k \mathrm{d}z = 2\pi \mathrm{i} a_{-1}$$

定义留数　$\mathrm{Res}f(z_0)$

$$\oint_{l} f(z)\mathrm{d}z = 2\pi \mathrm{i}\,\mathrm{Res} f(z_0)$$



例: $f(z) = ze^{\frac{1}{z}}, \quad z = 0$

$$ze^{\frac{1}{z}} = z + 1 + \frac{1}{2!}z^{-1} + \frac{1}{3!}z^{-2} + \cdots,$$

$$\mathrm{Res}\left[f(z), 0\right] = c_{-1} = \frac{1}{2}$$

例: $f(z) = \dfrac{e^{\frac{1}{z}}}{z^2 - z}, \quad z = 1$

$$\mathrm{Res}\left[\frac{e^{\frac{1}{z}}}{z^2 - z}, 1\right] = \frac{1}{2\pi \mathrm{i}} \oint_L \frac{e^{\frac{1}{z}}}{z^2 - z} \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \oint_L \frac{\frac{e^{\frac{1}{z}}}{z}}{z - 1} \mathrm{d}z$$

$$= \frac{1}{2\pi \mathrm{i}} \cdot 2\pi \mathrm{i} \left(\frac{e^{\frac{1}{z}}}{z} \right)\Bigg|_{z=1} = e$$

1. 可去奇点的留数为零

2. 极点的留数

$$(1)\quad \operatorname{Res} f(z_0)=\lim_{z\to z_0}(z-z_0)f(z)$$

$$(2)\quad \operatorname{Res} f(z_0)=\frac{P(z_0)}{Q'(z_0)},\quad \left(f(z)=\frac{P(z)}{Q(z)},P(z_0)\neq 0,Q(z_0)=0,Q'(z_0)\neq 0\right)$$

$$(3)\quad \operatorname{Res} f(z_0)=\frac{1}{(m-1)!}\lim_{z\to z_0}\frac{\mathrm{d}^{m-1}}{\mathrm{d}\,z^{m-1}}[(z-z_0)^m f(z)]$$

3. 本性奇点的留数:洛朗展开

例：

$$\operatorname{Res}[\frac{ze^z}{z^2-1},1]=\lim_{z\to 1}(z-1)\frac{ze^z}{z^2-1}=\lim_{z\to 1}\frac{ze^z}{z+1}=\frac{e}{2};$$

$$\operatorname{Res}\left[\frac{ze^z}{z^2-1},1\right]=\frac{P(1)}{Q'(1)}=\frac{e}{2}$$

$$\operatorname{Res}\left[\frac{1}{(z^2+1)^3},\mathrm{i}\right]=\frac{1}{(3-1)!}\lim_{z\to \mathrm{i}}\frac{\mathrm{d}^2}{\mathrm{d}\,z^2}\left((z-\mathrm{i})^3\cdot\frac{1}{(z-\mathrm{i})^3(z+\mathrm{i})^3}\right)$$

$$=\frac{1}{2}\lim_{z\to \mathrm{i}}[(-3)(-4)(z+\mathrm{i})^{-5}]=-\frac{3\mathrm{i}}{16}$$

留数定理　设函数 $f(z)$ 在区域 D 内除有限个孤立奇点 $z_1, z_2, \cdots, z_n$ 外处处解析，L 为区域内包围各奇点的一条正向简单闭曲线，则 $\oint_l f(z)\,\mathrm{d}z = 2\pi \mathrm{i}\sum_{k=1}^{n} \mathrm{Res}\, f(z_k)$

例:计算 $\oint_{|z|=2} \dfrac{5z-2}{z(z-1)^2}\,\mathrm{d}z$

$$\mathrm{Res}[f(z),0] = \lim_{z\to 0} z\frac{5z-2}{z(z-1)^2} = \lim_{z\to 0}\frac{5z-2}{(z-1)^2} = -2$$

$$\mathrm{Res}[f(z),1] = \lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z-1)^2\frac{5z-2}{z(z-1)^2}\right] = \lim_{z\to 1}\frac{2}{z^2} = 2$$

$$\oint_{|z|=2}\frac{5z-2}{z(z-1)^2}\,\mathrm{d}z = 2\pi\mathrm{i}(-2+2) = 0$$

例:计算 $\oint_{|z|=1} \frac{z\sin z}{(1-e^z)^3} \mathrm{d}\, z$

$$\frac{z\sin z}{(1-e^z)^3} \to z=0 \in C: |z|=1$$

$$\frac{z\sin z}{(1-e^z)^3} = \frac{z(z-\frac{z^3}{3!}+\cdots)}{-(z+\frac{z^2}{2!}+\cdots)^3} = -\frac{z^2}{z^3}\frac{(1-\frac{z^2}{3!}+\cdots)}{(1+\frac{z}{2!}+\cdots)^3} = -\frac{1}{z} - a_1 - \cdots$$

$$\mathrm{Res}\left[\frac{z\sin z}{(1-e^z)^3}, 0\right] = -1$$

$$\oint_{|z|=1} \frac{z\sin z}{(1-e^z)^3} \mathrm{d}\, z = -2\pi \mathrm{i}$$

设∞为$f(z)$的一个孤立奇点，即$f(z)$在去心邻域$R<|z|<+\infty$内解析，则定义函数$f(z)$在$z=\infty$处的留数为

$$\operatorname{Res} f(\infty)=\frac{1}{2\pi i}\oint_{L} f(z)\mathrm{d}z$$

其中L: 积分方向为顺时针方向（实际上是包含无穷远点的区域的正方向）.如果$f(z)$在$z=\infty$的去心邻域$R<|z|<+\infty$内的罗朗级数为

$$f(z)=\cdots+\frac{c_{-n}}{z^{n}}+\cdots+\frac{c_{-1}}{z}+c_{0}+c_{1}z+\cdots+c_{n}z^{n}+\cdots$$

由逐项积分定理及公式得到

$$\operatorname{Res} f(\infty)=\frac{1}{2\pi i}\oint_{L} f(z)\mathrm{d}z=-c_{-1}$$

当$f(z)$以$z=\infty$为可去奇点或解析点时,其留数可能不等于0.

例: $f(z)=1/z$以$z=\infty$为解析点，但留数 $\operatorname{Res} f(\infty)=-1$

函数$f(z)=(z-1)/z$以$z=\infty$为可去奇点，但留数 $\operatorname{Res} f(\infty)=1$

留数和定理 设函数$f(z)$在扩充复平面上除了有限远$z_k\ (k=1, 2, \ldots, n)$以及$z=\infty$以外处处解析，则

$$\sum_{k=1}^{n} \operatorname{Res} f(z_k)+\operatorname{Res} f(\infty)=0$$

1. $\operatorname{Res} f(\infty)=-\operatorname{Res}[f(\frac{1}{z})\frac{1}{z^2},0]$

2. $\lim\limits_{z\to\infty} f(z)=0 \mapsto \operatorname{Res} f(\infty)=-\lim\limits_{z\to\infty}[z\cdot f(z)]$

例：$\operatorname{Res}\left[\dfrac{e^{z}}{z^{2}-1},\infty\right]$

$$f(z)=1-\frac{1}{z}+\frac{2}{z^{2}}+\frac{3}{z^{3}}$$

$$\operatorname{Res} f(1)=\frac{e}{2},\operatorname{Res} f(-1)=-\frac{1}{2}e^{-1}$$

$$\operatorname{Res} f(\infty)=-\operatorname{Res}[f(\frac{1}{z})\cdot\frac{1}{z^{2}},0]=1$$

$$\operatorname{Res} f(\infty)=\frac{e^{-1}-e}{2}$$

例：$I=\oint_{|z|=4}\dfrac{5z^{27}}{(z^{2}-1)^{4}(z^{4}+2)^{5}}\mathrm{d}z$

$$\frac{5z^{27}}{(z^{2}-1)^{4}(z^{4}+2)^{5}}\mapsto\pm1,\sqrt[4]{2}e^{\frac{\pi+2k\pi}{4}\mathrm{i}}\quad(k=0,1,2,3)\in C:|z|=4$$

$$I=-\oint_{|z|=4}\frac{5z^{27}}{(z^{2}-1)^{4}(z^{4}+2)^{5}}\mathrm{d}z=-2\pi\mathrm{i}\operatorname{Res} f(\infty)$$

$$\operatorname{Res} f(\infty)=-\lim_{z\to\infty}[zf(z)]=-5$$

$$I=10\pi\mathrm{i}$$