(23) EPR and Bell's Inequality

1 Last Time

- Multiple Particle States
- Entaglement

2 This Time

- Einstein, Podolsky and Rosen (EPR)
- Bell's Inequalty

Final: Monday May 18, 9am to 12pm in Johnson Ice Rink

Final: Will cover all material, with emphasis on topics covered after Exam 2. Look over the notes, psets and previous exams. A lot of study material is on Stellar.

Final: Unphysical answers will cost you!

3 Problems with Locality

- Gr. 12.1, Sc. 8.8, Li 11.13, Ga 20.3

Last time I left you with a classic example of entanglement which leads to non-local effects (e.g., A changes B faster than the speed of light). We started with the decay of the neutral pi meson (spin 0) into a positron and an electron in the state

Entangled Spins

$$\begin{aligned} \pi^0 &\rightarrow e^- + e^+ \\ \psi_\pm &= \frac{1}{\sqrt{2}}\left(\uparrow_-\downarrow_+ \;-\; \downarrow_-\uparrow_+\right) \end{aligned}$$

If the pion was at rest in the lab, the electron and positron are produced with considerable kinetic energy, and go flying off in opposite directions.

The trouble comes here: imagine that we wait a while to measure the spins of the particles so that the electron and positron are very far apart... say 1 light-minute. If observer A measure the state of the electron, and observer B measure the state of the positron 1 second later, A knows what B will get. But how does the positron know what result to give B?

QM: wavefunction collapse is instantaneous (e.g., superluminal)!

Quantum Mechanics does not, however, suggest that this superluminal connection can be used to send information. That is, even though measurement A determines the outcome of measurement B, there is no way for A to send a signal to B since the result of A's measurement is random. The correlation between A's measurements and B's measurements is only evident if you have both sets of results in hand, so no information is transferred by the instantaneous wavefunction collapse.

None the less, if QM is correct, we are forced to give up a much cherished principle of physics: locality. Locality is the idea that distant regions cannot be causally connected faster than the speed of light.

---

4 Hidden Variables and Bell's Theorem

There is a simple out from this problem of non-locality: hidden variables. The idea is that our QM wavefunction maybe a sufficient description of the state of our system in that it let's us compute the probabilities of various results, but it is incomplete in that it lacks the state information which actually determines the outcomes of our measurements. If we knew the values of these "hidden variables", the randomness would go away. That is, things only appear to be random because we are ignorant, not because of some fundamental randomness of reality.

The analogy would be that of rolling dice. The outcome appears random only because we don't know well enough the parameters of the throw and bounce of the dice. If we knew all there was to know, we could predict which numbers would come up.

In terms of our 2 entangled particles, you can think of this as labeling the particles at the time of their creation as "up" and "down" in a particular direction so that there is no mystery to their anti correlation at the time of measurement: it was simply pre-determined, not communicated super-luminally. (Imagine that you have a red ball and a blue ball. I take one ball and leave, then at some later time you check which ball you have left. As soon as you see the color of your ball, you know the color of the ball I took, no matter where it is at that time.)

We can make this precise by creating a new parameter, say λ, which determines the outcome of both spin measurements. Let's choose $0 \leq \lambda < 1$, and $\mathbb{P}(\lambda) = 1$ on this interval such that $\int_0^1 \mathbb{P}(\lambda) = 1$ for simplicity. Now, to avoid any non-local effects, we assert that the probability of measuring the spin of our particles to be "up" in any particular direction θ is:

Hidden Variable: A and B anti-correlated

$$
\begin{aligned}
u(\theta, \lambda) &= \begin{cases} 1 & \uparrow\downarrow \text{ at } \theta \\ 0 & \downarrow\uparrow \text{ at } \theta \end{cases} \\
\mathbb{P}_{\uparrow A}(\theta) &= \int_0^1 u(\theta, \lambda)\, d\lambda \\
\mathbb{P}_{\uparrow B}(\theta) &= \int_0^1 1 - u(\theta, \lambda)\, d\lambda \\
\text{measure} & \quad \text{A up at } \theta_1 \text{ and B up at } \theta_2\text{:} \\
\Rightarrow \mathbb{P}_{\uparrow\uparrow}(\theta_1, \theta_2) &= \int_0^1 u(\theta_1, \lambda)(1 - u(\theta_2, \lambda))\, d\lambda
\end{aligned}
$$

I needn't specify what the function $u(\theta, \lambda)$ is, the only requirement is that if A and B make measurement at the same angle the results are anti-correlated such that angular momentum is conserved.

Now imagine that we make measurements at 3 values of $\theta_{\{1,2,3\}}$. We can say something about the relative probabilities of certain combinations of results

Bell's Theorem

$$\begin{aligned}
\mathbb{P}_{\uparrow\uparrow}(\theta_1,\theta_2)+\mathbb{P}_{\uparrow\uparrow}(\theta_2,\theta_3) &\geq \mathbb{P}_{\uparrow\uparrow}(\theta_1,\theta_3)\\
\text{Proof: (dealing only with the integrand)} &\\
\text{let}\quad u_n = u(\theta_n,\lambda) &\\
u_1(1-u_2)+u_2(1-u_3) &\geq u_1(1-u_3)\\
u_1(1-u_2)u_3+u_1(1-u_2)(1-u_3) &+ \cdots\\
u_1u_2(1-u_3)+(1-u_1)u_2(1-u_3) &\geq \cdots\\
u_1u_2(1-u_3)+u_1(1-u_2)(1-u_3) &\\
\Rightarrow u_1(1-u_2)u_3+(1-u_1)u_2(1-u_3) &\geq 0
\end{aligned}$$

Just to make sure that you don't think I'm trying to pull a fast one on you will all of this algebra, let me do this again another way. Imagine making a bunch of measurement pairs at $\{\theta_1,\theta_2\}$ $\{\theta_2,\theta_3\}$ and $\{\theta_1,\theta_3\}$ and we record the number of times we get both A and B up. Remember that at any given angle, if A is up B must be down to conserve momentum, and the result is determined by λ.

Bell's Theorem, Counting

$$\begin{aligned}
N(A_1,B_2)+N(A_2,B_3) &\geq N(A_1,B_3)\\
\text{use}\quad B_n = \bar{A}_n &\\
N(A_1,\bar{A}_2)+N(A_2,\bar{A}_3) &\geq N(A_1,\bar{A}_3)\\
\text{use}\quad A_n+\bar{A}_n = 1 &\\
N(A_1,\bar{A}_2,A_3)+N(A_1,\bar{A}_2,\bar{A}_3) &+ \cdots\\
N(A_1,A_2,\bar{A}_3)+N(\bar{A}_1,A_2,\bar{A}_3) &\geq \cdots\\
N(A_1,A_2,\bar{A}_3)+N(A_1,\bar{A}_2,\bar{A}_3) &\\
\Rightarrow N(A_1,\bar{A}_2,A_3)+N(\bar{A}_1,A_2,\bar{A}_3) &\geq 0
\end{aligned}$$

Show bar graph!

Bell's theorem gives us a very simple statement which must be obeyed by any theory which hopes to maintain locality. What does QM predict?

To make a comparison we must first compute $\mathbb{P}_{\uparrow\uparrow}(\theta_1, \theta_2)$ given the state $\psi_\pm$ we started the lecture with. We need the probability of getting both up, and we know that the measurement made by A has a 50% chance of giving up for any θ_1. After A measures $|\uparrow\rangle$ for the electron we know the positron's state is $|\downarrow\rangle$ along the same axis, but the measurement takes place along an axis which is rotated by $\theta_2 - \theta_1$. We can orient the positron state to make the computation of $\mathbb{P}_{\uparrow\uparrow}(\theta_1, \theta_2)$ easy.

Rotating a Spin

$$\begin{aligned}
|\uparrow_\theta\rangle &= \tfrac{1}{\sqrt{2}}(|\uparrow\rangle + e^{i\theta}|\downarrow\rangle) \\
\text{Recall: } |\uparrow_x\rangle &= |\uparrow_{\theta=0}\rangle = \tfrac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \\
\text{and } |\uparrow_y\rangle &= |\uparrow_{\pi/2}\rangle = \tfrac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle) \\
\Rightarrow \mathbb{P}_{\uparrow\uparrow}(\theta_1, \theta_2) &= \mathbb{P}_{\uparrow A}(\theta_1)\mathbb{P}_{\uparrow B}(\theta_2; \uparrow_{A\theta_1}) \\
&= \frac{1}{2}\,\mathbb{P}_{\uparrow B}(\theta_2; \uparrow_{A\theta_1}) = \frac{1}{2}\,|\langle\uparrow_{\theta_2}|\downarrow_{\theta_1}\rangle|^2
\end{aligned}$$

Rotating a Spin, continued

$$\begin{aligned}
\langle\uparrow_{\theta_2}|\downarrow_{\theta_1}\rangle &= \frac{1}{2}(\langle\uparrow| + e^{-i\theta_2}\langle\downarrow|)(|\uparrow\rangle + e^{i(\theta_1+\pi)}|\downarrow\rangle) \\
&= \frac{1}{2}\left(1 - e^{i(\theta_1-\theta_2)}\right) \\
&= -ie^{i(\theta_1-\theta_2)/2}\sin\left(\frac{\theta_1-\theta_2}{2}\right) \\
\Rightarrow \mathbb{P}_{\uparrow\uparrow}(\theta_1, \theta_2) &= \frac{1}{2}\sin^2\left(\frac{\theta_1-\theta_2}{2}\right)
\end{aligned}$$

So, Bell's inequality would require that

Bell's QM

$$\sin^2\left(\frac{\theta_1-\theta_2}{2}\right)+\sin^2\left(\frac{\theta_2-\theta_3}{2}\right) \;\geq\; \sin^2\left(\frac{\theta_1-\theta_3}{2}\right)$$

$$\text{Pick }\; \Delta\theta=\theta_1-\theta_2=\theta_2-\theta_3$$

$$\Rightarrow\;\; 2\sin^2(\Delta\theta/2) \;\geq\; \sin^2(\Delta\theta)$$

$$\text{False for any }\; \Delta\theta<\pi/2$$

$$\text{e.g., for }\; \Delta\theta\ll 1 \qquad \sin^2(\Delta\theta)\simeq\Delta\theta^2$$

$$\Rightarrow\;\; 2(\Delta\theta/4)^2 \;\geq\; \Delta\theta^2$$

$$\Rightarrow\;\; \frac{1}{2} \;\geq\; 1 \quad !!!$$

Which is to say that QM and *any theory which preserves locality* make different predictions about an experimental outcome.

Experimental tests of QM prove that the physical world is non-local. No theory which tries to avoid this by hiding more information in the state of our system that what is in the wavefunction can save you from this conclusion and match experiment. This is the end of reality as you know it (though you probably won't really notice).

5 Next Time

- Final on Monday at 9am in Johnson Ice Rink.