(17) Review and Clicker Bonanza

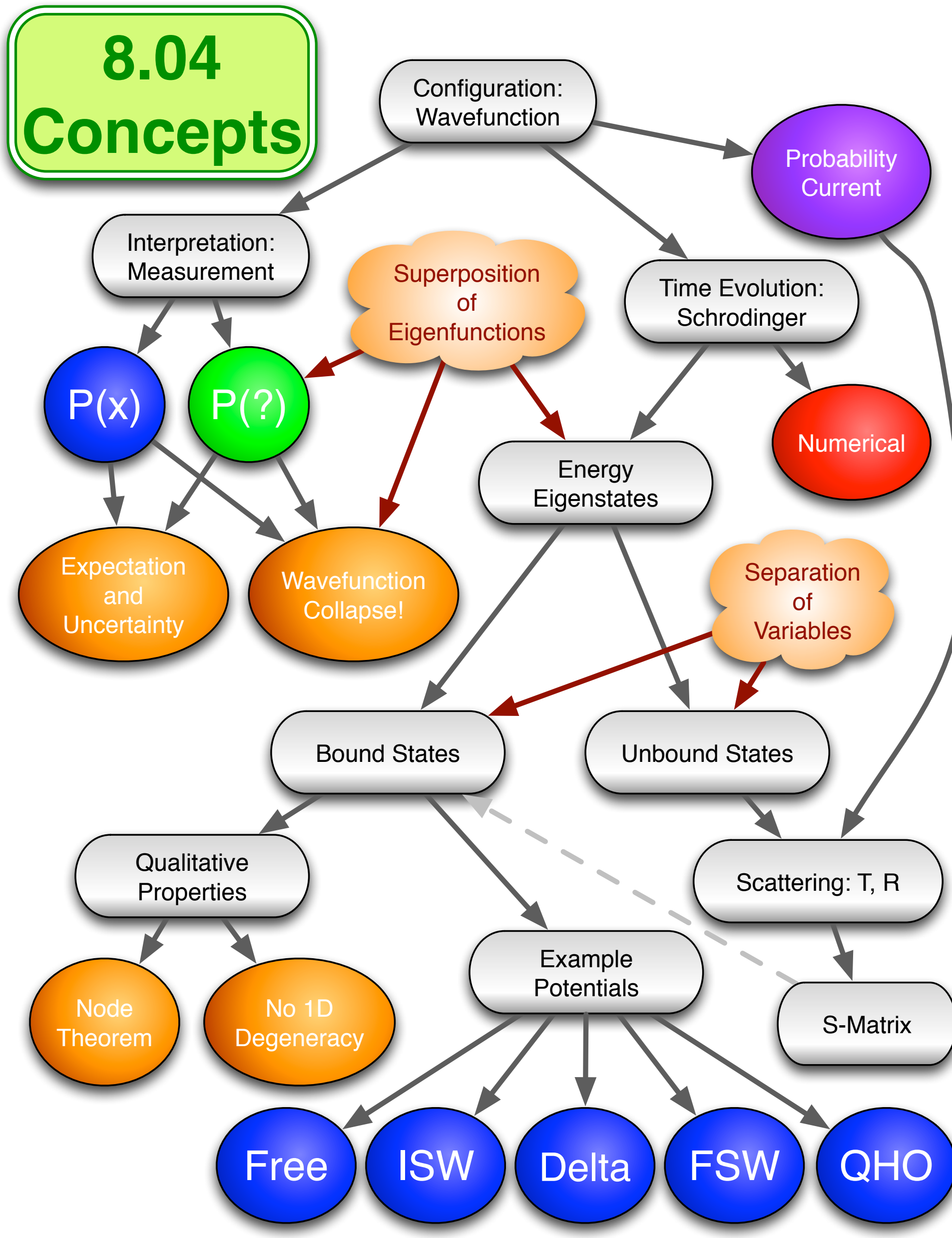
1 Announcements

No new pset posted... Exam on Thursday at class time: 11am to 12:30pm in 50-340 (Walker Gym)

Formula sheet on Stellar. (Want more?)

Thanks for the review suggestions!

2 Concept Graph



3 Qualitative Wavefunction Game

Draw some potentials and some WFs. Have the students identify what is wrong with them. Some of these can be "given this potential, what is wrong with the WF", and some can be the other way around.

4 Probability Current

The definition of probability current is

$$\mathcal{J}(x) = \frac{i\hbar}{2m}\left[\psi(x)\,\partial_x\,\psi^*(x) - \psi^*(x)\,\partial_x\,\psi(x)\right]$$

the implications of which which aren't necessarily very obvious.

For $\psi(x) = A(x)e^{i\theta(x)}$ where $A(x)$ and $\theta(x)$ are real,

$$\mathcal{J}(x) = |A(x)|^2\frac{\hbar}{m}\partial_x\theta(x) = \mathbb{P}(x)\frac{\hbar}{m}\partial_x\theta(x)$$

For example,

For $\psi(x) = e^{x^2/2x_0^2+ikx}$

$$\mathcal{J}(x) = \mathbb{P}(x)\frac{\hbar}{m}\partial_x(kx) = \frac{\hbar k}{m}e^{x^2/2x_0^2}$$

More generally, a boost applied to any WF gives

Boost of $\psi(x)$

$$\begin{aligned}
\hat{B}_q\psi &= e^{iqx}\psi \\
\mathcal{J}(x) &= \frac{i\hbar}{2m}\left(e^{iqx}\psi\,\partial_x\,e^{-iqx}\psi^* - e^{-iqx}\psi^*\,\partial_x\,e^{iqx}\psi\right) \\
&= \frac{i\hbar}{2m}\Big[e^{iqx}\psi\,(-iqe^{-iqx}\psi^* + e^{-iqx}\,\partial_x\,\psi^*) - \cdots \\
&\qquad e^{-iqx}\psi^*\,(iqe^{iqx}\psi + e^{iqx}\,\partial_x\,\psi)\Big] \\
&= \frac{i\hbar}{2m}\,(\psi\,\partial_x\,\psi^* - \psi^*\,\partial_x\,\psi) + \frac{\hbar q}{2m}\,(\psi\psi^* + \psi^*\psi) \\
&= \mathcal{J}_0(x) + \frac{\hbar q}{m}\mathbb{P}_0(x)
\end{aligned}$$

In a double well potential, with a sufficiently strong barrier between the wells the ground state and first excited state are very similar in energy. The wavefunctions for these states differ in that the first excited state WF crosses zero in the barrier, and has opposite sign in the 2 wells.

Double Well with Large Barrier

$$\phi_1(x) \simeq \begin{cases} +\phi_0(x) & x \ll 0 \\ -\phi_0(x) & x \gg 0 \end{cases}$$

$$E_1 = E_0 + \epsilon \qquad \Rightarrow \qquad \omega_1 - \omega_0 = \Delta\omega = \epsilon/\hbar$$

A particle which at time $t = 0$ is localized to the left well is well approximated by a superposition of these first 2 states with a positive sign.

Double Well: Starting Left

$$
\begin{aligned}
\text{for } \psi(x) &= \frac{1}{\sqrt{2}}\left[\phi_0(x)e^{-i\omega_0 t} + \phi_1(x)\right] \\
\psi(x,t) &= \frac{1}{\sqrt{2}}\left[\phi_0(x)e^{-i\omega_0 t} + \phi_1(x)e^{-i\omega_1 t}\right] \\
\psi(x,t) &= \frac{1}{\sqrt{2}}\left[\phi_0(x) + \phi_1(x)e^{-i\Delta\omega t}\right]e^{-i\omega_0 t} \\
\psi(x,t=\pi/2\Delta\omega) &= \frac{1}{\sqrt{2}}\left[\phi_0(x) - i\phi_1(x)\right]e^{-i\omega_0 t} \\
\psi(x,t=\pi/\Delta\omega) &= \frac{1}{\sqrt{2}}\left[\phi_0(x) - \phi_1(x)\right]e^{-i\omega_0 t}
\end{aligned}
$$

Note that the particle moves from the left well to the right well in time $\pi/\Delta\omega$.

The probability current for this situation is

Double Well: Moving Right

$$
\begin{aligned}
\mathcal{J}(x=0,t=0) &= 0 \quad \because \quad \psi(x) \in \mathbb{R} \\
\mathcal{J}(x=0,t=\pi/2\Delta\omega) &> 0 \\
\text{since} &\qquad \psi(x \ll 0) \simeq \frac{1}{\sqrt{2}}(1-i)\,\phi_0(x)e^{-i\omega_0 t} \\
\text{and} &\qquad \psi(x \gg 0) \simeq \frac{1}{\sqrt{2}}(1+i)\,\phi_0(x)e^{-i\omega_0 t} \\
&\Rightarrow \quad \partial_x\theta(x) > 0 \ \text{(phase is increasing)}
\end{aligned}
$$

which is what you should expect since at $t = \pi/\Delta\omega$ the particle will be localized to the right hand well.

5 Minimum Uncertainty States

Gaussian is good (compute Δx and Δp).

Gaussian WF

$$
\begin{aligned}
\psi(x) &= e^{-x^2/2\lambda^2} \\
\langle x \rangle = 0 & \qquad \langle p \rangle = 0 \quad \text{by symmetry} \\
\left\langle x^2 \right\rangle &= \int_{-\infty}^{\infty} \psi^* x^2 \psi \, dx = \int_{-\infty}^{\infty} x^2 e^{-x^2/\lambda^2} \, dx \\
&= \frac{\lambda^2}{2} \\
\left\langle p^2 \right\rangle &= \int_{-\infty}^{\infty} \psi^* (-i\hbar\partial_x)^2 \psi \, dx \\
&= -\hbar^2 \int_{-\infty}^{\infty} e^{-x^2/2\lambda^2} \, \partial_x^2 \, e^{-x^2/2\lambda^2} \, dx \\
&= \frac{\hbar^2}{2\lambda^2} \\
\Delta x^2 \Delta p^2 &= \left(\left\langle x^2 \right\rangle - \langle x \rangle^2 \right) \left(\left\langle p^2 \right\rangle - \langle p \rangle^2 \right) = \frac{\hbar^2}{4}
\end{aligned}
$$

Show spreading with $V(x) = 0$.

Note that ground state of the HO is simple example of minimum uncertainty state which stays that way. More generally, there exist states which move in the HO potential, called “coherent states”, which maintain minimum uncertainty. That are Gaussians with some phase, and we will encounter them again later in this lecture.

For the infinite square will, the ground state is NOT minimum uncertainty.

Infinite Square Well Ground State

$$\begin{aligned}
\psi(x) &= \sqrt{\frac{2}{L}}\sin\Big(\frac{\pi}{L}x\Big) \\
\langle x\rangle = 0 & \quad \langle p\rangle = 0 \text{ by symmetry} \\
\left\langle x^2\right\rangle &= \frac{2}{L}\int_0^L x^2\sin^2\left(\frac{\pi}{L}x\right)\,dx \\
&= L^2\left(\frac{1}{3}-\frac{1}{2\pi^2}\right)\sim\frac{L^2}{3.5} \\
\left\langle p^2\right\rangle &= \cdots = \frac{\pi^2\hbar^2}{L^2}\sim 10\frac{\hbar^2}{L^2} \\
\Delta x^2\Delta p^2 &= \hbar^2\left(\frac{\pi^2}{3}-\frac{1}{2}\right)\sim 11\frac{\hbar^2}{4}
\end{aligned}$$

---

6 Eigenfunctions and Commutators (by Paolo Glorioso)

Suppose we have a common eigenfunction ϕ_{ab} of 2 operators $\hat{A}$ and $\hat{B}$. From the following relations

$$\hat{A}\hat{B}\phi_{ab} = \hat{A}b\phi_{ab} = ab\phi_{ab} \tag{1}$$

$$\hat{B}\hat{A}\phi_{ab} = \hat{B}a\phi_{ab} = ba\phi_{ab} \tag{2}$$

we infer that the action of $\hat{A}$ and $\hat{B}$ on ϕ_{ab} can be interchanged, precisely because the eigenvalues a and b are complex numbers, and so they commute. We conclude that

$$[\hat{A},\hat{B}]\phi_{ab} = (ab-ba)\phi_{ab} = 0 \tag{3}$$

However, even if $\hat{A}$ and $\hat{B}$ are two commuting operators, it's not necessary that an eigenfunction of the former is also an eigenfunction of the latter. As a simple counterexample, take $\hat{A} = \hat{\mathbb{1}}$, $\hat{B} = \hat{x}$, and the function $f(x) = x^2$. Evidently, $[\hat{A},\hat{B}] = 0$, and x^2 is an eigenfunction of $\hat{A}$ with eigenvalue $\hat{\mathbb{1}}$. Nevertheless, x^2 is not an eigenfunction of $\hat{B}$.

As a common and concrete example, we have,

$$[\hat{x},\hat{p}] = i\hbar\hat{\mathbb{1}} \tag{4}$$

Since the commutator of $\hat{x}$ and $\hat{p}$ is nonvanishing on any function, they can't have any common eigenfunction. Indeed, suppose that $\psi(x)$ is an eigenfunction for both $\hat{x}$ and $\hat{p}$ with eigenvalues a_x, a_p, respectively. Then

$$\frac{\hbar}{i}\psi(x) = [\hat{x}, \hat{p}]\psi(x) = (a_x a_p - a_p a_x)\psi(x) = 0 \tag{5}$$

which gives a contradiction.

The operators $\hat{p}$ and $\hat{T}_L$ do share common eigenfunctions. Consider $\psi(x) = e^{ikx}$. Then

$$\hat{p}\psi(x) = \hbar k \psi(x) \tag{6}$$

$$\hat{T}_L\psi(x) = e^{-ikL}\psi(x). \tag{7}$$

Physically, this tells us that states with definite momentum are translationally invariant up to an overall phase.

According to classical mechanics, given sufficiently precise measurements, all observables can in principle be determined with total certainty. However, as we saw with the boxes, or the 2-slit experiment, this is empirically false: in some situations, knowledge of one observable can imply irreducible uncertainty about other observables.

In quantum mechanics, this remarkable fact is encoded by representing observables with operators and the state as a wavefunction. A quantum observable can thus only be said to have a well-defined value when the state (the wavefunction) is an eigenfunction of the corresponding operator.

When can two observables have definite values simultaneously? To have such certainly, the wavefunction must be a simultaneous eigenfunction of both of the corresponding operators. But such a shared eigenfunction can only exist if the commutator of the two operators can vanish. Thus, if the commutator of two observables does not vanish, there is an irreducible uncertainty in the values of those observables.

In short:

$$\left[\hat{A}, \hat{B}\right]\phi = 0 \ \forall \ \phi$$

implies that $\hat{A}$ and $\hat{B}$ *may* have common eigenfunctions.

$$\left[\hat{A}, \hat{B}\right]\phi \neq 0 \ \forall \ \phi$$

implies that $\hat{A}$ and $\hat{B}$ *do not* have common eigenfunctions.

For the harmonic oscillator, the failure of $\hat{a}$ to commute with $\hat{E}$, is a direct result of $\left[\hat{a}, \hat{a}^\dagger\right] = 1$, which comes from the failure of $\hat{x}$ to commute with $\hat{p}$. $[\hat{x}, \hat{p}] = i\hbar$ also leads to the ground state energy of the HO, which is an expression of the uncertainty principle.

Operators with $\hat{p}$ and $\hat{x}$

$$
\begin{aligned}
\hat{a} &\equiv \frac{\hat{x}}{x_0} + i\frac{\hat{p}}{p_0} \\
\Rightarrow \hat{a}^\dagger &= \frac{\hat{x}}{x_0} - i\frac{\hat{p}}{p_0} \qquad \because \quad \hat{x} = \hat{x}^\dagger, \;\; \hat{p} = \hat{p}^\dagger \\
\hat{a}^\dagger\hat{a} &= \left(\frac{\hat{x}}{x_0} - i\frac{\hat{p}}{p_0}\right)\left(\frac{\hat{x}}{x_0} + i\frac{\hat{p}}{p_0}\right) \\
&= \left(\frac{\hat{x}}{x_0}\right)^2 + \left(\frac{\hat{p}}{p_0}\right)^2 + i\frac{\hat{x}}{x_0}\frac{\hat{p}}{p_0} - i\frac{\hat{p}}{p_0}\frac{\hat{x}}{x_0} \\
&= \frac{\hat{E}}{\hbar\omega} + \frac{i}{2\hbar}[\hat{x}, \hat{p}] \qquad \because \quad x_0 p_0 = 2\hbar \\
&= \frac{\hat{E}}{\hbar\omega} - \frac{1}{2} \qquad \because \quad [\hat{x}, \hat{p}] = i\hbar \\
\Rightarrow \hat{E} &= \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)
\end{aligned}
$$

As a result, eigenstates of $\hat{a}$ are NOT energy eigenstates, and are not easily written in terms of energy eigenstates. It can, however, be done

Eigenstates of $\hat{a}$

$$\begin{aligned}
\psi_a &= \sum_{n=0}^{\infty} c_n \, \phi_n \\
\hat{a}\,\psi_a &= \hat{a} \sum_{n=0}^{\infty} c_n \, \phi_n \\
&= \sum_{n=1}^{\infty} c_n \sqrt{n} \, \phi_{n-1} \\
\text{for eigenfunction} & \quad \hat{a}\,\psi_a = \alpha \psi_a \\
\Rightarrow c_n \sqrt{n} &= \alpha \, c_{n-1} \\
\Rightarrow c_n &= \alpha \frac{1}{\sqrt{n}} \, c_{n-1} \\
\Rightarrow c_n &= \alpha^n \frac{1}{\sqrt{n!}} \, c_0
\end{aligned}$$

Note that if we choose

Coherent State, EF of $\hat{a}$

$$\begin{aligned}
\alpha &= \frac{x_0}{\sqrt{2}\,\lambda} \\
c_0 &= e^{-x_0^2/4\lambda^2}
\end{aligned}$$

we get the "coherent state" of the harmonic oscillator. This state is a gaussian centered at $x = x_0$ which oscillates around equilibrium without increasing x uncertainty. Recall that we generated our ground state by insisting that $\hat{a}\phi_0 = 0$. That is just particular eigenfunction of $\hat{a}$, which has $\alpha = 0$, and thus $x_0 = 0$.

---

7 Clicker Bonanza!

Ok, enough of me talking... time for clickers!