量子光学

习题解答集

二零零三年六月

量子光学习题解答

第一章

1.1 长 L 的立方腔内，$\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=0$，$\nabla\bullet\vec{A}=0$。求证满足边界条件的解包含分量 $A_x(\vec{r},t)=A_x(t)\cos(k_x x)\sin(k_y y)\sin(k_z z)$，

$$A_y(\vec{r},t)=A_y(t)\sin(k_x x)\cos(k_y y)\sin(k_z z)\text{，}$$

$$A_z(\vec{r},t)=A_z(t)\sin(k_x x)\sin(k_y y)\cos(k_z z)\text{，}$$

其中 $\vec{k}$ 的分量有 1.1.21 式决定。证明 1.1.21 式中 n_x，n_y，n_z 在某一时刻只有其中之一为零。

解：$\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=0$　　　（1）

在直角坐标系中，分离变量

$$A_i(\vec{r},t)=A_i(\vec{r})A_i(t)\qquad (i=x,y,z)\qquad （2）$$

代入（1）式，有

$$\frac{\nabla^2 A_i(\vec{r})}{A_i(\vec{r})}=\frac{\frac{1}{c^2}\frac{\partial^2 A_i(t)}{\partial t^2}}{A_i(t)}=-k^2$$

即　$\nabla^2 A_i(r)+k^2 A_i(r)=0$　　　（3）

再利用分离变量法，令

$$A_i(r)=X(x)Y(y)Z(z)\qquad （4）$$

则（3）式分解为

$$\begin{cases}\frac{d^2X}{dx^2}+k_x^2X=0\\ \frac{d^2Y}{dy^2}+k_y^2Y=0\\ \frac{d^2Z}{dz^2}+k_z^2Z=0\\ k_x^2+k_y^2+k_z^2=k^2\end{cases}\qquad （5）$$

由（5）式解得

$$A_i(r)=(C_1\cos k_x x+D_1\sin k_x x)(C_2\cos k_y y+D_2\sin k_y y)$$
$$\bullet(C_3\cos k_z z+D_3\sin k_z z)\qquad （6）$$

$$\because \vec{E}(\vec{r},t)=-\frac{\partial\vec{A}(\vec{r},t)}{\partial t}\qquad （7）$$

由边界条件$\begin{cases}\vec{n}\times\vec{E}=0\\ \frac{\partial E_n}{\partial n}\Big|_S=0\end{cases}$

把（7）式代入上式，得

$$\begin{cases}\vec{n}\times\vec{A}(\vec{r})=0\\ \dfrac{\partial A(\vec{r})_n}{\partial n}\Big|_S=0\end{cases}$$

$\Rightarrow$
$$A_x(\vec{r},t)=A_x(t)\cos(k_x x)\sin(k_y y)\sin(k_z z)\text{，}$$
$$A_y(\vec{r},t)=A_y(t)\sin(k_x x)\cos(k_y y)\sin(k_z z)\text{，}$$
$$A_z(\vec{r},t)=A_z(t)\sin(k_x x)\sin(k_y y)\cos(k_z z)$$

再考虑$x,y,z=L$时的边界条件，得

$$k_x=\frac{n_x\pi}{L}\text{，}\quad k_y=\frac{n_y\pi}{L}\text{，}\quad k_z=\frac{n_z\pi}{L}\text{，}\quad n_x,n_y,n_z=0,1,2,\cdots\cdots$$

若n_x,n_y,n_z中有两个或两个以上为零，则

$$A_x(\vec{r},t)=A_y(\vec{r},t)=A_z(\vec{r},t)=0$$

即$\vec{A}=0$，腔内没有电磁场，这个解没有意义。

$\therefore n_x,n_y,n_z$最多只有一个为零。

1.2 算符A,B不对易，但满足$[[A,B],A]=[[A,B],B]=0$，证明

$$e^{A+B}=e^{-\frac{1}{2}[A,B]}e^{A}e^{B}=e^{\frac{1}{2}[A,B]}e^{B}e^{A}\text{。}$$

证：记$[A,B]=C$， $f(\lambda)=e^{\lambda A}e^{\lambda B}$ （λ为参数），

则$f(0)=1$， $f(1)=e^{A}e^{B}$， $\frac{df}{d\lambda}=e^{\lambda A}(A+B)e^{\lambda B}$

$\because [A,B^n]=[A,B]B^{n-1}+B[A,B^{n-1}]=CB^{n-1}+B[A,B^{n-1}]$

$$=CB^{n-1}+B(CB^{n-2}+B[A,B^{n-2}])=\cdots\cdots=nCB^{n-1}\qquad(1)$$

$\therefore Ae^{\lambda B}=e^{\lambda B}(A+\lambda C)$

$\therefore \frac{df}{d\lambda}=e^{\lambda A}e^{\lambda B}(A+B+\lambda C)=f(\lambda)(A+B+\lambda C)$

$\therefore \ln f(\lambda)-\ln f(0)=(A+B)\lambda+\frac{1}{2}\lambda^2 C$

$\therefore f(\lambda)=e^{(A+B)\lambda+\frac{1}{2}\lambda^2 C}$

$\therefore e^{(A+B)\lambda}=e^{\lambda A}e^{\lambda B}e^{-\frac{1}{2}\lambda^2 C}=e^{-\frac{1}{2}\lambda^2 C}e^{\lambda A}e^{\lambda B}$

令$\lambda=1$，即$e^{A+B}=e^{-\frac{1}{2}C}e^{A}e^{B}=e^{-\frac{1}{2}[A,B]}e^{A}e^{B}$

$A\leftrightarrow B$，则有$e^{A+B}=e^{\frac{1}{2}[A,B]}e^{B}e^{A}$。

1.3 α 为参数，A,B 不对易，求证

$e^{-\alpha A}Be^{-\alpha A}=B-\alpha[A,B]+\frac{\alpha^2}{2!}[A,[A,B]]+\cdots$。

证：令 $f(\alpha)=e^{-\alpha A}Be^{-\alpha A}$，则 $\frac{df}{d\alpha}=-e^{-\alpha A}(AB-BA)e^{-\alpha A}=-e^{-\alpha A}[A,B]e^{-\alpha A}$

$\frac{d^2f}{d\alpha^2}=e^{-\alpha A}(A[A,B]-[A,B]A)e^{-\alpha A}=e^{-\alpha A}[A,[A,B]]e^{-\alpha A}$

……

所以 $f(\alpha)=f(0)+\sum_{n=1}^{\infty}\frac{\alpha^n}{n!}(\frac{d^nf}{d\alpha^n})_{\alpha=0}=B-\alpha[A,B]+\frac{\alpha^2}{2!}[A,[A,B]]+\cdots$

1.4 若 $f(a,a^+)$ 是一个可以展开成 a,a^+ 的幂级数的函数，证明：

（a） $[a,f(a,a^+)]=\frac{\partial f}{\partial a^+}$,(b) $[a^+,f(a,a^+)]=-\frac{\partial f}{\partial a}$，

(c) $e^{-\alpha a^+a}f(a,a^+)e^{\alpha a^+a}=f(ae^{\alpha},a^+e^{-\alpha})$，其中 α 为参数。

证：$\because[a,a^+]=1$， $[a^+,a\ ]=-1$

在（1.2）题中（1）式，$[A,B^n]=nCB^{n-1}=C\frac{\partial B^n}{\partial B}$

注意到 $f(a,a^+)$ 可以展开成 a,a^+ 的正序幂级数，也可以展开成 a,a^+ 的反序幂级数，

$\therefore[a,f(a,a^+)]=\frac{\partial f}{\partial a^+}$， $[a^+,f(a,a^+)]=-\frac{\partial f}{\partial a}$。

类似有 $[a,\frac{\partial f}{\partial a^+}]=\frac{\partial^2 f}{\partial a^{+2}}$， $[a,\frac{\partial f}{\partial a}]=\frac{\partial^2 f}{\partial a^+\partial a}$， $[a^+,\frac{\partial f}{\partial a^+}]=-\frac{\partial^2 f}{\partial a\partial a^+}$，

$[a^+,\frac{\partial f}{\partial a}]=-\frac{\partial^2 f}{\partial a^2}$，

f 关于 a,a^+ 的更高阶的偏导数也有类似的性质。

记 $a^+a=A$，则 $[A,f]=a^+\frac{\partial f}{\partial a^+}-a\frac{\partial f}{\partial a}$

$\therefore[A,[A,\cdots[A,f]]\cdots]$ （共 n 个 A） $=(a^+\frac{\partial}{\partial a^+}-a\frac{\partial}{\partial a})^nf$

利用 1.3 题的结论，有

$$e^{-\alpha a^+a}fe^{\alpha a^+a}=f-\alpha(a^+\tfrac{\partial}{\partial a^+}-a\tfrac{\partial}{\partial a})f+\frac{\alpha^2}{2!}(a^+\tfrac{\partial}{\partial a^+}-a\tfrac{\partial}{\partial a})^2f+\cdots$$

$\because f(ae^{\alpha},a^+e^{-\alpha})_{\alpha=0}=f(a,a^+)$，

$(\frac{\partial f(ae^{\alpha},a^+e^{-\alpha})}{\partial\alpha})_{\alpha=0}=-(a^+\frac{\partial}{\partial a^+}-a\frac{\partial}{\partial a})f(a,a^+)$，

$(\frac{\partial^2 f(ae^{\alpha},a^+e^{-\alpha})}{\partial\alpha^2})_{\alpha=0}=(a^+\frac{\partial}{\partial a^+}-a\frac{\partial}{\partial a})^2f(a,a^+)$， $\cdots$

$\therefore e^{-\alpha a^+ a} f(a,a^+) e^{\alpha a^+ a} = f(ae^{\alpha}, a^+ e^{-\alpha})$。

（c）的另一种证法：

由 1.2 题的（1）式，易得$[a^+, a^n] = -na^{n-1}$，$[a\ , a^{+n}] = na^{+n-1}$

$\therefore [a^+ a, a^n] = -na^n$，$[aa^+, a^{+n}] = na^{+n}$

由 1.3 题结果，易得$e^{-\alpha a^+ a} a^n e^{\alpha a^+ a} = a^n + \alpha n a^n + \frac{(\alpha n)^2}{2!} a^n + \cdots = (a\ e^{\alpha})^n$，

同理$e^{-\alpha a^+ a} a^{+n} e^{\alpha a^+ a} = (a^+ e^{-\alpha})^n$，

把$f(a,a^+)$展开成逆序形式

$$f(a,a^+) = \sum_{n,m=0}^{\infty} \frac{f^{(n,m)}(0,0)}{n!m!} a^n a^{+m}$$

则$e^{-\alpha a^+ a} f(a,a^+) e^{\alpha a^+ a} = \sum_{n,m=0}^{\infty} \frac{f^{(n,m)}(0,0)}{n!m!} e^{-\alpha a^+ a} a^n e^{\alpha a^+ a} e^{-\alpha a^+ a} a^{+m} e^{\alpha a^+ a}$

$$= \sum_{n,m=0}^{\infty} \frac{f^{(n,m)}(0,0)}{n!m!} (ae^{\alpha})^n (a^+ e^{-\alpha})^m$$

$$= f(ae^{\alpha}, a^+ e^{-\alpha})$$

（c）的第三种证法：

把f展开成正序形式，令$g(\alpha) = f(ae^{\alpha}, a^+ e^{-\alpha})$，$A = a^+ a$，

$B = f(a,a^+)$，

则$g(0) = B$，$g'(0) = \frac{\partial f}{\partial a} a - \frac{\partial f}{\partial a^+} a^+ = -[A,B] \cdots\cdots$

对照 1.3 题的结果，知$g(\alpha) = e^{-\alpha A} B e^{\alpha A}$，代入具体表达式即得到结论。

1.5 证明$[a, e^{-\alpha a^+ a}] = (e^{-\alpha} - 1) e^{-\alpha a^+ a} a$，$[a^+, e^{-\alpha a^+ a}] = (e^{\alpha} - 1) e^{-\alpha a^+ a} a^+$，$\alpha$为参数。

证法一：$\because e^{-\alpha a^+ a} = 1 - \alpha a^+ a + \frac{\alpha^2}{2!}(a^+ a)^2 + \cdots$，

$[a, (a^+ a)^n] = (a^+ a + 1)^n a - (a^+ a)^n a$，

$$\therefore [a, e^{-\alpha a^+ a}] = a - \alpha(a^+ a + 1)a + \frac{\alpha^2}{2!}(a^+ a + 1)^2 a + \cdots - (a - \alpha(a^+ a)a + \frac{\alpha^2}{2!}(a^+ a)^2 a + \cdots)$$

$$= e^{-\alpha(a^+ a + 1)} a - e^{-\alpha a^+ a} a$$

$$= (e^{-\alpha} - 1) e^{-\alpha a^+ a} a\ ;$$

$\because [a^+, (a\ a^+)^n] = (a^+ a)^n a^+ - (a^+ a + 1)^n a^+$，

$e^{-\alpha a^+ a} = e^{\alpha} e^{-\alpha a\ a^+}$，

$\therefore [a^+, e^{-\alpha a^+ a}] = e^{\alpha}[a^+, e^{-\alpha a\, a^+}] = e^{\alpha}(e^{-\alpha a^+ a}a^+ - e^{-\alpha(a^+ a+1)}a^+) = (e^{\alpha}-1)e^{-\alpha a^+ a}a^+$。

证法二：由 1.4 题结果，$e^{-\alpha a^+ a}a e^{\alpha a^+ a} = a e^{\alpha}$，
$\therefore e^{-\alpha}e^{-\alpha a^+ a}a = a e^{-\alpha a^+ a}$，
两边同减$e^{-\alpha a^+ a}a$，得$e^{-\alpha}e^{-\alpha a^+ a}a - e^{-\alpha a^+ a}a = a e^{-\alpha a^+ a} - e^{-\alpha a^+ a}a$
即$(e^{-\alpha}-1)e^{-\alpha a^+ a}a = [a, e^{-\alpha a^+ a}]$；
类似可证$[a^+, e^{-\alpha a^+ a}] = (e^{\alpha}-1)e^{-\alpha a^+ a}a^+$。

1.6 证明$H = \hbar\nu(a^+ a + \frac{1}{2})$可以写成$H = \sum_n E_n |n\rangle\langle n|$，从而
$e^{iHt/\hbar} = \sum_n e^{iE_n t/\hbar}|n\rangle\langle n|$。

证：在数态表象下，$H = \hbar\nu(a^+ a + \frac{1}{2})$的矩阵元
$H_{mn} = \langle m|H|n\rangle = (n+\frac{1}{2})\hbar\nu\delta_{mn} = E_n\delta_{mn}$，
$H = \sum_n E_n|n\rangle\langle n|$的矩阵元$H_{mk} = \sum_n E_n\langle m|n\rangle\langle n|k\rangle = E_k\delta_{mk}$，
$\therefore\ H = \hbar\nu(a^+ a + \frac{1}{2})$可以写成$H = \sum_n E_n|n\rangle\langle n|$。

同理比较$e^{iHt/\hbar}$和$\sum_n e^{iE_n t/\hbar}|n\rangle\langle n|$的矩阵元可得

$$e^{iHt/\hbar} = \sum_n e^{iE_n t/\hbar}|n\rangle\langle n|。$$

1.7 证明麦克斯韦方程组可以写成（1.5.27a）及（1.5.27b）的形式。首先证明$\frac{1}{c}\frac{\partial\tilde{E}}{\partial t} = \nabla\times\tilde{H}$，$\nabla\bullet\tilde{E} = 0$，$-\frac{1}{c}\frac{\partial\tilde{H}}{\partial t} = \nabla\times\tilde{E}$，$\nabla\bullet\tilde{H} = 0$，其中$\tilde{E} = \sqrt{\varepsilon_0}\vec{E}$，$\tilde{H} = \sqrt{\mu_0}\vec{H}$；然后证明$\vec{s}\bullet\nabla\vec{V} = \nabla\times\vec{V}$，
$s_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$，$s_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$，$s_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$。由此得到（1.5.27a）及（1.5.27b）。

证：（1）真空中的麦克斯韦方程组为
$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$，$\nabla\bullet\vec{E} = 0$，$\nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$，$\nabla\bullet\vec{B} = 0$。
把$\vec{E} = \frac{1}{\sqrt{\varepsilon_0}}\tilde{E}$，$\vec{B} = \mu_0\vec{H} = \sqrt{\mu_0}\tilde{H}$代入方程组，即可得到
$\frac{1}{c}\frac{\partial\tilde{E}}{\partial t} = \nabla\times\tilde{H}$，$\nabla\bullet\tilde{E} = 0$，$-\frac{1}{c}\frac{\partial\tilde{H}}{\partial t} = \nabla\times\tilde{E}$，$\nabla\bullet\tilde{H} = 0$。

（2）

$$\vec{s}\bullet\nabla\vec{V}=s_x\tfrac{\partial}{\partial x}\begin{pmatrix}V_x\\V_y\\V_z\end{pmatrix}+s_y\tfrac{\partial}{\partial y}\begin{pmatrix}V_x\\V_y\\V_z\end{pmatrix}+s_z\tfrac{\partial}{\partial z}\begin{pmatrix}V_x\\V_y\\V_z\end{pmatrix}=\begin{pmatrix}0\\-\tfrac{\partial}{\partial x}V_z\\\tfrac{\partial}{\partial x}V_y\end{pmatrix}+\begin{pmatrix}\tfrac{\partial}{\partial y}V_z\\0\\-\tfrac{\partial}{\partial y}V_x\end{pmatrix}+\begin{pmatrix}-\tfrac{\partial}{\partial z}V_y\\\tfrac{\partial}{\partial z}V_x\\0\end{pmatrix}=\nabla\times\vec{V}$$

（3）由 1.5.25 式，易得 $i\hbar\frac{\partial}{\partial t}\vec{\varphi}_\gamma=-c(-i\hbar\nabla)\times\vec{\chi}_\gamma=-c\vec{s}\bullet\vec{P}\vec{\chi}_\gamma$，

$i\hbar\frac{\partial}{\partial t}\vec{\chi}_\gamma=c(-i\hbar\nabla)\times\vec{\varphi}_\gamma=c\vec{s}\bullet\vec{P}\vec{\varphi}_\gamma$，

$\therefore i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\vec{\varphi}_\gamma\\\vec{\chi}_\gamma\end{pmatrix}=\begin{pmatrix}0&-c\vec{s}\bullet\vec{P}\\c\vec{s}\bullet\vec{P}&0\end{pmatrix}\begin{pmatrix}\vec{\varphi}_\gamma\\\vec{\chi}_\gamma\end{pmatrix}$，此即 1.5.27a；

$\because\nabla\bullet\vec{\varphi}_\gamma=0$， $\nabla\bullet\vec{\chi}_\gamma=0$， $\nabla\bullet\begin{pmatrix}\vec{\varphi}_\gamma\\\vec{\chi}_\gamma\end{pmatrix}=0$，这就是 1.5.27b。

1.8 推导 1.5.32。$\vec{\varphi}_\gamma$， $\vec{\chi}_\gamma$ 的动力学方程为 $\dot{\vec{\varphi}}_\gamma=c\vec{s}\bullet\nabla\vec{\chi}_\gamma$，

$\dot{\vec{\chi}}_\gamma=-c\vec{s}\bullet\nabla\vec{\varphi}_\gamma$， $\dot{\vec{\varphi}}^{+}{}_\gamma=c\nabla\vec{\chi}^{+}{}_\gamma\bullet\vec{s}^{+}$， $\dot{\vec{\chi}}^{+}{}_\gamma=-c\nabla\vec{\varphi}^{+}{}_\gamma\bullet\vec{s}^{+}$，注意 $\vec{s}^{+}=-\vec{s}$。

解：$\Psi^{+}{}_\gamma\Psi_\gamma=\vec{\varphi}^{+}{}_\gamma\vec{\varphi}_\gamma+\vec{\chi}^{+}{}_\gamma\vec{\chi}_\gamma$， $\Psi^{+}{}_\gamma\vec{v}\Psi_\gamma=c\vec{\chi}^{+}{}_\gamma\vec{s}\vec{\varphi}_\gamma-c\vec{\varphi}^{+}{}_\gamma\vec{s}\vec{\chi}_\gamma$

$\because\nabla\bullet(-c\vec{\chi}^{+}{}_\gamma\vec{s}\vec{\varphi}_\gamma)=-c(\nabla\vec{\chi}^{+}{}_\gamma\bullet\vec{s}\vec{\varphi}_\gamma+\vec{\chi}^{+}{}_\gamma\vec{s}\bullet\nabla\vec{\varphi}_\gamma)=\dot{\vec{\varphi}}^{+}{}_\gamma\vec{\varphi}_\gamma+\vec{\chi}^{+}{}_\gamma\dot{\vec{\chi}}_\gamma$，

同理 $\nabla\bullet(c\vec{\varphi}^{+}{}_\gamma\vec{s}\vec{\chi}_\gamma)=\vec{\varphi}^{+}{}_\gamma\dot{\vec{\varphi}}_\gamma+\dot{\vec{\chi}}^{+}{}_\gamma\vec{\chi}_\gamma$，

$$\begin{aligned}\therefore\tfrac{\partial}{\partial t}(\Psi^{+}{}_\gamma\Psi_\gamma)&=\dot{\vec{\varphi}}^{+}{}_\gamma\vec{\varphi}_\gamma+\vec{\chi}^{+}{}_\gamma\dot{\vec{\chi}}_\gamma+\vec{\varphi}^{+}{}_\gamma\dot{\vec{\varphi}}_\gamma+\dot{\vec{\chi}}^{+}{}_\gamma\vec{\chi}_\gamma\\&=\nabla\bullet(-c\vec{\chi}^{+}{}_\gamma\vec{s}\vec{\varphi}_\gamma+c\vec{\varphi}^{+}{}_\gamma\vec{s}\vec{\chi}_\gamma)\\&=-\nabla\bullet(\Psi^{+}{}_\gamma\vec{v}\Psi_\gamma)\end{aligned}$$

由于 $\frac{\partial}{\partial t}(\Psi^{+}{}_\gamma\Psi_\gamma)=-\nabla\bullet\vec{j}$，

因此，$\vec{j}=\Psi^{+}{}_\gamma\vec{v}\Psi_\gamma$，此即 1.5.32 式。

1.9 通过在两边用任意矢量 $\vec{v}$ 点积的方法证明 $\sum_i e_ie_i=1$。因此若 $\hat{e}_1=\hat{\varepsilon}_k^{(1)}$， $\hat{e}_2=\hat{\varepsilon}_k^{(2)}$， $\hat{e}_3=\frac{\vec{k}}{k}$，则得到 1.1.36。在极坐标中，$\vec{k}=k(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$，两横向偏振矢量表示为 $\hat{\varepsilon}_k^{(1)}=(\sin\phi,-\cos\phi,0)$， $\hat{\varepsilon}_k^{(2)}=(\cos\theta\cos\phi,\cos\theta\sin\phi,-\sin\theta)$，直接代入证明 $\varepsilon_{ki}^{(1)}\varepsilon_{kj}^{(1)}+\varepsilon_{ki}^{(2)}\varepsilon_{kj}^{(2)}=\delta_{ij}-\frac{k_ik_j}{k^2}$。

证：对任意矢量 $\vec{v}$， $\vec{v}\bullet(\sum_i e_ie_i)\bullet\vec{v}=\sum_i v_iv_i=\vec{v}\bullet\vec{v}$

$$\therefore\ \sum_i e_ie_i=1\ ;$$

$\because\sin^2\phi+\cos^2\theta\cos^2\phi+\sin^2\theta\cos^2\phi=1$ （$i=j=1$ 时）

$\cos^2\phi+\cos^2\theta\sin^2\phi+\sin^2\theta\sin^2\phi=1$　（$i=j=2$时）

$0+\sin^2\theta+\cos^2\theta=1$　（$i=j=3$时）

$-\sin\phi\cos\phi+\cos^2\theta\cos\phi\sin\phi+\sin^2\theta\cos\phi\sin\phi=0$　（$i=1,j=2$时）

$0-\cos\theta\sin\theta\cos\phi+\sin\theta\cos\phi\cos\theta=0$　（$i=1,j=3$时）

$0-\cos\theta\sin\theta\sin\phi+\sin\theta\sin\phi\cos\theta=0$　（$i=2,j=3$时）

$\therefore\varepsilon_{ki}^{(1)}\varepsilon_{kj}^{(1)}+\varepsilon_{ki}^{(2)}\varepsilon_{kj}^{(2)}+\frac{k_ik_j}{k^2}=\delta_{ij}$，

即$\varepsilon_{ki}^{(1)}\varepsilon_{kj}^{(1)}+\varepsilon_{ki}^{(2)}\varepsilon_{kj}^{(2)}=\delta_{ij}-\frac{k_ik_j}{k^2}$。

第二章

2.1 证明：$a^+\left|\alpha\right\rangle\left\langle\alpha\right|=(\alpha^*+\frac{\partial}{\partial\alpha})\left|\alpha\right\rangle\left\langle\alpha\right|$；$\left|\alpha\right\rangle\left\langle\alpha\right|a=(\alpha+\frac{\partial}{\partial\alpha^*})\left|\alpha\right\rangle\left\langle\alpha\right|$。

证：$\because\left|\alpha\right\rangle=e^{\alpha a^+}e^{-\frac{|\alpha|^2}{2}}\left|0\right\rangle$，

$$\therefore\frac{\partial}{\partial\alpha}\left|\alpha\right\rangle\left\langle\alpha\right|=\frac{\partial}{\partial\alpha}e^{\alpha a^+}e^{-\frac{|\alpha|^2}{2}}\left|0\right\rangle\left\langle0\right|e^{\alpha^*a}e^{-\frac{|\alpha|^2}{2}}$$

$$=(-\alpha^*+a^+)e^{\alpha a^+}e^{-|\alpha|^2}\left|0\right\rangle\left\langle0\right|e^{\alpha^*a}$$

$$=(-\alpha^*+a^+)\left|\alpha\right\rangle\left\langle\alpha\right|$$

$\therefore a^+\left|\alpha\right\rangle\left\langle\alpha\right|=(\alpha^*+\frac{\partial}{\partial\alpha})\left|\alpha\right\rangle\left\langle\alpha\right|$。

类似可证$\left|\alpha\right\rangle\left\langle\alpha\right|a=(\alpha+\frac{\partial}{\partial\alpha^*})\left|\alpha\right\rangle\left\langle\alpha\right|$。

（也可以把相干态展开成数态再证，比较麻烦。）

2.2 证明热光场中，$\left\langle D(\alpha)\right\rangle=\exp[-|\alpha|^2(\left\langle n\right\rangle+\frac{1}{2})]$，$\left\langle n\right\rangle$为场的平均光子数。

证：由（3.1.26），对热光场$P(\alpha,\alpha^*)=\frac{1}{\pi\left\langle n\right\rangle}e^{-|\alpha|^2/\left\langle n\right\rangle}$，

$$\therefore\left\langle D(\beta)\right\rangle=e^{-|\beta|^2/2}\left\langle e^{\beta a^+}e^{-\beta^*a}\right\rangle$$

$$=e^{-|\beta|^2/2}\frac{1}{\pi\langle n\rangle}\int e^{-|\alpha|^2/\langle n\rangle}e^{\beta\alpha^*-\beta^*\alpha}d^2\alpha$$

$$=e^{-|\beta|^2/2}\frac{1}{\pi\langle n\rangle}\iint e^{-(x_\alpha^2+y_\alpha^2)/\langle n\rangle}e^{2i(y_\beta x_\alpha-x_\beta y_\alpha)}dx_\alpha dy_\alpha$$

$$=e^{-|\beta|^2/2}\frac{1}{\pi\langle n\rangle}e^{-\langle n\rangle y_\beta^2}e^{-\langle n\rangle x_\beta^2}(\sqrt{\frac{\pi}{\frac{1}{\langle n\rangle}}})^2$$

$$=e^{-|\beta|^2/2}e^{-\langle n\rangle|\beta|^2},$$

$\therefore\ \langle D(\alpha)\rangle=\exp[-|\alpha|^2(\langle n\rangle+\frac{1}{2})]$。

2.3 证明：

$$\Psi(q,0)=\left(\frac{m\nu}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left[-\frac{m\nu}{2\hbar}(q-q_0)^2\right]$$

$$=D(q_0)\Phi_0(q)$$

$$=\exp\left(-iq_0\,\hat{p}/\hbar\right)\Phi_0(q)$$

其中，$\hat{p}=-i\hbar\frac{\partial}{\partial q}$，$\Phi_0(q)=\left(\frac{m\nu}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left[-\frac{m\nu}{2\hbar}q^2\right]$，

因而$D(q_0)=\exp\left(-iq_0\,\hat{p}/\hbar\right)$是平移算符。用$a^+,a$表示则为

$$D(q_0)=\exp\left(q_0\sqrt{\frac{m\nu}{2\hbar}}(a^+-a)\right),$$

所以 $\Psi(q,0)=\langle q|D(q_0,0)|0\rangle=\langle q|\exp\left(q_0\sqrt{\frac{m\nu}{2\hbar}}(a^+-a)\right)|0\rangle$，

令$\alpha=q_0\sqrt{\frac{m\nu}{2\hbar}}$，则

$\Psi(q,0)=\langle q|\exp\left(\alpha(a^+-a)\right)|0\rangle=\langle q|\alpha\rangle$，

即$\Psi(q,0)=\langle q|\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$，我们可以看出

$$a_n=\exp\left(-\frac{|\alpha|^2}{2}\right)\frac{\alpha^n}{\sqrt{n!}}。$$

因为 $H=\left(a^+a+\frac{1}{2}\right)\hbar\nu$，

所以 $\Psi(q,t)=\langle q|\exp\left(-\frac{iHt}{\hbar}\right)\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$

$$=\exp\left(-\frac{i\nu t}{2}\right)\langle q|\alpha\exp(-i\nu t)\rangle,$$

故而 $\Psi(q,t)=\exp\left(-\frac{i\nu t}{2}\right)\langle q|\alpha\exp(-i\nu t)\rangle$

$$=\exp\left(-\frac{i\nu t}{2}\right)\langle q|\exp[\alpha(t)a^+-\alpha^*(t)a]|0\rangle$$

把 $a=\frac{1}{\sqrt{2m\hbar\nu}}(m\nu q+ip)$， $a^+=\frac{1}{\sqrt{2m\hbar\nu}}(m\nu q-ip)$ 代入，

得

$$\Psi(q,0)=\exp\left(-\frac{i\nu t}{2}\right)\langle q|\exp[i\frac{q_0}{\hbar}(-m\nu q\sin\nu t-\hat{p}\cos\nu t)|0\rangle.$$

$$=\left(\frac{m\nu}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left(-\frac{i\nu t}{2}\right)\exp\left(\frac{i}{2\hbar}q_0{}^2 m\nu\sin\nu t\cos\nu t\right)\bullet$$

$$\exp\left(-\frac{i}{\hbar}q_0 m\nu q\sin\nu t\right)\exp\left(-\frac{m\nu(q-q_0\cos\nu t)^2}{2\hbar}\right)$$

所以 $|\Psi(q,t)|^2=\left(\frac{m\nu}{\pi\hbar}\right)^{\frac{1}{2}}\exp\left(-\frac{m\nu}{\hbar}(q-q_0\cos\nu t)^2\right)$。

2.4 证明：由 2.3 题我们知道，$a_n=\exp\left(-\frac{|\alpha|^2}{2}\right)\frac{\alpha^n}{\sqrt{n!}}$

所以有 $\Psi(q,0)=\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\Psi_n(q)$

$$=\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\langle q|n\rangle$$

2.5 证明：因

$$D^{-1}(\alpha)aD(\alpha)=a+\alpha$$

$$D^{-1}(\alpha)a^+D(\alpha)=a^++\alpha^*$$

$$S^+(\xi)a^+S(\xi)=a^+\cosh r-ae^{-i\theta}\sinh r$$

$$S^+(\xi)aS(\xi)=a\cosh r-a^+e^{i\theta}\sinh r$$

$$S^+(\xi)a^2S(\xi)=a^2\cosh^2 r-a^{+2}e^{2i\theta}\sinh r-(a^+a+aa^+)e^{i\theta}\sinh r\cosh r$$

所以 $\langle a\rangle=\langle 0|S^+(\xi)D^+(\alpha)aD(\alpha)S(\xi)|0\rangle$

$$=\langle 0|S^+(\xi)(a+\alpha)S(\xi)|0\rangle$$

$$=\alpha$$

$$\langle a^+\rangle=\alpha^*$$

$$\langle a^{+2}\rangle^*=\langle a^2\rangle$$

$$=\langle 0|S^{+}(\xi)D^{+}(\alpha)a^{2}D(\alpha)S(\xi)|0\rangle$$
$$=\langle 0|S^{+}(\xi)D^{+}(\alpha)aD(\alpha)D(\alpha)^{+}aD(\alpha)S(\xi)|0\rangle$$
$$=\alpha^{2}-e^{i\theta}\sinh r\cosh r$$
$$\left\langle a^{+}a\right\rangle=\langle 0|S^{+}(\xi)D^{+}(\alpha)a^{+}D(\alpha)D(\alpha)^{+}aD(\alpha)S(\xi)|0\rangle$$
$$=\langle 0|S^{+}(\xi)(a^{+}+\alpha^{*})(a+\alpha)S(\xi)|0\rangle$$
$$=\sinh^{2}r+|\alpha|^{2}$$
$$\left\langle aa^{+}\right\rangle=\cosh^{2}r+|\alpha|^{2}$$；

又 $Y_1=\dfrac{ae^{-\frac{i\theta}{2}}+a^{+}e^{\frac{i\theta}{2}}}{2}$，$Y_2=\dfrac{ae^{-\frac{i\theta}{2}}-a^{+}e^{\frac{i\theta}{2}}}{2i}$，

所以 $\langle Y_1\rangle^{2}=\left(\dfrac{\alpha e^{-\frac{i\theta}{2}}+\alpha^{*}e^{\frac{i\theta}{2}}}{2}\right)^{2}$

$$=\frac{\alpha^{2}e^{-i\theta}+\alpha^{*2}e^{i\theta}+2|\alpha|^{2}}{4}$$；

$$\langle Y_1^{2}\rangle=\frac{\alpha^{2}e^{-i\theta}-2\sinh r\cosh r+\alpha^{*2}e^{i\theta}+2|\alpha|^{2}+\sinh^{2}r+\cosh^{2}r}{4}$$

$$\langle\Delta Y_1\rangle^{2}=\langle Y_1^{2}\rangle-\langle Y_1\rangle^{2}=\frac{-2\sinh r\cosh r+\sinh^{2}r+\cosh^{2}r}{4}=\frac{1}{4}e^{-2r}$$

同理可得，$\langle\Delta Y_2\rangle^{2}=\dfrac{1}{4}e^{2r}$

2.6 证明：$S_{12}^{+}(\xi)=\exp(\xi^{*}a_1^{+}a_2^{+}+\xi a_1a_2)=S_{12}(\xi)$，这会导致$|\alpha_1,\alpha_2,\xi\rangle$不归一。查看有关资料可知，原题有错，可重新定义

$S_{12}(\xi)=\exp(-\xi\ a_1^{+}a_2^{+}+\xi^{*}a_1a_2)$，$\xi=r\exp(i\theta)$，则

$$S_{12}{}^{+}a_1S_{12}=a_1\cosh r-a_2{}^{+}e^{i\theta}\sinh r$$，
$$S_{12}{}^{+}a_2S_{12}=a_2\cosh r-a_1{}^{+}e^{i\theta}\sinh r$$，
$$\left\langle a_1\right\rangle=\langle 0|S_{12}{}^{+}D^{+}{}_2D^{+}{}_1a_1D_1D_2S_{12}|0\rangle$$
$$=\langle 0|S_{12}{}^{+}(a_1+\alpha_1)S_{12}|0\rangle$$
$$=\alpha_1$$，

同理，$\left\langle a_1^{2}\right\rangle=\alpha_1^{2}$，

$\left\langle a_1a_1{}^{+}\right\rangle=\cosh^{2}r+|\alpha_1|^{2}$，

$\left\langle a_1{}^{+}a_1\right\rangle=\sinh^{2}r+|\alpha_1|^{2}$，

所以 $\langle \Delta Y_1^{(1)} \rangle^2 = \left\langle \left(\dfrac{a_1 e^{-\frac{i\theta}{2}} + a_1^+ e^{\frac{i\theta}{2}}}{2} \right)^2 \right\rangle - \left\langle \dfrac{a_1 e^{-\frac{i\theta}{2}} + a_1^+ e^{\frac{i\theta}{2}}}{2} \right\rangle^2$

$$= \frac{1}{4}(2\sinh^2 r + 1) \ge \frac{1}{4}$$

同理，$\langle \Delta Y_2^{(1)} \rangle^2 = \dfrac{1}{4}(1 + 2\sinh^2 r) \ge \dfrac{1}{4}$，

所以模 a_1 没有压缩态。

经过同样的计算可得，模 a_2 也没有压缩态。

因此双模压缩态对两个单独模无压缩。

2.7 证明： 题目中有错，正确的 q^N 的表达式应该是 $q^N = \langle (\Delta X_1)^N \rangle - \left(\dfrac{1}{4} \right)^{\frac{N}{2}} (N-1)!!$，且只对 N 为偶数的情形成立。

因为 $[X_1, X_2] = \dfrac{i}{2}$，

故 $\Delta X_1 \Delta X_2 \ge \dfrac{1}{4} = C$，

由此易得 $\langle :(\Delta X_1)^2: \rangle = \langle (\Delta X_1)^2 \rangle - \dfrac{1}{4}$，

由 B-H 定理，$\langle e^{y\Delta X_1} \rangle = \langle : e^{y\Delta X_1} : \rangle e^{\frac{y^2}{8}}$，将此式两边同时按 y 的幂展开，取幂次相同的项比较系数得，

$$\langle (\Delta X_1)^N \rangle = \langle :(\Delta X_1)^N: \rangle + \frac{1}{2} CN \langle :(\Delta X_1)^N: \rangle + \cdots$$
$$+ \begin{cases} (N-1)!! C^{\frac{N}{2}} (N = 2k) \\ \dfrac{N! C^{\frac{N-3}{2}}}{3! 2^{\frac{N-3}{2}} \left(\frac{N-3}{2} \right)!} \langle :(\Delta X_1)^3: \rangle (N = 2k+1) \end{cases}$$

对于相干态，所有的 $\langle :(\Delta X_1)^N: \rangle$ 为零。因此，当 $\langle (\Delta X_1)^N \rangle < \left(\dfrac{1}{4} \right)^{\frac{N}{2}} (N-1)!!$ 时，即 $q^N = \langle (\Delta X_1)^N \rangle - \left(\dfrac{1}{4} \right)^{\frac{N}{2}} (N-1)!! < 0$ 时出现压缩。

（相干态的 $\langle (\Delta X_1)^N \rangle$ 的表达式也可以通过数学归纳法来证明，有些麻烦。）

2.8 证明：$[X_1, X_2] = \left[\dfrac{1}{2}(a^2 + a^{+2}), \dfrac{1}{2i}(a^2 - a^{+2}) \right]$

$$= 2i(a^+a + \frac{1}{2})$$

所以产生压缩的条件是 $\langle \Delta X_i \rangle^2 < \langle a^+a + \frac{1}{2} \rangle$ （i=1 or 2）。

由于上式中出现了平均光子数，而光场的光子数、因而这是一个非经典效应。

第三章

3.1 证明 $\frac{1}{2}\left\langle aa^+ + a^+a \right\rangle = \int W(\alpha,\alpha^*)|\alpha|^2 d^2\alpha$，其中 $W(\alpha,\alpha^*)$ 为 Wigner-Weyl 分布。

证：由（3.B.7）式，对 $O_1(a,a^+) = a^+a$， $O_{s1}(\alpha,\alpha^*) = |\alpha|^2 - \frac{1}{2}$

$O_2(a,a^+) = aa^+ = a^+a + 1$， $O_{s2}(\alpha,\alpha^*) = |\alpha|^2 + \frac{1}{2}$

所以对 $O(a,a^+) = \frac{1}{2}(aa^+ + a^+a)$， $O_s(\alpha,\alpha^*) = |\alpha|^2$

$\therefore \frac{1}{2}\left\langle aa^+ + a^+a \right\rangle = \int W(\alpha,\alpha^*)|\alpha|^2 d^2\alpha$

3.2 证明 $Tr[D(\alpha)] = \pi\delta^2(\alpha)$， $Tr[D(\alpha)D^+(\alpha')] = \pi\delta^2(\alpha - \alpha')$，其中 $D(\alpha)$ 为位移算符。用这一结果，证明

$Tr[\Delta^{(\Omega)}(\alpha - a, \alpha^* - a^+)\overline{\Delta}^{(\Omega)}(\alpha' - a, \alpha^{*\prime} - a^+)] = \frac{1}{\pi}\delta^2(\alpha - \alpha')$，其中 $\Delta^{(\Omega)}$ 和 $\overline{\Delta}^{(\Omega)}$ 分别在方程（3.4.2）和（3.4.9）中定义。

证：$Tr[D(\alpha)] = \frac{1}{\pi}\int d^2\beta \left\langle \beta \right| e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^+} e^{-\alpha^* a} \left| \beta \right\rangle$

$$= \frac{1}{\pi} e^{-\frac{|\alpha|^2}{2}} \int d^2\beta e^{\alpha\beta^* - \alpha^*\beta}$$

$$= \frac{1}{\pi} e^{-\frac{|\alpha|^2}{2}} \pi^2 \frac{1}{\pi^2} \int d^2\beta e^{\alpha\beta^* - \alpha^*\beta}$$

$$= \pi e^{-\frac{|\alpha|^2}{2}} \delta^2(\alpha)$$

$$= \pi\delta^2(\alpha)$$

$\because D(\alpha)D^+(\alpha') = e^{\alpha a^+ - \alpha^* a} e^{-\alpha' a^+ + \alpha^{*\prime} a}$

$$= e^{\alpha a^+ - \alpha^* a - \alpha' a^+ + \alpha^{*\prime} a} e^{\frac{1}{2}(\alpha^*\alpha' - \alpha\alpha^{*\prime})}$$

$$= D(\alpha - \alpha') e^{\frac{1}{2}(\alpha^*\alpha' - \alpha\alpha^{*\prime})}$$

$$\therefore Tr[D(\alpha)D^{+}(\alpha')] = \pi\delta^{2}(\alpha-\alpha')e^{\frac{1}{2}(\alpha^{*}\alpha'-\alpha\alpha^{*'})} = \pi\delta^{2}(\alpha-\alpha')$$

$$Tr[\Delta^{(\Omega)}(\alpha-a,\alpha^{*}-a^{+})\overline{\Delta}^{(\Omega)}(\alpha'-a,\alpha^{*'}-a^{+})]$$

$$= Tr[\tfrac{1}{\pi^{2}}\int e^{\Omega(\beta_{1},\beta_{1}^{*})}e^{-\beta_{1}(\alpha^{*}-a^{+})+\beta_{1}^{*}(\alpha-a)}d^{2}\beta_{1}\times\tfrac{1}{\pi^{2}}\int e^{-\Omega(\beta_{2},\beta_{2}^{*})}e^{\beta_{2}(\alpha^{*'}-a^{+})-\beta_{2}^{*}(\alpha'-a)}d^{2}\beta_{2}]$$

$$= Tr[\tfrac{1}{\pi^{4}}\int d^{2}\beta_{1}\int d^{2}\beta_{2}e^{\Omega(\beta_{1},\beta_{1}^{*})-\Omega(\beta_{2},\beta_{2}^{*})}e^{-\beta_{1}\alpha^{*}+\beta_{1}^{*}\alpha+\beta_{2}\alpha^{*'}-\beta_{2}^{*}\alpha'}$$

$$\times e^{\beta_{1}a^{+}-\beta_{1}^{*}a}e^{-\beta_{2}a^{+}+\beta_{2}^{*}a}]$$

$$= \tfrac{1}{\pi^{4}}\int d^{2}\beta_{1}\int d^{2}\beta_{2}e^{\Omega(\beta_{1},\beta_{1}^{*})-\Omega(\beta_{2},\beta_{2}^{*})}e^{-\beta_{1}\alpha^{*}+\beta_{1}^{*}\alpha+\beta_{2}\alpha^{*'}-\beta_{2}^{*}\alpha'}Tr[D(\beta_{1})D^{+}(\beta_{2})]$$

$$= \tfrac{1}{\pi^{4}}\int d^{2}\beta_{1}\int d^{2}\beta_{2}e^{\Omega(\beta_{1},\beta_{1}^{*})-\Omega(\beta_{2},\beta_{2}^{*})}e^{-\beta_{1}\alpha^{*}+\beta_{1}^{*}\alpha+\beta_{2}\alpha^{*'}-\beta_{2}^{*}\alpha'}\pi\delta^{2}(\beta_{1}-\beta_{2})$$

$$= \frac{1}{\pi^{3}}\int d^{2}\beta_{1}e^{-\beta_{1}(\alpha^{*}-\alpha^{*'})+\beta_{1}^{*}(\alpha-\alpha')}$$

$$= \tfrac{1}{\pi}\delta^{2}(\alpha-\alpha')$$

3.3 证明$W(\alpha,\alpha^{*}) = \frac{2}{\pi}\int P(\beta,\beta^{*})\exp(-2|\alpha-\beta|^{2})d^{2}\beta$。

$\because C^{(s)}(\beta,\beta^{*}) = e^{-\frac{1}{2}|\beta|^{2}}C^{(n)}(\beta,\beta^{*})$，

$C^{(s)}(\beta,\beta^{*})$， $e^{-\frac{1}{2}|\beta|^{2}}$， $C^{(n)}(\beta,\beta^{*})$的傅立叶变换分别为

$4W(\alpha,\alpha^{*})$， $4P(\alpha,\alpha^{*})$， $4e^{-2|\alpha|^{2}}$

由卷积定理，$W(\alpha,\alpha^{*}) = 4\frac{1}{2\pi}\int P(\beta,\beta^{*})\exp(-2|\alpha-\beta|^{2})d^{2}\beta$

$$= \frac{2}{\pi}\int P(\beta,\beta^{*})\exp(-2|\alpha-\beta|^{2})d^{2}\beta$$

3.4 确定相干态光场和热光场的$Q(\alpha,\alpha^{*})$和$W(\alpha,\alpha^{*})$。

解：对于相干态$\rho = |\alpha_{0}\rangle\langle\alpha_{0}|$，

$Q(\alpha,\alpha^{*}) = \frac{1}{\pi}\langle\alpha|\alpha_{0}\rangle\langle\alpha_{0}|\alpha\rangle = \frac{1}{\pi}\exp(-|\alpha-\alpha_{0}|^{2})$，

$$W(\alpha,\alpha^{*}) = \frac{2}{\pi^{2}}e^{2|\alpha|^{2}}\int\langle-\beta|\alpha_{0}\rangle\langle\alpha_{0}|\beta\rangle e^{-2(\beta\alpha^{*}-\beta^{*}\alpha)}d^{2}\beta$$

$$= \frac{2}{\pi^{2}}e^{2|\alpha|^{2}}\int e^{-\frac{|\beta|^{2}}{2}-\beta^{*}\alpha_{0}-\frac{|\alpha_{0}|^{2}}{2}}e^{-\frac{|\beta|^{2}}{2}+\beta\alpha_{0}^{*}-\frac{|\alpha_{0}|^{2}}{2}}e^{-2(\beta\alpha^{*}-\beta^{*}\alpha)}d^{2}\beta$$

$$=\frac{2}{\pi^2}e^{2|\alpha|^2}\int e^{-|\beta|^2-|\alpha_0|^2+(\alpha_0{}^*-2\alpha^*)\beta+(2\alpha-\alpha_0)\beta^*}d^2\beta$$

$$=\frac{2}{\pi}e^{2|\alpha|^2}e^{-|\alpha_0|^2+(\alpha_0{}^*-2\alpha^*)(2\alpha-\alpha_0)}$$

$$=\frac{2}{\pi}e^{-2|\alpha-\alpha_0|^2}$$

对于热光场，$\rho=\sum_n\frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}}|n\rangle\langle n|$，

$$Q(\alpha,\alpha^*)=\frac{1}{\pi}\sum_n\frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}}\langle\alpha|n\rangle\langle n|\alpha\rangle$$

$$=\frac{1}{\pi}\sum_n\frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}}e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}$$

$$=\frac{1}{\pi}\frac{1}{1+\langle n\rangle}e^{-|\alpha|^2}\sum_n(\frac{\langle n\rangle|\alpha|^2}{1+\langle n\rangle})^n\frac{1}{n!}$$

$$=\frac{1}{\pi}\frac{1}{1+\langle n\rangle}e^{-|\alpha|^2}e^{\frac{\langle n\rangle|\alpha|^2}{1+\langle n\rangle}}$$

$$=\frac{1}{\pi}\frac{1}{1+\langle n\rangle}e^{-\frac{|\alpha|^2}{1+\langle n\rangle}}$$

$$W(\alpha,\alpha^*)=\frac{2}{\pi^2}e^{2|\alpha|^2}\int\sum_n\frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}}\langle-\beta|n\rangle\langle n|\beta\rangle e^{-2(\beta\alpha^*-\beta^*\alpha)}d^2\beta$$

$$=\frac{2}{\pi^2}e^{2|\alpha|^2}\int e^{-|\beta|^2}e^{-2(\beta\alpha^*-\beta^*\alpha)}\sum_n\frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}}\frac{(-|\beta|^2)^n}{n!}d^2\beta$$

$$=\frac{2}{\pi^2}e^{2|\alpha|^2}\int e^{-|\beta|^2}e^{-2(\beta\alpha^*-\beta^*\alpha)}\frac{1}{1+\langle n\rangle}e^{-\frac{\langle n\rangle|\beta|^2}{1+\langle n\rangle}}d^2\beta$$

$$=\frac{2}{\pi^2}e^{2|\alpha|^2}\frac{1}{1+\langle n\rangle}\frac{\pi}{1+\frac{\langle n\rangle}{1+\langle n\rangle}}e^{-\frac{4|\alpha|^2}{1+\frac{\langle n\rangle}{1+\langle n\rangle}}}$$

$$=\frac{1}{\pi(\langle n\rangle+\frac{1}{2})}e^{-\frac{|\alpha|^2}{\langle n\rangle+\frac{1}{2}}}$$

第四章

4.1 证明真空态和单光子态的叠加态$|\psi\rangle = a_0|0\rangle + a_1|0\rangle$为非经典态。

证：$\because g^{(2)}(0)=0<1$

$\therefore |\psi\rangle$是一个非经典态。

4.2（没有证明出来。）

4.3 证明：

（a） $TR(\rho)=1$

$$= TR(Na^{+m}e^{-ka^+a}a^m)$$

$$= N\sum_{n=0}^{\infty}\left\langle n\left|Na^{+m}e^{-ka^+a}a^m\right|n\right\rangle$$

$$= N\sum_{n=m}^{\infty}\left\langle n-m\left|e^{-ka^+a}\right|n-m\right\rangle\frac{n!}{(n-m)!}$$

$$= N\sum_{n=m}^{\infty}e^{-k(n-m)}\frac{n!}{(n-m)!}$$

$$= Nm!\sum_{n=m}^{\infty}C_n^m(e^{-k})^{n-m}$$

$$= \frac{Nm!}{(1-e^{-k})^{m+1}}$$

$$\therefore \quad N = \frac{(1-e^{-k})^{m+1}}{m!}$$

$$\rho = \frac{(1-e^{-k})^{m+1}}{m!}a^{+m}e^{-ka^+a}a^m$$

$\rho_{nm}=0$ 当 $n\neq m$

而 $\rho_{nn} = \begin{cases} 0(n<m) \\ (1-e^{-k})^{m+1}C_n^m(e^{-k})^{n-m}(else) \end{cases}$

（1） 当 $n=m$时， $\rho_{nn}=(1-e^{-k})^{n+1}$， $\lim\limits_{k\to\infty}\rho_{nn}=1$；

当 $n\neq m$时，$\lim\limits_{k\to\infty}\rho_{nn}=0$。

$\therefore$ $\lim\limits_{k\to\infty}\rho_{nn}=\delta_{nm}$，此时的态为 Fock 态$|m\rangle$。

（2） $\rho_{nn}=C_n^m(1-e^{-k})^{m+1}e^{-k(n-m)}=C_n^m(e^k-1)^m(1-e^{-k})e^{-kn}$

此式与热光场$\rho'_{nn}=\dfrac{e^k-1}{e^{k(n+1)}}$之间的关系，无论是否$k\to 0$，都很难看出来。

（b） $\langle a^+a\rangle = Tr(\rho a^+a)$

$$=Tr(\rho aa^{+})-Tr(\rho)$$
$$=Tr(a^{+}\rho a)-1$$
$$=\frac{(1-e^{-k})^{m+1}}{m!}\bullet\frac{(m+1)!}{(1-e^{-k})^{m+2}}-1$$
$$=\frac{m+1}{1-e^{-k}}-1$$
$$\left\langle a^{+2}a^{2}\right\rangle=Tr(\rho a^{+2}a^{2})$$
$$=Tr(\rho a^{2}a^{+2})-4Tr(\rho aa^{+})+2Tr(\rho)$$
$$=\frac{(m+1)(m+2)}{(1-e^{-k})^{2}}-\frac{4(m+1)}{1-e^{-k}}+2$$
对热光场， $e^{-k}=\dfrac{\bar{n}}{\bar{n}+1}$，则
$$\left\langle a^{+}a\right\rangle=m(\bar{n}+1)+\bar{n}$$
$$\left\langle a^{+2}a^{2}\right\rangle=(m+2)(m+1)(\bar{n}+1)^{2}-4(m+1)(\bar{n}+1)+2$$
所以$\left\langle a^{+2}a^{2}\right\rangle-\left\langle a^{+}a\right\rangle^{2}=(m+1)(\bar{n}+1)^{2}-2(m+1)(\bar{n}+1)+1$

当 $\bar{n}+1<\dfrac{2+\sqrt{4-\dfrac{4}{m+1}}}{2}=\sqrt{\dfrac{m}{m+1}}+1$ ，

即 $\bar{n}<\sqrt{\dfrac{m}{m+1}}$时，

$\left\langle a^{+2}a^{2}\right\rangle-\left\langle a^{+}a\right\rangle^{2}<1$，即$g^{(2)}(0)<1$，为亚泊松分布。

4.4 解：（1）对于相干态$|\alpha\rangle$，
$$P(\alpha',\alpha'^{*})=\delta^{(2)}(\alpha'-\alpha)$$
$$\therefore P_{m}=\int d\alpha'^{2}P(\alpha',\alpha'^{*})\frac{(\eta|\alpha'|^{2})^{m}}{m!}e^{-\eta|\alpha'|^{2}}$$
$$=\frac{(\eta|\alpha|^{2})^{m}}{m!}e^{-\eta|\alpha|^{2}}\text{；}$$
（2） 对数态$|n\rangle$，
$$P_{m}=\begin{cases}\begin{pmatrix}n\\m\end{pmatrix}\eta^{m}(1-\eta)^{n-m}(n\geq m)\\ \qquad=0(n<m)\end{cases}\text{；}$$
（3）对热光场，利用 3.1.26 式，
$$P_{m}=\int d\alpha^{2}P(\alpha,\alpha^{*})\frac{(\eta|\alpha|^{2})^{m}}{m!}e^{-\eta|\alpha|^{2}}$$
$$=\frac{1}{\pi\langle n\rangle m!}\int d\alpha^{2}(\eta|\alpha|^{2})^{m}e^{-\left(\eta+\frac{1}{\langle n\rangle}\right)|\alpha|^{2}}$$

$$=\frac{(\langle n\rangle\eta)^m}{(\langle n\rangle\eta+1)^{m+1}}$$

（也可通过对数态的求和来求得，但是有点麻烦。）

第五章

5.1 证明：

$$i\hbar\frac{\partial}{\partial t}\left(\Psi e^{i\chi}\right)=i\hbar\frac{\partial\Psi}{\partial t}e^{i\chi}-\hbar\Psi\frac{\partial\chi}{\partial t}e^{i\chi}$$

$$=\left\{-\frac{\hbar^2}{2m}\left[\nabla-i\frac{e}{\hbar}\vec{A}\right]^2+eu-\hbar\frac{\partial\chi}{\partial t}\right\}\Psi e^{i\chi}\qquad\cdots\cdots(1)$$

$$\left[\nabla-i\frac{e}{\hbar}\left(\vec{A}+\frac{\hbar}{e}\nabla\chi\right)\right]^2\Psi e^{i\chi}=\left[\nabla-i\frac{e}{\hbar}\vec{A}\right]^2\Psi e^{i\chi}-i\nabla\chi\Psi e^{i\chi}+\Psi\nabla e^{i\chi}$$

$$=\left[\nabla-i\frac{e}{\hbar}\vec{A}\right]^2\Psi e^{i\chi}\qquad\cdots\cdots(2)$$

$$e\left(u-\frac{\hbar}{e}\frac{\partial\chi}{\partial t}\right)\Psi e^{i\chi}=\left(eu-\hbar\frac{\partial\chi}{\partial t}\right)\Psi e^{i\chi}\qquad\cdots\cdots(3)$$

综合上面三式，即可得

$$i\hbar\frac{\partial}{\partial t}\left(\Psi e^{i\chi}\right)=\left\{-\frac{\hbar^2}{2m}\left[\nabla-i\frac{e}{\hbar}\left(\vec{A}+\frac{\hbar}{e}\nabla\chi\right)\right]^2+eu-\hbar\frac{\partial\chi}{\partial t}\right\}\Psi e^{i\chi}$$

所以 *schrödinger* 方程是规范不变的。

5.2 解：

令 $C_a=e^{-\frac{\gamma t}{2}}A$， $C_b=e^{-\frac{\gamma t}{2}}B$，则

$$\dot{A}=\frac{i\Omega_R}{2}e^{-i\Phi}B$$

$$\dot{B}=\frac{i\Omega_R}{2}e^{i\Phi}A$$

对应于方程 5.2.12 和 5.2.13 在 $\Delta=0$ 时的情况，所以由 5.2.27 式可知，

$$|A|^2-|B|^2=\cos\Omega_R t$$

因而 $W(t)=e^{-\gamma t}\cos\Omega_R t$

5.3 解：

$$\begin{cases}\dot{R}_1=-\Delta R_2\\ \dot{R}_2=\Delta R_1+\Omega_R R_3\\ \dot{R}_3=-\Omega_R R_2\end{cases}$$

由上面的方程组可推出

$\ddot{R}_2 = -\Delta^2 R_2 - \Omega_R{}^2 R_2 = -\Omega^2 R_2$　其中，$\Omega^2 = \Delta^2 + \Omega_R{}^2$。

解之得

$$R_2 = a_1 e^{-i\Omega t} + a_2 e^{i\Omega t}$$

由初始条件 $R_2(0) = 0$ 有

$$R_2 = a_3 \sin \Omega t 。$$

将上式代回方程组可解得

$$\begin{cases} R_1 = \dfrac{\Delta}{\Omega} a_3 (\cos \Omega t - 1) \\ R_3 = \dfrac{\Omega_R}{\Omega} a_3 (\cos \Omega t - 1) \end{cases}$$

由于 $\dot{R}_2 = \Omega a_3 \cos \Omega t = \Delta R_1 + \Omega_R R_3 = \Omega a_3 (\cos \Omega t - 1)$

$\therefore a_3 = 0$

$\therefore \vec{R}(t) = 0$

$\therefore \rho_{ab} = \rho_{ba} = 0$，　$\rho_{aa} = \rho_{bb} = \frac{1}{2}$

物理意义是，在没有弛豫过程时，如果初始状态为上下能态布局相同，则会一直保持布局相同，不发生跃迁。

5.4 证明：ρ是厄米算符，一定可以通过一个幺正变换对角化，

$\rho' = S\rho S^{+} = S\rho S^{-1}$，

$\therefore Tr\rho^2 = Tr\rho'^2 = \sum_n (\rho_n')^2 \le (\sum_n \rho_n')^2 = 1$。

5.5 解：令$|\Psi\rangle = C_a|\Psi\rangle + C_b|\Psi\rangle + C_c|\Psi\rangle$

由 *schrödinger* 方程$|\dot{\Psi}\rangle = -\frac{i}{\hbar} H|\Psi\rangle$，

$H = H_0 + H_1$，

$H_0 = \hbar\omega_a |a\rangle\langle a| + \hbar\omega_b |b\rangle\langle b| + \hbar\omega_c |c\rangle\langle c|$，

$H_1 = -(P_{ab}|a\rangle\langle b| + P_{ba}|b\rangle\langle a| + P_{bc}|b\rangle\langle c| + P_{cb}|c\rangle\langle b|)\varepsilon\cos\nu t$　；

将上述方程写成矩阵形式

$$\begin{pmatrix} \dot{C}_a \\ \dot{C}_b \\ \dot{C}_c \end{pmatrix} = -\frac{i}{\hbar}\begin{pmatrix} \hbar\omega_a & -P_{ab}\varepsilon\cos\nu t & 0 \\ -P_{ba}\varepsilon\cos\nu t & \hbar\omega_b & -P_{bc}\varepsilon\cos\nu t \\ 0 & -P_{cb}\varepsilon\cos\nu t & \hbar\omega_c \end{pmatrix}\begin{pmatrix} C_a \\ C_b \\ C_c \end{pmatrix}$$

也即

$$\dot{C}_a = -i\omega_a C_a + \frac{i}{\hbar} P_{ab} \varepsilon \cos \nu t C_b \text{；} \quad \cdots\cdots\cdots(1)$$

$$\dot{C}_b = -i\omega_b C_b + \frac{i}{\hbar} P_{ba} \varepsilon \cos \nu t C_a + \frac{i}{\hbar} P_{bc} \varepsilon \cos \nu t C_c \text{；} \quad \cdots\cdots\cdots(2)$$

$$\dot{C}_c = -i\omega_c C_c + \frac{i}{\hbar} P_{cb} \varepsilon \cos \nu t C_c \text{。} \quad \cdots\cdots\cdots(3)$$

做慢变振幅及旋转波近似

$C_a = \tilde{C}_a e^{-i\omega_a t}$，

$C_b = \tilde{C}_b e^{-i\omega_b t}$，

$C_c = \tilde{C}_c e^{-i\omega_c t}$ ；

代入（1），（2），（3），由于$\nu = \omega_a - \omega_b = \omega_b - \omega_c$得

$$\dot{\tilde{C}}_a = \frac{iP_{ab}\varepsilon}{2\hbar} \tilde{C}_b$$

$$\dot{\tilde{C}}_b = \frac{iP_{ba}\varepsilon}{2\hbar} \tilde{C}_a + \frac{iP_{bc}\varepsilon}{2\hbar} \tilde{C}_c$$

$$\dot{\tilde{C}}_c = \frac{iP_{cb}\varepsilon}{2\hbar} \tilde{C}_b$$

令$\Omega_{R1} = \frac{|P_{ab}|\varepsilon}{\hbar}$， $\Omega_{R2} = \frac{|P_{bc}|\varepsilon}{\hbar}$ ；

$P_{ab} = |P_{ab}| e^{-i\Phi_1}$， $P_{bc} = |P_{bc}| e^{-i\Phi_2}$ ；

则上三式变为

$$\dot{\tilde{C}}_a = \frac{i\Omega_{R1}}{2} e^{-i\Phi_1} \tilde{C}_b \quad \cdots\cdots\cdots(4)$$

$$\dot{\tilde{C}}_b = \frac{i\Omega_{R1}}{2} e^{i\Phi_1} \tilde{C}_a + \frac{i\Omega_{R2}}{2} e^{-i\Phi_2} \tilde{C}_c \quad \cdots\cdots\cdots(5)$$

$$\dot{\tilde{C}}_c = \frac{i\Omega_{R2}}{2} e^{i\Phi_2} \tilde{C}_b \quad \cdots\cdots\cdots(6)$$

上三方程的解为

$$\tilde{C}_a = \frac{\Omega_{R2}\Omega_{R1}}{\Omega^2} e^{-i(\Phi_2+\Phi_1)} (\cos\frac{\Omega t}{2} - 1)$$

$$\tilde{C}_b = \frac{i\Omega_{R2}}{\Omega} e^{-i\Phi_2} \sin\frac{\Omega t}{2}$$

$$\tilde{C}_c = \frac{\Omega^2{}_{R2}}{\Omega^2} \cos\frac{\Omega t}{2} + \frac{\Omega^2{}_{R1}}{\Omega^2}$$

其中，$\Omega^2 = \Omega_{R1}^2 + \Omega_{R2}^2$。

所以

$$\begin{cases} P_a = |\tilde{C}_a|^2 = 4(\frac{\Omega_{R2}\Omega_{R1}}{\Omega^2})^2 \sin^4 \frac{\Omega t}{4} \\ P_c = \left(\frac{\Omega^2{}_{R2}}{\Omega^2} \cos \frac{\Omega t}{2} + \frac{\Omega^2{}_{R1}}{\Omega^2} \right)^2 \end{cases}$$

5.6 证明：（a）

将 $E = U_k(z) e^{-i\nu_k t}$ 代入

$$\frac{\partial^2 E}{\partial z^2} - \mu_0 \varepsilon_0 [1 + \eta \delta(z)] \frac{\partial^2 E}{\partial t^2} = 0$$

得 $\frac{\partial^2 U_k}{\partial z^2} + k^2 [1 + \eta \delta(z)] U_k = 0$ ………(1)

其中 $\nu_k = kC$，方程（1）的边界条件为 $U_k(L) = U_k(-L_0) = 0$。

将（1）式在 $z = 0$ 附近积分得

$U_k'(0^+) - U_k'(0^-) = -k^2 \eta U_k(0)$ ………(2)

对 $z \neq 0$，方程（1）的解为

$U_k(z) = \begin{cases} M_k \sin k(z - L)(z > 0) \\ \xi_k \sin k(z + L_0)(z < 0) \end{cases}$ ………(3)

将方程（3）代入方程（2），并由 $U_k(0^+) = U_k(0^-)$ 的连接条件可得

$$\begin{cases} M_k \sin kL = -\xi_k \sin kL_0 \\ M_k (\cos kL - k\eta \sin kL) = \xi_k \cos kL_0 \end{cases}$$

将上式代入（1）式得

$\tan kL_0 = \frac{\tan kL}{\Lambda \tan kL - 1}$

其中 $\Lambda = \frac{\mu_0 \varepsilon_0 \eta \nu^2}{k} \simeq \eta k$，

又 $\frac{M_k}{\xi_k} = \frac{-\sin kL_0}{\sin kL}$，

所以 $\frac{M^2{}_k}{\xi^2{}_k} = \frac{\tan^2 kL + 1}{\tan^2 kL + (\Lambda \tan kL - 1)^2}$ ………(4)

从上式可知，当 $\tan k_0 L = \frac{1}{\Lambda}$ 时，有极大值 $\left(\frac{M^2{}_k}{\xi^2{}_k} \right)_{k=k_0} = \Lambda^2 + 1$。

因而 kL 可写成 $kL = n\pi + \theta_k (\theta_k \ll 1)$，所以

$\tan kL \simeq \theta_k$，

代回（4）式可得

$$\frac{M^2{}_k}{\xi^2{}_k}=\frac{\Gamma^2\Lambda^2}{(\nu_k-\nu)^2+\Gamma^2}$$

其中 $\nu_k=Ck=\frac{(n\pi+\theta_k)C}{L}$

$$\nu=Ck_0=\frac{(n\pi+\frac{1}{\Lambda})C}{L}$$

$$\Gamma=\frac{C}{\Lambda^2 L}$$ ；

在 $L_0 \gg L \gg \eta$ 时， $\xi^2{}_k \simeq 1$，

所以 $M_k=\frac{\Gamma\Lambda}{\sqrt{(\nu_k-\nu)^2+\Gamma^2}}$。

(b)

因为 $\frac{\partial^2 U_k}{\partial z^2}+k^2\varepsilon(z)U_k$

$$\frac{\partial^2 U'_k}{\partial z^2}+k'^2\varepsilon(z)U'_k=0$$

联合可得

$$(k^2-k'^2)\varepsilon(z)U'_k(z)U_k(z)=\frac{\partial}{\partial z}\left(U_k(z)\frac{\partial U_{k'}(z)}{\partial z}-U_{k'}(z)\frac{\partial U_k(z)}{\partial z}\right)$$

积分上式，

$$(k^2-k'^2)\int_{L_0-}^{L}\varepsilon(z)U'_k(z)U_k(z)dz=\int_{L_0-}^{L}\frac{\partial}{\partial z}\left(U_k(z)\frac{\partial U_{k'}(z)}{\partial z}-U_{k'}(z)\frac{\partial U_k(z)}{\partial z}\right)dz$$

由边界条件 $U_k(L)=U_k(-L_0)=U_{k'}(L)=U_{k'}(-L_0)=0$ 可知

$$\int_{L_0-}^{L}\frac{\partial}{\partial z}\left(U_k(z)\frac{\partial U_{k'}(z)}{\partial z}-U_{k'}(z)\frac{\partial U_k(z)}{\partial z}\right)dz$$

$$=U_k(z)\frac{\partial U_{k'}(z)}{\partial z}-U_{k'}(z)\frac{\partial U_k(z)}{\partial z}\Big|_{-L_0}^{L}$$

$$=0$$

所以， $(\nu^2-\nu'^2)\int_{L_0-}^{L}\varepsilon(z)U'_k(z)U_k(z)dz=0$。

5.7 证明：

（a） 由 5.2.56 式可知

$V_{a'b}(t)=-e\varepsilon(x_{a'b}\cos\nu t+y_{a'b}\sin\nu t)e^{i\omega' t}$，

其中

$$ex_{a'b}=\int\Psi_{a'}^{*}x\Psi_{b}dr=\wp$$

$$ey_{a'b}=\int\Psi^{*}{}_{a'}y\Psi_{b}dr=i\wp$$

所以 $V_{a'b}(t)=-\wp\varepsilon(\cos\nu t+i\sin\nu t)e^{i\omega' t}$

$$=-\wp\varepsilon e^{i(\omega'+\nu)t}$$

(b)

在没有外加磁场时，$m=0,\pm1$的三个能级是简并，对 Rydberg 原子，$\omega'=\omega\cong10^{9}Hz$。外加磁场后，由于 Zeeman 分裂，$\omega'=10^{10}Hz$，$\omega\cong10^{9}Hz$，

在近共振的情况下，由于外加光场的频率$\nu\approx\omega$，故而 $\frac{1}{\nu+\omega}=\frac{1}{2\times1000}\gg\frac{1}{\nu+\omega'}\cong\frac{1}{10^{10}}$。

第六章

6.1 解：$H=H_0+H_1$，$H_0=\hbar\nu a^{+}a+\hbar\omega\sigma_z$，

$H_1=\hbar g[\sigma_{+}a(a^{+}a)^{\frac{1}{2}}+(a^{+}a)^{\frac{1}{2}}a^{+}\sigma_{-}]$，

由（6.2.6）、（6.2.7）易得

$e^{i\nu a^{+}at}a(a^{+}a)^{\frac{1}{2}}e^{-i\nu a^{+}at}=a(a^{+}a)^{\frac{1}{2}}e^{-i\nu t}$，

$e^{i\omega\sigma_z t}\sigma_{+}e^{-i\omega\sigma_z t}=\sigma_{+}e^{2i\omega t}$，

$\therefore V=e^{\frac{iH_0t}{\hbar}}H_1e^{-\frac{iH_0t}{\hbar}}=\hbar g[\sigma_{+}a(a^{+}a)^{\frac{1}{2}}e^{i\Delta t}+(a^{+}a)^{\frac{1}{2}}a^{+}\sigma_{-}e^{-i\Delta t}]$，

$\Delta=2\omega-\nu$，

假设$|\psi(t)\rangle=\sum_n[C_{a,n}(t)|a,n\rangle+C_{b,n}(t)|b,n\rangle]$，

则由薛定鄂方程得$\dot{C}_{a,n}=-ig(n+1)e^{i\Delta t}C_{b,n+1}$，

$\dot{C}_{b,n+1}=-ig(n+1)e^{-i\Delta t}C_{a,n}$，$\dot{C}_{b,0}=0$。

对照（6.2.13）——（6.2.15），有

$$C_{a,n}(t)=\{C_{a,n}(0)[\cos\frac{\Omega_n t}{2}-\frac{i\Delta}{\Omega_n}\sin\frac{\Omega_n t}{2}]-\frac{2ig(n+1)}{\Omega_n}C_{b,n+1}(0)\sin\frac{\Omega_n t}{2}\}e^{\frac{i\Delta t}{2}}$$

$$C_{b,n+1}(t)=\{C_{b,n+1}(0)[\cos\frac{\Omega_n t}{2}+\frac{i\Delta}{\Omega_n}\sin\frac{\Omega_n t}{2}]-\frac{2ig(n+1)}{\Omega_n}C_{a,n}(0)\sin\frac{\Omega_n t}{2}\}e^{\frac{-i\Delta t}{2}}$$

$C_{b,0}(t)=0$，

$\Omega_n^2=\Delta^2+4g^2(n+1)^2$。

假设初始时刻原子处于上能态，

（a）对相干态光场，$\left|\psi(0)\right\rangle = \sum_{n=0}^{\infty} e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}} \left|a,n\right\rangle$，

$$C_{a,n}(t) = e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}} (\cos\frac{\Omega_n t}{2} - \frac{i\Delta}{\Omega_n}\sin\frac{\Omega_n t}{2}) e^{\frac{i\Delta t}{2}},$$

$$C_{b,n+1}(t) = e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}} [-\frac{2ig(n+1)}{\Omega_n}\sin\frac{\Omega_n t}{2}] e^{\frac{-i\Delta t}{2}}$$

$$\therefore W(t) = \sum_{n=0}^{\infty} [\left|C_{a,n}(t)\right|^2 + \left|C_{b,n}(t)\right|^2]$$

$$= e^{-|\alpha|^2} \{\cos^2\frac{\Omega_0 t}{2} + \frac{\Delta^2}{\Omega_0^2}\sin^2\frac{\Omega_0 t}{2} + \sum_{n=1}^{\infty} \frac{|\alpha|^{2n}}{n!} [\cos^2\frac{\Omega_n t}{2} + \frac{\Delta^2}{\Omega_n^2}\sin^2\frac{\Omega_n t}{2} - \frac{4g^2(n+1)^2}{\Omega_n^2}\sin^2\frac{\Omega_n t}{2}]\}$$

当 $\left\langle n\right\rangle = |\alpha|^2 \gg 1$ 时，

$$t_R \sim \frac{1}{\Omega_{\langle n\rangle}} = \frac{1}{[\Delta^2 + 4g^2(|\alpha|^2+1)^2]^{\frac{1}{2}}},$$

$$t_c \sim \frac{1}{\Omega_{\langle n\rangle + \sqrt{\langle n\rangle}} - \Omega_{\langle n\rangle - \sqrt{\langle n\rangle}}}$$

$$\simeq \frac{1}{[\Delta^2 + 4g^2(|\alpha|^2 + |\alpha|)^2]^{\frac{1}{2}} - [\Delta^2 + 4g^2(|\alpha|^2 - |\alpha|)^2]^{\frac{1}{2}}}$$

$$\simeq \frac{1}{8g|\alpha|}(1 + \frac{\Delta^2}{4g^2|\alpha|^4})^{\frac{1}{2}}$$

$$t_r = \frac{2\pi m}{\Omega_{\langle n\rangle} - \Omega_{\langle n\rangle - 1}} = \frac{\pi m}{2g}(1 + \frac{\Delta^2}{4g^2|\alpha|^4})^{\frac{1}{2}}。$$

（b）热光场，$\rho(0) = \sum_{n=0}^{\infty} \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}} \left|a,n\right\rangle\left\langle a,n\right|$，

$$C_{a,n}(t) = \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}} (\cos\frac{\Omega_n t}{2} - \frac{i\Delta}{\Omega_n}\sin\frac{\Omega_n t}{2}) e^{\frac{i\Delta t}{2}},$$

$$C_{b,n+1}(t) = \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}} [-\frac{2ig(n+1)}{\Omega_n}\sin\frac{\Omega_n t}{2}] e^{\frac{-i\Delta t}{2}},$$

$$W(t) = \frac{1}{1+\langle n\rangle} \{\cos^2\frac{\Omega_0 t}{2} + \frac{\Delta^2}{\Omega_0^2}\sin^2\frac{\Omega_0 t}{2} + \sum_{n=1}^{\infty} \frac{\langle n\rangle^n}{(1+\langle n\rangle)^n} [\cos^2\frac{\Omega_n t}{2} + \frac{\Delta^2}{\Omega_n^2}\sin^2\frac{\Omega_n t}{2} - \frac{4g^2(n+1)^2}{\Omega_n^2}\sin^2\frac{\Omega_n t}{2}]\}$$

6.2 解：利用（6.5）题的结果，由于

$$\sigma_+(t)\sigma_-(t)=[\sigma_+(0)\left(\cos kt-iC\frac{\sin kt}{k}\right)+ig\frac{\sin kt}{k}a^+(0)]\left[\left(\cos kt+iC\frac{\sin kt}{k}\right)\sigma_-(0)-ig\frac{\sin kt}{k}a(0)\right],$$

$$\left[\left(\cos kt+iC\frac{\sin kt}{k}\right)\sigma_-(0)-ig\frac{\sin kt}{k}a(0)\right]\left|a,\alpha\right\rangle=\left(\cos kt-\tfrac{i\Delta}{2}\frac{\sin kt}{k}\right)\left|b,\alpha\right\rangle,$$

$$\left[\left(\cos kt+iC\frac{\sin kt}{k}\right)\sigma_-(0)-ig\frac{\sin kt}{k}a(0)\right]\left|b,\alpha\right\rangle=-ig\frac{\sin kt}{k}\alpha\left|b,\alpha\right\rangle,$$

$$\left|\psi\right\rangle=\tfrac{1}{\sqrt{2}}(\left|a,\alpha\right\rangle+e^{-i\phi}\left|b,\alpha\right\rangle,$$

$$\therefore\left[\left(\cos kt+iC\frac{\sin kt}{k}\right)\sigma_-(0)-ig\frac{\sin kt}{k}a(0)\right]\left|\psi\right\rangle=\tfrac{1}{\sqrt{2}}[\cos kt-i(\tfrac{\Delta}{2}+g\alpha e^{-i\phi})\frac{\sin kt}{k}]\left|b,\alpha\right\rangle,$$

$$\therefore\left\langle\sigma_+(t)\sigma_-(t)\right\rangle=\tfrac{1}{2}\left\langle b,\alpha\right|[\cos kt+i(\tfrac{\Delta}{2}+g\alpha^*e^{i\phi})\frac{\sin kt}{k}][\cos kt-i(\tfrac{\Delta}{2}+g\alpha e^{-i\phi})\frac{\sin kt}{k}]\left|b,\alpha\right\rangle$$

$$=\tfrac{1}{2}e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}\{\cos^2\frac{\Omega_n t}{2}+\left|\Delta+2g\alpha e^{-i\phi}\right|^2\frac{\sin^2\frac{\Omega_n t}{2}}{\Omega_n^2}+\frac{4ig(\alpha^*e^{i\phi}-\alpha e^{-i\phi})}{\Omega_n^2}\cos\frac{\Omega_n t}{2}\sin\frac{\Omega_n t}{2}\}$$

$$W(t)=2\left\langle\sigma_+(t)\sigma_-(t)\right\rangle-1$$

$$=e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}\{\cos^2\frac{\Omega_n t}{2}+\left|\Delta+2g\alpha e^{-i\phi}\right|^2\frac{\sin^2\frac{\Omega_n t}{2}}{\Omega_n^2}+\frac{4ig(\alpha^*e^{i\phi}-\alpha e^{-i\phi})}{\Omega_n^2}\cos\frac{\Omega_n t}{2}\sin\frac{\Omega_n t}{2}\}-1$$

6.3 解：

$$V=\hbar\sum_{\vec{k}}[g_{\vec{k}}^{(ab)}a_{\vec{k}}^+\left|b\right\rangle\left\langle a\right|e^{-i(\omega_{ab}-v_{\vec{k}})t}+g_{\vec{k}}^{(a'b)}a_{\vec{k}}^+\left|b\right\rangle\left\langle a\right|e^{-i(\omega_{a'b}-v_{\vec{k}})t}]+H.c.,$$

设 $\left|\psi\right\rangle=\sum_{n_{\vec{k}}}[c_{a,n_{\vec{k}}}\left|a,n_{\vec{k}}\right\rangle+c_{a',n_{\vec{k}}}\left|a',n_{\vec{k}}\right\rangle+c_{b,n_{\vec{k}}}\left|b,n_{\vec{k}}\right\rangle]$

则

$$V\left|\psi\right\rangle=\hbar\sum_{n_{\vec{k}}}\{g_{\vec{k}}^{(ab)}e^{-i(\omega_{ab}-v_{\vec{k}})t}\sqrt{n_{\vec{k}}+1}c_{a,n_{\vec{k}}}\left|b,n_{\vec{k}}+1\right\rangle+g_{\vec{k}}^{(a'b)}e^{-i(\omega_{a'b}-v_{\vec{k}})t}\sqrt{n_{\vec{k}}+1}c_{a',n_{\vec{k}}}\left|b,n_{\vec{k}}+1\right\rangle$$

$$+g_{\vec{k}}^{(ab)}e^{i(\omega_{ab}-v_{\vec{k}})t}\sqrt{n_{\vec{k}}}c_{b,n_{\vec{k}}}\left|a,n_{\vec{k}}-1\right\rangle+g_{\vec{k}}^{(a'b)}e^{i(\omega_{a'b}-v_{\vec{k}})t}\sqrt{n_{\vec{k}}}c_{b,n_{\vec{k}}}\left|b,n_{\vec{k}}-1\right\rangle\}$$

代入薛定鄂方程，得

$$\dot{c}_{a,n_{\vec{k}}}=-ig_{\vec{k}}^{(ab)}e^{i(\omega_{ab}-v_{\vec{k}})t}\sqrt{n_{\vec{k}}+1}c_{b,n_{\vec{k}}+1}\quad(1)$$

$$\dot{c}_{a',n_{\vec{k}}} = -ig_{\vec{k}}^{(a'b)} e^{i(\omega_{a'b}-\nu_{\vec{k}})t} \sqrt{n_{\vec{k}}+1} c_{b,n_{\vec{k}}+1} \quad (2)$$

$$\dot{c}_{b,n_{\vec{k}}+1} = -i\sqrt{n_{\vec{k}}+1}[g_{\vec{k}}^{(ab)} e^{-i(\omega_{ab}-\nu_{\vec{k}})t} c_{a,n_{\vec{k}}} + g_{\vec{k}}^{(a'b)} e^{-i(\omega_{a'b}-\nu_{\vec{k}})t} c_{a',n_{\vec{k}}}] \quad (3)$$

由方程（1）、（2）可知两个上能级在跃迁时共用一个下能级，因此会发生量子干涉效应。

6.4 证：$\because \sigma_z^2 = 1$， $\sigma_z\sigma_+ = \sigma_+$， $\sigma_+\sigma_z = -\sigma_+$， $\sigma_z\sigma_- = -\sigma_-$， $\sigma_-\sigma_z = \sigma_-$，
$\sigma_+^2 = \sigma_-^2 = 0$， $\sigma_+\sigma_- + \sigma_-\sigma_+ = 1$，

$\therefore (\frac{1}{2}\Delta\sigma_z)^2 = \frac{\Delta^2}{4}$，

$\frac{1}{2}\Delta\sigma_z \bullet g(\sigma_+ a + a^+\sigma_-) + g(\sigma_+ a + a^+\sigma_-)\bullet\frac{1}{2}\Delta\sigma_z = 0$，

$$\begin{aligned}[g(\sigma_+ a + a^+\sigma_-)]^2 &= g^2(\sigma_+ a a^+ \sigma_- + a^+\sigma_-\sigma_+ a) \\ &= g^2(aa^+\sigma_+\sigma_- + a^+ a\sigma_-\sigma_+) \\ &= g^2(a^+ a + \sigma_+\sigma_-) \\ &= g^2 N\end{aligned}$$

$\therefore C^2 = \frac{\Delta^2}{4} + g^2 N$

6.5 证：

$$\because \sigma_-(t) = [\sigma_+(t)]^+ = e^{-i\nu t} e^{iCt}\left[\left(\cos kt + iC\frac{\sin kt}{k}\right)\sigma_-(0) - ig\frac{\sin kt}{k}a(0)\right],$$

$$\therefore \sigma_+(t)\sigma_-(t+\tau) = e^{-i\nu\tau}\left[\sigma_+(0)\left(\cos kt - iC\frac{\sin kt}{k}\right) + ig\frac{\sin kt}{k}a^+(0)\right]e^{iC\tau}$$
$$\times\left[\left(\cos k(t+\tau) + iC\frac{\sin k(t+\tau)}{k}\right)\sigma_-(0) - ig\frac{\sin k(t+\tau)}{k}a(0)\right],$$

$\because C = \frac{1}{2}\Delta\sigma_z + g(\sigma_+ a + a^+\sigma_-)$，

$\therefore C|b,\alpha\rangle = -\frac{\Delta}{2}|b,\alpha\rangle + g\alpha|a,\alpha\rangle$， $C|a,\alpha\rangle = \frac{\Delta}{2}|a,\alpha\rangle + ga^+|b,\alpha\rangle$

$$\therefore \left[\left(\cos k(t+\tau) + iC\frac{\sin k(t+\tau)}{k}\right)\sigma_-(0) - ig\frac{\sin k(t+\tau)}{k}a(0)\right]|a,\alpha\rangle$$

$$= \left(\cos k(t+\tau) - \frac{i\Delta}{2}\frac{\sin k(t+\tau)}{k}\right)|b,\alpha\rangle,$$

同理$\langle a,\alpha|\left[\sigma_+(0)\left(\cos kt - iC\frac{\sin kt}{k}\right) + ig\frac{\sin kt}{k}a^+(0)\right] = \langle b,\alpha|\left(\cos kt + \frac{i\Delta}{2}\frac{\sin kt}{k}\right)$；

由于$\langle b,n|C^2|b,n\rangle = \frac{\Delta^2}{4} + g^2 n = \frac{\Omega_{n-1}^2}{4}$，

因此$\langle b,n|e^{iC\tau}|b,n\rangle = \cos\frac{\Omega_{n-1}\tau}{2} + \frac{\sin\frac{\Omega_{n-1}\tau}{2}}{\frac{\Omega_{n-1}}{2}}\langle b,n|C|b,n\rangle$

$$=\cos\frac{\Omega_{n-1}\tau}{2}-\tfrac{i\Delta}{\Omega_{n-1}}\sin\frac{\Omega_{n-1}\tau}{2},$$

又由于$\langle b,n|k|b,n\rangle=\tfrac{\Omega_n}{2}$，

$\therefore\langle a,\alpha|\sigma_+(t)\sigma_-(t+\tau)|a,\alpha\rangle$

$$=e^{-i\nu\tau}e^{-|\alpha|^2}\sum_n\{\frac{|\alpha|^{2n}}{n!}(\cos\frac{\Omega_{n-1}\tau}{2}-\tfrac{i\Delta}{\Omega_{n-1}}\sin\frac{\Omega_{n-1}\tau}{2})$$

$$\times\left(\cos\tfrac{\Omega_n t}{2}+\tfrac{i\Delta}{\Omega_n}\sin\tfrac{\Omega_n t}{2}\right)\left(\cos\tfrac{\Omega_n(t+\tau)}{2}-\tfrac{i\Delta}{\Omega_n}\sin\tfrac{\Omega_n(t+\tau)}{2}\right)\}$$

把$\left(\cos\frac{\Omega_n t}{2}+\frac{i\Delta}{\Omega_n}\sin\frac{\Omega_n t}{2}\right)\left(\cos\frac{\Omega_n(t+\tau)}{2}-\frac{i\Delta}{\Omega_n}\sin\frac{\Omega_n(t+\tau)}{2}\right)$

$$=\cos\tfrac{\Omega_n t}{2}\cos\tfrac{\Omega_n(t+\tau)}{2}+\tfrac{\Delta^2}{\Omega_n^2}\sin\tfrac{\Omega_n t}{2}\sin\tfrac{\Omega_n(t+\tau)}{2}+\tfrac{i\Delta}{\Omega_n}(\sin\tfrac{\Omega_n t}{2}\cos\tfrac{\Omega_n(t+\tau)}{2}-\cos\tfrac{\Omega_n t}{2}\sin\tfrac{\Omega_n(t+\tau)}{2})$$

$$=\frac{1-\frac{\Delta^2}{\Omega_n^2}}{2}(\cos\tfrac{\Omega_n t}{2}\cos\tfrac{\Omega_n(t+\tau)}{2}-\sin\tfrac{\Omega_n t}{2}\sin\tfrac{\Omega_n(t+\tau)}{2})$$

$$+\frac{1+\frac{\Delta^2}{\Omega_n^2}}{2}(\cos\tfrac{\Omega_n t}{2}\cos\tfrac{\Omega_n(t+\tau)}{2}+\sin\tfrac{\Omega_n t}{2}\sin\tfrac{\Omega_n(t+\tau)}{2})-\tfrac{i\Delta}{\Omega_n}\sin\tfrac{\Omega_n\tau}{2}$$

$$=\frac{1-\frac{\Delta^2}{\Omega_n^2}}{2}\cos\tfrac{\Omega_n(2t+\tau)}{2}+\frac{1+\frac{\Delta^2}{\Omega_n^2}}{2}\cos\tfrac{\Omega_n\tau}{2}-\tfrac{i\Delta}{\Omega_n}\sin\tfrac{\Omega_n\tau}{2}$$

$$=\frac{4g^2(n+1)}{2\Omega_n^2}\cos\tfrac{\Omega_n(2t+\tau)}{2}+\frac{4g^2(n+1)+2\Delta^2}{2\Omega_n^2}\frac{e^{\frac{i\Omega_n\tau}{2}}+e^{\frac{-i\Omega_n\tau}{2}}}{2}+\tfrac{\Delta}{\Omega_n}\frac{-e^{\frac{i\Omega_n\tau}{2}}+e^{\frac{-i\Omega_n\tau}{2}}}{2}$$

$$=\frac{4g^2(n+1)}{2\Omega_n^2}\cos\tfrac{\Omega_n(2t+\tau)}{2}+[\frac{2g^2(n+1)+\Delta^2}{\Omega_n^2}-\tfrac{\Delta}{\Omega_n}]\frac{e^{\frac{i\Omega_n\tau}{2}}}{2}+[\frac{2g^2(n+1)+\Delta^2}{\Omega_n^2}+\tfrac{\Delta}{\Omega_n}]\frac{e^{\frac{-i\Omega_n\tau}{2}}}{2}$$

$$=\frac{4g^2(n+1)}{2\Omega_n^2}\cos\tfrac{\Omega_n(2t+\tau)}{2}+\frac{\Omega_n^2+\Delta^2-2\Delta\Omega_n}{4\Omega_n^2}e^{\frac{i\Omega_n\tau}{2}}+\frac{\Omega_n^2+\Delta^2+2\Delta\Omega_n}{4\Omega_n^2}e^{\frac{-i\Omega_n\tau}{2}}$$

$$=\frac{1}{4\Omega_n^2}[8g^2(n+1)\cos\tfrac{\Omega_n(2t+\tau)}{2}+(\Omega_n-\Delta)^2e^{\frac{i\Omega_n\tau}{2}}+(\Omega_n+\Delta)^2e^{\frac{-i\Omega_n\tau}{2}}]$$

代入上式，即得 6.2.44。书中把$\cos\frac{\Omega_{n-1}\tau}{2}-\frac{i\Delta}{\Omega_{n-1}}\sin\frac{\Omega_{n-1}\tau}{2}$误为$\cos\frac{\Omega_{n-1}\tau}{2}-\frac{i\Delta}{2\Omega_{n-1}}\sin\frac{\Omega_{n-1}\tau}{2}$。

第八章

8.1 证明：由$e^ABe^{-A}=B+[A,B]+\frac{1}{2!}[A,[A,B]]+\cdots$

可得

$$S^{+}{}_{k-k_0} b_k S_{k-k_0} = e^{\xi b^{+}{}_{k} b^{+}{}_{2k_0-k} - \xi^{*} b_k b_{2k_0-k}} b_k e^{\xi^{*} b_k b_{2k_0-k} - \xi b^{+}{}_{k} b^{+}{}_{2k_0-k}}$$

$$= b_k - \xi b^{+}{}_{2k_0-k} + \frac{|\xi|^2}{2!} b_k - \frac{\xi |\xi|^2}{3!} b^{+}{}_{2k_0-k} \cdots$$

$$= b_k \cosh(r) - b^{+}{}_{2k_0-k} e^{-i\theta} \sinh(r) ,$$

$$S^{+}{}_{k-k_0} b^{+}{}_{k} S_{k-k_0} = b^{+}{}_{k} - \xi b_{2k_0-k} + \frac{|\xi|^2}{2!} b^{+}{}_{k} - \frac{\xi |\xi|^2}{3!} b_{2k_0-k} \cdots$$

$$= b^{+}{}_{k} \cosh(r) - b_{2k_0-k} e^{i\theta} \sinh(r)$$

$$\langle b_k \rangle = \prod_q \langle 0_q | S^{+}{}_q b_k S_q | 0_q \rangle$$

$$= \langle 0_k | b_k \cosh(r) - b^{+}{}_{2k_0-k} e^{-i\theta} \sinh(r) | 0_k \rangle$$

$$= 0$$

$$\langle b^{+}{}_{k} \rangle = \prod_q \langle 0_q | S^{+}{}_q b^{+}{}_{k} S_q | 0_q \rangle$$

$$= \langle 0_k | b^{+}{}_{k} \cosh(r) - b_{2k_0-k} e^{i\theta} \sinh(r) | 0_k \rangle$$

$$= 0$$

$$\langle b^{+}{}_{k} b_{k'} \rangle = \prod_q \langle 0_q | S^{+}{}_q b^{+}{}_{k} S_q S^{+}{}_q b_{k'} S_q | 0_q \rangle$$

$$= \langle 0_k | (b^{+}{}_{k} \cosh(r) - b_{2k_0-k} e^{i\theta} \sinh(r))(b_{k'} \cosh(r) - b^{+}{}_{2k_0-k'} e^{-i\theta} \sinh(r)) | 0_k \rangle$$

$$= \langle 0_k | b_{2k_0-k} b^{+}{}_{2k_0-k'} \sinh^2(r) | 0_k \rangle$$

$$= \sinh^2(r) \delta_{kk'}$$

$$\langle b_k b^{+}{}_{k'} \rangle = \langle 0_k | (b_k \cosh(r) - b^{+}{}_{2k_0-k} e^{-i\theta} \sinh(r))(b^{+}{}_{k'} \cosh(r) - b_{2k_0-k'} e^{i\theta} \sinh(r)) | 0_k \rangle$$

$$= \langle 0_k | b_k b^{+}{}_{k'} \cosh^2(r) | 0_k \rangle$$

$$= \cosh^2(r) \delta_{kk'}$$

$$\langle b_k b_{k'} \rangle = \langle 0_k | (b_k \cosh(r) - b^{+}{}_{2k_0-k} e^{i\theta} \sinh(r))(b_{k'} \cosh(r) - b^{+}{}_{2k_0-k'} e^{i\theta} \sinh(r)) | 0_k \rangle$$

$$= -e^{i\theta} \sinh(r) \cosh(r) \langle 0_k | b_{k'} b^{+}{}_{2k_0-k} | 0_k \rangle$$

$$= -e^{i\theta} \sinh(r) \cosh(r) \delta_{2k_0-k,k'}$$

$$\langle b^{+}{}_{k} b^{+}{}_{k'} \rangle = \langle 0_k | (b^{+}{}_{k} \cosh(r) - b_{2k_0-k} e^{-i\theta} \sinh(r))(b^{+}{}_{k'} \cosh(r) - b_{2k_0-k'} e^{-i\theta} \sinh(r)) | 0_k \rangle$$

$$= -e^{-i\theta} \sinh(r) \cosh(r) \langle 0_k | b_{2k_0-k} b^{+}{}_{k'} | 0_k \rangle$$

$$= -e^{-i\theta} \sinh(r) \cosh(r) \delta_{2k_0-k,k'} 。$$

8.2 解： $\frac{d}{dt}\langle a \rangle = \langle \dot{a} \rangle$

$$= tr(\dot{\rho} a)$$

$$= -\frac{\ell}{2} tr(a^{+} a \rho a - 2 a \rho a^{+} a + \rho a^{+} a a)$$

$$=-\frac{\ell}{2}tr(\rho a)$$

$$=-\frac{\ell}{2}\langle a\rangle,$$

同理，$\frac{d}{dt}\langle a^{+}\rangle=-\frac{\ell}{2}\langle a^{+}\rangle$，

$$\frac{d}{dt}\langle a^{2}\rangle=-\frac{\ell}{2}tr(a^{+}a\rho a^{2}-2a\rho a^{+}a^{2}+\rho a^{+}aa^{2})$$

$$=-\ell tr(\rho a^{2})$$

$$=-\ell\langle a^{2}\rangle,$$

$$\frac{d}{dt}\langle a^{+2}\rangle=-\ell\langle a^{+2}\rangle,$$

$$\frac{d}{dt}\langle aa^{+}\rangle=-\ell\langle aa^{+}\rangle,$$

$$\frac{d}{dt}\langle a^{+}a\rangle=-\ell\langle a^{+}a\rangle$$

所以

$$(\Delta X_{1})_{t}{}^{2}=\langle X_{1}{}^{2}\rangle-\langle X_{1}\rangle^{2}$$

$$=\frac{1}{4}\langle a^{2}+a^{+2}+aa^{+}+a^{+}a\rangle_{t}-\frac{1}{4}\langle a^{+}+a\rangle_{t}{}^{2}$$

$$=\frac{1}{4}e^{-\ell t}\langle a^{2}+a^{+2}+aa^{+}+a^{+}a\rangle_{0}-\frac{1}{4}e^{-\ell t}\langle a+a^{+}\rangle_{0}{}^{2}$$

$$=e^{-\ell t}(\Delta X_{1})_{0}{}^{2}$$

同理，$(\Delta X_{2})_{t}{}^{2}=e^{-\ell t}(\Delta X_{2})_{0}{}^{2}$。

8.3 证明：由 8.3.4 式，

$$\frac{d}{dt}\langle a\rangle=tr(a\dot{\rho})$$

$$=tr[-\frac{\ell}{2}(N+1)(aa^{+}a\rho-2a^{2}\rho a^{+}+a\rho aa^{+})$$

$$-\frac{\ell}{2}N(a^{2}a^{+}\rho-2aa^{+}\rho a+a\rho aa^{+})+\frac{\ell}{2}M(a^{3}\rho-2a^{2}\rho a+a\rho a)$$

$$+\frac{\ell}{2}M^{*}(aa^{+2}\rho-2aa^{+}\rho a^{+}+a\rho a^{+2})$$

$$=-tr(\frac{\ell}{2}a\rho)$$

$$=-\frac{\ell}{2}\langle a\rangle$$

同理， $\frac{d}{dt}\langle a^{+}\rangle=-\frac{\ell}{2}\langle a^{+}\rangle$，

$$\frac{d}{dt}\langle a^{2}\rangle=tr[-\frac{\ell}{2}(N+1)(a^{2}a^{+}a\rho-2aa^{2}\rho a^{+}+a^{2}\rho aa^{+})$$

$$-\frac{\ell}{2}N(aa^{2}a^{+}\rho-2a^{2}a^{+}\rho a+a^{2}\rho aa^{+})+\frac{\ell}{2}M(a^{4}\rho-2a^{3}\rho a+a^{2}\rho aa$$

$$+\frac{\ell}{2}M^{*}(a^{2}a^{+2}\rho-2a^{2}a^{+}\rho a^{+}+a^{2}\rho a^{+2})$$

$$=-\ell\langle a^{2}\rangle+\ell M^{*}$$

$$\frac{d}{dt}\langle a^{+2}\rangle=-\ell\langle a^{+2}\rangle+\ell M$$

$$\frac{d}{dt}\langle aa^{+}\rangle=-\ell\langle aa^{+}\rangle+\ell(N+1)$$

$$\frac{d}{dt}\langle a^{+}a\rangle=-\ell\langle a^{+}a\rangle+\ell N$$

我们有

$$\langle a\rangle_{t}=\langle a\rangle_{0}e^{-\frac{\ell}{2}t}$$

$$\langle a^{+}\rangle_{t}=\langle a^{+}\rangle_{0}e^{-\frac{\ell}{2}t}$$

$$\langle a^{2}\rangle_{t}=(\langle a^{2}\rangle_{0}-M^{*})e^{-\ell t}+M^{*}$$

$$\langle a^{2+}\rangle_{t}=(\langle a^{2+}\rangle_{0}-M)e^{-\ell t}+M$$

$$\langle a^{+}a\rangle_{t}=(\langle a^{+}a\rangle_{0}-N)e^{-\ell t}+N$$

$$\langle aa^{+}\rangle_{t}=(\langle aa^{+}\rangle_{0}-N-1)e^{-\ell t}+N+1$$

$$\therefore\langle X_{1}\rangle_{t}=\langle X_{1}\rangle_{0}e^{-\frac{\ell}{2}t}$$

$$\langle X^{2}{}_{1}\rangle_{t}=\frac{1}{4}\Big(\langle a^{+2}\rangle_{0}+\langle a^{2}\rangle_{0}+\langle aa^{+}\rangle_{0}+\langle aa^{+2}\rangle_{0}-M-M^{*}-2N-1\Big)e^{-\ell t}$$

$$+\frac{1}{4}(M+M^{*}+2N+1)$$

令 $S=M+M^{*}+2N+1$，

$$\therefore \langle \Delta X_1\rangle^2{}_t = (\langle \Delta X_1\rangle^2{}_t - \frac{S}{4})e^{-\ell t} + \frac{S}{4}$$

同理，$\langle \Delta X_2\rangle^2{}_t = (\langle \Delta X_2\rangle^2{}_t + \frac{\Gamma}{4})e^{-\ell t} - \frac{\Gamma}{4}$

其中，$\Gamma = M + M^* - 2N - 1$。

8.4 解： $\dot{\rho} = -\frac{\ell}{2}(\bar{n}+1)(a^+a\rho - 2a\rho a^+ + \rho a^+ a) - \frac{\ell}{2}\bar{n}(aa^+\rho - 2a^+\rho a + \rho aa^+)$

因为

$$Q(\alpha,\alpha^*) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle = \frac{1}{\pi}Tr(\rho|\alpha\rangle\langle\alpha|)$$

$\frac{d}{dt}Q(\alpha,\alpha^*) = \frac{1}{\pi}Tr(\dot{\rho}|\alpha\rangle\langle\alpha|)$，

所以，第一项

$$Tr[(a^+a\rho - 2a\rho a^+ + \rho a^+ a)|\alpha\rangle\langle\alpha|]$$

$$= Tr[(\rho|\alpha\rangle\langle\alpha|a^+a - 2\rho a^+|\alpha\rangle\langle\alpha|a + \rho a^+ a|\alpha\rangle\langle\alpha|]$$

$$= Tr\left[(\rho|\alpha\rangle\langle\alpha|\alpha^*\left(\frac{\partial}{\partial\alpha^*}+\alpha\right) - 2\rho|\alpha\rangle\langle\alpha|\left(\frac{\partial}{\partial\alpha}+\alpha^*\right)\left(\frac{\partial}{\partial\alpha^*}+\alpha\right) + \rho|\alpha\rangle\langle\alpha|\alpha\left(\frac{\partial}{\partial\alpha}+\alpha^*\right)\right]$$

$$= \left[\alpha^*\left(\frac{\partial}{\partial\alpha^*}+\alpha\right) - 2\left(\frac{\partial}{\partial\alpha}+\alpha^*\right)\left(\frac{\partial}{\partial\alpha^*}+\alpha\right) + \alpha\left(\frac{\partial}{\partial\alpha}+\alpha^*\right)\right]Q$$

$$= \left[-\alpha^*\frac{\partial}{\partial\alpha^*} - 2\frac{\partial^2}{\partial\alpha\partial\alpha^*} - \alpha\frac{\partial}{\partial\alpha} - 2\right]Q$$

第二项

$$= Tr\left[(\rho|\alpha\rangle\langle\alpha|aa^+ - 2\rho a|\alpha\rangle\langle\alpha|a^+ + \rho a^+|\alpha\rangle\langle\alpha|\right]$$

$$= Tr\left[(\rho|\alpha\rangle\langle\alpha|\alpha^*\left(\frac{\partial}{\partial\alpha^*}+\alpha\right) - 2|\alpha|^2\rho|\alpha\rangle\langle\alpha| + \rho|\alpha\rangle\langle\alpha|\alpha\left(\frac{\partial}{\partial\alpha}+\alpha^*\right)\right]$$

$$= \left[2 + \alpha^*\frac{\partial}{\partial\alpha^*} + \alpha\frac{\partial}{\partial\alpha}\right]Q$$

所以

$$\dot{Q} = -\frac{\ell}{2}(\bar{n}+1)\left[-\alpha^*\frac{\partial}{\partial\alpha^*} - 2\frac{\partial^2}{\partial\alpha\partial\alpha^*} - \alpha\frac{\partial}{\partial\alpha} - 2\right]Q - \frac{\ell}{2}\bar{n}\left[2 + \alpha^*\frac{\partial}{\partial\alpha^*} + \alpha\frac{\partial}{\partial\alpha}\right]Q$$

$$= \frac{\ell}{2}\left(\alpha^*\frac{\partial}{\partial\alpha^*} + \alpha\frac{\partial}{\partial\alpha}\right)Q + \ell(\bar{n}+1)\frac{\partial^2}{\partial\alpha\partial\alpha^*}Q$$

初始态为$|\alpha_0\rangle$，

$$\therefore Q(\alpha,\alpha^*,0) = \frac{1}{\pi}\langle\alpha|\alpha_0\rangle\langle\alpha_0|\alpha\rangle$$

$= \frac{1}{\pi}\exp(-|\alpha-\alpha_0|)$，

设 $Q(\alpha,\alpha^*,t)=\exp[-a(t)+b(t)\alpha+c(t)\alpha_0-d(t)\alpha\alpha^*]$ $\cdots\cdots\cdots(1)$

初始条件为

$$\begin{cases} a(0)=|\alpha_0|^2+\ln\pi \\ b(0)=\alpha^*{}_0 \\ c(0)=\alpha_0 \\ d(0)=1 \end{cases}$$

将（1）代入运动方程

$$-\dot{a}+\dot{b}\alpha+\dot{c}\alpha_0-\dot{d}\alpha\alpha^*$$
$$=\ell[1+(\overline{n}+1)(bc-d)+(\frac{b}{2}-(\overline{n}+1)bd)\alpha+(\frac{1}{2}-(\overline{n}+1)cd)\alpha^*-(d-(\overline{n}+1)d^2)|\alpha|^2]$$

解之得

$$a(t)=\frac{|\alpha_0|^2 e^{-\ell t}}{(\overline{n}+1)(1-e^{-\ell t})+e^{-\ell t}}+\ln\left[\pi(\overline{n}+1)(1-e^{-\ell t})+e^{-\ell t}\right]$$

$$b(t)=\frac{\alpha_0{}^* e^{-\frac{\ell t}{2}}}{(\overline{n}+1)(1-e^{-\ell t})+e^{-\ell t}}$$

$$c(t)=\frac{\alpha_0 e^{-\frac{\ell t}{2}}}{(\overline{n}+1)(1-e^{-\ell t})+e^{-\ell t}}$$

$$d(t)=\frac{1}{(\overline{n}+1)(1-e^{-\ell t})+e^{-\ell t}}$$

8.5 证明：

$$\langle\dot{\sigma}_x\rangle=tr(\dot{\rho}\sigma_x)$$
$$=tr\{-\frac{\Gamma}{2}\cosh^2(r)(\sigma_+\sigma_-\rho\sigma_x-2\sigma_-\rho\sigma_+\sigma_x+\rho\sigma_+\sigma_-\sigma_x)$$
$$-\frac{\Gamma}{2}\sinh^2(r)(\sigma_-\sigma_+\rho\sigma_x-2\sigma_+\rho\sigma_-\sigma_x+\rho\sigma_-\sigma_+\sigma_x)$$
$$-\Gamma\sinh(r)\cosh(r)\sigma_-\rho\sigma_-\sigma_x-\Gamma\sinh(r)\cosh(r)\sigma_+\rho\sigma_+\sigma_x\}$$
$$=-\frac{\Gamma}{2}tr\{[\cosh^2(r)+\sinh^2(r)]\sigma_x\rho+2\sinh(r)\cosh(r)\sigma_x\rho\}$$
$$=-\frac{\Gamma}{2}e^{2r}\langle\sigma_x\rangle，$$

同理可证 $\langle\dot{\sigma}_y\rangle==-\frac{\Gamma}{2}e^{-2r}\langle\sigma_y\rangle$，

$$\langle\dot{\sigma}_z\rangle=tr(\dot{\rho}\sigma_z)=tr(2\sigma_+\sigma_-\dot{\rho})-tr(\dot{\rho})$$

$$=2tr\{-\frac{\Gamma}{2}\cosh^2(r)(\sigma_+\sigma_-\sigma_+\sigma_-\rho-2\sigma_+\sigma_-\sigma_-\rho\sigma_++\sigma_+\sigma_-\rho\sigma_+\sigma_-)$$
$$-\frac{\Gamma}{2}\sinh^2(r)(\sigma_+\sigma_-\sigma_-\sigma_+\rho-2\sigma_+\sigma_-\sigma_+\rho\sigma_-+\sigma_+\sigma_-\rho\sigma_-\sigma_+)$$
$$-\Gamma\sinh(r)\cosh(r)\sigma_+\sigma_-\sigma_-\rho\sigma_--\Gamma\sinh(r)\cosh(r)\sigma_+\sigma_-\sigma_+\rho\sigma_+\}$$
$$=-\frac{\Gamma}{2}tr\{[\cosh^2(r)+\sinh^2(r)]\sigma_x\rho+2\sinh(r)\cosh(r)\sigma_x\rho\}$$
$$=2tr[-\Gamma\cosh^2(r)\sigma_+\sigma_-+\Gamma\sinh^2(r)\sigma_-\sigma_+]$$
$$=-\Gamma[2\sinh^2(r)+1]\langle\sigma_z\rangle-\Gamma$$

第九章

9.1 解：$\dot{\tilde{a}}=-\frac{1}{2}\ell\tilde{a}+F_{\tilde{a}}(t)$，

$\dot{\tilde{a}}=-\frac{1}{2}\ell\tilde{a}+F_{\tilde{a}}(t)$，

$\therefore \dot{X}_1=-\frac{1}{2}\ell X_1+\frac{1}{2}(F_{\tilde{a}}(t)+F_{\tilde{a}^+}(t))$，

$\therefore \frac{d\langle X_1\rangle}{dt}=\langle\dot{X}_1\rangle=-\frac{1}{2}\ell\langle X_1\rangle$，

$\therefore \frac{d\langle X_1\rangle^2}{dt}=2\langle X_1\rangle\langle\dot{X}_1\rangle=-\ell\langle X_1\rangle^2$，

$$\frac{d\langle X_1^2\rangle}{dt}=\langle\dot{X}_1^2\rangle=\langle X_1\dot{X}_1+\dot{X}_1X_1\rangle$$
$$=\langle-\tfrac{1}{2}\ell X_1^2+\tfrac{1}{2}X_1(F_{\tilde{a}}(t)+F_{\tilde{a}^+}(t))-\tfrac{1}{2}\ell X_1^2+\tfrac{1}{2}(F_{\tilde{a}}(t)+F_{\tilde{a}^+}(t))X_1\rangle$$
$$=-\ell\langle X_1^2\rangle+\frac{1}{4}\langle\tilde{a}F_{\tilde{a}}+\tilde{a}^+F_{\tilde{a}}+\tilde{a}F_{\tilde{a}^+}+\tilde{a}^+F_{\tilde{a}^+}+F_{\tilde{a}}\tilde{a}+F_{\tilde{a}}\tilde{a}^++F_{\tilde{a}^+}\tilde{a}+F_{\tilde{a}^+}\tilde{a}^+\rangle$$

由于$\langle\tilde{a}F_{\tilde{a}}\rangle=\langle\tilde{a}^+F_{\tilde{a}^+}\rangle=\langle F_{\tilde{a}}\tilde{a}\rangle=\langle F_{\tilde{a}^+}\tilde{a}^+\rangle=0$，

$\langle\tilde{a}F_{\tilde{a}^+}\rangle=\langle F_{\tilde{a}}\tilde{a}^+\rangle=\frac{1}{2}\ell(\bar{n}_{th}+1)$，

$\langle\tilde{a}^+F_{\tilde{a}}\rangle=\langle F_{\tilde{a}^+}\tilde{a}\rangle=\frac{1}{2}\ell\bar{n}_{th}$

易得$\langle\dot{X}_1^2\rangle=-\ell\langle X_1^2\rangle+\frac{1}{4}\ell(2\bar{n}_{th}+1)$，

$$\frac{d(\Delta X_1)^2}{dt}=\langle\dot{X}_1^2\rangle-\frac{d\langle X_1\rangle^2}{dt}=-\ell(\Delta X_1)^2+\tfrac{1}{4}\ell(\bar{n}_{th}+1)$$

解这个微分方程得

$$(\Delta X_1)^2=\tfrac{1}{4}(\bar{n}_{th}+1)+[(\Delta X_1)^2(0)-\tfrac{1}{4}(\bar{n}_{th}+1)]e^{-\ell t}\text{。}$$

9.2 解：$f(t_i,t,\tau)=\begin{cases}e^{-\Gamma(t-t_i)}(t_i\le t\le t_i+\tau)\\ 0(\text{其它})\end{cases}$，

$$f(t_i,t',\tau)=\begin{cases} e^{-\Gamma(t'-t_i)}(t_i \le t' \le t_i+\tau) \\ 0(\text{其它}) \end{cases},$$

$$f(t_i,t,\tau)f(t_i,t',\tau)=\begin{cases} e^{-\Gamma(t-t_i)}e^{-\Gamma(t'-t_i)}(t_i \le t \le t_i+\tau\text{且}t_i \le t' \le t_i+\tau) \\ 0(\text{其它}) \end{cases}$$

$$=\begin{cases} e^{-\Gamma(t-t_i)}e^{-\Gamma(t'-t_i)}(t_i \le t \le t' \le t_i+\tau\text{且}t+\tau \ge t'\text{，或}t_i \le t' \le t \le t_i+\tau\text{且}t'+\tau \ge t) \\ 0(\text{其它}) \end{cases} \quad (1)$$

代入 9.2.13 式$\langle F_a^+(t)F_a(t')\rangle = g^2[1+\exp(\frac{\hbar\nu}{k_BT})]^{-1}\sum_i f(t_i,t,\tau)f(t_i,t',\tau)$即可求出结果；

由 9.2.16 式$\langle F_a^+(t)F_a(t')\rangle = r_a g^2[1+\exp(\frac{\hbar\nu}{k_BT})]^{-1}\int_{-\infty}^t dt_i f(t_i,t,\tau)f(t_i,t',\tau)$，

由于

$$\int_{-\infty}^t dt_i f(t_i,t,\tau)f(t_i,t',\tau)=\begin{cases} \int_{t'-\tau}^{t} e^{-\Gamma(t-t_i)}e^{-\Gamma(t'-t_i)}dt_i = \frac{1}{2\Gamma}e^{-\Gamma(t+t')}(e^{2\Gamma t}-e^{2\Gamma(t'-\tau)})(t+\tau \ge t' \ge t) \\ \int_{t-\tau}^{t'} e^{-\Gamma(t-t_i)}e^{-\Gamma(t'-t_i)}dt_i = \frac{1}{2\Gamma}e^{-\Gamma(t+t')}(e^{2\Gamma t'}-e^{2\Gamma(t-\tau)})(t'+\tau \ge t \ge t') \\ 0(\text{其它}) \end{cases}$$

$$=\begin{cases} e^{\Gamma(|t-t'|-\tau)}(e^{-2\Gamma(|t-t'|-\tau)}-1)(|t-t'| \le \tau) \\ 0(\text{其它}) \end{cases}$$

$$\langle F_a^+(t)F_a(t')\rangle = \begin{cases} r_a g^2[1+\exp(\frac{\hbar\nu}{k_BT})]^{-1}e^{\Gamma(|t-t'|-\tau)}(e^{-2\Gamma(|t-t'|-\tau)}-1)(|t-t'| \le \tau) \\ 0(\text{其它}) \end{cases}\text{。}$$

9.3 解：多模压缩真空库，由 8.2.12，8.2.16a—8.2.16e，

$\rho_R = |\xi\rangle\langle\xi| = \prod_k S_k(\xi)|0_k\rangle\langle 0_k|S_k^+(\xi)$，

$\langle b_k\rangle = \langle b_k^+\rangle = 0$， $\langle b_k^+ b_{k'}\rangle = \sinh^2(r)\delta_{kk'}$，

$\langle b_k b_{k'}^+\rangle = \cosh^2(r)\delta_{kk'}$，

$\langle b_k b_{k'}\rangle = -e^{i\theta}\sinh(r)\cosh(r)\delta_{2k_0-k,k'}$，

$\langle b_k^+ b_{k'}^+\rangle = -e^{-i\theta}\sinh(r)\cosh(r)\delta_{2k_0-k,k'}$，

由此结合 9.1.22、9.1.23 式，易得$\langle F_{\tilde{a}}(t)\rangle_R = \langle F_{\tilde{a}}^+(t)\rangle_R = 0$，

$$\langle F_{\tilde{a}}^+(t)F_{\tilde{a}}(t')\rangle_R = \sum_k\sum_{k'} g_k g_{k'}\langle b_k^+ b_{k'}\rangle_R \exp[i(\nu_k-\nu)t - i(\nu_{k'}-\nu)t']$$

$$=\sum_k g_k^2 \sinh^2(r)\exp[i(\nu_k-\nu)t - i(\nu_k-\nu)t']$$

$$=\int_0^\infty D(\nu_k)[g(\nu_k)]^2\sinh^2(r)\exp[i(\nu_k-\nu)t-i(\nu_k-\nu)t']d\nu_k$$

$$=\ell\sinh^2(r)\delta(t-t')$$

同理$\left\langle F_{\tilde a}(t)F_{\tilde a}^+(t')\right\rangle_R=\ell\cosh^2(r)\delta(t-t')$；

$$\left\langle F_{\tilde a}(t)F_{\tilde a}(t')\right\rangle_R=\sum_k\sum_{k'}g_k g_{k'}\left\langle b_k b_{k'}\right\rangle_R\exp[-i(\nu_k-\nu)t-i(\nu_{k'}-\nu)t']$$

$$=-\sum_k g_k g_{2k_0-k}e^{i\theta}\sinh(r)\cosh(r)\exp[-i(\nu_k-\nu)t-i(\nu_{2k_0-k}-\nu)t']$$

把$\nu_{2k_0-k}=2\nu-\nu_k$代入上式，得

$$\left\langle F_{\tilde a}(t)F_{\tilde a}(t')\right\rangle_R=-\sum_k g_k g_{2k_0-k}e^{i\theta}\sinh(r)\cosh(r)\exp[i(\nu_k-\nu)(t'-t)]$$

$$=-\int_0^\infty D(\nu_k)g_k g_{2k_0-k}e^{i\theta}\sinh(r)\cosh(r)\exp[i(\nu_k-\nu)(t'-t)]d\nu_k$$

$$=-2\pi D(\nu)g_{k_0}g_{2k_0-k_0}e^{i\theta}\sinh(r)\cosh(r)\delta(t-t')$$

$$=-\ell e^{i\theta}\sinh(r)\cosh(r)\delta(t-t')\text{，}$$

同理$\left\langle F_{\tilde a}^+(t)F_{\tilde a}^+(t')\right\rangle_R=-\ell e^{-i\theta}\sinh(r)\cosh(r)\delta(t-t')$。

9.4 解：

$$\frac{d}{dt}\langle\tilde a^{+m}\tilde a^n\rangle_R=m\langle\tilde a^{+(m-1)}\frac{d}{dt}\tilde a^+\tilde a^n\rangle_R+n\langle\tilde a^{+m}\tilde a^{n-1}\frac{d}{dt}\tilde a\rangle_R$$

$$=-\frac{\ell}{2}(m+n)\langle\tilde a^{+m}\tilde a^n\rangle_R+m\langle\tilde a^{+(m-1)}F_a^+\tilde a^n\rangle_R+n\langle\tilde a^{+m}F_a\tilde a^{n-1}\rangle_R$$

（1）热光场

$$m\langle\tilde a^{+(m-1)}F_a^+\tilde a^n\rangle_R=n\langle\tilde a^{+m}F_a\tilde a^{n-1}\rangle_R=\frac{\ell}{2}nmn_{th}\langle\tilde a^{+(m-1)}\tilde a^{n-1}\rangle_R$$

所以

$$\frac{d}{dt}\langle\tilde a^{+m}\tilde a^n\rangle_R=-\frac{\ell}{2}(m+n)\langle\tilde a^{+m}\tilde a^n\rangle_R+\ell nmn_{th}\langle\tilde a^{+(m-1)}\tilde a^{n-1}\rangle_R$$

（2）压缩真空场

$$\langle b_k(0)\rangle_R=\langle b^+{}_k(0)\rangle_R=0$$

$$\langle b^+{}_k(0)b_{k'}(0)\rangle_R=\sinh^2(r)\delta_{kk'}$$

$$\langle b_k(0)b^+{}_{k'}(0)\rangle_R=\cosh^2(r)\delta_{kk'}$$

$$\langle b^+{}_k(0)b^+{}_{k'}(0)\rangle_R==-e^{-i\theta}\sinh(r)\cosh(r)\delta_{2k_0-k,k'}$$

$$\langle b_k(0)b_{k'}(0)\rangle_R==-e^{-i\theta}\sinh(r)\cosh(r)\delta_{2k_0-k,k'}$$

$$\therefore\langle F(t)\rangle_R=\langle F^+(t)\rangle_R=0$$

$$\langle F^+(t)F(t')\rangle_R=\ell\sinh^2(r)\delta(t-t')$$

$$\langle F(t')F^+(t)\rangle_R=\ell\cosh^2(r)\delta(t-t')$$

$$\langle F^+(t)\tilde{a}\rangle_R = \frac{\ell}{2}\sinh^2(r)$$

$$\langle \tilde{a}^+ F(t)\rangle_R = \frac{\ell}{2}\cosh^2(r)$$

$$\frac{d}{dt}\langle \tilde{a}^{+m}\tilde{a}^n\rangle_R = -\frac{\ell}{2}(m+n)\langle \tilde{a}^{+m}\tilde{a}^n\rangle_R + \frac{\ell}{2}nm[\sinh^2(r)+\cosh^2(r)]\langle \tilde{a}^{+(m-1)}\tilde{a}^{n-1}\rangle_R$$

9.5 解：对压缩真空库，由（9.1.15）式

$\left\langle \dot{a} \right\rangle = -\frac{1}{2}\ell\left\langle a\right\rangle$，

$\left\langle \dot{a}^+ \right\rangle = -\frac{1}{2}\ell\left\langle a^+\right\rangle$，

$$\frac{d}{dt}\left\langle aa^+\right\rangle = \left\langle \dot{a}\,a^+\right\rangle + \left\langle a\,\dot{a}^+\right\rangle = -\ell\left\langle aa^+\right\rangle + \left\langle F_{\tilde{a}}(t)a^+\right\rangle + \left\langle aF_{\tilde{a}}^+(t)\right\rangle$$

由 9.1.32 式以及 9.3 题的结论$\left\langle F_{\tilde{a}}(t)F_{\tilde{a}}^+(t')\right\rangle_R = \ell\cosh^2(r)\delta(t-t')$，

有$\left\langle F_{\tilde{a}}(t)a^+\right\rangle + \left\langle aF_{\tilde{a}}^+(t)\right\rangle = \ell\cosh^2(r)$ （好像少了一个 2，不知道怎么回事），

$\therefore \frac{d}{dt}\left\langle aa^+\right\rangle = -\ell\left\langle aa^+\right\rangle + \ell\cosh^2(r)$，

由于 $\tilde{a} = ae^{ivt}$， $F_{\tilde{a}}(t) = F_a(t)e^{ivt}$，由 9.4.4 式，

$$2\left\langle D_{\tilde{a}\tilde{a}^+}\right\rangle_R = \frac{d}{dt}\left\langle aa^+\right\rangle - \left\langle \tilde{a}[\dot{\tilde{a}}^+ - F_{\tilde{a}}^+(t)]\right\rangle_R - \left\langle [\dot{\tilde{a}} - F_{\tilde{a}}(t)]\tilde{a}^+\right\rangle_R$$

$$= \frac{d}{dt}\left\langle aa^+\right\rangle - \left\langle a[\dot{a}^+ - F_a^+(t)]\right\rangle_R - \left\langle [\dot{a} - F_a(t)]a^+\right\rangle_R$$

把 9.4.8、9.4.9 式代入上式，得

$2\left\langle D_{\tilde{a}\tilde{a}^+}\right\rangle_R = \ell\cosh^2(r)$。

用朗之万方程，由 9.1.27、9.1.29 知$\left\langle D_{\tilde{a}\tilde{a}^+}\right\rangle_R$的定义式为

$2\left\langle D_{\tilde{a}\tilde{a}^+}\right\rangle_R \delta(t-t') = \left\langle F_{\tilde{a}}(t)F_{\tilde{a}}^+(t')\right\rangle_R$，

由 9.3 题，$\left\langle F_{\tilde{a}}(t)F_{\tilde{a}}^+(t')\right\rangle_R = \ell\cosh^2(r)\delta(t-t')$，

因此有$2\left\langle D_{\tilde{a}\tilde{a}^+}\right\rangle_R = \ell\cosh^2(r)$，和上面的结论一致。