§3.几种重要随机变量的数学期望及方差

1.两点分布

X	0	1
p_k	$1-p$	p

EX=p， $DX = EX^2 - (EX)^2 = p - p^2 = pq$ 。

2. 二项分布

方法1：

$$P\{X = k\} = C_n^k p^k q^{n-k}, k = 0,1,\cdots,n 。$$

$$EX = \sum_{k=0}^{n} k \cdot C_n^k p^k q^{n-k} = \sum_{k=0}^{n} k \cdot \frac{n!}{k!(n-k)!} p^k q^{n-k}$$

$$= np\sum_{k=1}^{n}\frac{(n-1)!}{(k-1)!(n-1-(k-1))!} p^{k-1} q^{n-1-(k-1)}$$

$$EX = np\sum_{k=1}^{n} C_{n-1}^{k-1} p^{k-1} q^{n-1-(k-1)} = np\sum_{i=0}^{n-1} C_{n-1}^{i} p^{i} q^{n-1-i}$$

$$= np(p+q)^{n-1} = np$$

$$EX^{2} = \sum_{k=0}^{n} k^{2} \cdot C_{n}^{k} p^{k} q^{n-k} = \sum_{k=0}^{n} k^{2} \cdot \frac{n!}{k!(n-k)!} p^{k} q^{n-k}$$

$$= p\sum_{k=1}^{n} k \frac{n!}{(k-1)!(n-k)!} p^{k-1} q^{n-k}$$

$$= p\sum_{k=1}^{n} (k-1) \frac{n!}{(k-1)!(n-k)!} p^{k-1} q^{n-k} + p\sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^{k-1} q^{n}$$

$$= n(n-1)p^{2}\sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!(n-2-(k-2))!} p^{k-2} q^{n-2-(k-2)} + np$$

$$= n(n-1)p^{2}(p+q)^{n-2} + np = n^{2}p^{2} - np^{2} + np$$

$$DX = EX^{2} - (EX)^{2} = n^{2}p^{2} - np^{2} + np - n^{2}p^{2} = np(1-p) = np$$

方法2:

X_i 服从（0-1）分布， $P\{X_i = 0\} = q, P\{X_i = 1\} = p, i = 1,2,\cdots,n$

且 $X_1,\cdots,X_n$ 独立，令 $X = X_1 + \cdots + X_n$， 则 X 的可能取值为 0,1,…n，

$$P\{X = k\} = C_n^k p^k q^{n-k}, k = 0,\cdots,n$$

$$EX = \sum_{i=1}^{n} EX_i = np, \qquad DX = \sum_{i=1}^{n} DX_i = npq,$$

3．泊松分布

设 X 服从参数为λ泊松分布，

其分布律为 $P\{X = k\} = \frac{\lambda^k}{k!}e^{-\lambda}$， k=0,1,...

$$EX = \sum_{k=0}^{\infty} k\frac{\lambda^k}{k!}e^{-\lambda} = \lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda}e^{\lambda} = \lambda$$

$$EX^2=\sum_{k=0}^{\infty}k^2\frac{\lambda^k}{k!}e^{-\lambda}=\sum_{k=1}^{\infty}k\frac{\lambda^k}{(k-1)!}e^{-\lambda}$$

$$=\sum_{k=1}^{\infty}(k-1)\frac{\lambda^k}{(k-1)!}e^{-\lambda}+\sum_{k=1}^{\infty}\frac{\lambda^k}{(k-1)!}e^{-\lambda}$$

$$=\lambda^2e^{-\lambda}\sum_{k=2}^{\infty}\frac{\lambda^{k-2}}{(k-2)!}+\lambda e^{-\lambda}e^{\lambda}=\lambda^2+\lambda$$

$$DX=EX^2-(EX)^2=\lambda^2+\lambda-\lambda^2=\lambda$$

4.均匀分布

$$f(x)=\begin{cases}1/(b-a), a<x<b\\ 0,\text{其它}\end{cases}。$$

$$EX=\int_{-\infty}^{\infty}xf(x)dx=\int_a^b x\frac{1}{b-a}dx=\frac{a+b}{2}$$

$$DX = EX^2 - (EX)^2 = \int_a^b x^2 \frac{1}{b-a} dx - (\frac{a+b}{2})^2 = \frac{(b-a)^2}{12}$$

5. 正态分布　$X \sim N(\mu, \sigma^2)$

$$EX = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma t + \mu) e^{-\frac{t^2}{2}} dt, (\frac{x-\mu}{\sigma} =$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t e^{-\frac{t^2}{2}} dt + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt = \mu$$

$$DX = E(X-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx, (\frac{x-\mu}{\sigma} = t)$$

$$= \int_{-\infty}^{\infty} \frac{\sigma^2 t^2}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^2 e^{-\frac{t^2}{2}} dt = -\frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t de^{-\frac{t^2}{2}}$$

$$= -\frac{\sigma^2}{\sqrt{2\pi}} t e^{-\frac{t^2}{2}} \Big|_{-\infty}^{\infty} + \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt = \sigma^2$$

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$$P\{|X-\mu|\le\sigma\}=P\{\mu-\sigma\le X\le\mu+\sigma\}$$

$$=\Phi(\frac{\mu+\sigma-\mu}{\sigma})-\Phi(\frac{\mu-\sigma-\mu}{\sigma})\;=\Phi(1)-\Phi(-1)=2\Phi(1)-1=0.682$$

$$P\{|X-\mu|\le 2\sigma\}=P\{\mu-2\sigma\le X\le\mu+2\sigma\}$$

$$=2\Phi(2)-1=0.9544$$

$$P\{|X-\mu|\le 3\sigma\}=\;P\{\mu-3\sigma\le X\le\mu+3\sigma\}$$

$$=2\Phi(3)-1=0.9974$$

因此，对于正态随机变量来说，它的值落在区间 $[\mu-3\sigma,\mu+3\sigma]$ 内几乎是肯定的。

在上一节用切比晓夫不等式估计概率有：

$$P\{|X-\mu|<3\sigma\}\ge 0.8889$$