定义：二元实函数的可微性.

考虑 D 上二元函数 u. $(x,y)\in D$. 若

$$u(x+\Delta x, y+\Delta y)-u(x,y) = A(x,y)\Delta x + B(x,y)\Delta y + o(\sqrt{\Delta x^2+\Delta y^2})$$

其中 $o(\rho)$ 是 $\rho\to 0$ 的高阶无穷小. 则称 f 在 (x,y) 处可微.

定理：设 $f=u(x,y)+iv(x,y)$ 在区域 D 上定义. $\underset{x+iy}{\overset{}{z}}\in D$. 则

$$\left\{\begin{array}{l} u(x,y),\ v(x,y) \text{ 在 } (x,y) \text{ 处可微} \\ u(x,y),\ v(x,y) \text{ 在 } (x,y) \text{ 满足 CR} \end{array}\right. \Longleftrightarrow f(z) \text{ 在 } z=x+iy \text{ 处可导.}$$

证明：先证 $f(z)$ 可导 $\Rightarrow$ u,v 可微且在 (x,y) 处满足 CR

$f(z)$ 在 z 处可导 $\Rightarrow$ $f(z+\Delta z)-f(z)=\boxed{f'(z)\Delta z+\eta(z,\Delta z)\Delta z}$.

其中 $\lim\limits_{\Delta z\to 0}\eta(z,\Delta z)=0$.

令 $f'(z)=\alpha+i\beta$, $f(z)=u(x,y)+iv(x,y)$ (1)

则 $\Delta u+i\Delta v=\boxed{(\alpha+i\beta)(\Delta x+i\Delta y)+\eta(z,\Delta z)\Delta z}$

$$=(\alpha\Delta x-\beta\Delta y)+i(\beta\Delta x+\alpha\Delta y)+\eta(z,\Delta z)\Delta z \quad (2)$$

比较左右.

$$\Rightarrow \Delta u(x,y)=\alpha\Delta x-\beta\Delta y+\mathrm{Re}(\eta(z,\Delta z)\Delta z) \quad (3)$$

$$\Delta v(x,y)=\beta\Delta x+\alpha\Delta y+\mathrm{Im}(\eta(z,\Delta z)\Delta z) \quad (4)$$

又由于 $|\mathrm{Re}\,\eta(z,\Delta z)\Delta z|<|\eta(z,\Delta z)\Delta z|$, 因此.

$$\lim_{\Delta z\to 0} \operatorname{Re}\eta(z,\Delta z) = 0, \quad (5)$$

类似地有

$$\lim_{\Delta z\to 0} \operatorname{Im}\eta(z,\Delta z) = 0 \quad (6)$$

$\Rightarrow$ $\operatorname{Re}(\eta\Delta z)$ 与 $\operatorname{Im}(\eta\Delta z)$ 为高阶无穷小

$$\Rightarrow \quad \Delta u(x,y) = \alpha\Delta x - \beta\Delta y + \text{高阶无穷小} \quad (7)$$

$$\Delta v(x,y) = \beta\Delta x + \alpha\Delta y + \text{高阶无穷小}. \quad (8)$$

$$\Rightarrow \begin{cases} u \text{ 和 } v \text{ 在 } (x,y) \text{ 处可微. 以及} \\ \alpha = \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \qquad \beta = -\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}. \end{cases} \quad (9)$$

再证 u,v 可微且 CR $\Rightarrow$ f 可导.

过程把上面证明过程反过来即可，关键是

$$\Delta u(x,y) = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \eta_u$$ 中 $\lim_{\rho\to 0}\frac{\eta_u}{\rho} = 0$, $\rho = \sqrt{\Delta x^2+\Delta y^2}$

定理. 设 $f = u(x,y) + iv(x,y)$ 在区域 D 上定义. $z = x+iy \in D$. 则

$$\begin{cases} u(x,y),\ v(x,y) \text{ 在 } (x,y) \text{ 处有连续1阶偏导} \\ u(x,y),\ v(x,y) \text{ 在 } (x,y) \text{ 处满足 CR} \end{cases} \Leftrightarrow f(z) \text{ 在 } z \text{ 处可导}.$$

注意

u,v 有<u>连续</u>1阶偏导数 —— 强

u,v 可微 —— 弱

$$\begin{cases} u,v \text{ 有}\underline{\text{连续}}\text{1阶偏导数} \\ \text{CR 条件} \end{cases} \quad \underline{\text{等价}} \quad \begin{cases} u,v \text{ 可微} \\ \text{CR 条件}. \end{cases}$$

例：$f(z)=|z|^2 = x^2+y^2 \Rightarrow$ 各1阶偏导数连续，于是 u,v 可微

但是 CR条件不满足．($v=0$)

解析函数物理意义

u,v 共轭调和函数．

$$CR \Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

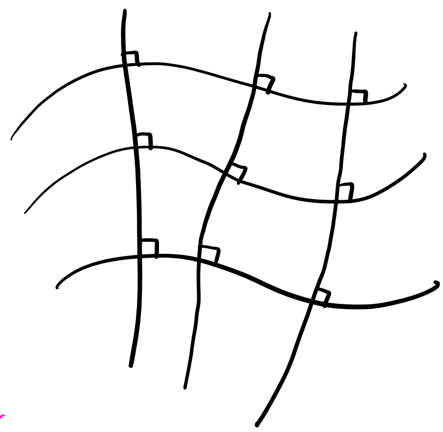
$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)\cdot\left(\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right)=0.$$

u(或 v) 可描述 无电荷区域的静电场势．

$u=\text{const}$ (或 $v=\text{const}$) 为等势线，

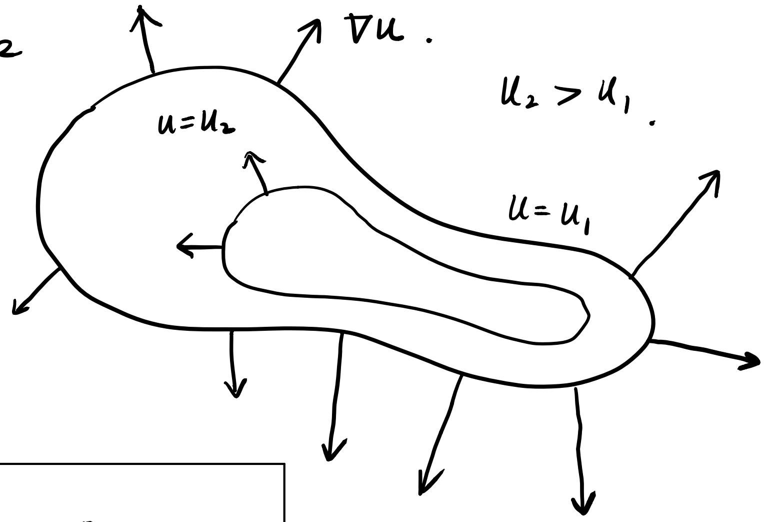
u的法线 与 v的法线垂直

$\Rightarrow$ 等u线 $\perp$ 等v线



$\Rightarrow$ $v=\text{const}$ (或 $u=\text{const}$) 为电场线

⑪ 无电荷静电场与共轭调和



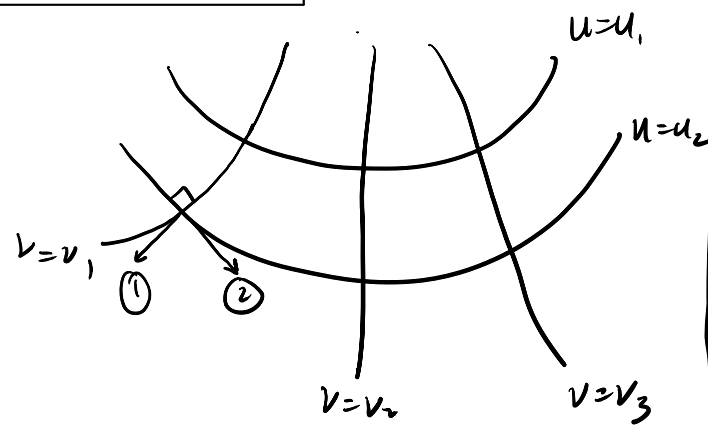
u, v 的梯度

$$\boxed{\begin{aligned} \nabla u &= \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \quad u \text{ 增长最快方向.} \\ \nabla v &= \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) \quad v \text{ 增长最快方向} \end{aligned}}$$

$$\boxed{\nabla u \cdot \nabla v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \overset{CR}{=\!=} 0.}$$

$\Rightarrow$ u, v 等值线 正交

（因 $\nabla u \perp u$ 等值线
$\nabla v \perp v$ 等值线 ）



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$\Rightarrow$ 两个论断

(a) 等 u 线 可作为 无电荷 平面 静电场 等势线

(b) 等 v 线. 是无电荷 平面 静电场 电场线

(a): 来自 Maxwell 方程 $\nabla \cdot E(x,y,z) \propto \rho(x,y,z)$ ← 电荷密度.

$$\rho \propto \nabla \cdot \underline{\underline{E}} \propto \nabla \cdot \underline{\underline{\nabla \varphi}} = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

若无电荷

若 φ 与 z 无关 （沿 z 平移不变）

$$0 = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \equiv \nabla_{2d}^2 \varphi$$

无荷区 电势 调和! 可用 $u = \mathrm{Re} f$ 来表示

解析函数 $u+iv$ → 共轭调和函数 u,v → 调和函数 u, 或 v.

调和函数 u 或 v → 共轭调和函数 u,v → 解析函数 $u+iv$

CR.

说明:

已知 $u(x,y)$ 为调和函数, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

希望找到解析函数 $u+iv$. 以 u 为实部 $\Longrightarrow$ 最重要找 $v(x,y)$

设这样的 $v(x,y)$ 存在. 则

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy$$

$\parallel$ CR

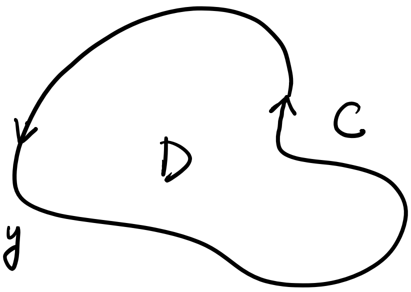
$$dv=-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy$$ 是全微分!!

因 $\left(\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)-\frac{\partial}{\partial x}\frac{\partial u}{\partial x}=0\right)$

$$\Longrightarrow v(x,y)=\int_{(x_0,y_0)}^{(x,y)}-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy$$

路径无关

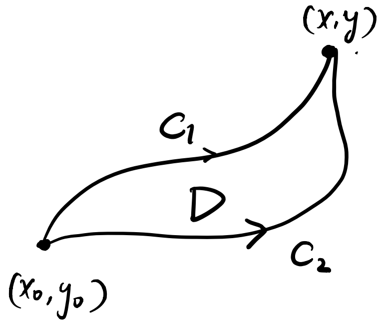
Green's 公式.



$$\oint_C P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\left(\int_{C_2\,(x_0,y_0)}^{(x,y)} - \int_{-C_1\,(x,y)}^{(x_0,y_0)}\right)\left(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\right) = \oint_{C_2-C_1}\left(\underbrace{\left(-\frac{\partial u}{\partial y}\right)}_{P}dx + \underbrace{\frac{\partial u}{\partial x}}_{Q}dy\right)$$

$$\overset{\text{Green's}}{=\!=}\ \iint_D \underbrace{\left[\frac{\partial}{\partial x}\left(\overset{Q}{\frac{\partial u}{\partial x}}\right) - \frac{\partial}{\partial y}\left(\overset{P}{-\frac{\partial u}{\partial y}}\right)\right]}_{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0.} dx\,dy = 0.$$



积分技巧：凑全微分: $\frac{\partial u}{\partial y}dx - \frac{\partial u}{\partial x}dy = d[V(x,y)]$

$$\Rightarrow \int \frac{\partial u}{\partial y}dx - \frac{\partial u}{\partial x}dy = \int_{x_0,y_0}^{x,y} dV(x,y) = V(x,y) - V(x_0,y_0)$$

例: $u = x^2 - y^2 \Rightarrow$

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy. \qquad (1)$$

$$= 2y\,dx + 2x\,dy \qquad (2)$$

$$\Rightarrow v(x,y) = \int_{(x_0,y_0)}^{(x,y)} 2y'dx' + 2x'dy' \qquad (3)$$

凑全微分.

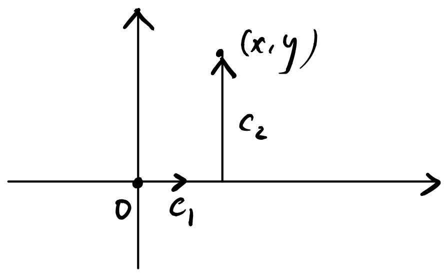
$$= \int_{(x_0,y_0)}^{(x,y)} d(2x'y') = 2xy - 2x_0y_0. \qquad (4)$$

$\Rightarrow$ 共轭调和函数 $u = x^2 - y^2$ $\quad v = 2xy + \text{Const}$.

$\Rightarrow$ 解析函数 $x^2 - y^2 + i(2xy + \text{const}) = (x+iy)^2 + \text{Const}$

$= z^2 + \text{Const}$.

不想凑全微分：选路径 死算



$$\int_C 2y'dx' + 2x'dy' = \int_{C_1} (2y'dx' + 2x'dy') + \int_{C_2} (2y'dx' + 2x'dy')$$

$$= \int_{C_1} \underbrace{2y'dx'}_{y'|_{C_1}=0} + \int_{C_2} 2x'dy'$$

$$= 0 + 2x \int_{C_2} dy' = 2xy \longrightarrow \text{添上积分常数 } 2xy + C.$$

例：$v = \frac{y}{x^2+y^2}$，找 u s.t. $u+iv$ 解析.

$$\Rightarrow du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy = \frac{x^2-y^2}{(x^2+y^2)^2} dx + \frac{2xy}{(x^2+y^2)^2} dy.$$

凑全微分

$$\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{x^2-y^2}{(x^2+y^2)^2}, \quad \text{找 } u, \text{ s.t. } \frac{\partial}{\partial x} u = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = -\frac{2xy}{(x^2+y^2)^2} \qquad \frac{\partial u}{\partial y} u = -\frac{2xy}{(x^2+y^2)^2}$$

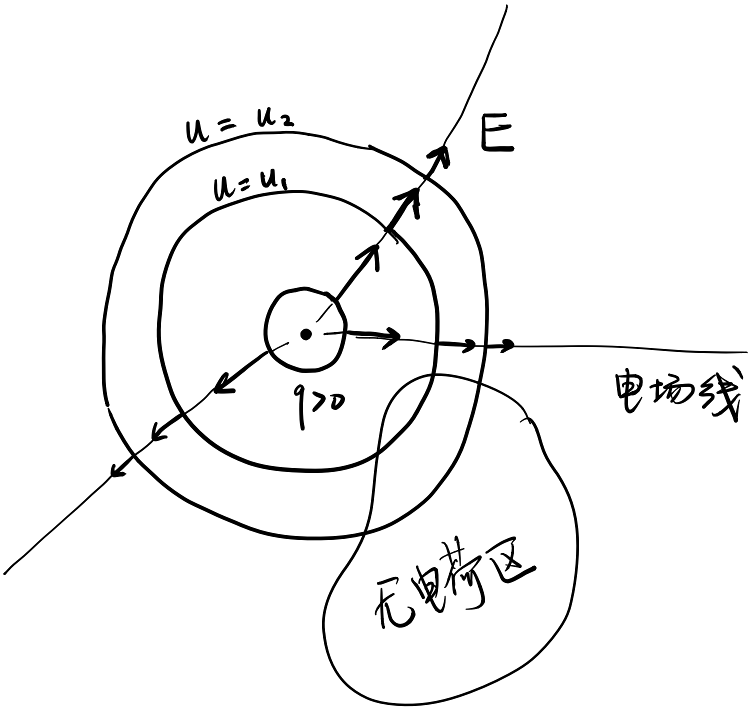
$x \leftrightarrow y$?

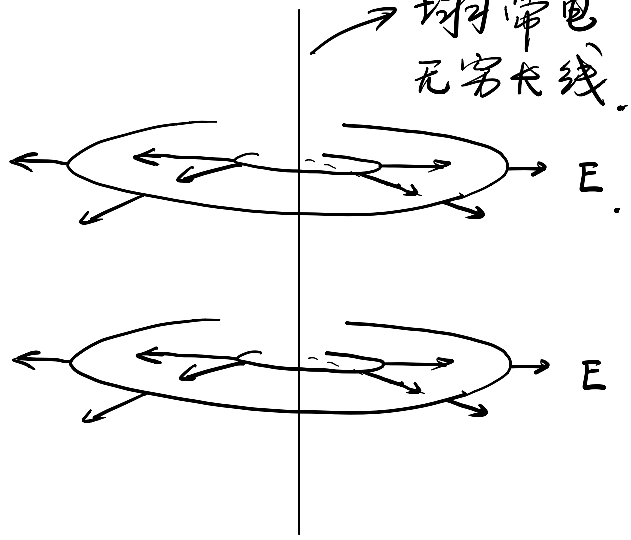
$$\frac{x^2-y^2}{(x^2+y^2)^2} = -\frac{y^2-x^2}{(y^2+x^2)^2} = -\frac{\partial}{\partial x}\left(\frac{x}{y^2+x^2}\right)$$

$$\Rightarrow u = -\frac{x}{x^2+y^2} + C$$

$$\frac{2xy}{(x^2+y^2)^2} = \frac{2yx}{(y^2+x^2)^2} = -\frac{\partial}{\partial y}\left(\frac{x}{y^2+x^2}\right)$$

$$\Rightarrow u + iv = -\frac{x}{x^2+y^2} + C + i\frac{y}{x^2+y^2} = \frac{-\bar{z}}{z\bar{z}} + C = -\frac{1}{z} + C.$$

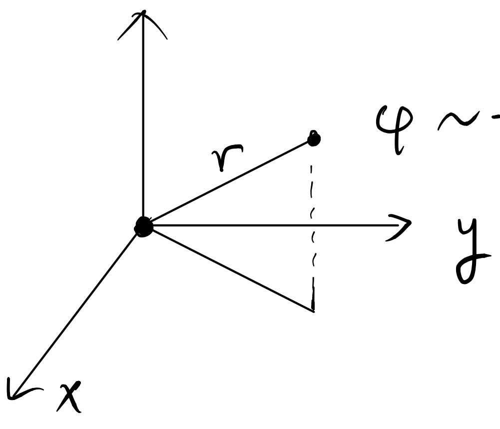




(b) 来自"E是势的梯度" $\Rightarrow$ E⊥等势线 $u=\varphi=\text{const}$

$\Rightarrow$ 电场线⊥等势线

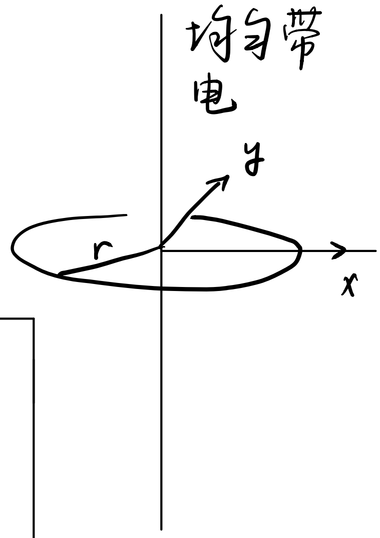
等v线、等u线



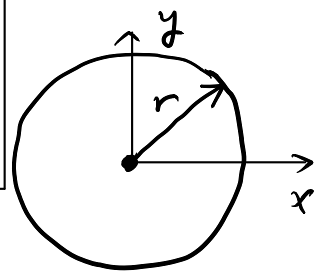
$\varphi \sim \frac{1}{r}$ 3维电势.

⑫ "平面点电荷"(用无穷长均匀带电线 构造)

$$\varphi \propto \ln r \sim \ln(z\bar{z}) \overset{俑}{\underset{讲}{=}} \ln z + \ln \bar{z}$$



$$\boxed{\begin{aligned} \underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi}_{\sim \nabla\cdot E} &= \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\varphi \\ &\propto \frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}\varphi \\ &= \frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}\left(\ln z + \ln \bar{z}\right) \end{aligned}}$$

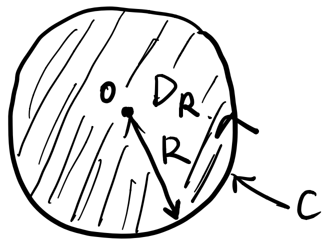


$$\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}\ln z = \frac{\partial}{\partial \bar{z}}\left(\frac{1}{z}\right) \overset{?}{=} 0 \begin{cases} \text{当 } z\neq 0\text{，的确为 } 0 \\ z = 0\,? \end{cases}$$

降维打击.

$$\text{Maxwell 方程} \xrightarrow{\text{预言}} \nabla^2_{2d}\varphi \sim \frac{\partial}{\partial \bar{z}}\left(\frac{1}{z}\right) \propto \rho \sim \delta^2(x,y)$$

$$\iint_{D_R} dx\,dy\ \delta^2(x,y) = 1$$



大致证明：关键思想 $\iint_{D_R} \frac{\partial}{\partial \bar z}\left(\frac{1}{z}\right) dxdy \neq 0$ 且与R无关
(类似高斯定理).

$$\frac{\partial}{\partial \bar z}\left(\frac{1}{z}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\bar z}{z\bar z}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{x-iy}{x^2+y^2}\right) \quad (1)$$

$$= \frac{\partial}{\partial x}\underbrace{\left(\frac{1}{2}\frac{x-iy}{x^2+y^2}\right)}_{Q} - \frac{\partial}{\partial y}\underbrace{\left(-\frac{i}{2}\frac{x-iy}{x^2+y^2}\right)}_{P} \quad (2)$$

$$\equiv \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}. \quad (3)$$

$$\Rightarrow \int_{D_R} \frac{\partial}{\partial \bar z}\left(\frac{1}{z}\right) dxdy = \int_{D_R}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \quad (4)$$

$$\overset{\text{Green's}}{=\!=} \oint_{C} (Pdx + Qdy) = \oint_C\left(-\frac{i}{2}\frac{x-iy}{x^2+y^2}dx + \frac{1}{2}\frac{x-iy}{x^2+y^2}dy\right) \quad (5)$$

$|z|=R$.

令 $x = r\cos\theta \qquad y = r\sin\theta. \Rightarrow dx = \cos\theta dr - r\sin\theta d\theta, dy = \cdots$

$$= \int_C -\frac{i}{2}\frac{re^{-i\theta}}{r^2}(\cos\theta dr - r\sin\theta d\theta) + \frac{1}{2}\frac{re^{-i\theta}}{r^2}(\sin\theta dr + r\cos\theta d\theta) \quad (6)$$

$$= \frac{1}{2}\int_C \frac{e^{-i\theta}}{r}\left[(-i\cos\theta + \sin\theta)dr + (-i(-r\sin\theta) + r\cos\theta)d\theta\right] \quad (7)$$

$$= \frac{1}{2}\int_C \frac{e^{-i\theta}}{r}\left[(-i)e^{i\theta}dr + re^{i\theta}d\theta\right] \quad (8)$$

$$= \frac{1}{2}\int_C \frac{1}{r}\left[(-i)dr + rd\theta\right] = \cancel{\frac{-i}{2}\int_C \frac{1}{r}dr}^{\,0} + \underbrace{\frac{1}{2}\int_C d\theta}_{\frac{1}{2}(2\pi)}. \quad (9)$$

$$\Rightarrow \int_{D_R} \frac{\partial}{\partial \bar{z}}\left(\frac{1}{z}\right) dxdy = \pi, \ \forall R>0. \Rightarrow \begin{cases} \frac{\partial}{\partial \bar{z}}\left(\frac{1}{z}\right) \text{几乎处处为0} \\ \frac{\partial}{\partial \bar{z}}\left(\frac{1}{z}\right) \text{在 } z=0 \text{ 处不为0.} \\ \frac{\partial}{\partial \bar{z}}\left(\frac{1}{z}\right) \text{的面积分为常数} \end{cases}$$

$$\Rightarrow \quad \frac{\partial}{\partial \bar{z}}\left(\frac{1}{z}\right) = \pi \delta^2(x,y) \xrightarrow{\text{复共轭.}} \frac{\partial}{\partial z}\left(\frac{1}{\bar{z}}\right) = \pi \delta^2(x,y) \quad \blacksquare$$

$\Rightarrow$ Maxwell is right,

$$\partial_z \partial_{\bar{z}} \ln(z\bar{z}) = \partial_z \partial_{\bar{z}}(\ln z + \ln \bar{z}) = \partial_{\bar{z}}\left(\frac{1}{z}\right) + \partial_z\left(\frac{1}{\bar{z}}\right)$$
$$= 2\pi \delta^2(x,y)$$

$\Rightarrow$ $\frac{1}{z}$ 是 $\partial_{\bar{z}}$ 的格林函数之一（课程结尾）.

$\ln(z\bar{z})$ 是 Laplace 算符 $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\partial_z \partial_{\bar{z}}$ 的格林函数.

初等单值函数

- 常数、幂函数、指数函数、对数函数、三角函数、反三角函数

  及有限次加减乘除、复合所构成的函数。

定义：常数 $f(z) = c \in \mathbb{C}$. 全 $\mathbb{C}$ 解析. $f'(z) = 0$.

定义：幂函数 $f(z) = z^n$, $n = 1, 2, 3, \cdots$ $f'(z) = nz^{n-1}$, 全 $\mathbb{C}$ 解析.

定义：多项式. $P_n(z) = \sum_{k=0}^{n} \underset{\in \mathbb{C}}{a_k} z^k$. $n = 0, 1, 2, \cdots$, 全 $\mathbb{C}$ 解析、$P_n(z)' = \sum_{k=0}^{n} k a_k z^{k-1}$

定义：有理式 即两个多项式之商 $\dfrac{P_n(z)}{Q_m(z)} \rightarrow$ $\mathbb{C}$上的亚纯函数.

$Q_m(z)$ 的根，即 $Q_m(z_i) = 0$ 的 z_i 为奇点，，$\dfrac{P}{Q}$ 在 $\mathbb{C} \setminus \{z_i\}$ 解析.

有时 z_i 是 $P_n(z)$ 的零点，同时也是 Q_m 的零点，，

有可能 可以补上函数定义 使有理式在 z_i 也解析 如

$$f(z) = \frac{(z-1)(z-3)}{(z-1)(z-4)},$$

可以补上 $f(1) = \dfrac{1-3}{1-4} = \dfrac{-2}{-3}$ 使 f 在 $z=1$ 处也解析.

定义：指数函数为 $e^{z} = e^{x+iy} \equiv e^{x} e^{iy} = e^{x}(\cos y + i\sin y)$

复自变量

$(e^{z})' = e^{z}$，在全$\mathbb{C}$解析

$e^{z_1} e^{z_2} = e^{z_1+z_2}$ ， $|e^{z}|>0$

$e^{z+2\pi i n} = e^{z}$

定义：三角函数 $\cos z \equiv \frac{1}{2}(e^{iz}+e^{-iz})$ $\sin z \equiv \frac{1}{2i}(e^{iz}-e^{-iz})$.

$(\cos z)' = -\sin z$. $(\sin z)' = \cos z$.

$\cos^2 z + \sin^2 z = 1$、 $\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

$\sin(z_1+z_2) = \sin z_1 \cos z_2 + \sin z_2 \cos z_1$.

证明：

$$\cos^2 z + \sin^2 z = \frac{1}{4}(e^{iz}+e^{-iz})^2 - \frac{1}{4}(e^{iz}-e^{-iz})^2$$

$$= \frac{1}{4}(e^{2iz} + e^{-2iz} + 2e^{iz}e^{-iz})$$

$$-\frac{1}{4}(e^{2iz} + e^{-2iz} - 2e^{iz}e^{-iz})$$

$$= \frac{1}{4}\cdot 4 = 1.$$

$$\cos(z_1+z_2) = \frac{1}{2}\left(e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}\right)$$

$$= \frac{1}{2}\left(e^{iz_1}e^{iz_2} + e^{-iz_1}e^{-iz_2}\right)$$

$$= \frac{1}{4}\left(e^{iz_1}e^{iz_2} + e^{-iz_1}e^{-iz_2} + e^{iz_1}e^{-iz_2} + e^{iz_2}e^{-iz_1}\right)$$

$$+\frac{1}{4}\left(e^{iz_1}e^{iz_2} + e^{-iz_1}e^{-iz_2} - e^{iz_1}e^{-iz_2} + e^{iz_2}e^{-iz_1}\right)$$

$$= \frac{1}{2}(e^{iz_1}+e^{-iz_1})\frac{1}{2}(e^{iz_2}+e^{-iz_2})$$

$$+\frac{1}{2}(e^{iz_1}-e^{-iz_1})\frac{1}{2}(e^{iz_2}-e^{-iz_2})$$

$$= \cos z_1 \cos z_2 - \sin z_1 \sin z_2 .$$

定义：双曲函数． $\cosh z \equiv \frac{1}{2}(e^z + e^{-z})$ $\sinh z \equiv \frac{1}{2}(e^z - e^{-z})$.

$(\cosh z)' = \sinh z$ $(\sinh z)' = \cosh z$

$\cos(iz) = \cosh z$ $\cosh(iz) = \cos z$ $\sin(iz) = i \sinh z$ $\sinh iz = i \sin z$

$(\cosh z)^2 - (\sinh z)^2 = 1$

初等多值函数

① 根式函数 $w=\sqrt{z-a}$. 是最简单的多值函数

② $z=re^{i\theta} = re^{i(\theta+2\pi k)}$

$\Rightarrow w=\sqrt{\rho}e^{i\frac{\theta}{2}+ik\pi} = \sqrt{\rho}e^{i\frac{\theta}{2}}\cdot(-)^{k}$, $k\in\mathbb{Z}$

③ $\forall z\in\mathbb{C}$. $w(z)$ 有两个可能值. 相差 -1 因子. [redacted]

④ 多值性 = 源于宗量 幅角多值性. 如,

$$\boxed{\begin{array}{l} f(z) = \sqrt{\underbrace{z-a}_{z'\text{ 宗量}}} = \sqrt{\underbrace{\rho e^{i\varphi}}_{z\text{ 自变量}}-a} \\ \quad\quad\quad \| \\ \quad\quad \sqrt{\rho' e^{i\theta}} \end{array}}$$

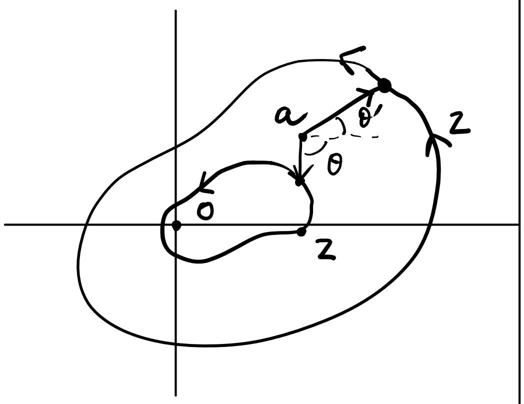
当 $z-a$ 的辐角 θ 取不同值 $\theta, \theta+2\pi, \cdots, \theta+2\pi k$

$$\sqrt{\rho' e^{i\theta+2\pi k i}} = \sqrt{\rho}\ e^{i\frac{\theta}{2}}\ e^{\pi k i}$$
$$= \sqrt{\rho}\ e^{\frac{i\theta}{2}}\ (-1)^{k}.$$

即 $f(z)$ 的宗量 $z-a$ 取不同主值分支时. 可对应

$\sqrt{\rho'}e^{i\frac{\theta}{2}}$ 及 $-\sqrt{\rho'}e^{i\frac{\theta}{2}}$ 两值.

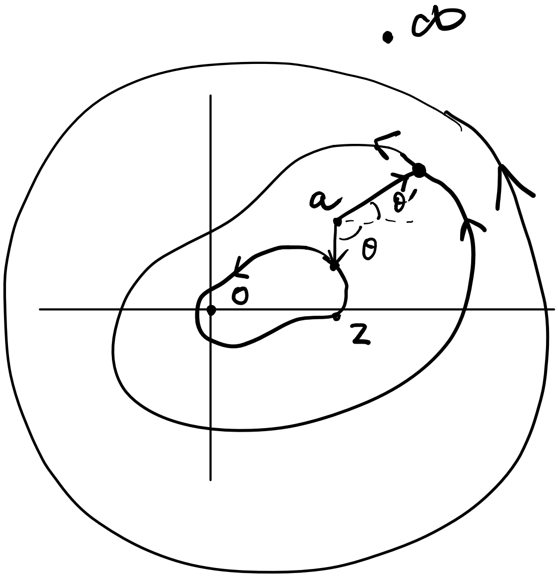
几何上，让自变量 z 沿包围原点的曲线走一圈.



包围 a $\begin{cases} z \quad \text{幅角} \quad \phi \to \phi + 2\pi \\ z-a \text{ 的幅角} \quad \theta \to \theta + 2\pi \\ \sqrt{z-a} \to -\sqrt{z-a} \end{cases}$

不包围 a $\begin{cases} z \quad \text{幅角} \quad \phi \to \phi + 2\pi \\ z-a \text{ 的幅角} \quad \theta \to \theta \\ \sqrt{z-a} \to \sqrt{z-a} \end{cases}$

$\boxed{a \text{ 称为 } \sqrt{z-a} \text{ 支点 (branch point)}}$



大圆包围 ∞ (绕 ∞ 一周). $\begin{cases} z \quad \text{幅角} \quad \phi \to \phi + 2\pi \\ z-a \text{ 的幅角} \quad \theta \to \theta + 2\pi \\ \sqrt{z-a} \to -\sqrt{z-a} \end{cases}$

$\boxed{\infty \text{ 也是 } \sqrt{z-a} \text{ 的支点}}$

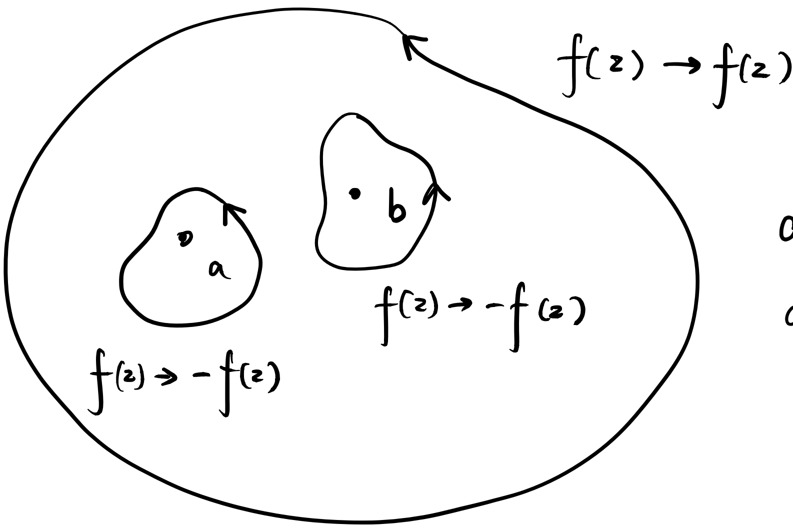
当 z 沿包支点 a 曲线 2 周，$\sqrt{z-a}$ 回复原值

⇒ 定义：一阶支点

k 周　$f(z)$ 回复 ...

⇒ 定义：(k-1)-阶支点

例：$f(z)=\sqrt{(z-a)(z-b)}$：



a, b 均为一阶支点

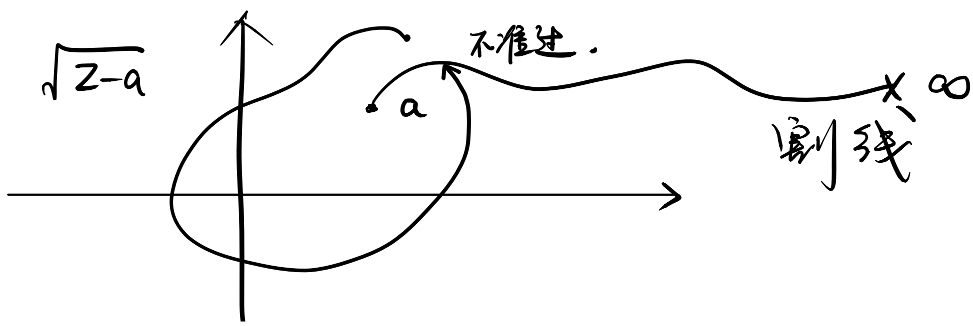
∞ 不是支点

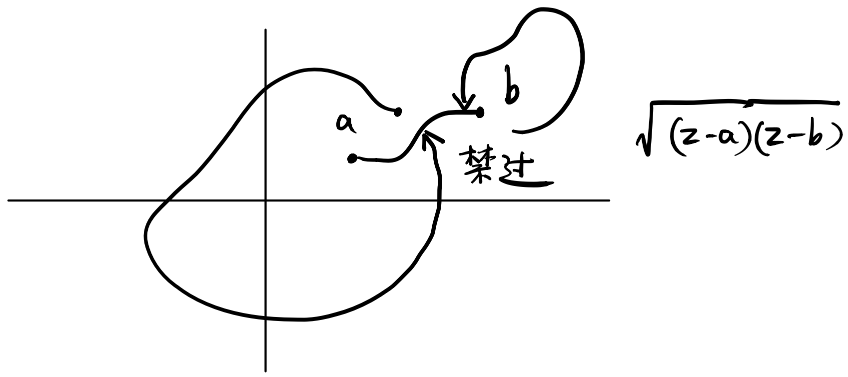
⑤ 割线：从支点出发的线割破 $\mathbb{C}$，禁止自变量越过

禁止绕支点作完整一圈转动

⇒ 多值函数的"单值分支"，该线为<u>割线</u>

例：$\sqrt{z-a}$





⑥ 对数函数.

若 $e^{w}=z \Longrightarrow$ 定义反函数. $w=\ln z$

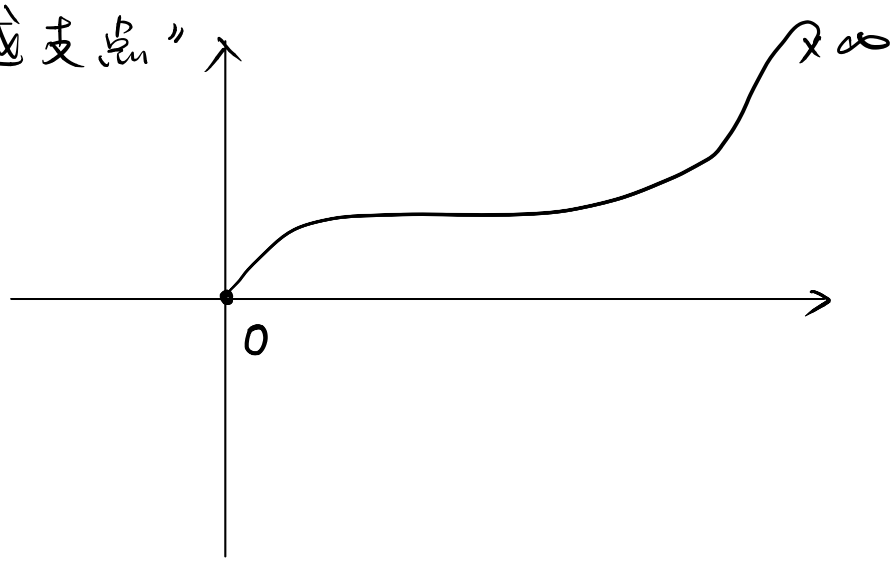
注意到 $e^{2\pi i k}=1 \Rightarrow e^{w+2\pi k i}=z$.

即给定 z，所对应的 W 不唯一. 可以相差 $2\pi k i$.

即 $z \longrightarrow \ln z$ 是多值函数. $z=0$，$z=\infty$ 为支点.

且不管绕 $z=0$，$z=\infty$ 多少圈. $\ln z$ 均不回复原值

$\Rightarrow$ 定义"超越支点"



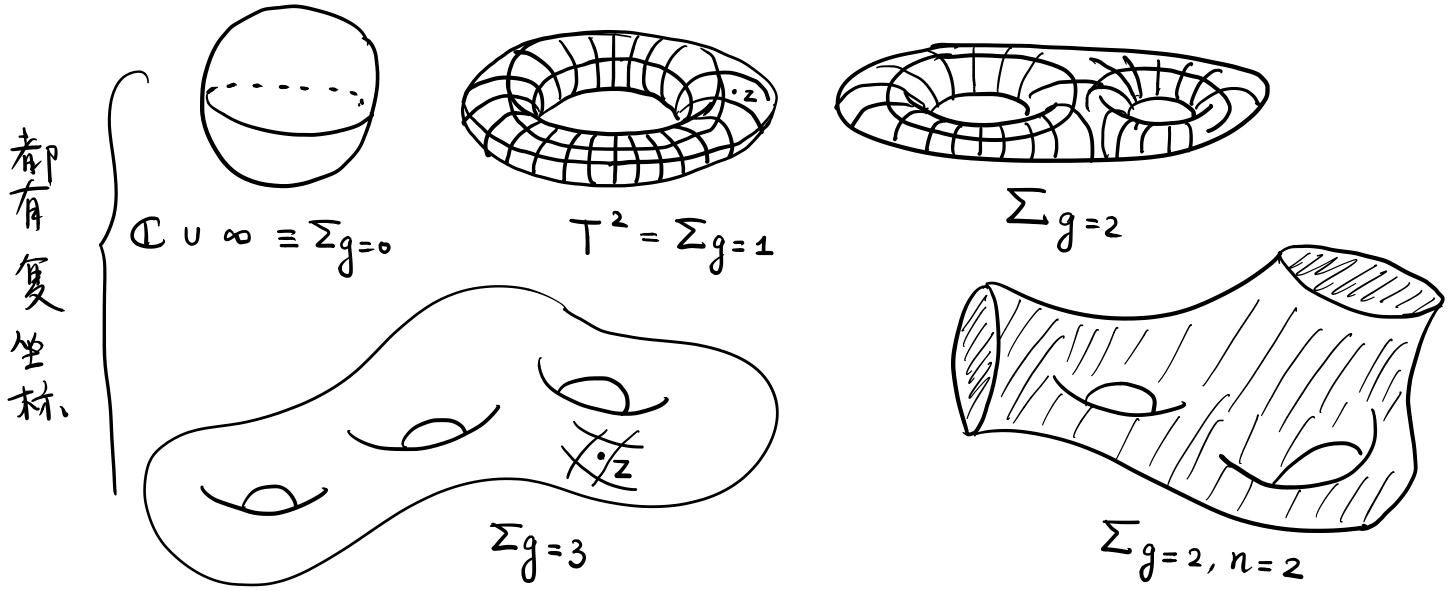
⑦ 对数函数导数 $f(z)=\ln z \Rightarrow \frac{df}{dz}=\frac{1}{z}$.

⑧ 支点，邻域函数值不确定，不可导.

其它空间的复分析.

- $\mathbb{C}$ 是最简单的"复空间"，即存在 well-defined 的复坐标 $z, \bar{z}$

- 可推广所有我们所学概念到"黎曼曲面"上.

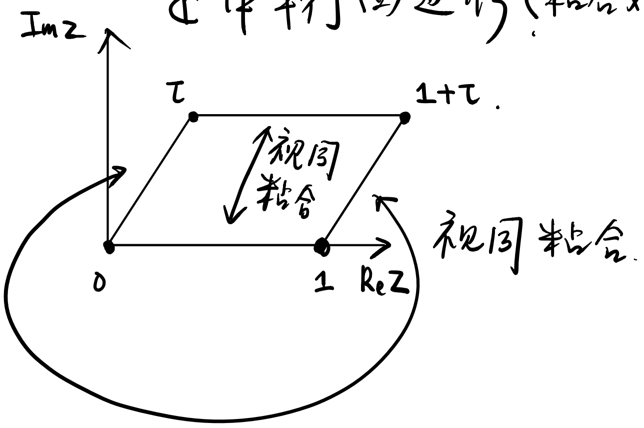


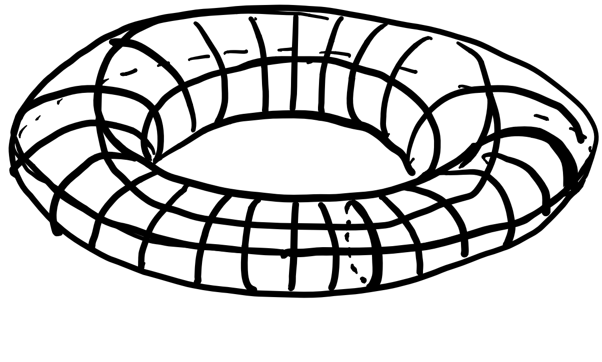
- 闭合黎曼曲面：代数曲线 complex algebraic curves.

$$\Sigma_g = \left\{ (z,w) \in \underset{\mathbb{R}^4}{\underset{\|}{\mathbb{C}^2}} \;\middle|\; w^2 = \sum_{k=0}^{2g+1} \underset{\text{给定复系数}}{\underset{\uparrow}{a_k}}\, z^k \right\}.$$

- 如：

$\mathbb{C}$ 中平行四边形(粘合对边) ＝ T^2





环面 T^2 上的全纯函数 f $\Leftrightarrow$ $\mathbb{C}$ 上全纯函数 f s.t.

$f(z+1) = f(z)$ $\qquad$ $f(z+\tau) = f(z)$, $\tau \in \mathbb{C}$

($z, z+1, z+\tau, z+1+\tau, z+1+2\tau, \cdots z+m+n\tau$ 均粘成 T^2 的一个点.)

$\mathbb{C}$, T^2 上的复变函数均在共形场论中有重要应用.

"$\partial_{\bar{z}} G = \delta^2(z, \bar{z})$, s.t. $G(z+m+n\tau) = G(z)$." 有解吗?

与"T^2上自由标量场传播子."有关

"高维虚数单位"

- 高维复空间: $(M^n, J^2 = -1)$ (不一定有复坐标)

高维流形 $\qquad$ 张量

- 复几何、代数几何