1 Thus Far

- A wavefunction describes our system
- The Schrödinger equation tells us how our system evolves
- Writing the wavefunction as a superposition of EESs (Energy EigenStates) makes life easier by removing time dependence
- Piecewise flat 1D potentials are nice because we already know the EESs in the flat regions, so we just need to match them at the boundaries
- The Probability Current associated with a wavefunction describes the motion of the particle
- The S-Matrix presents a general framework for investigating localized potentials

2 This Time

- Why are Energy Eigenstates so interesting?
- Bound vs. Unbound States (a.k.a., Scattering States)

3 Why Energy Eigenstates

In the last lecture before the break, I drew this diagram on the board

I was hoping to explain why we are talking about energy eigenstates (EES), and how we go from these "stationary states" to something that moves (superposition!).

Energy Eigenstates "Don't Move" : $\partial_t \, \mathbb{P}(x,t) = 0$
Superpositions of EES Move!

For an EES

$$\partial_t \, \mathbb{P}(x,t) = \partial_t \left| \phi_E(x) e^{-i\omega t} \right|^2 = \partial_t \left| \phi_E(x) \right|^2 = 0$$

For a superposition of EES

$$\begin{aligned} \partial_t \, \mathbb{P}(x,t) &= \tfrac{1}{2}\partial_t \left| \phi_0(x) e^{-i\omega_0 t} + \phi_1(x) e^{-i\omega_1 t} \right|^2 \\ &= \tfrac{1}{2}\partial_t \left| \phi_0(x) + \phi_1(x) e^{-i(\omega_1 - \omega_0)t} \right|^2 \neq 0 \end{aligned}$$

Unless $\omega_1 = \omega_0$ (e.g., they are degenerate states)!!

In the following section I'll give a quick overview of what we have learned about EES in the last few weeks.

4   Bound vs. Unbound Energy Eigenstates

We have spent the last several lectures talking about energy eigenstates of various potentials. Depending on the potential of the day, we tended to talk about "bound states", or "unbound states", or one and then the other, but not about both together.


Bound vs. Unbound
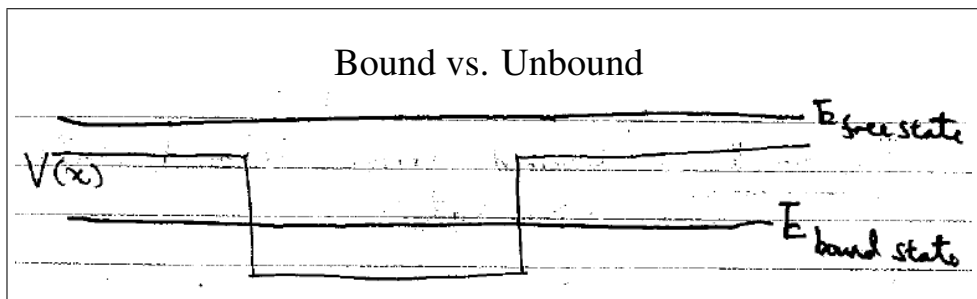
Roughly speaking, bound states represent a localized particle, while unbound states represent a particle that can escape to $\pm\infty$.

2

This distinction is conceptually similar in Classical and QM, with the subtle difference that a classical particle can be trapped in a local minimum, which a quantum wavefunction cannot (thanks to tunneling).
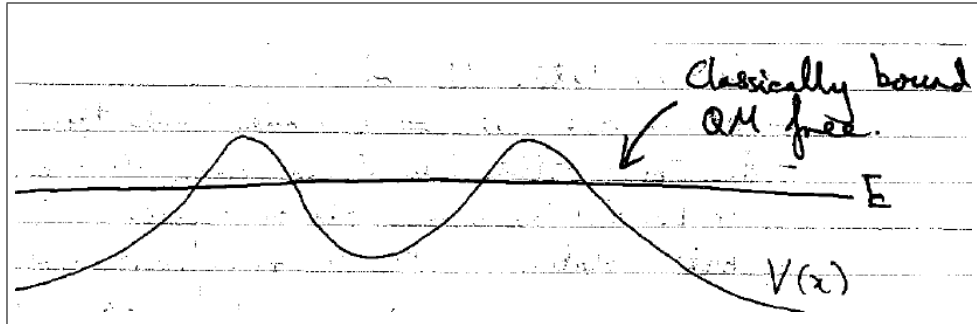

In order for a state to be bound in QM, its energy must be less than the potential at both $+\infty$ and $-\infty$.
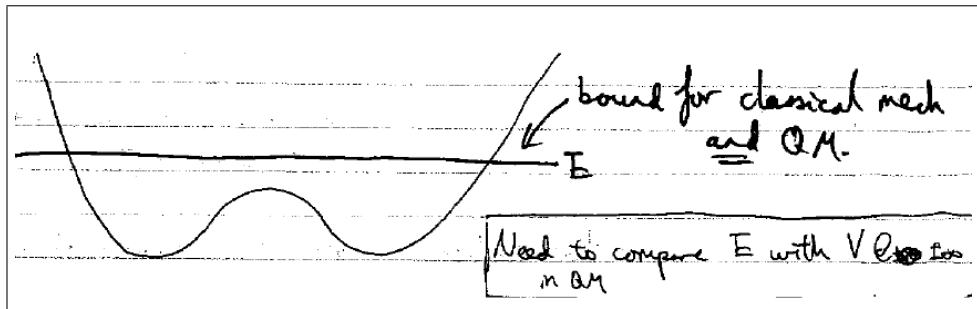
The distinction between these two types of states arises from their somewhat different behavior, that is in the different sorts of solutions which result from solving the equations of motion. In classical mechanics, using $F = ma$, you find that a bound particle will execute some periodic motion, while an unbound particle will not. In QM, using the Schrödinger equation, you find that bound states have discrete energies, while unbound states do not (they have a continuous range of energies).

With the infinite square-well, for instance, we can safely focus on bound state solutions.

3
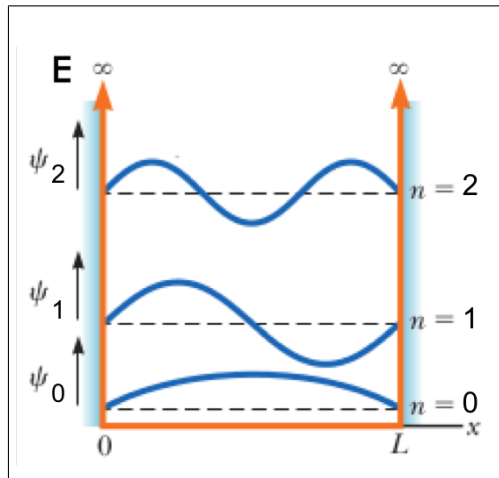
In general, bound states have some important features

Bound States

- Discrete energies
- Constant phase
    - more precisely: $\mathcal{J}(x) = 0 \; \forall \; x$
    - might as well make them real (easy to draw)
    - known as a "standing wave" solution

- Oscillatory where $E > V(x)$
- Exponential where $E < V(x)$
- Node theorem applies
    - ground state has zero nodes
    - each higher energy state has one more node

For a wide range of step and barrier potentials, there are only unbound states.

Unbound States (a.k.a., "Scattering States")

- Continuous energies
- Possible degeneracy
  - can have $\mathcal{J}(x) \neq 0$
  - you can make them real, but this is not always a good idea
  - there are "traveling wave" solutions
- "Oscillatory" where $E > V(x)$ (may be e^{ikx})
- Exponential where $E < V(x)$
- Node theorem DOES NOT apply

For the finite square well (and similar potentials), on the other hand, both kinds of solutions exist. We solved for the bound states and found that they are similar to the infinite square well, and we treated the unbound states as a scattering problem.

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5 Changing Potentials

A classic type of exam problem involves potentials which suddenly change. For instance, what happens if you have particle in the ground state of an infinite square well, and the well suddenly gets a factor of 2 wider?

Expanding Well

- What is the wave function of the particle immediately after the change in $V(x)$?
- How does it change after that?
  - expand in terms of EES of new potential
  - add $e^{-i\omega_n t}$ terms to get time evolution
- let's compute $\langle \phi_1^{new} | \phi_0^{old} \rangle$

Recall

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin(k_n x) \tag{1}$$

with

$$k_n = \frac{(n+1)\pi}{L} \quad n \in \{0, 1, \ldots\} \tag{2}$$

and

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 \pi^2}{2mL^2}(n+1)^2 \tag{3}$$

then

$$
\begin{aligned}
\langle \phi_1^{new} | \phi_0^{old} \rangle &= \int_0^L \left(\sqrt{\frac{2}{2L}} \sin\left(\frac{2\pi}{2L}x\right) \right)^* \left(\sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L}x\right) \right) \\
&= \frac{2}{\sqrt{2L}} \int_0^L \sin^2\left(\frac{\pi}{L}x\right) \\
&= \frac{1}{\sqrt{2}} \qquad \text{(many others are harder!)}
\end{aligned}
$$

So, if you were asked what is the probability of measuring the particle which was in the ground state of the old well to be in the first excited state of the new well at some time $t > 0$, what would you say? How does this answer change in time?

Was energy conserved during the change of $V(x)$?

Yes. There is no reason for $\langle E \rangle$ to have changed since the WF is the same and the potential in the well is zero.

Don't we violate energy conservation by populating higher energy modes in the new well? How large can the values of c_n in the new ϕ_n^{new} superposition be for $n > 1$?

Since
$$\langle E \rangle = E_0^{old} = \frac{\hbar^2 \pi^2}{2mL^2}$$

we know that

$$
\begin{aligned}
\langle E \rangle &= E_0^{old} = \frac{\hbar^2 \pi^2}{2mL^2} = 4E_0^{new} \\
&= \sum_{n=0}^{\infty} |c_n|^2 \, E_n^{new} \\
&= E_0^{new} \sum_{n=0}^{\infty} |c_n|^2 \, (n+1)^2 \\
\Rightarrow \quad 4 &= \sum_{n=0}^{\infty} |c_n|^2 \, (n+1)^2
\end{aligned}
$$

and we found that $|c_1|^2 = \frac{1}{2}$, so considering just one other mode, we have

$$
\begin{aligned}
4 &> 2 + |c_n|^2 \, (n+1)^2 \\
\Rightarrow \quad |c_n| &< \frac{\sqrt{2}}{n+1}
\end{aligned}
$$

So it is ok to populate these modes, just not too much.

Note that $c_n = 0$ for $n = 4m + 3$ and $m \in \{0, 1, 2, \ldots\}$ so some modes are not populated.

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6   Next Time

- Exam 2
- Then back to the Quantum Harmonic Oscillator, this time with Operators!