Problem Sheet 7.

Problem 1. In three warehouses A_1, A_2 and A_3 there are one type products. In A_1 there are 140 units, in $A2$ there are 180 units, and in $A3$ there are 160 units. These products should be shipped by truck to four distributing warehouses B_1, B_2, B_3, B_4, and B_5, in such a way as to minimize the total distribution cost. Each destination has a fixed demand for units, correspondingly, are 60, 70, 120, 130, and 100 units. Moreover, this entire demand must be received from the sources. The shipping costs from A_i to B_j you might find in the table below. Using the Northwest Corner Rule and the minimal element method determine any plan of shipments.

Source	Destination					Supply
	B_1	B_2	B_3	B_4	B_5	
A_1	2	3	4	2	4	140
A_2	8	4	1	4	1	180
A_3	9	7	3	7	2	160
Demand	60	70	120	130	100	480

Solution

the Northwest Corner Rule

Source	Destination					Supply
	B_1	B_2	B_3	B_4	B_5	
A_1	2 (60)	3 (70)	4 (10)	2	4	140
A_2	8	4	1 (110)	4 (70)	1	180
A_3	9	7	3	7 (60)	2 (100)	160
Demand	60	70	120	130	100	480

Therefore, the total distribution cost

$$S = 2 \cdot 60 + 3 \cdot 70 + 4 \cdot 10 + 1 \cdot 110 + 4 \cdot 70 + 7 \cdot 60 + 2 \cdot 100 = 1380.$$

the minimal element method

Source	Destination					Supply
	B_1	B_2	B_3	B_4	B_5	
A_1	2 (60)	3	4	2 (80)	4	140
A_2	8	4	1 (120)	4	1 (60)	180
A_3	9	7 (70)	3	7 (50)	2 (40)	160
Demand	60	70	120	130	100	480

$$S = 2 \cdot 60 + 7 \cdot 70 + 1 \cdot 120 + 2 \cdot 80 + 7 \cdot 50 + 1 \cdot 60 + 2 \cdot 40 = 1380.$$

Problem 2.

In three warehouses A_1, A_2 and A_3 there are one type products. In A_1 there are 160 units, in $A2$ there are 140 units, and in $A3$ there are 170 units. These products should be shipped by truck to four distributing warehouses B_1, B_2, B_3, and B_4, in such a way as to minimize the total distribution cost. Each destination has a fixed demand for units, correspondingly, are 120, 50, 190, and 110 units. Moreover, this entire demand must be received from the sources. The shipping costs from A_i to B_j you might find in the table below. Using the Northwest Corner Rule and the minimal element method determine any plan of shipments.

Source	Destination				Supply
	B_1	B_2	B_3	B_4	
A_1	7	8	1	2	160
A_2	4	5	9	8	140
A_3	9	2	3	6	170
Demand	120	50	190	110	470

Solution

the Northwest Corner Rule

Source	Destination				Supply
	B_1	B_2	B_3	B_4	
A_1	7 (120)	8 (40)	1	2	160
A_2	4	5 (10)	9 (130)	8	140
A_3	9	2	3 (60)	6 (110)	170
Demand	120	50	190	110	470

$$S = 7 \cdot 120 + 8 \cdot 40 + 5 \cdot 10 + 9 \cdot 130 + 3 \cdot 60 + 6 \cdot 100 = 3230.$$

The minimal element method

Source	Destination				Supply
	B_1	B_2	B_3	B_4	
A_1	7	8	1 (160)	2	160
A_2	4 (120)	5	9	8 (20)	140
A_3	9	2 (50)	3 (30)	6 (90)	170
Demand	120	50	190	110	480

$$S = 1 \cdot 160 + 4 \cdot 120 + 8 \cdot 20 + 2 \cdot 50 + 3 \cdot 30 + 6 \cdot 90 = 1530.$$

Problem 3. Using Vogel's approximation method determine any plan of shipments.

Source	Destination				Supply
	B_1	B_2	B_3	B_4	
A_1	7	8	1	2	160
A_2	4	5	9	8	140
A_3	9	2	3	6	170
Demand	120	50	190	110	470

Solution

Source	Destination				Supply	By rows					
	B_1	B_2	B_3	B_4							
A_1	7	8	1 (50)	2 (110)	160	1	$\boxed{6}$	-	-	-	-
A_2	4 (120)	5 (20)	9	8	140	1	1	1	1	1	0
A_3	9	2 30	3 (140)	6	170	1	1	1	$\boxed{7}$	-	-
Demand	120	50	190	110	480						
By columns	3	3	2	$\boxed{4}$							
	3	3	2	-							
	5	3	$\boxed{6}$	-							
	5	3	-	-							
	-	0	-	-							
	-	0	-	-							

$$S = 1 \cdot 50 + 2 \cdot 110 + 4 \cdot 120 + 5 \cdot 20 + 2 \cdot 30 + 3 \cdot 140 = 1330.$$

Problem 4.
Show that solution of problem 3 is optimal.

Solution

We need to find solution of the following system of linear equations.

$$\begin{cases} \beta_3 - \alpha_1 = 1 \\ \beta_4 - \alpha_1 = 2 \\ \beta_1 - \alpha_2 = 4 \\ \beta_2 - \alpha_2 = 5 \\ \beta_2 - \alpha_3 = 2 \\ \beta_3 - \alpha_3 = 3 \end{cases}$$

The solution is

$$\alpha_1 = 0, \alpha_2 = -5, \alpha_3 = -2, \beta_1 = -1, \beta_2 = 0, \beta_3 = 1, \beta_4 = 2$$

Consequently, since

$$\alpha_{ij} = \beta_j - \alpha_i - c_{ij}$$

then

$$\alpha_{11} = \beta_1 - \alpha_1 - c_{11} = -1 - 0 - 7 = -8 < 0;$$

$$\alpha_{12} = \beta_2 - \alpha_1 - c_{12} = 0 - 0 - 8 = -8 < 0;$$

$$\alpha_{23} = \beta_3 - \alpha_2 - c_{24} = 1 - (-5) - 9 = -3 < 0;$$

$$\alpha_{24} = \beta_4 - \alpha_2 - c_{24} = 1 - 0 - 8 = -7 < 0;$$

$$\alpha_{31} = \beta_1 - \alpha_2 - c_{31} = -1 - (-5) - 9 = -3 < 0;$$

$$\alpha_{34} = \beta_4 - \alpha_3 - c_{34} = 1 - (-2) - 6 = -3 < 0;$$

Therefore, this plan is optimal.

Problem 5. Find the optimal solution for the problem in the following table

Source	Destination				Supply
	B_1	B_2	B_3	B_4	
A_1	1 (30)	2 (20)	4	1	50
A_2	2	3 (10)	1 (10)	5 (10)	30
A_3	3	2	4	4 (10)	10
Demand	30	30	10	20	0

Solution

Source	Destination				Supply
	B_1	B_2	B_3	B_4	
A_1	1 (30)	2 (20)	4	1	50
A_2	2	3 (10)	1 (10)	5 (10)	30
A_3	3	2	4	4 (10)	10
Demand	30	30	10	20	0

We need to find solution of the following system of linear equations.

$$\begin{cases} \beta_1 - \alpha_1 = 1 \\ \beta_2 - \alpha_1 = 2 \\ \beta_2 - \alpha_2 = 3 \\ \beta_3 - \alpha_2 = 1 \\ \beta_4 - \alpha_2 = 5 \\ \beta_4 - \alpha_3 = 4 \end{cases}$$

The solution is

$$\alpha_1 = 0, \beta_1 = 1, \beta_2 = 2, \alpha_2 = -1, \beta_3 = 0, \beta_4 = 4, \alpha_3 = 0$$

Consequently, since

$$\alpha_{ij} = \beta_j - \alpha_i - c_{ij}$$

then

$$\alpha_{13} = -4 < 0;$$

$$\alpha_{14} = 3 > 0;$$

$$\alpha_{21} = \alpha_{32} = 0;$$

$$\alpha_{31} = -2 < 0;$$

$$\alpha_{33} = -4 < 0;$$

Therefore, this plan is not optimal.

We have

Source	Destination				Supply
	B_1	B_2	B_3	B_4	
A_1	1 (30)	2 (20) -	4 $\boxed{-4}$	1 $\boxed{3}$ +	50
A_2	2 $\boxed{0}$	3 (10)+	1 (10)	5 (10) -	30
A_3	3 $\boxed{-2}$	2 $\boxed{0}$	4 $\boxed{-4}$	4 (10)	10
Demand	30	30	10	20	0

Consequently, we have

Source	Destination				Supply
	B_1	B_2	B_3	B_4	
A_1	1 (30)	2 (10)	4	1 (10)	50
A_2	2	3 (20)	1 (10)	5	30
A_3	3	2	4	4 (10)	10
Demand	30	30	10	20	0

We need to find solution of the following system of linear equations.

$$\begin{cases} \beta_1 - \alpha_1 = 1 \\ \beta_2 - \alpha_1 = 2 \\ \beta_4 - \alpha_1 = 1 \\ \beta_2 - \alpha_2 = 3 \\ \beta_3 - \alpha_2 = 1 \\ \beta_4 - \alpha_3 = 4 \end{cases}$$

The solution is

$$\alpha_1 = 0, \beta_1 = \beta_4 = 1, \beta_2 = 2, \beta_3 = 0, \alpha_3 = -3, \alpha_2 = -1, \alpha_3 = -3$$

Consequently, since

$$\alpha_{ij} = \beta_j - \alpha_i - c_{ij}$$

then

$$\alpha_{13} = -2 < 0;$$

$$\alpha_{21} = 0;$$

$$\alpha_{24} = -3;$$

$$\alpha_{32} = 3 > 0;$$

$$\alpha_{33} = -1 < 0;$$

Therefore, this plan is not optimal.

Source	Destination				Supply
	B_1	B_2	B_3	B_4	
A_1	1 (30)	2 (10) -	4 $\boxed{-2}$	1 (10)+	50
A_2	2 $\boxed{0}$	3 (20)	1 (10)	5 $\boxed{-3}$	30
A_3	3,$\boxed{1}$	2, $\boxed{3}$+	4 $\boxed{-1}$	4 (10) -	10
Demand	30	30	10	20	0

Hence,

Source	Destination				Supply
	B_1	B_2	B_3	B_4	
A_1	1 (30)	2 (0)	4 $\boxed{-4}$	1 (20)	50
A_2	2 $\boxed{0}$	3 (20)	1 (10)	5 $\boxed{-3}$	30
A_3	3 $\boxed{-2}$	2(10)	4 $\boxed{-4}$	4 $\boxed{-3}$	10
Demand	30	30	10	20	0

Therefore,

$$S = 1 \cdot 30 + 2 \cdot 0 + 1 \cdot 20 + 3 \cdot 20 + 1 \cdot 10 + 2 \cdot 10 = 140$$