CS6730: Probabilistic Graphical Models
Homework-2 answers

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Answers

Q-1

X_1 \ X_2X_3	**00**	**01**	**10**	**11**
0	0	0	0	0
1	0	0	0	0

Z = 0

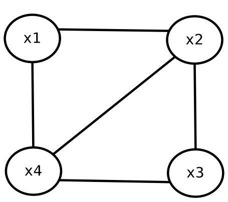
X_1 \ X_2X_3	**00**	**01**	**02**	**10**	**11**	**12**	**20**	**21**	**22**
0	0	0	0	0	0	1	0	1	0
1	0	0	1	0	0	1	0	0	0
2	0	1	0	1	0	0	0	0	0

Z = 6

Q-2

$P(X1 = x_1, X2 = x_2, X3 = x_3, X4 = x_4) = \frac{1}{Z}\psi_1(x_1, x_2).\psi_2(x_2, x_3).\psi_3(x_3, x_4).\psi_4(x_1, x_4).\psi_5(x_2, x_4)$

Markov network $\mathcal{H}$



$Q(X1 = x_1, X2 = x_2, X3 = x_3, X4 = x_4) = \frac{1}{Z}\psi_1(x_1, x_2, x_4).\psi_2(x_2, x_3, x_4)$

The number of parameters required to represent distribution $Q = 2a^3$ (max clique = 3)

The number of parameters required to represent distribution $P = 5a^2$ (max clique = 2)

For large values of a, $2a^3 > 5a^2$

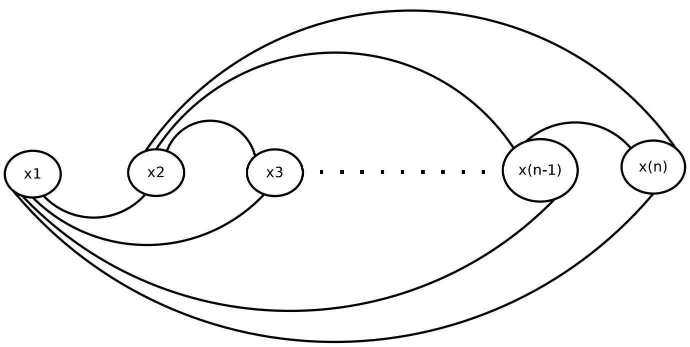
Q-3

two distinct factor settings $P(X_i = 1|Y = j) = 0.9$ if $i = j$ and 0.1 otherwise

1. $\psi(X_i = 1, Y = j) = 9 \quad \forall i = \{1, 2, \dots n\} \quad \& \quad (i = j)$
   $\psi(X_i = 1, Y = j) = 1 \quad \forall i = \{1, 2, \dots n\} \quad \& \quad (i \neq j)$
   $\psi(X_i = 0, Y = j) = 9 \quad \forall i = \{1, 2, \dots n\} \quad \& \quad (i \neq j)$
   $\psi(X_i = 0, Y = j) = 1 \quad \forall i = \{1, 2, \dots n\} \quad \& \quad (i = j)$

2. $\psi(X_i = 1, Y = j) = 90 \quad \forall i = \{1, 2, \dots n\} \quad \& \quad (i = j)$
   $\psi(X_i = 1, Y = j) = 10 \quad \forall i = \{1, 2, \dots n\} \quad \& \quad (i \neq j)$
   $\psi(X_i = 0, Y = j) = 90 \quad \forall i = \{1, 2, \dots n\} \quad \& \quad (i \neq j)$
   $\psi(X_i = 0, Y = j) = 10 \quad \forall i = \{1, 2, \dots n\} \quad \& \quad (i = j)$

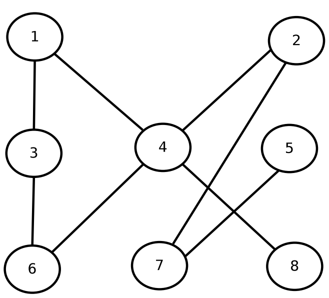
Markov network for just the features $x_1, x_2 \dots x_n$



factors $= \psi(x_1, x_2, x_3, \dots x_n)$

Q-4

Markov network $\mathcal{H}'$



$\mathcal{I}(\mathcal{H}')\backslash\mathcal{I}(\mathcal{G})$

- $x_2 \perp\!\!\!\perp x_5 | x_7 \in \mathcal{I}(\mathcal{H}'), \notin \mathcal{I}(\mathcal{G})$
- $x_3 \perp\!\!\!\perp x_4 | x_6 \in \mathcal{I}(\mathcal{H}'), \notin \mathcal{I}(\mathcal{G})$
- $x_1 \perp\!\!\!\perp x_2 | x_4 \in \mathcal{I}(\mathcal{H}'), \notin \mathcal{I}(\mathcal{G})$

Moralized Markov network $\mathcal{H}$ equivalent to $\mathcal{G}$

Please refer figure 1.

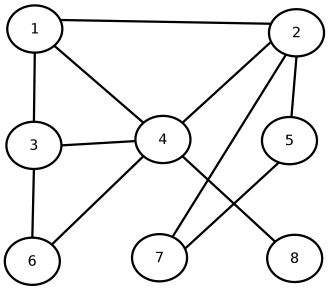


Figure 1: Moralized markov network

Largest conditional independence in Moralized Markov network $\mathcal{H}$

$x_3, x_6 \perp\!\!\!\perp x_2, x_5, x_7, x_8 | x_1, x_4$

$\mathcal{I}(\mathcal{G}) \backslash \mathcal{I}(\mathcal{H}')$

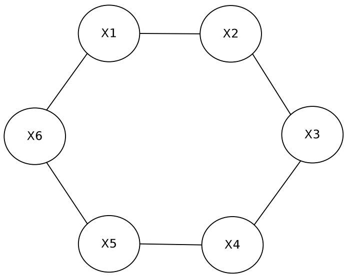
- $x_1 \perp\!\!\!\perp x_2 \in \mathcal{I}(\mathcal{G}), \notin \mathcal{I}(\mathcal{H}')$
- $x_3 \perp\!\!\!\perp x_4 | x_1 \in \mathcal{I}(\mathcal{G}), \notin \mathcal{I}(\mathcal{H}')$
- $x_5 \perp\!\!\!\perp x_2 \in \mathcal{I}(\mathcal{G}), \notin \mathcal{I}(\mathcal{H}')$
- $x_2 \perp\!\!\!\perp x_3 | x_1 \in \mathcal{I}(\mathcal{G}), \notin \mathcal{I}(\mathcal{H}')$
- $x_6 \perp\!\!\!\perp x_7 | x_4 \in \mathcal{I}(\mathcal{G}), \notin \mathcal{I}(\mathcal{H}')$

Q-5

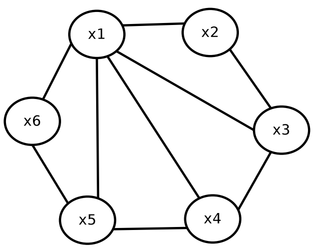
Q-6

$P(X_1, X_2, X_3, X_4, X_5, X_6) = \frac{1}{Z}\psi_1(X_1, X_2).\psi_2(X_2, X_3).\psi_3(X_3, X_4).\psi_4(X_4, X_5).\psi_5(X_5, X_6).\psi_6(X_6, X_1)$

Markov network $\mathcal{H}$



Triangulated graph $\mathcal{H}'$



Distribution factors according to $\mathcal{H}' = \frac{1}{Z}\psi_1^{\mathcal{H}'}(X_1, X_2, X_3).\psi_2^{\mathcal{H}'}(X_1, X_3, X_4).\psi_3^{\mathcal{H}'}(X_1, X_4, X_5).\psi_4^{\mathcal{H}'}(X_1, X_5, X_6)$

Distribution P using the form of factors in the triangulated graph $\mathcal{H}'$

$$P(X_1, X_2, X_3, X_4, X_5, X_6) = \frac{1}{Z}(\sum_{X_2}\psi_1^{\mathcal{H}'}(X_1, X_2, X_3)).(\sum_{X_1}\psi_1^{\mathcal{H}'}(X_1, X_2, X_3)).(\sum_{X_1}\psi_2^{\mathcal{H}'}(X_1, X_3, X_4)).$$
$$(\sum_{X_1}\psi_3^{\mathcal{H}'}(X_1, X_4, X_5)).(\sum_{X_1}\psi_4^{\mathcal{H}'}(X_1, X_5, X_6)).(\sum_{X_5}\psi_4^{\mathcal{H}'}(X_1, X_5, X_6))$$

$\mathcal{I}(\mathcal{H})\backslash\mathcal{I}(\mathcal{H}')$

- $x_1 \perp\!\!\!\perp x_3, x_4, x_5 | x_2, x_6 \in \mathcal{I}(\mathcal{H}), \notin \mathcal{I}(\mathcal{H}')$
- $x_3 \perp\!\!\!\perp x_5, x_6, x_1 | x_2, x_4 \in \mathcal{I}(\mathcal{H}), \notin \mathcal{I}(\mathcal{H}')$
- $x_4 \perp\!\!\!\perp x_6, x_1, x_2 | x_3, x_5 \in \mathcal{I}(\mathcal{H}), \notin \mathcal{I}(\mathcal{H}')$
- $x_5 \perp\!\!\!\perp x_1, x_2, x_3 | x_6, x_4 \in \mathcal{I}(\mathcal{H}), \notin \mathcal{I}(\mathcal{H}')$
- $x_4 \perp\!\!\!\perp x_1 | x_5, x_2 \in \mathcal{I}(\mathcal{H}), \notin \mathcal{I}(\mathcal{H}')$
- $x_4 \perp\!\!\!\perp x_1 | x_6, x_3 \in \mathcal{I}(\mathcal{H}), \notin \mathcal{I}(\mathcal{H}')$

Q-7

Triangulated graph

A triangulated graph is a graph in which for every cycle of length L ¿ 3, there is an edge joining two nonconsecutive vertices. This property is satisfied for the graph in Figure 2. Hence the given graph is triangulated.

An MC-ordering : 6, 5, 4, 3, 2, 1

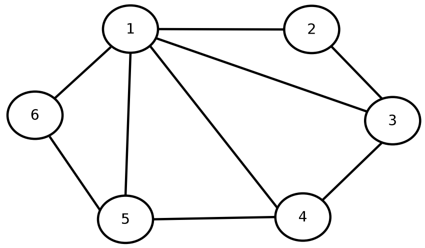


Figure 2: Markov graph $\mathcal{H}$

Variable eliminated	**Variables involved**	**Resulting function**	**Computation time**
6	1,5,6	$\tau_1(1,5)$	a^3
5	1,4,5	$\tau_1(1,4)$	a^3
4	1,3,4	$\tau_1(1,3)$	a^3
3	1,2,3	$\tau_1(1,2)$	a^3
2	1,2	$\tau_1(1)$	a^2
1	1	-	a

Table 1: Elimination order : 6,5,4,3,2,1

Computation steps required to compute the partition function

Consider the elimination ordering : 6, 5, 4, 3, 2, 1. Refer table 1. Computation steps: $O(a^3)$

Computation steps required to compute the partition function in order 1,2,3,4,5,6

Refer table 2.

Variable eliminated	**Variables involved**	**Resulting function**	**Computation time**
1	1,2,3,4,5,6	$\tau_1(2,3,4,5,6)$	a^6
2	2,3,4,5,6	$\tau_1(3,4,5,6)$	a^5
3	3,4,5,6	$\tau_1(4,5,6)$	a^4
4	4,5,6	$\tau_1(5,6)$	a^3
5	5,6	$\tau_1(6)$	a^2
6	6	-	a

Table 2: Elimination order : 1,2,3,4,5,6

Computation steps: $O(a^6)$