Homework 2 Solutions

2.1 In this homework, we will consider "ordinary" decks of playing cards which have 52 cards, with 13 of each of the four suits (Hearts, Spades, Diamonds and Clubs), with each suit having the 13 ranks (Ace, 2, 3, ..., 9, 10, Jack, Queen, King).

(a) How many 5-card hands are there where all the cards have the same suit?

(b) How many 5-card hands are there where all the cards belong to at most two suits?

(c) How many 5-card hands are there where all the cards have at most 2 different ranks?

(a) You can choose 1 of the 4 suits, and then you choose 5 of the 13 cards in that suit.
$\Rightarrow \binom{4}{1}\binom{13}{5}$.

(b) There are two cases for this question: all the cards could belong to one suit, or all the cards could belong to two suits. The number of ways that all the cards could belong to one suit is given in part (a). Otherwise, we could have a hand with 2 cards in one suit, and 3 cards in the other suit. Or, 1 card in one suit, and 4 cards in the other suit. With each of the suits, there are 4 ways to choose the first suit, and 3 ways to choose the second suit.
$\Rightarrow 4 * 3[\binom{13}{2}\binom{13}{3} + \binom{13}{4}\binom{13}{1}] + \binom{4}{1}\binom{13}{5}$.

(c) Since you cannot have a hand made of 1 rank, we only need to consider the case where there are exactly 2 different ranks. There two templates: $XXXXY$ and $XXXYY$. There are 13 ways to choose X, and then 12 ways to choose Y. Once these are chosen, we have to choose the suits.
$\Rightarrow 13 * 12[\binom{4}{3}\binom{4}{2} + \binom{4}{1}\binom{4}{4}]$

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2.2 How many 8 digit numbers formed from the digits $\{0, 1, ..., 9\}$ have some digit occurring **at least** 3 times.

Count the complement. So there are a total of 10^8 ways to form a string of length 8. Therefore, we have to subtract the ways to form a string where all the digits are distinct, and the ways to form a digit where each digit appears at most twice.

- The number of ways to form a digit where all the digits are distinct is: 10*9*8*7*6*5*4*3.
- Let D denote a distinct digit, and let $S_1, S_2, ...$ be a digit that is the same.
  Then the cases are:

$$S_1S_1DDDDDD = \binom{10}{1} * \frac{8!}{2! * 6!} * 9 * 8 * 7 * 6 * 5 * 4$$
$$S_1S_1S_2S_2DDDD = \binom{10}{2} * \frac{8!}{2! * 2! * 4!} * 8 * 7 * 6 * 5$$
$$S_1S_1S_2S_2S_3S_3DD = \binom{10}{3} * \frac{8!}{2! * 2! * 2! * 2!} * 7 * 6$$
$$S_1S_1S_2S_2S_3S_3S_4S_4 = \binom{10}{4} * \frac{8!}{2! * 2! * 2! * 2!}$$

For example, for the $S_1S_1S_2S_2DDDD$ case, we first choose the two ranks which will be duplicated ($\binom{10}{2}$ ways), then choose the number of ways of placing them ($\frac{8!}{2!*2!*4!}$), and finally choose the (distinct) ranks of the digits that go into the remaining 4 places ($8 * 7 * 6 * 5$ choices). The other values are computed in a similar way.
Thus, the number we want is 10^8 *minus* the sum of all the numbers above (which may be $36, 269, 200$ if I calculated correctly!) Thus, the probability that a random bill has some digit occurring at least 3 times is about 0.363 which is greater than $\frac{1}{3}$ of the time. ∎

2.3

(a) How many different 5-letter strings can be formed from *distinct* letters from the word **ABRACADABRA**.

(b) How many different 5-letter strings can be formed using the letters from the word **ABRACADABRA** if duplicated letters are allowed but no letter can be used more times than it occurs in the word?

(a) 5! The set of distinct letters: (A, B, R, C, D)

(b) 1271

We consider the possible letter patterns that are possible. For example, YYVWX denotes the possibility that one letter (denoted by Y) occurs twice, and the other 3 letters each occur once. Here are the possible patterns, and their counts. We have the letter counts in **ABRACADABRA** as:

A = 5, B = 2, R = 2, C = 1, D = 1

Pattern	*Count*
$VWXYZ = 5!$	$= 120$
$YYVWX = 3 * 4 * (5!/2!)$	$= 720$
$YYXXV = 3 * 3 * (5!/(2! * 2!)$	$= 270$
$YYYVX = 1 * 6 * (5!/3!)$	$= 120$
$YYYXX = 1 * 2 * (5!/(3! * 2!)$	$= 20$
$YYYYV = 1 * 4 * (5!/4!)$	$= 20$
$YYYYY = 1 * (5!/5!)$	$= 1$

One way to think about this problem is one of putting balls in bins, where the "balls" are the positions and the "bins" are the letters. Let's take the first pattern VWXYZ, for example. We then have 5 bins (since we use all 5 of the different letters) and so the number of ways of distributing the 5 positions $\{1, 2, 3, 4, 5\}$ is 5!.
Now take the second pattern YYVWX. Here, we have only 4 bins, labeled Y, V, W and X. We are going to put 2 balls (i.e., positions) into

the Y bin, and one each into the other 3 bins. Now we have to count how many ways we can label the bins with letters. Since only 3 of the letters (A, B, R) occur twice, then there are only 3 choices for Y. Once that choice is made, then there are $\binom{4}{3} = 4$ choices for the other labels. (Keep in mind that the *order* of the labels doesn't matter). Thus, the total for this pattern is $3 * 4 * \frac{5!}{2!*1!*1!*1!} = 720$.
The next pattern YYXXV is computed the same way. Now there are 3 bins labeled Y, X and V (and as usual, the *order* of the bins doesn't matter, just the labels on them). So, there are $\binom{3}{1} = 3$ ways to choose the labels for Y and X, and then 3 ways to choose the remaining label for V. Since there are then $\frac{5!}{2!*2!*1!}$ ways to put the balls in these 3 bins, then the total in this case is $3 * 3 * 30 = 270$.
Finally, let's do one more pattern, say YYYXX. Since only the letter A occurs more than 2 times, then the label for bin Y must be A. There are then 2 choices for X, namely B or R. The balls (positions) can be placed into the bins in $\binom{5}{3}$ ways, so the total in this case is $2 * 2 * 10 = 40$.
Carrying similar calculations for the otter patterns and adding up the numbers, you get a grand total of 1271.

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2.4

(a) How many 5-card hands (from an ordinary deck) have at least one card of each suit?

(b) How many 6-card hands have at least one card of each suit?

(a) $\binom{4}{1}\binom{13}{2}\binom{13}{1}\binom{13}{1}\binom{13}{1}$;
(First choose the suit that will have 2 cards ($\binom{4}{1} = 4$ ways), then choose the 2 cards from that suit ($\binom{13}{2}$ ways), and then choose one card from each of the the remaining 3 suits.)

(b) There are 2 possibilities for the suits: 3 of one suit and 1 of each of the other three, or 2 suits that each have 2, and the remaining 2 suits that just have one.
For the first case, the number of choices is $\binom{4}{1}\binom{13}{3}\binom{13}{1}\binom{13}{1}\binom{13}{1}$.

For the second case, the number of choices is $\binom{4}{2}\binom{13}{2}\binom{13}{2}\binom{13}{1}\binom{13}{1}$.
The answer is the total of these the numbers.

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2.5

(a) In how many different ways can the letters in **MISSISSIPPIPOKER** be arranged (using all the letters)?

(b) In how many different ways can all the letters in the word **supercalifragilisticexpialidocious** be arranged?

(a) 16! / (4! * 4! * 3!). Count of letters. M = 1, I = 4, S= 4, P = 3, O = 1, K = 1, E = 1, R = 1

(b) 34! / (3! * 2! * 2! * 2! * 2! * 3! * 3! * 3! * 7! * 2!). Counts of all the letters. s = 3, u = 2, p = 2, e = 2, r = 2, c = 3, a = 3, l = 3, i = 7, f = 1, g = 1, t = 1, x = 1, d = 1, o = 2

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2.6 Show that the number of subsets of an n-set S having an *odd* number of elements is exactly the same as the number of subsets of S having an even number of elements (where we assume $n > 0$).

The easiest way to do this is to apply the binomial theorem:
$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$. Plugging in $x = -1, y = 1$, this becomes:
$(-1+1)^n = 0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$, i.e., $\sum_j \binom{n}{2j} = \sum_j \binom{n}{2j+1}$ and we are done since the left hand sum counts the number of even-sized subsets of S while the right hand sum counts the number of odd-sized subsets of S.

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