On the Game Push

Max Fan

January 21, 2021

For the following lemmas, I'm considering all games in which left has a winning strategy (values > 0). By swapping $\underline{\text{L}}$ and $\underline{\text{R}}$, you can get the same results for $\underline{\text{R}}$ and thus for all valid positions of the game.

Notation:

- Each literal symbol will be underlined ($\underline{\text{L}}$ or $\underline{\text{R}}$)
- $\mathbb{X}_n$ represents, without loss of generality any sequence of characters or spaces of length n. $\mathbb{X}'_n$ represents the sequence $\mathbb{X}$ with the rightmost space removed, if one exists. If a rightmost space does not exist, then $\mathbb{X}_n = \mathbb{X}'_n$
- Upper case symbols (like G, A, and B) represent sequences/game states.
- Greek symbols will be used to denote a single character, that can either be $\underline{L}$, $\underline{R}$, or $\_$.
- The entire game state will be completely underlined (e.g. $\underline{\underline{\underline{LLR}}}$)
- "+" means concatenation of a sequence

The evaluation of a sequence will be denoted like so, where G is the current game state and the number preceding the colon is the square being pushed:

$$G := \underline{\underline{\underline{LLR}}}$$

$$G_0 : \underline{\underline{\underline{LLR}}} \rightarrow \underline{\underline{\underline{\_LR}}}$$

$$G_1 : \underline{\underline{\underline{LLR}}} \rightarrow \underline{\underline{\underline{LR}}}$$

$$G_2 : \underline{\underline{\underline{LLR}}} \rightarrow \underline{\underline{\underline{LR}}}$$

The game is zero-indexed and always ends in $\underline{R}$ or $\underline{L}$.
Different game positions can be compared like so:

$$G > G_1 \geq G_2$$

Lemmas

Left Accessibility Axiom (done)

If one game position Q is accessible from another game position P, ending in $\underline{L}$, by a series of moves, then $P > Q$.
Rationale: Whenever a move is played, by $\underline{L}$ or $\underline{R}$, the value of the game is always closer to zero.

Right Accessibility Axiom (done)

If one game position Q is accessible from another game position P, ending in $\underline{R}$, by a series of moves, then $P < Q$.
Rationale: Whenever a move is played, by $\underline{L}$ or $\underline{R}$, the value of the game is always closer to zero.

Negation Axiom (done)

Swapping all values of $\underline{L}$ and $\underline{R}$ in the game state is logically equivalent to negating the value of the game.

Left Reduction Lemma (done)

If $\mathbb{X}_n$ has spaces and ends in $\underline{L}$, then:

$$\mathbb{X}_n > \mathbb{X}'_n$$

If $\mathbb{X}_n$ has no spaces and ends in $\underline{L}$, then:

$$\mathbb{X}_n = \mathbb{X}'_n$$

Right Reduction Lemma (done)

If $\mathbb{X}_n$ has spaces and ends in $\underline{R}$, then:

$$\mathbb{X}_n < \mathbb{X}'_n$$

If $\mathbb{X}_n$ has no spaces and ends in $\underline{R}$, then:

$$\mathbb{X}_n = \mathbb{X}'_n$$

Same-Length Concat Lemma

If $P > Q$ and P and Q are of the same length and A ends in $\underline{L}$, then $P + A > Q + A$. Proof.

Starting from the rightmost square, if $P > Q$ then $P \neq Q$ and P and Q must differ at a location n. Let $P = P_1 + \underline{\alpha} + A$ and $Q = Q_1 + \underline{\beta} + A$ where β is the symbol at the location n where P and Q differ. By construction, $\underline{\alpha} \neq \underline{\beta}$. Suppose for contradiction that $\underline{\alpha} < \underline{\beta}$. Then,
Suppose P is positive. Q is either positive, zero, or negative. If zero, done .

General Prepend Lemma (done)

If A ends in $\underline{L}$, then for any characters $\underline{\alpha}$ and $\underline{\beta}$ such that $\underline{\alpha} > \underline{\beta}$, then $P := \underline{\underline{\alpha} + A} > Q := \underline{\underline{\beta} + A}$. Proof.
There are six cases to examine since $\underline{\alpha}$ and $\underline{\beta}$ can either be $\underline{L}$, $\underline{R}$, $\underline{\ }$:

- $\underline{\alpha} = \underline{L}$, $\underline{\beta} = \underline{\underline{\ }}$ In this case, $P_0 : \underline{\underline{\alpha} + A} \to \underline{\underline{\beta} + A}$, so $P > Q$ by the left accessibility axiom.

- $\underline{\alpha} = \underline{\underline{\ }}$, $\underline{\beta} = \underline{R}$ In this case, we want to show that $P > Q$. It is equivalent to show $-Q > -P$. By the negation axiom, $-Q$ and -P both end in $\underline{R}$. $-Q$ and $-P$ only differ at the zeroth place. $$

Previous proof of this point: In this case, Q has more $\underline{R}$ values than P and the same number and location of $\underline{L}$ values. Since right has more moves available in Q than in P, $P > Q$ (TODO: better justification).

- $\underline{\alpha} = \underline{L}$, $\underline{\beta} = \underline{R}$ From the first case and the second case and the transitive property, $P > Q$ follows.

In all three cases, $P > Q$, so necessarily, $P > Q$.

Append Lemma (unproven)

For any sequence of characters (empty or not), $\mathbb{X}_n$:

$$\underline{\mathbb{X}_n \underline{L}} > \underline{\mathbb{X}_n}$$

Prepend Lemma (WIP)

For any sequence of characters (empty or not), $\mathbb{X}_n$:

$$P := \underline{\_\mathbb{X}_n\underline{L}} > Q := \underline{\mathbb{X}_n\underline{L}}$$

Proof.

Suppose that $\mathbb{X}_n$ does not contain spaces. Then,

$$P_n : \underline{\_\mathbb{X}_n\underline{L}} \to \underline{\mathbb{X}_n\underline{L}}$$

So, $P_n = Q$ and Q is accessible from P. Therefore, by the left accessibility axiom, $P > Q$.

Suppose that $\mathbb{X}_n$ does contain spaces. Then, $\underline{\mathbb{X}_n\underline{L}} \to \underline{\mathbb{X}'_n\underline{L}}$. Without loss of generality, split $\mathbb{X}_n$ by its rightmost space. Define $\mathbb{X}_n = A + \_ + B$ where $\mathbb{X}_b$ contains no spaces. Then, $\mathbb{X}'_n = A + B$

$\mathbb{X}_n$ either contains or does not contain spaces, so by disjunction elimination, the lemma holds.
Scratch (problematic because it depends on the concat lemma): Suppose that $\mathbb{X}_n$ does contain spaces. Then, $\underline{\mathbb{X}_n\underline{L}} \to \underline{\mathbb{X}'_n\underline{L}}$. By the left reduction lemma, since the sequence $\mathbb{X}_n$ has spaces, $\mathbb{X}_n > \mathbb{X}'_n$.

The Fundamental Theorem of Push

Statement: For any given game, pushing the leftmost piece is always the optimal move.

A valid game ends in either $\underline{L}$ or $\underline{R}$. We first induct over all games that end in $\underline{L}$.

Base case: Pushing the first $\underline{L}$ is the optimal move in the game: $\underline{\underline{L}\underline{L}}$.
Inductive hypothesis: Given any sequence of character(s) $\mathbb{X}$, the most

Acknowledgements

Sophie Vulpe provided excellent feedback and encouragement.