# CHAPTER 1 Real Numbers

## 1.1 Introduction

"God made the integers. All else is the work of man" – Leopold Kronecker.

Life is full of numbers. Imagine the moment you were born. Your parents probably noted the time you were born, your weight, your length and the most important, counted your fingers and toes. From then, numbers accompany you throughout life.

What are the other contexts where you deal with numbers?

We use the numbers to tell our age to keep track of our income and to find the savings after spending certain money. We measure our wealth also.

In this chapter we are going to explore the notion of the numbers. Numbers play a fundamental role within the realm of mathematics. We will come to see the richness of numbers and delve into their surprising traits. Some collection of numbers fit so well together that they actually lead to notions of aesthetics and beauty.

Let us look in to a puzzle.

In a garden of flowers, a swarm of bees is setting in equal number on flowers. When they settle on two flowers, one bee will be left out. When they settle on three flowers, two bees will be left out. When they settle on four flowers, three bees will be left out. Similarly, when they settle on five flowers no bee will be left out. If there are at most fifty bees, how many bees are there in the swarm?

Let us analyse and solve this puzzle.

Let the number of bees be '$x$'. Then working backwards we see that $x \leq 50$.

If the swarm of bees is divided into 5 equal groups no bee will be left, which translates to $x = 5a + 0$ for some natural number '$a$'.

If the swarm is divided in to 4 equal groups 3 bees will be left out and it translates to $x = 4b + 3$ for some natural number $b$.

If the swarm is divided into 3 equal groups 2 bees will be left out and it translates to $x = 3c + 2$ for some natural number $c$.

If the swarm is divided into 2 equal groups 1 bee will be left out and it translates to $x = 2d + 1$ for some natural number $d$.

That is, in each case we have a positive integer $y$ (in this example $y$ takes values 5, 4, 3 and 2 respectively) which divides $x$ and leaves remainder '$r$' (in our case r is 0, 3, 2 and 1 respectively), that is **smaller than** $y$. In the process of writing above equations, unknowingly, we have used Division Algorithm.

Getting back to our puzzle. To solve it we must look for the multiples of 5, which satisfy all the conditions, because $x = 5a + 0$.

If a number leaves remainder 1 when it is divided by 2 we must consider odd multiples only. In this case we have 5, 15, 25, 35 and 45. Similarly if we check for the remaining two conditions you will get 35.

Therefore, the swarm of bees contains 35 bees.

Let us verify the answer.

When 35 is divided by 2, the remainder is 1. That can be written as

$$35 = 2 \times 17 + 1$$

When 35 is divided by 3, the remainder is 2. That can be written as

$$35 = 3 \times 11 + 2$$

When 35 is divided by 4, the remainder is 3. That can be written as

$$35 = 4 \times 8 + 3$$

and when 35 is divided by 5, the remainder is '0'. That can be written as

$$35 = 5 \times 7 + 0$$

Let us generalise this. For each pair of positive integers $a$ and $b$ (dividend and divisor respectively), we can find the whole numbers $q$ and $r$ (quotient and remainder respectively) satisfying the relation

$$a = bq + r, \quad 0 \leq r < b$$

## Do This

Find $q$ and $r$ for the following pairs of positive integers $a$ and $b$, satisfying $a = bq + r$.

(i) $a = 13, \quad b = 3$  (ii) $a = 80, \; b = 8$  (iii) $a = 125, \; b = 5$

(iv) $a = 132, \; b = 11$

## Think and Discuss

In questions of above "DO THIS", what is the nature of $q$ and $r$?

**Theorem-1.1 : (Division Algorithm) :** Given positive integers $a$ and $b$, there exist unique pair of integers $q$ and $r$ satisfying $a = bq + r, \; 0 \leq r < b$.

This result, was first recorded in Book VII of Euclid's Elements. Euclid's algorithm is based on this division algorithm.

Euclid's algorithm is a technique to compute the Highest common factor (HCF) of two given integers. Recall that the HCF of two positive integers $a$ and $b$ is the greatest positive integer $d$ that divides both $a$ and $b$.

Let us find the HCF of 60 and 100, through the following activity.

## ACTIVITY

Take two paper strips of equal width and having lengths 60 cm, and 100 cm long. Our task is to find the maximum length of a strip which can measure both the strips without leaving any part.

Take 60 cm strip and measure the 100 cm strip with it. Cut off the left over 40 cm. Now, take this 40 cm strip and measure the 60 cm strip with it. Cut off the left over 20 cm. Now, take this 20 cm strip and measure the 40cm with it.

Since nothing is left over, we may conclude that 20cm strip is the longest strip which can measure both 60 cm and 100 cm strips without leaving any part.

Let us link the process we follwed in the "Activity" to Euclid's algorithm to get HCF of 60 and 100.

When 100 is divided by 60, the remainder is 40

$$100 = (60 \times 1) + 40$$

Now consider the division of 60 with the remainder 40 in the above equation and apply the division algorithim

$$60 = (40 \times 1) + 20$$

Now consider the division of 40 with the remainder 20, and apply the division lemma

$$40 = (20 \times 2) + 0$$

Notice that the remainder has become zero and we cannot proceed any further. We claim that the HCF of 60 and 100 is the divisor at this stage, i.e. 20. (You can easily verify this by listing all the factors of 60 and 100.) We observe that it is a series of well defined steps to find HCF of 60 and 100. So, let us state **Euclid's algorithm** clearly.

To obtain the HCF of two positive integers, say $c$ and $d$ with $c > d$, follow the steps below:

**Step 1** : Apply Euclid's division lemma, to $c$ and $d$. So, we find unique pair of whole numbers, $q$ and $r$ such that $c = dq + r$, $0 \leq r < d$.

**Step 2** : If $r = 0$, $d$ is the HCF of $c$ and $d$. If $r \neq 0$, apply the division lemma to $d$ and $r$.

**Step 3** : Continue the process till the remainder is zero. The divisor at this stage will be the required HCF.

This algorithm works because HCF $(c, d)$ = HCF $(d, r)$ where the symbol HCF $(m, n)$ denotes the HCF of any two positive integers $m$ and $n$.

## Do This

Find the HCF of the following by using Euclid algorithm.

(i) 50 and 70 (ii) 96 and 72 (iii) 300 and 550

(iv) 1860 and 2015

### Think and Discuss

Can you find the HCF of 1.2 and 0.12? Justify your answer.

Euclid's algorithm is useful for calculating the HCF of very large numbers, and it was one of the earliest examples of an algorithm that a computer had been programmed to carry out.

**Remarks :**

1. Euclid's algorithm and division algorithm are so closely interlinked that people often call former as the division algorithm also.

2. Although Division Algorithm is stated for only positive integers, it can be extended for all integers $a$ and $b$ where $b \neq 0$. However, we shall not discuss this aspect here.

Division algorithm has several applications related to finding properties of numbers. We give some examples of these applications below:

**Example 1 :** Show that every positive even integer is of the form $2q$, and that every positive odd integer is of the form $2q + 1$, where $q$ is some integer.

**Solution :** Let $a$ be any positive integer and $b = 2$. Then, by division algorithm, $a = 2q + r$, for some integer $q \geq 0$, and $r = 0$ or $r = 1$, because $0 \leq r < 2$. So, $a = 2q$ or $2q + 1$.

If $a$ is of the form $2q$, then $a$ is an even integer. Also, a positive integer can be either even or odd. Therefore, any positive odd integer is of the form $2q + 1$.

**Example 2 :** Show that every positive odd integer is of the form $4q + 1$ or $4q + 3$, where $q$ is some integer.

**Solution :** Let $a$ be a positive odd integer, and $b = 4$. We apply the division algorithm for $a$ and $b = 4$.

We get $a = 4q + r$, for $q \geq 0$, and $0 \leq r < 4$. The possible remainders are 0, 1, 2 and 3.

That is, $a$ can be $4q$, $4q + 1$, $4q + 2$, or $4q + 3$, where q is the quotient. However, since $a$ is odd, $a$ cannot be $4q = 2\,(2q)$ or $4q + 2 = 2(2q+1)$ (since they are both divisible by 2).

Therefore, any odd integer is of the form $4q + 1$ or $4q + 3$.

## Exercise - 1.1

1. Use Euclid's algorithm to find the HCF of

   (i) 900 and 270 (ii) 196 and 38220 (iii) 1651 and 2032

2. Use division algorithm to show that any positive odd integer is of the form $6q + 1$, or $6q + 3$ or $6q + 5$, where $q$ is some integer.

3. Use division algorithm to show that the square of any positive integer is of the form $3p$ or $3p + 1$.

4. Use division algorithm to show that the cube of any positive integer is of the form $9m$, $9m + 1$ or $9m + 8$.

5. Show that one and only one out of $n$, $n + 2$ or $n + 4$ is divisible by 3, where $n$ is any positive integer.

## 1.2 The Fundamental Theorem of Arithmetic

We know from Division Algorithm that for given positive integers $a$ and $b$ there exist unique pair of integers $q$ and $r$ satifying

$$a = bq + r, \qquad 0 \leq r < b$$

### Think - Discuss

If $r = 0$, then what is the relationship between $a$, $b$ and $q$ in $a = bq + r$ ?

From the above discussion you might have concluded that if $a = bq$, '$a$' is divisible by '$b$' then we can say that '$b$' is a factor of '$a$'.

For example we know that $24 = 2 \times 12$

$$24 = 8 \times 3$$
$$= 2 \times 2 \times 2 \times 3$$

We know that, if $24 = 2 \times 12$ then we can say that 2 and 12 are factors of 24. We can also write $24 = 2 \times 2 \times 2 \times 3$ and you know that this is the prime factorisation of 24.

Let us take any collection of prime numbers, say 2, 3, 7, 11 and 23. If we multiply some or all of these numbers, allowing them to repeat as many times as we wish, we can produce infinitely many large positive integers. Let us observe a few :

$2 \times 3 \times 11 = 66$ $\qquad$ $7 \times 11 \times 23 = 1771$

$3 \times 7 \times 11 \times 23 = 5313$ $\qquad$ $2 \times 3 \times 7 \times 11 \times 23 = 10626$

$2^3 \times 3 \times 7^3 = 8232$ $\qquad$ $2^2 \times 3 \times 7 \times 11 \times 23 = 21252$

Now, let us suppose your collection of primes includes all the possible primes. What is your guess about the size of this collection? Does it contain only a finite number of primes or infinitely many? In fact, there are infinitely many primes. So, if we multiply all these primes in all possible ways, we will get an infinite collection of composite numbers.

Now, let us consider the converse of this statement i.e. if we take a composite number can it be written as a product of prime numbers? The following theorem answers the question.

**Theorem-1.2 : (Fundamental Theorem of Arithmetic) : Every composite number can be expressed (factorised) as a product of primes, and this factorization is unique, apart from the order in which the prime factors occur.**

This gives us the Fundamental Theorem of Arithmetic which says that every composite number can be factorized as a product of primes. Actually, it says more. It says that any given composite number can be factorized as a product of prime numbers in a **'unique'** way, except for the order in which the primes occur. For example, when we factorize 210, we regard $2 \times 3 \times 5 \times 7$ as same as $3 \times 5 \times 7 \times 2$, or any other possible order in which these primes are written. That is, given any composite number there is one and only one way to write it as a product of primes, as long as we are not particular about the order in which the primes occur.

In general, given a composite number $x$, we factorize it as $x = p_1.p_2.p_3 .....p_n$, where $p_1, p_2, p_3,...., p_n$ are primes and written in ascending order, i.e., $p_1 \leq p_2 \leq ... \leq p_n$. If we combine the equal primes, we will get powers of primes. Once we have decided that the order will be ascending, then the way the number is factorised, is unique. For example,

$$27300 = 2 \times 2 \times 3 \times 5 \times 5 \times 7 \times 13 = 2^2 \times 3 \times 5^2 \times 7 \times 13$$

## Do This

Express 2310 as a product of prime factors. Also see how your friends have factorized the number. Have they done it same as you? Verify your final product with your friend's result. Try this for 3 or 4 more numbers. What do you conclude?

Let us apply fundamental theorem of arithmetic

**Example 3.** Consider the numbers of the form $4^n$ where $n$ is a natural number. Check whether there is any value of $n$ for which $4^n$ ends with zero?

**Solution :** If $4^n$ is to end with zero for a natural number $n$, it should be divisible by 2 and 5. This means that the prime factorisation of $4^n$ should contain the prime number 5 and 2. But it is not possible because $4^n = (2)^{2n}$ so 2 is the only prime in the factorisation of $4^n$. Since 5 is not present in the prime factorization, there is no natural number n for which $4^n$ ends with the digit zero.

You have already learnt how to find the HCF (Highest Common Factor) and LCM (Lowest Common Multiple) of two positive integers using the Fundamental Theorem of Arithmetic in earlier classes, without realizing it! This method is also called the *prime factorization method*. Let us recall this method through the following example.

**Example-4.** Find the HCF and LCM of 12 and 18 by the prime factorization method.

**Solution :** We have $12 = 2 \times 2 \times 3 = 2^2 \times 3^1$

$18 = 2 \times 3 \times 3 = 2^1 \times 3^2$

Note that HCF $(12, 18) = 2^1 \times 3^1 = 6$ = **Product of the smallest power of each common prime factor of the numbers.**

LCM $(12, 18) = 2^2 \times 3^2 = 36$ = **Product of the greatest power of each prime factor of the numbers.**

From the example above, you might have noticed that HCF $(12, 18) \times$ LCM $[12, 18] = 12 \times 18$. In fact, we can verify that for any two positive integers $a$ and $b$, HCF $(a, b) \times$ LCM $[a, b] = a \times b$. We can use this result to find the LCM of two positive integers, if we have already found the HCF of the two positive integers.

## DO THIS

Find the HCF and LCM of the following given pairs of numbers by prime factorisation method.

(i) 120, 90 (ii) 50, 60 (iii) 37, 49

## TRY THIS

Show that $3^n \times 4^m$ cannot end with the digit 0 or 5 for any natural numbers '$n$' and '$m$'

## EXERCISE - 1.2

1. Express each of the following numbers as a product of its prime factors.

   (i) 140 (ii) 156 (iii) 3825 (iv) 5005 (v) 7429

2. Find the LCM and HCF of the following integers by the prime factorization method.

   (i) 12, 15 and 21 (ii) 17, 23, and 29 (iii) 8, 9 and 25

   (iv) 72 and 108 (v) 306 and 657

3. Check whether $6^n$ can end with the digit 0 for any natural number $n$.

4. Explain why $7 \times 11 \times 13 + 13$ and $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 + 5$ are composite numbers.

5. How will you show that $(17 \times 11 \times 2) + (17 \times 11 \times 5)$ is a composite number? Explain.

6. What is the last digit of $6^{100}$.

Now, let us use the Fundamental Theorem of Arithmetic to explore real numbers further. First, we apply this theorem to find out when the decimal from of a rational number is terminating and when it is non-terminating, repeating. Second, we use it to prove the irrationality of many numbers such as $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$.

### 1.2.1 RATIONAL NUMBERS AND THEIR DECIMAL EXPANSIONS

Till now we have discussed some properities of integers. How can you find the preceeding or the succeeding integers for a given integer? You might have recalled that the difference between an integer and its preceeding or succeding integer is 1. And by this property only you might have found required integers.

In calss IX, you learned that the rational numbers would be in either a terminating decimal form or a non-terminating, repeating decimal form. In this section, we are going to consider a rational number, say $\frac{p}{q} (q \neq 0)$ and explore exactly when the number $\frac{p}{q}$ is a terminating decimal, and when it is a non-terminating repeating (or recurring) decimal. We do so by considering certain examples

Let us consider the following terminating decimals.

(i) 0.375 (ii) 1.04 (iii) 0.0875 (iv) 12.5

Now let us express them in $\frac{p}{q}$ form.

(i) $0.375 = \frac{375}{1000} = \frac{375}{10^3}$ (ii) $1.04 = \frac{104}{100} = \frac{104}{10^2}$

(iii) $0.0875 = \frac{875}{10000} = \frac{875}{10^4}$ (iv) $12.5 = \frac{125}{10} = \frac{125}{10^1}$

We see that all terminating decimals taken by us can be expressed in $\frac{p}{q}$ form whose denominators are powers of 10. Let us now factorize the numerator and denominator and then express then in the simplest form :

Now (i) $0.375 = \frac{375}{10^3} = \frac{3 \times 5^3}{2^3 \times 5^3} = \frac{3}{2^3} = \frac{3}{8}$

(ii) $1.04 = \frac{104}{10^2} = \frac{2^3 \times 13}{2^2 \times 5^2} = \frac{26}{5^2} = \frac{26}{25}$

(iii) $0.0875 = \frac{875}{10^4} = \frac{5^3 \times 7}{2^4 \times 5^4} = \frac{7}{2^4 \times 5} = \frac{7}{80}$

(iv) $12.5 = \frac{125}{10} = \frac{5^3}{2 \times 5} = \frac{25}{2}$

Have you observed any pattern in the denominators of the above numbers? It appears that when the decimal is expressed in its simplest rational form then $p$ and $q$ are coprime and the denominator (i.e., $q$) has only powers of 2, or powers of 5, or both. This is because 2 and 5 are the only prime factors of powers of 10.

From the above examples, you have seen that any rational number that terminates in its decimal form can be expressed in a rational form whose denominator is a power of 2 or 5 or both. So, when we write such a rational number, in $\frac{p}{q}$ form, the prime factorization of $q$ will be in $2^n5^m$, where $n, m$ are some non-negative integers.

We can write our result formally :

**Theorem-1.3 : Let $x$ be a rational number whose decimal form terminates. Then $x$ can be expressed in the form of $\frac{p}{q}$, where $p$ and $q$ are coprime, and the prime factorization of $q$ is of the form $2^n5^m$, where $n$, $m$ are non-negative integers.**

### Do This

Write the following terminating decimals in the form of $\frac{p}{q}$, $q \neq 0$ and $p$, $q$ are co-primes

(i) 15.265 (ii) 0.1255 (iii) 0.4 (iv) 23.34 (v) 1215.8

Write the denominators in $2^n5^m$ form.

You are probably wondering what happens the other way round. That is, if we have a rational number in the form of $\frac{p}{q}$ and the prime factorization of $q$ is of the form $2^n5^m$, where n, m are non-negative integers, then does $\frac{p}{q}$ have a terminating decimal expansion?

So, it seems to make sense to convert a rational number of the form $\frac{p}{q}$, where $q$ is of the form $2^n5^m$, to an equivalent rational number of the form $\frac{a}{b}$, where $b$ is a power of 10. Let us go back to our examples above and work backwards.

(i) $\frac{3}{8} = \frac{3}{2^3} = \frac{3\times 5^3}{2^3\times 5^3} = \frac{375}{10^3} = 0.375$ (ii) $\frac{26}{25} = \frac{26}{5^2} = \frac{13\times 2^3}{2^2\times 5^2} = \frac{104}{10^2} = 1.04$

(iii) $\frac{7}{80} = \frac{7}{2^4\times 5} = \frac{7\times 5^3}{2^4\times 5^4} = \frac{875}{10^4} = 0.0875$ (iv) $\frac{25}{2} = \frac{5^3}{2\times 5} = \frac{125}{10} = 12.5$

So, these examples show us how we can convert a rational number of the form $\frac{p}{q}$, where $q$ is of the form $2^n5^m$, to an equivalent rational number of the form $\frac{a}{b}$, where $b$ is a power of 10. Therefore, the decimal forms of such a rational number terminate. We find that a rational number of the form $\frac{p}{q}$, where $q$ is a power of 10, is a terminating decimal.

So, we conclude that the converse of theorem 1.3 is also true which can be formally stated as :

**Theorem 1.4 :** **Let $x = \frac{p}{q}$ be a rational number, such that the prime factorization of $q$ is of the form $2^n5^m$, where $n$ and $m$ are non-negative integers. Then $x$ has a decimal expansion which terminates.**

## Do This

Write the denominator of the following rational numbers in $2^n5^m$ form where $n$ and $m$ are non-negative integers and then write them in their decimal form

(i) $\frac{3}{4}$ (ii) $\frac{7}{25}$ (iii) $\frac{51}{64}$ (iv) $\frac{14}{25}$ (v) $\frac{80}{100}$

## 1.2.2 Non-terminating, recurring decimals in rational numbers

Let us now consider rational numbers whose decimal expansions are non-terminating and recurring.

Let us look at the decimal conversion of $\frac{1}{7}$.

$\frac{1}{7} = 0.1428571428571$ ..... which is a non-terminating and recurring decimal. Notice, the block of digits '142857' is repeating in the quotient.

$$\begin{array}{r} 0.1428571 \\ 7\overline{)1.0000000} \\ 7 \\ \hline 30 \\ 28 \\ \hline 20 \\ 14 \\ \hline 60 \\ 56 \\ \hline 40 \\ 35 \\ \hline 50 \\ 49 \\ \hline 10 \\ 7 \\ \hline 30 \end{array}$$

**Notice that the denominator i.e., 7 can't be written in the form $2^n5^m$.**

## Do This

Write the following rational numbers in the decimal form and find out the block of repeating digits in the quotient.

(i) $\frac{1}{3}$ (ii) $\frac{2}{7}$ (iii) $\frac{5}{11}$ (iv) $\frac{10}{13}$

From the 'Do this' exercise and from the example taken above, we can formally state:

**Theorem-1.5 :** **Let $x = \frac{p}{q}$ be a rational number, such that the prime factorization of $q$ is not of the form $2^n5^m$, where $n$ and $m$ are non-negative integers. Then, $x$ has a decimal expansion which is non-terminating repeating (recurring).**

From the above discussion, we can conclude that the decimal form of every rational number is either terminating or non-terminating repeating.

**Example-5.** Using the above theorems, without actual division, state whether decimal form of the following rational numbers are terminating or non-terminating, repeating decimals.

(i) $\frac{16}{125}$ (ii) $\frac{25}{32}$ (iii) $\frac{100}{81}$ (iv) $\frac{41}{75}$

**Solution :** (i) $\frac{16}{125} = \frac{16}{5\times5\times5} = \frac{16}{5^3}$ has a terminating decimal form.

(ii) $\frac{25}{32} = \frac{25}{2\times2\times2\times2\times2} = \frac{25}{2^5}$ has a terminating decimal form.

(iii) $\frac{100}{81} = \frac{100}{3\times3\times3\times3} = \frac{10}{3^4}$ has a non-terminating, repeating decimal form.

(iv) $\frac{41}{75} = \frac{41}{3\times5\times5} = \frac{41}{3\times5^2}$ has a non-terminating, repeating decimal form.

**Example-6.** Write the decimal form of the following rational numbers without actual division.

(i) $\frac{35}{50}$ (ii) $\frac{21}{25}$ (iii) $\frac{7}{8}$

**Solution :** (i) $\frac{35}{50} = \frac{7\times5}{2\times5\times5} = \frac{7}{2\times5} = \frac{7}{10^1} = 0.7$

(ii) $\frac{21}{25} = \frac{21}{5\times5} = \frac{21\times2^2}{5\times5\times2^2} = \frac{21\times4}{5^2\times2^2} = \frac{84}{10^2} = 0.84$

(iii) $\frac{7}{8} = \frac{7}{2\times2\times2} = \frac{7}{2^3} = \frac{7\times5^3}{(2^3\times5^3)} = \frac{7\times25}{(2\times5)^3} = \frac{875}{(10)^3} = 0.875$

## Exercise - 1.3

1. Write the following rational numbers in their decimal form and also state which are terminating and which are non-terminating, repeating decimal.

(i) $\frac{3}{8}$ (ii) $\frac{229}{400}$ (iii) $4\frac{1}{5}$ (iv) $\frac{2}{11}$ (v) $\frac{8}{125}$

2. Without performing division, state whether the following rational numbers will have a terminating decimal form or a non-terminating, repeating decimal form.

(i) $\frac{13}{3125}$ (ii) $\frac{11}{12}$ (iii) $\frac{64}{455}$ (iv) $\frac{15}{1600}$ (v) $\frac{29}{343}$

(vi) $\frac{23}{2^3\cdot5^2}$ (vii) $\frac{129}{2^2\cdot5^7\cdot7^5}$ (viii) $\frac{9}{15}$ (ix) $\frac{36}{100}$ (x) $\frac{77}{210}$

3. Write the following rationals in decimal form using Theorem 1.4.

(i) $\frac{13}{25}$ (ii) $\frac{15}{16}$ (iii) $\frac{23}{2^3.5^2}$ (iv) $\frac{7218}{3^2.5^2}$ (v) $\frac{143}{110}$

4. Express the following decimals in the form of $\frac{p}{q}$, and write the prime factors of $q$. What do you observe?

(i) 43.123 (ii) 0.1201201 (iii) $43.\overline{12}$ (iv) $0.\overline{63}$

## 1.3 Irrational numbers

In class IX, you were introduced to irrational numbers and some of their properties. You studied about their existence and how the rationals and the irrationals together made up the real numbers. You even studied how to locate irrationals on the number line. However, we did not prove that they were irrationals. In this section, we will prove that $\sqrt{2}, \sqrt{3}, \sqrt{5}$ and $\sqrt{p}$ in general is irrational, where $p$ is a prime. One of the theorems, we use in our proof, is the fundamental theorem of Arithmetic.

Recall, a real number is called *irrational* if it cannot be written in the form $\frac{p}{q}$, where $p$ and $q$ are integers and $q \neq 0$. Some examples of irrational numbers, with which you are already familiar, are :

$\sqrt{2}, \sqrt{3}, \sqrt{15}, \pi$, 0.10110111011110…, etc.

Before we prove that $\sqrt{2}$ is irrational, we will look at a theorem, the proof of which is based on the Fundamental Theorem of Arithimetic.

**Theorem-1.6 : Let $p$ be a prime number. If $p$ divides $a^2$, (where a is a positive integer), then $p$ divides $a$.**

**Proof :** Let the prime factorization of $a$ be as follows :

$a = p_1 p_2 \ldots p_n$, where $p_1, p_2, \ldots, p_n$ are primes, not necessarily distinct.

Therefore $a^2 = (p_1p_2 \ldots p_n)(p_1p_2 \ldots p_n) = p^2_1p^2_2 \ldots p^2_n$.

Now, we are given that $p$ divides $a^2$. Therefore, from the Fundamental Theorem of Arithmetic, it follows that $p$ is one of the prime factors of $a^2$. However, using the uniqueness part of the Fundamental Theorem of Arithmetic, we realise that the only prime factors of $a^2$ are $p_1, p_2, \ldots p_n$. So $p$ is one of $p_1, p_2, \ldots p_n$.

Now, since $a = p_1p_2 \ldots p_n$, $p$ divides $a$.

## Do This

Verify the theorem proved above for p= 2, p = 5 and for $a^2 = 1, 4, 9, 25, 36, 49, 64$ and 81.

We are now ready to give a proof that $\sqrt{2}$ is irrational. We will use a technique called proof by contradiction.

**Example 7.** Show that $\sqrt{2}$ is irrational.

**Solution :** Let us assume that $\sqrt{2}$ is rational.

If it is rational, then there must exist two integers $r$ and $s$ $(s \neq 0)$ such that $\sqrt{2} = \frac{r}{s}$.

If $r$ and $s$ have a common factor other than 1, then, we divide $r$ and $s$ by their highest common factor to get $\sqrt{2} = \frac{a}{b}$, where $a$ and $b$ are co-prime. So, $b\sqrt{2} = a$.

On squaring both sides and rearranging, we get $2b^2 = a^2$. Therefore, 2 divides $a^2$.

Now, by Theorem 1.6, it follows that since 2 is dividing $a^2$, it also divides $a$.

So, we can write $a = 2c$ for some integer $c$.

Substituting for $a$, we get $2b^2 = 4c^2$, that is, $b^2 = 2c^2$.

This means that 2 divides $b^2$, and so 2 divides $b$ (again using Theorem 1.6 with $p = 2$).

Therefore, both $a$ and $b$ have 2 as a common factor.

But this contradicts the fact that $a$ and $b$ are co-prime.

This contradiction has arisen because of our assumption that $\sqrt{2}$ is rational. Thus our assumption is false. So, we conclude that $\sqrt{2}$ is irrational.

In general, it can be shown that $\sqrt{d}$ is irrational whenever $d$ is a positive integer which is not the square of another integer. As such, it follows that $\sqrt{6}$, $\sqrt{8}$, $\sqrt{15}$ $\sqrt{24}$ etc. are all irrational numbers.

In class IX, we mentioned that :

- the sum or difference of a rational and an irrational number is irrational and
- the product or quotient of a non-zero rational and irrational number is irrational.

We prove some particular cases here.

**Example-8.** Show that $5-\sqrt{3}$ is irrational.

**Solution :** Let us assume that $5-\sqrt{3}$ is rational.

That is, we can find coprimes $a$ and $b$ ($b \neq 0$) such that $5-\sqrt{3}=\frac{a}{b}$.

Therefore, $5-\frac{a}{b}=\sqrt{3}$

we get $\sqrt{3}=5-\frac{a}{b}$

Since $a$ and $b$ are integers, $5-\frac{a}{b}$ is rational, and $\sqrt{3}$ is also rational.

But this contradicts the fact that $\sqrt{3}$ is irrational.

This contradiction has arisen because of our assumption that $5-\sqrt{3}$ is rational.

So, we conclude that $5-\sqrt{3}$ is irrational.

**Example-9.** Show that $3\sqrt{2}$ is irrational.

**Solution :** Let us assume, the contrary, that $3\sqrt{2}$ is rational.

i.e., we can find co-primes $a$ and $b$ ($b \neq 0$) such that $3\sqrt{2}=\frac{a}{b}$.

we get $\sqrt{2}=\frac{a}{3b}$.

Since 3, $a$ and $b$ are integers, $\frac{a}{3b}$ is rational, and so $\sqrt{2}$ is rational.

But this contradicts the fact that $\sqrt{2}$ is irrational.

So, we conclude that $3\sqrt{2}$ is irrational.

**Example-10.** Prove that $\sqrt{2}+\sqrt{3}$ is irrational.

**Solution :** Let us suppose that $\sqrt{2}+\sqrt{3}$ is rational.

Let $\sqrt{2}+\sqrt{3}=\frac{a}{b}$, where $a$, $b$ are integers and $b \neq 0$

Therefore, $\sqrt{2}=\frac{a}{b}-\sqrt{3}$.

Squarring on both sides, we get

$$2=\frac{a^2}{b^2}+3-2\frac{a}{b}\sqrt{3}$$

Rearranging

$$\frac{2a}{b}\sqrt{3}=\frac{a^2}{b^2}+3-2$$

$$=\frac{a^2}{b^2}+1$$

$$\sqrt{3}=\frac{a^2+b^2}{2ab}$$

Since $a$, $b$ are integers, $\frac{a^2+b^2}{2ab}$ is rational, and so $\sqrt{3}$ is rational.

This contradicts the fact that $\sqrt{3}$ is irrational. Hence $\sqrt{2}+\sqrt{3}$ is irrational.

**Note :**

1. The sum of two irrational numbers need not be irrational.

   For example, if $a=\sqrt{2}$ and $b=-\sqrt{2}$, then both $a$ and $b$ are irrational, but $a+b=0$ which is rational.

2. The product of two irrational numbers need not be irrational.

   For example, $a=\sqrt{2}$ and $b=3\sqrt{2}$, where both a and b are irrational, but $ab=6$ which is rational.

## EXERCISE - 1.4

1. Prove that the following are irrational.

   (i) $\frac{1}{\sqrt{2}}$ (ii) $\sqrt{3}+\sqrt{5}$ (iii) $6+\sqrt{2}$ (iv) $\sqrt{5}$ (v) $3+2\sqrt{5}$

2. Prove that $\sqrt{p}+\sqrt{q}$ is an irrational, where $p$, $q$ are primes.

## 1.4 EXPONENTIALS REVISTED

We know the power '$a^n$' of a number '$a$' with natural exponent '$n$' is the product of '$n$' factors each of which is equal to '$a$' i.e $a^n=\underbrace{a\cdot a\cdot a\cdots\cdots a}_{n-\text{factors}}$

$2^0, 2^1, 2^2, 2^3$ ............. are powers of 2

$3^0, 3^1, 3^2, 3^3$ ............... are powers of 3

We also know that when 81 is written as $3^4$, it is said to be in exponential form. The number '4' is the 'exponent' or 'index' and 3 is the 'base'. we read it as " 81 is the 4th power of base 3".

Recall the laws of exponents

If $a$, $b$ are real numbers, where $a \neq 0$, $b \neq 0$ and $m$, $n$ are integers, then

(i) $a^m . a^n = a^{m+n}$; (ii) $(ab)^m = a^m . b^m$ (iii) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

(iv) $(a^m)^n = a^{mn}$

(v) $a^0 = 1$ (vi) $a^{-m} = \frac{1}{a^m}$

**Do – This**

1. Evaluate

(i) $2^1$ (ii) $(4.73)^0$ (iii) $0^3$ (iv) $(-1)^4$ (v) $(0.25)^{-1}$ (vi) $\left(\frac{5}{4}\right)^2$ (vii) $\left(1\frac{1}{4}\right)^2$

2. (a) Express 10, 100, 1000, 10000, 100000 is exponential form

(b) Express in simplest exponential form

(i) $16 \times 64$ (ii) $25 \times 125$ (iii) $128 \div 32$

## Exponential and Logarithims

Let us Observe the following

$$2^x = 4 = 2^2 \text{ gives } x = 2$$

$$3^y = 81 = 3^4 \text{ gives } y = 4$$

$$10^z = 100000 = 10^5 \text{ gives } z = 5$$

Can we find the values of $x$ for the following?

$$2^x = 5, \qquad 3^x = 7, \qquad 10^x = 5$$

If so, what are the values of $x$?

For $2^x = 5$, What should be the power to which 2 must be raised to get 5?

Therefore, we need to establish a new relation between $x$ and 5.

In such situation, a new relation logarithm is introduced.

Consider $y = 2^x$ , we need that value of $x$ for which $y$ becomes 5 from the facts that if $x = 1$ then $y = 2^1 = 2$, if $x = 2$ then $y = 2^2 = 4$ , if $x = 3$ then $y = 2^3 = 8$, we observe that $x$ lies between 2 and 3.

We will now use the graph of $y=2^x$ to locate such a $'x'$ for which $2^x = 5$.

## GRAPH OF EXPONENTIAL $2^x$

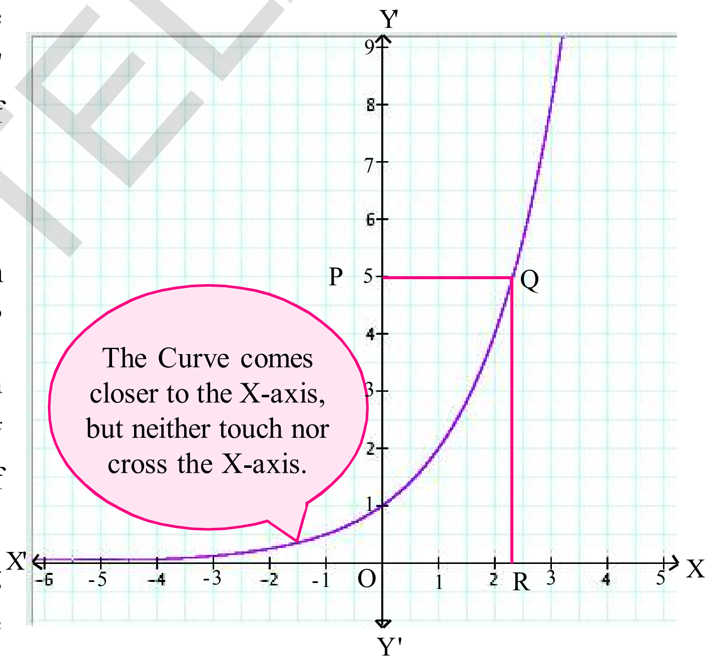
Let us draw the graph of $y = 2^x$

For this we compute the value of '$y$' by choosing some values for '$x$'.

| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y=2^x$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | 8 |

We plot the points and connect them with smooth curve.

Note that as $x$ increases, the value of $y = 2^x$ increases. As '$x$' decreases the value of $y = 2^x$ decreases very close to 0, but never attains the value 0.




Let us think, if $y = 2^x$ then for which value of $x$, $y$ becomes 5?

We know that, in the graph Y- axis represents the value of $2^x$ and X- axis represents the value of $x$. Locate the value of 5 on Y - axis, and represent it as a corresponding point "P" on Y- axis. Draw a line parallel to X- axis through P, which meets the graph at the point Q.

Now draw QR perpendicular to X - axis. Can we find the length of OR approximately from the graph? or where does it lie? Thus, we know that the $x$ coordinate of the point R is the required value of $x$, for which $2^x=5$.

This value of $x$ is called the logarithm of 5 to the base 2, written as $\log_2 5$.

### Think and Discuss

Let us observe the scale factor in the graph of $y = 2^x$

On X - axis (Refer Ratio - proportion)

If 10 places = 1 unit

20 places = 2 units

40 places = 4 units, then

Can you imagine the corresponding value on X-axis, with reference to the 5 on Y-axis?

We rewrite the above table as follows

| $x$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|
| $y = 2^x$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | 8 |

| $y$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | 8 |
|---|---|---|---|---|---|---|
| $x = \log_2 y$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |

Observe the graph $y = 2^x$ in the light of our definition of logarithm

If $y = \frac{1}{4}$ ; $x = -2$ i.e. $2^{-2} = \frac{1}{4}$ and $-2 = \log_2 \frac{1}{4}$

$y = \frac{1}{2}$ ; $x = -1$ i.e. $2^{-1} = \frac{1}{2}$ and $-1 = \log_2 \frac{1}{2}$

$y = 2$ ; $x = 1$ i.e. $2^1 = 2$ and $1 = \log_2 2$

$y = 4$ ; $x = 2$ i.e. $2^2 = 4$ and $2 = \log_2 4$

$y = 8$ ; $x = 3$ i.e. $2^3 = 8$ and $3 = \log_2 8$

i.e. $y$ – cordinate of any point on the curve is the $x$th power of 2, and $x$ – coordinate of any point on the curve is the logarithm of $y$ – coordinate to the base 2.

Let us consider one more example :

If $10^y = 25$ then it can be represented as $y = \log_{10} 25$ or $y = \log 25$,

Logarithms of a number to the base 10 are also called common logarithms. In this case, we generally omit the base i.e. $\log_{10} 25$ is also written as log 25.

**In general, $a$ and N are positive real numbers such that $a \neq 1$ we define $\log_a N = x \Leftrightarrow a^x = N$.**

**Do – This**

(1) Write the following in logarithmic form.

(i) $7 = 2^x$ (ii) $10 = 5^b$ (iii) $\frac{1}{81} = 3^c$ (iv) $100 = 10^z$ (iv) $\frac{1}{257} = 4^a$

(2) Write the following in exponential form.

(i) $\log_{10} 100 = 2$ (ii) $\log_5 25 = 2$ (iii) $\log_2 2 = 1$

**Try This**

Solve the following

(i) $\log_2 32 = x$ (ii) $\log_5 625 = y$ (iii) $\log_{10} 10000 = z$

(iv) $\log_x 16 = 2$ $\therefore$ $x^2 = 16$ $\Rightarrow$ $x = \pm 4$, Is it correct or not?

Can we say "exponential form and logarithm" form are inverses of one another?

Also, observe that every positive real number has a unique logarithmic value, because any horizontal line touches the graph at only one point.

**Think and Discuss**

(1) Does $\log_2 0$ exist? Give reasons.

(2) Prove (i) $\log_b b = 1$ (ii) $\log_b 1 = 0$ (iii) $\log_x b^x = x$

## Properties of Logarithms

Logarithms are more important in many applications, and also in advanced mathematics. We now estiablish some basic properties useful in manipulating expressions involving logarithms.

**(i) The Product Rule**

The properties of exponents correspond to properties of logarithms. For example when we multiply with the same base, we add exponents

i.e. $a^x . a^y = a^{x+y}$

This property of exponents coupled with an awareness that a logarithm is an exponent suggest the **Product Rule.**

**Theorem: (Product Rule) Let *a, x* and *y* be positive real numbers with $a \neq 1$.**

**Then $\log_a xy = \log_a x + \log_a y$**

**i.e., The logarithm of a product is the sum of the logarithms**

***Proof:***

Let $\log_a x = m$ and $\log_a y = n$ then we have $a^m = x$ and $a^n = y$

Now

$$xy = a^m a^n = a^{m+n}$$

$$\therefore \quad \log_a xy = m + n = \log_a x + \log_a y$$

**Try This**

We know that $\log_{10} 100000 = 5$

Show that you get the same answer by writing $100000 = 1000 \times 100$ and then using the product rule. Verify the answer.

**Do This**

Express the logarithms of the following as the sum of the logarithm

(i) $35 \times 46$ (ii) $235 \times 437$ (iii) $2437 \times 3568$

**(ii) The Quotient Rule**

When we divide with the same base, we subtract exponents

i.e. $$\frac{a^x}{a^y} = a^{x-y}$$

This property suggests the **quotient rule.**

**Theorem: (Quotient Rule) Let $a$, $x$ and $y$ be positive real numbers where $a \neq 1$.**

**Then $\log_a \frac{x}{y} = \log_a x - \log_a y$**

**i.e. the logarithm of a quotient is the difference of the logarithms of the two numbers taken in the same order.**

**Try This**

We know $\log_2 32 = 5$. Show that we get the same answer by writing 32 as $\frac{64}{2}$ and then using the product rule. Verify your answer .

**Do This**

Express the logarithms of the following as the difference of logarithms

(i) $\frac{23}{34}$ (ii) $\frac{373}{275}$ (iii) $4325 \div 3734$ (iv) $5055 \div 3303$

**THINK AND DISCUSS**

Prove the quotient rule using $\frac{a^m}{a^n} = a^{m-n}$ .

**(iii) The Power Rule**

When an exponential expression is raised to a power, we multiply the exponents

i.e. $(a^m)^n = a^{m.n}$

This property suggests the **power rule.**

**Theorem: (Power Rule) Let $a$ and $x$ be positive real numebrs with $a \neq 0$ and $n$ be any real number**

**then,** $\log_a x^n = n \log_a x$

**i.e. the logarithm of a number with an exponent is the product of the exponent and the logerithm of that number.**

**TRY THIS**

We have $\log_2 32 = 5$. Show that we get the same result by writing $32 = 2^5$ and then using power rules. Verify the answer.

Can we find the value of $x$ such that $2^x = 3^5$? In such cases we find the value of $3^5 = 243$. Then we can evaluate the value of $x$, for which the value of $2^x$ equals to 243.

Applying the logarithm and using the formula $\log_a x^n = n \log_a x$, Easily we can find the values of $3^{25}$, $3^{33}$ etc.

$2^x = 3^5$

writing in logarthmic form

$$\log_2 3^5 = x$$

$$5 \log_2 3 = x \quad (\because \log_a x^n = n \log_a x)$$

We observe that the value of $x$ is the product of 5 and the value of $\log_2 3$.

**DO THIS**

Using $\log_a x^n = n \log_a x$, expand the following

(i) $\log_2 7^{25}$ (ii) $\log s_5 8^{50}$ (iii) $\log 5^{23}$ (iv) $\log 1024$

**Note:** $\log x = \log_{10} x$

### Think and Discuss

We can write $\log \frac{x}{y} = \log (x.y^{-1})$. Can you prove that $\log \frac{x}{y} = \log x - \log y$ using product and power rules.

### Try This

(i) Find the value of $\log_2 32$

(ii) Find the value of $\log_c \sqrt{c}$

(iii) Find the value of $\log_{10} 0.001$

(iv) Find the value of $\log_{\frac{2}{3}} \frac{8}{27}$

### Think - Discuss

We know that, if $7 = 2^x$ then $x = \log_2 7$. Then, what is the value of $2^{\log_2 7}$ ? Justify your answer. Generalise the above by taking some more examples for $a^{\log_a N}$

**Example-11.** Expand $\log \frac{343}{125}$

**Solution :** As you know, $\log_a \frac{x}{y} = \log_a x - \log_a y$

$$\text{So, } \log \frac{343}{125} = \log 343 - \log 125$$
$$= \log 7^3 - \log 5^3$$
$$= 3\log 7 - 3\log 5 \qquad (\text{Since, } \log_a x^n = \text{m} \log_a x)$$

$$\text{So } \log \frac{343}{125} = 3(\log 7 - \log 5).$$

**Example-12.** Write $2\log 3 + 3\log 5 - 5\log 2$ as a single logarithm.

**Solution :** $2\log 3 + 3\log 5 - 5\log 2$

$= \log 3^2 + \log 5^3 - \log 2^5$ ( Since in $n \log_a x = \log_a x^n$)

$= \log 9 + \log 125 - \log 32$

$= \log (9 \times 125) - \log 32$ ( Since $\log_a x + \log_a y = \log_a xy$ )

$= \log 1125 - \log 32$

$= \log \frac{1125}{32}$ (Since $\log_a x - \log_a y = \log_a \frac{x}{y}$ )

**Example-13.** Solve $3^x = 5^{x-2}$.

**Solution :** $x \log_{10}3 = (x-2)\log_{10}5$

$$x \log_{10}3 = x\log_{10}5 - 2\log_{10}5$$

$$x\log_{10}5 - 2\log_{10}5 = x\log_{10}3$$

$$x\log_{10}5 - x\log_{10}3 = 2\log_{10}5$$

$$x(\log_{10}5 - \log_{10}3) = 2\log_{10}5$$

$$x = \frac{2\log_{10}5}{\log_{10}5 - \log_{10}3}$$

**Example-14.** Find $x$ if $2\log 5 + \frac{1}{2}\log 9 - \log 3 = \log x$

**Solution :** $\log x = 2\log 5 + \frac{1}{2}\log 9 - \log 3$

$$= \log 5^2 + \log 9^{\frac{1}{2}} - \log 3$$

$$= \log 25 + \log\sqrt{9} - \log 3$$

$$= \log 25 + \log 3 - \log 3$$

$$\log x = \log 25$$

$$\therefore \quad x = 25$$

## Exercise - 1.5

1. Determine the value of the following.

| | | | | | |
|---|---|---|---|---|---|
| (i) | $\log_{25}5$ | (ii) | $\log_{81}3$ | (iii) | $\log_2\left(\frac{1}{16}\right)$ |
| (iv) | $\log_7 1$ | (v) | $\log_x\sqrt{x}$ | (vi) | $\log_2 512$ |
| (vii) | $\log_{10}0.01$ | (viii) | $\log_{\frac{2}{3}}\left(\frac{8}{27}\right)$ | (ix) | $2^{2+\log_2 3}$ |

2. Write the following expressions as $\log N$ and find their values.

| | | | | |
|---|---|---|---|---|
| (i) | $\log 2 + \log 5$ | (ii) | $\log_2 16 - \log_2 2$ | (iii) $3\log_{64}4$ |
| (iv) | $2\log 3 - 3\log 2$ | (v) | $\log 10 + 2\log 3 - \log 2$ | |

3. Evaluate each of the following in terms of $x$ and $y$, if it is given that $x = \log_2 3$ and $y = \log_2 5$

   (i) $\log_2 15$ (ii) $\log_2 7.5$ (iii) $\log_2 60$ (iv) $\log_2 6750$

4. Expand the following.

   (i) $\log 1000$ (ii) $\log\left(\frac{128}{625}\right)$ (iii) $\log x^2y^3z^4$ (iv) $\log\left(\frac{p^2q^3}{r^4}\right)$ (v) $\log\sqrt{\frac{x^3}{y^2}}$

5. If $x^2 + y^2 = 25xy$, then prove that $2\log(x+y) = 3\log 3 + \log x + \log y$.

6. If $\log\left(\frac{x+y}{3}\right) = \frac{1}{2}(\log x + \log y)$, then find the value of $\frac{x}{y} + \frac{y}{x}$.

7. If $(2.3)^x = (0.23)^y = 1000$, then find the value of $\frac{1}{x} - \frac{1}{y}$.

8. If $2^{x+1} = 3^{1-x}$ then find the value of $x$.

9. Is (i) log 2 rational or irrational? Justify your answer.

   (ii) log 100 rational or irrational? Justify your answer.

## OPTIONAL EXERCISE

**[For extensive learning]**

1. Can the number $6^n$, n being a natural number, end with the digit 5? Give reason.

2. Is $7 \times 5 \times 3 \times 2 + 3$ a composite number? Justify your answer.

3. Prove that $\left(2\sqrt{3} + \sqrt{5}\right)$ is an irrational number. Also check whether $\left(2\sqrt{3} + \sqrt{5}\right)\left(2\sqrt{3} - \sqrt{5}\right)$ is rational or irrational.

4. If $x^2 + y^2 = 6xy$, prove that $2\log(x+y) = \log x + \log y + 3\log 2$

5. Find the number of digits in $4^{2013}$, if $\log_{10} 2 = 0.3010$.

**Note :** Ask your teacher about integral part and decimal part of the logarithm of a number.

## Suggested Projects

**Euclid Algorithm**

- Find the H.C.F by Euclid Algorithm by using colour ribbon band or grid paper.

## What we have discussed

1. Division Algorithm: Given positive integers $a$ and $b$, there exist whole numbers $q$ and $r$ satisfying $a = bq + r$, $0 \leq r < b$.
2. The Fundamental Theorem of Arithmetic states that every composite number can be expressed (factorized) as a product of its primes, and this factorization is unique, apart from the order in which the prime factors occur.
3. If $p$ is a prime and $p$ divides $a^2$, where $a$ is a positive integer, then $p$ divides $a$.
4. Let $x$ be a rational number whose decimal expansion terminates. Then we can express $x$ in the form of $\frac{p}{q}$, where $p$ and $q$ are coprime, and the prime factorization of $q$ is of the form $2^n5^m$, where $n$ and $m$ are non-negative integers.
5. Let $x = \frac{p}{q}$ be a rational number, such that the prime factorization of $q$ is of the form $2^n5^m$, where $n$ and $m$ are non-negative integers. Then $x$ has a decimal expansion which terminates.
6. Let $x = \frac{p}{q}$ be a rational number, such that the prime factorization of $q$ is not of the form $2^n5^m$, where $n$ and $m$ are non-negative integers. Then $x$ has a decimal expansion which is non-terminating and repeating (recurring).
7. We define $\log_a x = n$, if $a^n = x$, where $a$ and $x$ are positive numbers and $a \neq 1$.
8. Laws of logarithms :

   If $a$, $x$ and $y$ are positive real numbers and $a \neq 1$, then

   (i) $\log_a xy = \log_a x + \log_a y$ (ii) $\log_a \frac{x}{y} = \log_a x - \log_a y$

   (iii) $\log_a x^m = m \log_a x$ (iv) $a^{\log_a N} = N$ (v) $\log_a 1 = 0$

   (vi) $\log_a a = 1$
9. Logarithms are used for calculations in engineering, science, business and economics.

CHAPTER

# 2 Sets

## 2.1 INTRODUCTION

How would you describe a person, when you are asked to do?

Let us see some examples.

**Ramanujan was a mathematican, interested in number theory.**

**Dasarathi was a telugu poet, and also a freedom fighter.**

**Albert Eienstein German by birth was a physicist and music was his hobby.**

**Maryam Mirzakhan is the only women mathematican to win Fields medal.**

We classify the individuals with specific character and interest, then as a member of larger more recogizable group. People classify and category the world around in order to make sense of their environment and their relationship to other.

Books in the library are arranged according to the subject, so that we can find the required quickly.

In chemistry the elements are categorized in groups and classes to study their general properties

Your mathematics syllabus for tenth class has been divided into 14-Chapters under different headings.

### DENTAL FORMULA

Observe set of human teeth is of classified into 4-types according to their functions.

(i) Incisors (ii) Canines

(iii) Premolars and (iv) Molars

For each quadrant, for the sequence of incisors, canines, premolars and molars we write the dental formula 2, 1, 2, 3 specifying their number.