1 Space of increasing trees

1.1

Let L_n be the set of labeled trees on n vertices. We can make l_n into a topological space by putting the discrete topology on L_n. We can build the infinite product space, L, where L is the cartesian product of topological spaces L_n :

$$L = \Pi_{n=1}^{\infty} L_n. \tag{1}$$

Any $l \in L$ can be written as $(l_1, l_2, l_3 \dots)$ where each $l_i \in L_i$. Therfore there exists a subspace $A \subset L$ of attachment trees so that $a \in A$ if and only if l_1 is the tree on one vertex with no edges and each l_i can be built from l_{i-1} by attaching a vertex via an edge.

1.2

Another space we could consider is the infinite product space of $N_n := \{0, 1, 2, 3, \dots, n\}$ for $n = 1, 2 \dots \infty$ such that each $\mathbb{N}_n$ is equipt with the discrete topology.

$$\mathcal{L} = \Pi_{i=1}^{\infty} \mathbb{N}_n. \tag{2}$$

We claim that any $l \in \mathcal{L}$ corrosponds to an attachment tree since such a tree is built iteratively by attaching vertex 2 to vertex 1 then attaching vertex 3 to vertex 1 with probability 0.5 or to vertex 0 with probability 0.5 etc. so that each $N \in \mathbb{N}_n$ represents the single vertex with a higher label that vertex $n+1$ is attached to. We can think of this as an infinite sequence of independen random variables.

2 Infinite sequences of independent random variables

A probability space $(\Omega, \mathcal{F}, P)$ is a measure space such that Ω is a sample space, $\mathcal{F}$ is a $\sigma-$algebra event space (i.e. there exist events $E \in \mathcal{F}$), $P(E)$ is the probability of event E occuring and that $P(\Omega) = 1$.

Theorem 1. *Let A be an arbitary set. For each $\alpha \in A$ $(\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha)$ is a probability space such that Ω_α is a locally compact, σ - compact metric space with Borel σ - algebra $\mathcal{F}_\alpha$ then there exists a unique probability measure*

$$P_A = \Pi_{\alpha \in A} P_\alpha \text{ on } (\Pi_{\alpha \in A}(\Omega_\alpha, \Pi_{\alpha \in A}\mathcal{F}_\alpha,)$$

Such that $P_A\left(\Pi_{alpha \in A} E_\alpha\right) = \Pi_{a\alpha \in A} P_\alpha(E_\alpha)$.

Furthermore, whenever $E_\alpha \in \mathcal{F}_\alpha$ one has $E_\alpha = \Omega_\alpha$ for all but finitely many α.

We aim to use Theorem 1 to prove that there exists a unique probability measure on the space $\mathcal{L}$ defined in section 1.2. Let $A = \mathbb{N}$ then for all $\alpha \in A$ we wish to prove the following Lemmas:

Lemma 2.1. $(\mathbb{N}_\alpha, \mathcal{P}(\mathbb{N}_\alpha), \mu_\alpha)$ *is a probability space where* μ_α *is the uniform probability measure and* $\mathcal{P}(\mathbb{N}_\alpha)$ *is the power set of* $\mathbb{N}_\alpha$.

Lemma 2.2. $\mathbb{N}_\alpha$ *is a locally compact space.*

Lemma 2.3. $\mathbb{N}_\alpha$ *is a* σ *- compact metric space.*

Here I summarise the two most recent supervisory meetings:

Meeting on 15/10/13 We discussed appropriate topologies for the space of labeled trees and teh space of attachment trees. Our discourse also encompassed the relationship between Prufer sequences and the space of attachment trees including a discussion on our previous work detecting increasing trees in Prufer sequences [insert joke about not being able to see the forest for the trees here].

Future Goals: To investigate appropriate measures for the spaces mentioned above.

Meeting on 22/10/13:

We discussed the product topology and product measures as a method of formalising the work we've done so far.

Future Goals:

1)Short term Re-read Janson's paper on Polya Urns and the expected number of leaves that a random recursive tree has in light of our recent measure theoretical discusssions

2) Medium Term

Read about probability from a measure theoretic perspective, clarifying the notions of expectation etc.

3) Long Term

Return to Ben's Conjecture with this measure theoretic footing.