1 Introduction

2 Path Induction

3 Functions are Functors

4 Chapter 2.3 Type Families are fibrations

Every proof in the relevant section in the book is proved by path induction.

Let $P : A \to U$ be a type family which we can consider as a fibration.

Consider some $x, y : A$ and a path $p : x =_A y$ this will be our base space. The total space is $\sum_{x:A} P(x)$ which is the dependent pair types. This means that given any $x : A$ there exists a fibre $P(x) : \sum_{x:A} P(x)$ of elements that map to $x : A$ via the first projection $\pi(x, u) = x$.

Lemma 4.1. *Transport There exists a function* $transport^P(p,_) := p_* : P(x) \to P(y)$.

Proof. Let p be $refl_x$ and use path induction. Note that $(refl_x)_* = id_{P(x)}$. □

Homotopy Version: Given any two points, x, y in base space A and fibers $P(x)$ and $P(y)$ above x and y respectively then there exists a function from $P(x)$ to $P(y)$

Lemma 4.2. *Path Lifting Property Fix some* $u : P(x)$ *then there exists a path*

$$lift(u, p) : (x, u) = (y, p_*(u))$$

Proof. Assume that p is $refl_x$. By the proof of lemma 4.1 $(refl_x)_* = id_{P(x)}$, hence;

$$lift(u, refl_x) := ((x, u) = (x, u)) := refl_{(x,u)}$$

Now path inductions give us the existence of $lift$. □

So given a path, $p : x = y$, in the base space and and any element of the fibre $P(x)$ there exists a path, $p' : (x, u) = (y, p_*(u)$ such that $(x, u) : P(x)$ the fibre over x and $(y, p_*(u) : P(y)$ the fibre over y. I.e. we have lifted the path p.

Given some $f : \Pi_{x:A} P(x)$ we can define a non-dependent function $f' : A \to \sum_{x:A} P(x)$ by setting $p'(x) := (x, f(x))$. In homotopological terms f' is a right continuous inverse to pi_1 i.e. a section.

Lemma 4.3. *Fix* f *be as above, then there exists a dependent map:*

$$apd_f \Pi_{p:x=y} p_*(f(x)) =_{P_y} f(y)$$

Proof. So suppose (as usual) that p is $refl_x$. Then

$$(refl_x)_*(f(x)) =_{P(y)} f(x)$$

$$:= refl_{f(x)}$$

By path induction apd_f exists. □

So, given some section, $f' := (x, f(x)$, there exists a map over the section $P(y)$ from $p_*(f(x))$ to $f(y))$.

Lemma 4.4. *Fix some $A, B : U$ and $b{:}B$. If the type family $P : A \to U$ is defined to be $P(x) := B$ (not dependant on A). Then there exists a path:*

$$transportconst^B_p(b) : transport^P(p, b) = b.$$

Proof. We can assume that $p := refl_x$ so that we must consider $transport^P(refl_x.b) = b$. By lemma 4.1 $transport^P(refl_x.b) := b$ so we just need to construct a witness of type $refl_b$ and apply path induction. □

Lemma 4.4 highlights the subtle difference between a *function* and a path: just compare *transport* which is a function and tells us that a point u goes to point v and *transportconst* which is a path between u and v. So if P is a type family independent from A and given $x, y : A$ and a path between them $p : x =_A y$, then for any b in the total space B we have $transport^P(p, b)$ is a function from $P(x) := B$ to $P(y) := B$. In particular, it sends $transport^B(p, b) : B$ to $b : B$. Furthermore $transportconst^P_p(b)$ is a *path* between $transport^P(p, b)$ and b. We will see that *transportconst* makes sense when mapping *within* a particular fibre $P(x)$.

Now we wish to link the dependent and independent functions $ap_f(p)$ and $apd_f(p)$.

Lemma 4.5. *Let $f : A \to B$, then*

$$apd_f(p) = transportconst^B_p(f(x)) \Box ap_f(p)$$

Proof. By path induction it is enough to assume that p is $refl_x$. So we wish to construct a witness to type $apd_f(refl_x) = transportconst^B_{refl_x}(f(x))\dot{a}p_f(refl_x)$. By lemma **??** $transportconst^B_p(b) = refl_b$, by lemma **??** $ap_f(refl_x) = refl_{f(x)}$ and by lemma **??** $refl_x \Box refl_x = refl_x$. Putting this together we need to prove $refl_{f(x)} = refl_{f(x)}$: we have $refl_{refl_{f(x)}}$ for this. Path induction gives the required result. □

In topological terms, given a fibration $\pi_1 : \sum_{x:A} P(x)$ and a function $f : A \to B$ such that $f'(x) = (x, f(x))$ and a path $x = y$ in space A we know there exists the following three maps in B:

(i) $apd_f(p)$ is a path in the fibre, $P(y) := B$ over y from $transport^P(p, f(x))$ to $f(y)$

(ii) $transportconst^B_p(f(x))$ is a path in B from $transport^P(p, f(x)) : B$ to $f(x) : B$.

(iii) $ap_f(p)$ is the path from f_x to $f(y)$ in B.

Drawing these paths in B assures us that Lemma 4.5 is well-typed.