Summary

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A *random recursive tree* (RRT), T, with vertices $V(T) = \{v_1, \dots, v_n\}$ is a labeled, rooted tree obtained by assigning a root vertex v_1 then adding $n-1$ vertices one by one such that each new vertex is joined by an edge to a randomly and uniformly chosen existing vertex. A random recursive q-ary tree is a labeled, rooted tree built in the same way as a random recursive tree except each new vertex can only be attached to an existing vertex that has outdegree less than q.

There exists a geometric decomposition of $\mathrm{Aut}(T)$ into a direct product of symmetric groups and wreath products of symmetric groups which correspond to simple and complex symmetric branches of T. We can use this decomposition to calculate $\mathrm{Aut}(T)$ by counting (n,k)- stars. In particular we want to calculate $\mathrm{Aut}_1(T)$ which is the part of the automorphism group coming from $(1,k)$-stars. Note that each $(1,k)$-star contributes $k!$ to the size of $\mathrm{Aut}(T)$

In order to calculate the expected number of $(1,k)$-stars in a random recursive tree we have set up a Pólya urn process. In this process we associate an infinite square matrix, A with a random recursive tree such that the (right) eigenvector corresponding to the largest eigenvalue of A, $v = (v_1, v_2, \dots)$, (normalised such that $\sum v_i = 1$) gives us information about $(1,k)$-stars. In particular, given a random recursive tree T, the component v_k of the associated eigenvector v is the proportion of vertices in T which are the vertices of an induced subtree of T isomorphic to a $(1,k)$-star. We need to prove that these two paragraphs combined mean that.

$$\lim_{n\to\infty} |\,\mathrm{Aut}_1(T_n))|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\prod_{k=1}^{\infty} \frac{nv_i(i!)}{i+1} \right)^{\frac{1}{n}}$$

If we build such a matrix, A, we have a system of equations which define v:

(i) $v_1 = 4v_0$

(ii) $v_2 = 2v_0 - v_1 + \frac{8/3}{v}{}_2 + \sum_{k=3}^{\infty} \frac{2k}{k+1} v_k$

(ii) $v_p = \frac{(p+1)^2}{p} v_{p-1} - \frac{p+1}{pp-1} v_{p-2}$ for all $p > 2$

Either we solve the above equations to find analytic results for the v_i or we follow the following, more long-winded process. (Or we follow some third method of proof that we haven't thought of yet).

Given some random recursive q-ary tree we can again use a Pólya urn process to describe the limiting distribution of $(1,k)$-stars for $k = 0, 1, \dots, q$. We can define a sequence of matrices $A_1, A_2, A_3, \dots$ where A_{q-1} is a $q \times q$ matrix that describes a q-ary tree.

For example

$$A_5 = \begin{pmatrix} -1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 2 & -1 & \frac{8}{3} & \frac{6}{4} & \frac{8}{5} & \frac{10}{6} \\ 0 & \frac{3}{2} & -3 & \frac{9}{4} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & -4 & \frac{16}{5} & 0 \\ 0 & 0 & 0 & \frac{5}{4} & -5 & \frac{25}{6} \\ 0 & 0 & 0 & 0 & \frac{6}{5} & -5 \end{pmatrix} \tag{1}$$

To each A_q we can find the (normalised) principal eigenvector $v_q = (v_{0,q}, v_{1,q}, \ldots, v_{q,q})$. We can use A_q to again form governing equations:

(i) $v_{1,q} = 4v_{0,q}$

(ii) $v_{p,q} = \frac{(p+1)^2}{p} v_{p,q-1} - \frac{p+1}{p(p-1)} v_{p,q-2}$ for $2 < p < q$.

(iii) $v_{qq} = \frac{v_{q-1,q}}{q}$.

QUESTION: Given any $n \in \mathbb{N}$ does the sequence $v_{n,n}, v_{n,n+1}, v_{n,n+2}, \ldots$ converge?
If the above sequence does converge then define $\tilde{v}_j := \lim_{i\to\infty} v_{ji}$ and define

$$\tilde{v} = (\tilde{v}_0, \tilde{v}_1, \ldots)^T$$

Now we ask: are the following three equal:

$$\lim_{n\to\infty} \left(\prod_{i=1}^{n} \frac{nv_i(i!)}{(i+1)} \right)^{\frac{1}{n}} \tag{2}$$

$$\lim_{n\to\infty} \left(\prod_{i=1}^{n} \frac{n\tilde{v}_i(i!)}{(i+1)} \right)^{\frac{1}{n}} \tag{3}$$

$$\lim_{n\to\infty} \left(\prod_{i=1}^{n} \frac{nv_{in}(i!)}{(i+1)} \right)^{\frac{1}{n}} \tag{4}$$

We have already discussed the relationship between equation (1) and the expected size of $|\operatorname{Aut}_1(T_n)|^{\frac{1}{n}}$ and that we have not been able to analytically solve this. However, if equations (2),(3) and (4) are equal then equation (4) may be much easier to solve than equation (2) and provide an easier route to calculate the expected $|\operatorname{Aut}_1(T)|^{\frac{1}{n}}$. These results can be extended using a more complicated Pólya urn process to calculate the whole automorphism group.