1 Complex automorphisms are nontrivial

Let $\{T_n\}_{n=1}^{\infty}$ be a random recursive tree and X_n be the number of trees with leaves isomorphic to a $(2,2)$-star. If we assume that almost surely $lim_{n\to\infty} \frac{X_n}{n} \to \epsilon_{2,2}$ for some $\epsilon_{2,2} > 0$. This means that except for a set of measure 0 exceptions for all $\delta > 0$ there exists $N_\delta \in \mathbb{N}$ such that for all $n > N_\delta$

$$|\frac{X_n}{n} - \epsilon| < \delta$$

Therefore we can conclude that:

$$\begin{aligned} |X_n - n\epsilon| &< n\delta && (1)\\ n\epsilon - n\delta < X_n &< n\delta + n\epsilon && (2)\\ n(\epsilon - \delta) < X_n &< n(\epsilon + \delta) && (3) \end{aligned}$$

The part of the (complex) automorphism group coming from $(2,2)$-stars can therefore be estimated as follows. For $n > N_\delta$,

$$8^{n(\epsilon-\delta)} < |Aut_{2,2}(t_N)| = 8^{X_n} < 8^{n(\epsilon+\delta)} \qquad (4)$$

$$1 < 8^{(\epsilon-\delta)} < |Aut_{2,2}(t_N)|^{\frac{1}{n}} = 8^{X_n^{\frac{1}{n}}} < 8^{(\epsilon+\delta)} \qquad (5)$$

Since we can choose our $\delta << \epsilon$ and there exists a Polya urn model which shows that for all m and n there exists an $\epsilon_{n,m}$ such that $lim_{n\to\infty} \frac{X_n}{n} \to \epsilon_{n,m}$. This disproves Ben's conjecture that almost surely, in the limit as $n \to \infty$:

$$lim_{n\to\infty} |Aut_{\text{Complex}}(T_n)|^{\frac{1}{n}} \to 1$$

2 Convergence of the Automorphism group

In this section we will prove that $\lim_{n\to\infty} Aut(T_n)$ converges.

Let $\{T_n\}_{n=1}^{\infty}$ be a random recursive tree and X_{ni} be the number of vertices of degree i in T_n.

Theorem 2.1. *In the limt as $n \to \infty$, almost surely $\frac{X_{n,i}}{n} \to 2^{-i}$.*

Proof. See Janson **??** □

In other words, except ofr a measure 0 set of exceptions for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$

$$|\frac{X_{ni}}{n} - 2^{-i}| < \epsilon.$$

We can play the same game as section **??** and write:

$$\begin{aligned} |X_{ni} - n2^{-i}| &< n\epsilon && (6)\\ n2^{-i} - n\epsilon < X_{ni} &< n\epsilon + n2^{-i} && (7)\\ n(2^{-i} - \epsilon) < X_{ni} &< n(\epsilon + 2^{-i}) && (8)\\ & && (9) \end{aligned}$$

For each i case we can choose ϵ to be as small as possible so let each $\epsilon_i = 2^{-i}$

Recall that there exists a geometric decomposition of $\mathrm{Aut}(T_n)$ into (p,k)-stars corresponing to a direct product decomposition of $\mathrm{Aut}\,T_n$ into subgroups isomorphic to symmetric groups or wreath products of symmetric groups. The stars corresponding to a wreath product $G_1 = S_{m_1} \wr S_{m_2} \wr \cdots \wr S_{m_x}$ contribute $|S_{m_1} \wr S_{m_2} \wr \cdots \wr S_{m_x}| = (\ldots(m_1!^{m_2} m_2!)^{m_3} \ldots m_{x-1}!)^{m_x} m_x!$ to the automorphism group $\mathrm{Aut}(T_n)$. The star corresponding to G_1 is isomorphic to the graph depicted in figure **??**. Notice that $|G_1| = \prod_{v \in V} \deg(v)!$. Therefore given some instance, T_n, of random recursive tree $\{T_n\}_{n=1}^{\infty}$, $\mathrm{Aut}(T_n)$ is bounded above by $\prod_{v \in V(T_n)} deg(v)!$, hence Equation 6 gives us the following bound in the limit as $n \to \infty$ almost surely:

$$\mathrm{Aut}(T_n) < \prod_{i=2}^{\infty} (i!)^{X_{ni}} \tag{10}$$

$$< \prod_{i=2}^{\infty} (i!)^{n(\epsilon_i + 2^{-i})} \quad < \prod_{i=2}^{\infty} (i!)^{n(2^{-i}+2^{-i})} < \prod_{i=2}^{\infty} (i!)^{n2^{-i+1}} \tag{11}$$

This implies that $\mathrm{Aut}(T_n^{\frac{1}{n}} < \prod_{i=2}^{\infty} (i!)^{2^{-i+1}} := X$. It remains to check that the convergence of X for which we will need the following theorem.

Theorem 2.2. *If $b_n \neq 0$ for all n then $\prod_{n=0}^{\infty} b_n$ converges if and only if $\sum_{n=0}^{\infty} Log(b_n)$ converges.*

Proof. See the proof of Theorem 3.8.1 in **??**. $\square$

Therefore it suffices to prove that $\sum_{n=0}^{\infty} \frac{Log(i!)}{2^{i-1}}$ converges. By Stirling's approximation and the comparison test $\sum_{n=0}^{\infty} \frac{Log(i!)}{2^{i-1}}$ converges.

Remark 1. *Assume that in the limit as $n \to \infty$ $\frac{X_n}{n} \to X$ almost surely. This means that apart from a measure zero set of sample of RRTs $\forall \epsilon > 0 \exists N_\epsilon$ such that $\forall n > N_\epsilon |\frac{X_n}{n} - X| < \epsilon$.*

This does not *mean that apart from a measure zero set of RRTs that $\lim_{n \to \infty} X_n = Xn$. On the other hand we can say that given any $\delta > 0$ there exists n' such that $|X_n - Xn| < \delta$ (just take $n' = N_{\delta / N_\delta}$).*