1 Space of increasing trees

1.1

Let L_n be the set of labeled trees on n vertices. We can make l_n into a topological space by putting the discrete topology on L_n. We can build the infinite product space, L, where L is the cartesian product of topological spaces L_n :

$$L = \Pi_{n=1}^{\infty} L_n. \tag{1}$$

Any $l \in L$ can be written as $(l_1, l_2, l_3 \dots)$ where each $l_i \in L_i$. Therfore there exists a subspace $A \subset L$ of attachment trees so that $a \in A$ if and only if l_1 is the tree on one vertex with no edges and each l_i can be built from l_{i-1} by attaching a vertex via an edge.

1.2

Another space we could consider is the infinite product space of $N_n := \{0, 1, 2, 3, \dots, n\}$ for $n = 1, 2 \dots \infty$ such that each $\mathbb{N}_n$ is equipt with the discrete topology.

$$\mathcal{L} = \Pi_{i=1}^{\infty} \mathbb{N}_n. \tag{2}$$

We claim that any $l \in \mathcal{L}$ corrosponds to an attachment tree since such a tree is built iteratively by attaching vertex 2 to vertex 1 then attaching vertex 3 to vertex 1 with probability 0.5 or to vertex 0 with probability 0.5 etc. so that each $N \in \mathbb{N}_n$ represents the single vertex with a higher label that vertex $n+1$ is attached to. We can think of this as an infinite sequence of independen random variables.

2 Infinite sequences of independent random variables

A probability space $(\Omega, \mathcal{F}, P)$ is a measure space such that Ω is a sample space, $\mathcal{F}$ is a $\sigma-$algebra event space (i.e. there exist events $E \in \mathcal{F}$), $P(E)$ is the probability of event E occuring and that $P(\Omega) = 1$.

Theorem 1. *Let A be an arbitary set. For each $\alpha \in A$ $(\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha)$ is a probability space such that Ω_α is a locally compact, σ - compact metric space with Borel σ - algebra $\mathcal{F}_\alpha$ then there exists a unique probability measure*

$$P_A = \Pi_{\alpha \in A} P_\alpha \text{ on } (\Pi_{\alpha \in A}(\Omega_\alpha, \Pi_{\alpha \in A}\mathcal{F}_\alpha,)$$

Such that $P_A\left(\Pi_{alpha \in A} E_\alpha\right) = \Pi_{a\alpha \in A} P_\alpha(E_\alpha)$.

Furthermore, whenever $E_\alpha \in \mathcal{F}_\alpha$ one has $E_\alpha = \Omega_\alpha$ for all but finitely many α.

We aim to use Theorem 1 to prove that there exists a unique probability measure on the space $\mathcal{L}$ defined in section 1.2. Let $A = \mathbb{N}$ then for all $\alpha \in A$ we wish to prove the following Lemmas:

Lemma 2.1. *$(\mathbb{N}_\alpha, \mathcal{P}(\mathbb{N}_\alpha), \mu_\alpha)$ is a probability space where μ_α is the uniform probability measure and $\mathcal{P}(\mathbb{N}_\alpha)$ is the power set of $\mathbb{N}_\alpha$.*

Lemma 2.2. *$\mathbb{N}_\alpha$ is a locally compact space.*

Lemma 2.3. *$\mathbb{N}_\alpha$ is a σ - compact metric space.*

3 Properties of a Probability Measure

Let $(\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha)$ be a probability space then:

(i) $P_\alpha(\emptyset) = 0$

(ii) $P_\alpha(\Omega_\alpha) = 1$

(iii) P_α satisfies a countably additive propertry for all countable sets I,

$$P_\alpha\left(\bigcup_{i\in I} E_i\right) = \sum_{i\in I} P_\alpha E_i$$

So consider the space of labeled trees, L, built in the way described in section 1.1 and let L be equipt with the product measure, μ described in section 2. Similarly recall that A be the subspace of L consisting of attachment trees.

Lemma 3.1. *$\mu A = 0$ in L.*

Proof. Let A_i be the event that a tree $l_i \in L_i$ is an attachment trees on i vertices.

Given a tree $l = (l_1, l_2, \dots) \in L$ consider the first i entries $l^i = (l_1, l_2, \dots, l_i)$ and let $mathcal A_i$ be the event that l is chosen such that l^i corrosponds to an attachment tree.

By definition of the product measure

$$\mu(\mathcal{A}_i = \mu(\Pi_{a\in\mathbb{N}} E_a = \Pi_{a\in\mathbb{N}} \mu_a E_a.$$

Where E_a is the event that $l_i \in A_i$ for $a = 1, 2, \dots, i$ and $E_a = \Omega_a$ for all other a.

Since $|A_i| = (i-1)!$ and $|L_i| = i^{i-2}$ we can write

$$\mu(\mathcal{A}_i) = \Pi_{n=1}^{i} \frac{(i-1)!}{i^{i-2}}$$

Given any $\epsilon > 0$ there exists an i such that $\mu(\mathcal{A}_i) < \epsilon$.

Since $\mathcal{A}_i \subset A$ for every i, $\mu(A) = 0$. ☐