1 Introduction

Cayley was first to prove that there are n^{n-2} labeled, rooted trees on n vertices [1]. A later proof due to Prüfer uses a correspondence between particular sequences called *Prüfer sequences* and labeled trees [2].

In Section 2 we will characterize the subset of Prüfer sequences that correspond to a family of trees called *increasing trees*.

2 Prüfer Sequences

2.1 Notation

For succinctness we will use $\mathbb{N}_n := \{1, 2, 3, \ldots, n\}$ and by *tree* we mean labelled, rooted, non embedded tree throughout this section. If A and B are sequences we say that $A \backslash B$ consists of all the distinct entries of A that are not in B. Given some graph G we denote $V(G)$ to be set of vertices of G and we say that $E(G)$ is the set of edges of G. If $v, w \in V(G)$ and there is a edge, $e \in E(G)$, joining v and w then we write $e = (v, w)$ amd say that v is connected via an edge to w. Given some $v \in V(G)$ the degree of v is denoted $\deg(v)$ and we say that vertices v such that $\deg(v) = 1$ are *leaves*. A tree is an *increasing tree* if every vertex other than the root is adjacent to precisely one other vertex with a lower label. Note that we will write a vertex i interchangeably with a vertex labelled i throughout.

2.2 The Prüfer Correspondence

Any sequence $P_n = (a[1], a[2], \ldots, [n-2])$ such that every $a[i] \in \mathbb{N}_n$ is a Prüfer sequence. For example the following sequence is a valid Prüfer sequence:

$$P_{10} = (4, 3, 2, 8, 1, 1, 2, 4).$$

We will use the notation $P_n[i : j]$ to mean the $i^{th} - j^{th}$ entries (inclusive) of a given Prüfer sequence. In terms of the example above $P_{10}[3 : 7] = (2, 8, 1, 1, 2)$. We define $P_n[x : y]$ to be the empty sequence if $y < x$.

Let X_v denote the multiplicity of the label v in P_n. For example, if we consider P_{10} defined above:

$$X_1 = 2,\ X_2 = 2,\ X_3 = 1,\ X_4 = 2,\ X_5 = 0 \ldots$$

Given some Prüfer sequence P_n we can associate a sequence

$$\mathrm{Pos}_v = (\mathrm{Pos}_{v,1}, \mathrm{Pos}_{v,2}, \ldots, \mathrm{Pos}_{v,X_v})$$

to every $v \in P_n$ such that $\mathrm{Pos}_{v,j}$ is the position of the j^{th} occurrence of v. For example, entry 4 occurs in the first and eighth position of P_{10} so $\mathrm{Pos}_4 = (1, 8)$. For any $v \in P_n$ we say that an occurrence of v is *final* if that occurrence of v is in position Pos_{v,X_v}. We define $L = \{v : v \in \mathbb{N}_n \backslash P_n\}$. In our example $L = \{5, 6, 7, 9, 10\}$.

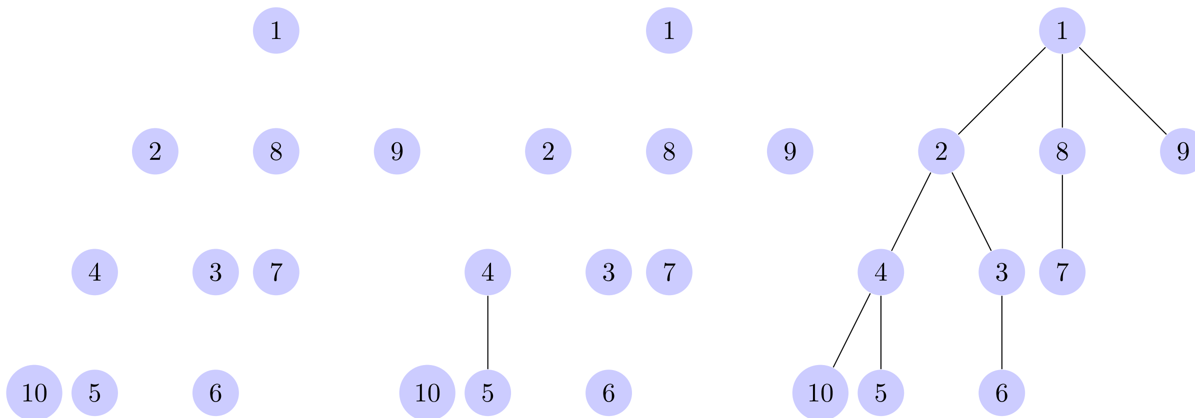

Figure 1: G_0 is on the far left above. G_1 is central above and $T_n(= G_{n-1})$ is on the right above.

We can associate a tree witha Prüfer sequence $P_n = (a[1], a[2], \ldots, a[n-2])$ via a dynamic correspondence. At each time, t an appropriate vertex is chosen from a set of available vertices and is attached via an edge to $a[i]$. At time $t = 0$ set $L_0 = L$ and define G_0 to be the graph on n vertices labelled $1, 2, \ldots, n$ with no edges. At time $t = 1$ the vertex with label $a[1]$ is attached via an edge to $\min L_0$ to form G_1. More generally, at time t we form L_t from L_{t-1} by removing $\min L_{t-1}$ and adding $a[t-1]$ if $a[t-1]$ is final and form G_t by connecting via an edge the labelled vertex $a[t]$ to vertex $\min L_t$ for $t = 2, 3, 4, \ldots, n-2$. Let D be the number of distinct entries of P_n. Since D elements are added to L_0 and $n - 2$ elements are removed to form L_{t-1}:

$$|L_{n-1}| = |L_0| + D - (n - 2).$$

Clearly $D + |L_0| = n$ therefore $|L_{n-1}| = 2$. Finally, G_{n-1} is formed by adding an edge between these vertices.

It is known that G_{n-1} is always a tree on n nodes so we write $T_n := G_{n-1}$. It is also known that $\deg(v) = X_v + 1$ for all $v \in V(T_n)$. Therefore L corresponds to the leaves of T_n.

Example 2.1. Let $P = (4, 3, 2, 8, 1, 1, 2, 4)$. We build the corresponding tree, T, as follows. Let G_0 be the graph on 10 vertices with 0 edges. At time $t = 0$ $L_0 = L = \{5, 6, 7, 9, 10\}$ therefore we build G_1 from G_0 by adding an edge between the vertex labelled 4 and the vertex labelled $\min L_0 = 5$. Since 4 is not final $L_1 = \{6, 7, 9, 10\}$ and we build G_2 by adding edge $(3, 6)$ to G_1. Since 3 is final $L_2 = \{3, 7, 9, 10\}$ so we form G_3 by adding the edge $(2, 3)$ to G_2. This process continues until we build T_n shown in Figure 1:

2.3 Increasing Trees

We can associate another sequence $B_n = (b[1], b[2], \ldots, b[n-2])$ of length $n-2$ with P_n as follows: firstly, for all $v \in P_n[1 : n-3]$ we set $b[\mathrm{Pos}_{v,X_v} + 1] = a[\mathrm{Pos}_{v,X_v}]$.

In our example:

$$B_{10} = (\; , \; , 3, \; , 8, \; , 1, 2)$$

We then order L from lowest to highest and fill the remaining empty entries of B_n with the first $|L| - 1$ entries from L ordered from lowest to highest. For example,

$$B_{10} = (5, 6, 3, 7, 8, 9, 1, 2)$$

Let $m_1 = \min P_n$. If $1 \in P_n$ then $m_1 = 1$. If $1 \notin P_n$ then the vertex labelled 1 must be a leaf of T_n, if $n > 2$ then the vertex labelled 2 cannot be a leaf so $m_1 = 2$.

We say that the (possibly empty) set $E = \{e_1, e_2, \ldots, e_m\}$ is the set of *exceptional Prüfer positions* ordered such that $e_1 < e_2 < \cdots < e_m$ and let E be defined as follows.

Case 1 $(\min P_n = 1)$**.** If $\mathrm{Pos}_{1,X_1} \neq n-2$ then $\mathrm{Pos}_{1,X_1} + 1 = e_1$ we then set $m_2 = \min P_n[e_1 : n-2]$ and if $\mathrm{Pos}_{m_2 X_{m_2}} \neq n-2$ then $e_2 = \mathrm{Pos}_{m_2 X_{m_2}} + 1$. In general $m_i = \min P_n[e_{i-1} : n-2]$ and, if $\mathrm{Pos}_{m_i X_{m_i}} \neq n-2$ then $e_i = \mathrm{Pos}_{m_i X_{m_i}} + 1$.

Case 2 $(\min P_n = 2)$**.** We can imagine that $Pos_{1,X_1} = 0$ so $1 \in E$. We then set $m_2 = \min P_n$ and the other elements of E are as in Case 1.

Given any $v \in P_n$ we say v is an *exceptional Prüfer entry* if there exists a label v in an exceptional Prüfer position, i.e. there is a $j \in \{1, 2, \ldots X_v\}$ such that $\mathrm{Pos}_{v,j} \in E$. In the case of P_{10} above $m_1 = 1$, $m_2 = 2$ and $E = \{7, 8\}$. Vertices 2 and 4 are exceptional Prüfer entries.

For intuition behind the difference behaviours of exceptional Prüfer entries and entries that are not exceptional we might think of the Prüfer process as a "directed" process. Given some Prüfer sequence P_n each entry $a[i] \in P_n$ corresponds to an edge $e \in T_n$ we can build $\tilde{T}_n$ such that $V(\tilde{T}_n) = V(T_n)$ and $E(\tilde{T}_n) = E(T_n)$ with the additional information that each edge in $\tilde{T}_n$ is directed *from* $\min L_i$ *to* $a[i]$.

So let T_n be an increasing tree and consider the corresponding Prüfer sequence P_n.

Case 3 $(\min P_n = 1)$**.** The lowest valued leaf, l, is chosen first and attached to $a[1]$. By the definiton of an increasing tree $a[1] < l$ therefore the edge $(a[1], l)$ is directed from l to $a[1]$ and more generally is directed towards the root node. Subsequently chosen leaves will also be attached to vertices with larger labels inducing edges directed towards the root vertex. However, consider $a[\mathrm{Pos}_{1,X_1} + 1]$, clearly $1 \in L_{\mathrm{Pos}_{1,X_1} + 1}$ and therefore $a[\mathrm{Pos}_{1,X_1} + 1]$ will be attached to vertex 1 corresponding to an edge directed away from the root vertex. In fact we will see that all vertices will be directed towards the root apart from a path from vertex 1 to vertex n. Intuitively the endpoints of edges directed away from the root are the exceptional Prüfer entries.

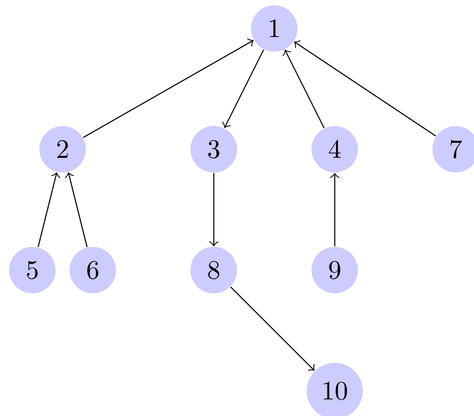

Figure 2

Case 4 ($\min P_n = 2$)**.** We will see that the vertices directed away from the root are the exceptional Prüfer entries (again). Then $a[1] = 2$ and 2 is attached to 1. Therefore edge $(1, 2)$ is directed away from 1 which is expected since if $\min P_n = 2$ then 1 is an exceptional Prüfer entry.

Example 2.2. The Prüfer sequence, $P_{10} = (2, 2, 1, 1, 4, 1, 3, 8)$ corresponds to the random recursive tree T_{10}. We have drawn $\tilde{T}_{10}$ in figure 2 (below) with directed edges which point from the element $\min L_i$ to $a[i]$ as described above.

2.4 Increasing Prüfer Sequences

Definition 2.3 (Increasing Prüfer sequence)**.** Given a Prüfer sequence, P_n, we can always build B_n as described in Section 2.3. Let $C_n = (c[1], c[2], \ldots, c[n-2])$ be the sequence defined componentwise by $c[i] = b[i] - a[i]$ for $i = 1, 2, 3, \ldots, n-2$. if C_n satisfies:

$$\begin{cases} c[i] < 0 & \text{if } i \in E \\ c[i] > 0 & \text{otherwise} \end{cases}$$

and $n \notin P_n$ then we say that P_n is an *increasing Prüfer sequence.*

This section culminates in Theorem 2.11 we will prove that if P_n is an increasing Prüfer sequence then P_n corresponds to an increasing tree.

Throughout the remainder of this section we assume that P_n is an increasing Prüfer sequence.

Lemma 2.4. *If $a[i] \in P_n[1 : e_1 - 1]$ then the vertices with label $a[i]$ are adjacent to vertices with label $b[i]$.*

Proof of Lemma 2.4. Consider $P_n[1 : e_1 - 1]$. If $e_1 = 1$ then there is nothing to do, so assume $e_1 > 1$. Then the Prüfer process dictates that the vertex labelled $a[1]$ is connected to the lowest leaf which must be $b[1]$ since $b[1] = \min L$. Let j be the smallest value such that $a[j]$ is final. Then $L_1 = L_0 \backslash \{b[1]\}$ and

$b[2] = \min L_1$ so the vertex labelled $a[2]$ is joined to $b[2]$. Similarly the vertex $a[3]$ is connected to $b[3]$ until vertex $a[j]$ which is attached to $b[j]$. Since $a[j]$ is final $a[j] \in L_{j+1}$. If $j \in E$ then we're done so assume $j \notin E$. By the condition on C_n, $a[j] < b[j] = \min L_j$ so $a[j]$ is not just available it is minimal ($\min L_{j+1} = a[j]$). Therefore at time $j+1$ the edge $(a[j+1], a[j])$ is added to make G_{j+1}. By the definition of B_n this means that $a[j+1]$ is adjacent to $b[j+1]$. $\square$

If $|E| \geq 2$ then recall that we recursively defined $m_i = \min P_n[e_{i-1} : n-2]$ for $i = 2, 3 \ldots, n-2$. For simplicity we define $P'_n = P_n[e_1 : n-2]$ and $P''_n = P_n[e_1 + 1 : n-2]$.

Given any graph G_n on $n \geq 2$ vertices the number of vertices $v \in V(G)$ such that $\deg(v) = 1$ is at least 2. In particular $|L| \geq 2$. We can think of L_i as the set of leaves of (not necessarily connected) subgraphs of T_n for $i = 1, 2, \ldots n-2$ so $|L_i| \geq 2$ for $i = 1, 2, \ldots n-2$. We know that $|L_{n-1}| = 2$. We also know that $a[n-2] \in L_{n-1}$ and $\max\{P_n[1 : n-3], L\} \in L_{n-1}$. Since we assumed that $n \notin P_n$ it must be the case that $n \in L_{n-1}$.

Lemma 2.5. *Let $|E| \geq 1$ and $e_1 < n-2$ then there does not exist a vertex $v \in L_{e_1+1}$ such that $v < m_2$.*

Proof. Let $Y_1 = a[e_1 + 1]$. Assume that there exists a vertex labelled $v < m_2$ that is attached to some $a[i] \in P''_n$. In the language of the Prüfer process v must become available to be chosen before at time $e_1 + 1$. More precisely $v \in L_{e_1+1}$.

By the Prüfer process Y_1 is attached to $v' = \min L_{e_1+1} \leq v$.

Case 5 (e_1 is final)**.** Let $m > e_1$ be the least integer such that $a[m]$ is not final. By Lemma 2.4 $a[i]$ is adjacent to $b[i]$ for $i = 1, 2, \ldots e_1 - 1$. Therefore $b[m+1] = v' \leq v < m_2$ this contradicts C_n so there does not exist such an m.

Therefore either every $a[i] \in P'_n$ is final. If this is the case then by C_n: $a[e_1] < a[e_1+1] < \cdots < a[n-2] < n$ and there cannot be a path between v' and 1.

Case 6 (e_1 is not final)**.** By Lemma 2.4 $a[i]$ is adjacent to $b[i]$ for $i = 1, 2, \ldots e_1 - 1$ therefore $b[e_1 + 1] = v' < Y_1$ contradicting C_n.

$\square$

Lemma 2.6. *Let P_n be defined as in Lemma 2.5. Then $e_1 = m_2$.*

Proof. Assume for a contradiction that $e_1 \neq m_2$.

By Lemma 2.4 all vertices adjacent to m_2 which were joined to $a[i]$ for $i = 1, 2, 3, \ldots, e_1$ have higher labels than m_2. These vertices are further only attached to vertices with labels higher than m_2 in addition to the edges joining them to these vertices etc.. By Lemma 2.5 every $m_2 \in P'_n$ is adjacent to some vertex with a higher label. Any of these vertices must be joined to vertices with label larger than m_2 by the same argument. Therefore there does not exist a path from m_1 to m_2. This cannot be the case since a Prüfer sequence always gives rise to a tree. Therefore $m_2 = Y_1$. $\square$

Corollary 2.7. *Every Prüfer position corresponds to exactly one distinct Prüfer entry.*

Lemma 2.8. *Assume $n > 3$, if $e_1 = n-2$ or $E = \emptyset$ then vertices with label $a[i]$ are adjacent to vertices with label $b[i]$ for $i = 1, 2, 3, \ldots, n-2$.*

Proof. By Lemma 2.4 vertex $a[i]$ is adjacent to vertex $b[i]$ for $i = 1, 2, \ldots, \mathrm{Pos}_{1X_1}$.

(i) If $E = \emptyset$ then, by Lemma 2.4 there is nothing to check.

(ii) If $e_1 = n-2$ then $a[n-3] = m_1$. If $m_1 = 1$ then $1 = \min L_{n-2}$ and $1 = b[n-2]$ as required. If $m_1 = 2$ then 1 must have been attached to $a[1]$, since $n > 3$ this is not the case.

□

Lemma 2.9. *Every vertex $a[i]$ is attached via an edge to $b[i]$ in T_n.*

Proof. By Lemma 2.8 we're done if $e_1 = n-2$ or $E = \emptyset$ so assume that $E \geq \emptyset$ and $e_1 \neq n-2$.

By Lemma 2.4 all $a[i] \in P_n[1 : e_1 - 1]$ are attached via an edge to vertices $b[i]$. So $a[e_1]$ must be attached to vertex labelled 1 which is $b[e_1]$.

Using a similar argument to Lemma 2.4 the subsequent $a[i]$ are attached to $b[i]$ until $a[\mathrm{Pos}_{m_2, X_{m_2}}]$ but by Lemma 2.6 $m_2 = e_1$. So we can use the same argument as Lemma 2.6 to say that e_2 is attached to $b[e_2]$ as required. Therefore if P_n is a Prüfer sequence with the condition C_n then at time i we construct G_i by adding $(a[i-1], b[i-1])$ to G_{i-1}.

□

Lemma 2.10. *If $b[i] = \min L_i$ and $E \neq \emptyset$ then $a[n-2]$ is an exceptional Prüfer entry.*

Note that if $b[i] = \min L_i$ then at time i the vertex labelled $a[i]$ is attached via an edge to vertex labelled $b[i]$.

Proof of 2.10. Since $E \neq \emptyset$ there exists $e_1 \in E$. If $e_1 = n-2$ the we are done. If $e_1 \neq n-2$ then there exists some $m_2 = \min P_n[e_1 : n-2]$ and $e_2 = Pos_{m_2, X_{m_2}+1}$. Again if $e_2 = n-2$ we are done. After enough iterations of this process it must be the case that either $e_i = n-2$ or $a[n-2] = a[n-3] = \ldots a[n-m]$ and $e_i = a[n-m]$ in either case $a[n-2]$ is an exceptional Prüfer entry. □

Theorem 2.11. *If P_n adheres to condition C_n then P_n corresponds to a random recursive tree.*

Proof. Consider any $v \in P_n$ that is not an exceptional Prüfer entry then v occurs X_v times in P_n. If $v \neq m_1$ this corresponds to X_v edges with v as an endpoint such that the other endpoints are vertices with higher labels than v (in particular these vertices are $b[\mathrm{Pos}_{v,1}], b[\mathrm{Pos}_{v,2}], \ldots, b[Pos_{v,X_v}]$). By Lemma 2.10 v is not in position $n-2$ so v is also attached to one vertex with a lower vertex: $a[Pos_{v,X_v} + 1]$. If $m_1 = 1$ then we do not need to worry since for an increasing tree only the *non-root* vertices are required to be attached to precisely one vertex with higher label. If $m_1 = 2$ then m_1 is exceptional.

If v is an exceptional Prüfer entry then v occurs X_v times in P_n which, by Corollary 2.7, corresponds to $X_v - 1$ edges adjacent to v such that the adjacent vertices have a higher label than v and 1 edge adjacent to a vertex with a lower label.

If $\text{Pos}_{v,X_v} \neq n-2$ then v is also attached to another vertex with higher label. If the final $a[\text{Pos}_{v,X_v}]$ is in position $n-2$ then, by the discussion before Lemma 2.5 v is attached to vertex n and clearly $n > v$. Therefore all vertices other than 1 are attached to exactly 1 vertex with lower label than themselves so T_n is an increasing tree. $\square$

2.5 Further Directions

Consider the set of RRTs on n vertices: $\mathcal{T}_n$. For each tree $T_n \in \mathcal{T}$ if we send every label i to $n-i$ then we form a new set of trees $\tilde{\mathcal{T}}_n$.

The allowable Prüfer sequences are $P_n = (a[1], a[2], \ldots, a[n-2])$ such that $a[i] \in \{2, 3, \ldots, n-i\}$. Clearly then there are $n-1!$ possible such Prüfer sequences.

Let T_n be a tree chosen at random from the set of n^{n-2} labeled trees. Let L_n be the number of leaves of T_n, Renyi [3] proved that almost surely:

$$E(L_n) \ \frac{n}{e} \tag{1}$$

$$var(L_n) \ \frac{(e-2)n}{e^2} \tag{2}$$

Renyi's proof of these theorems relies upon the Prüfer correspondence discussed here. Our correspondence between increasing Prüfer sequences and increasing trees may yield similar theorems for increasing trees.

3 Counting Leaves

Let $T(n,k)$ be the number of increasing trees on n vertices with exactly k leaves then $T(n,k) > 0$ if and only if $1 < k < n$. Given some increasing tree T_n with l_n leaves we can build n new trees by attaching a vertex labeled $n+1$ via an edge to any $n \in V(T_n)$. The total number of nonisomorphic increasing trees on n vertices is $n-1!$ and since each tree must have some number of leaves we form the following relation:

$$\sum_{k=2}^{n-1} T(n,k) = (n-1)! \tag{3}$$

Let l_{n+1} be the number of leaves of any tree T_{n+1} built from T_n in the way that we have described, then:

$$l_{n+1} = \begin{cases} l_n & \text{if } n+1 \text{ is attached to a leaf} \\ l_n + 1 & \text{otherwise} \end{cases}$$

Therefore we can recursively define $T(n,k)$.

Example 3.1.

$$\begin{aligned} T(6,5) &= T(5,4) \\ T(6,4) &= 4T(5,4) + 2T(5,3) \\ T(6,3) &= 3T(5,3) + 3T(5,2) \\ T(6,2) &= 2T(5,2) \end{aligned}$$

In general if $1 < k < n$ then:

$$T(n,k) = \begin{cases} T(n-1,n-2) & \text{if } k = n-1 \\ kT(n-1,k) + (n-k)T(n-1,k-1) & \text{if } 2 < k < n-1 \\ 2T(n-1,2) & \text{if } k = 2 \end{cases} \tag{4}$$

Therefore $T(n,n-1) = 2$ for $n > 2$. Similarly we can see by induction that $T(n,2) = 2^{n-2}$.

Let μ_n be the expected proportion of vertices of an increasing tree, T_n, with degree 1. In addition, let ν_n be the expected number of vertices that are leaves for an increasing tree on n leaves.

$$\nu_n = \frac{1}{n-1!} \sum_{k=2}^{k=n-1} kT(n,k) \tag{5}$$

We can write that $\mu_n = \frac{\nu_n}{n}$.

Notice that the recurrence relation for Equation 4 also holds for stirling numbers of the second kind.

By combining equation 4 with equation 5 and simple manipulation of summations we find that

$$\mu_n = \frac{1}{n!}\left(T(n-1,n-2) + 2T(n-1,2) + \sum_{k=3}^{n-2} k\left(kT(n-1,k) + (n-k)T(n-1,k-1)\right)\right) \tag{6}$$

$$= \frac{1}{n!}\left(\sum_{k=2}^{n-2} k^2 T(n-1,k) + \sum_{k=3}^{n-1} \left(k(n-k)T(n-1,k-1)\right)\right) \tag{7}$$

$$= \frac{1}{n!}\left(\sum_{k=2}^{n-2} k^2 T(n-1,k) + \sum_{k=2}^{n-2} (k+1)(n-(k+1))T(n-1,k)\right) \tag{8}$$

$$\tag{9}$$

The two sums in equation 6 can be combined since they sum between the same limits (2 and $n-2$) and involve $T(n-1,k)$. In order to determine the asymptotic value of μ_n we wish to find a simple recurrence relation. Prüfer

$$\mu_n = \frac{1}{n!}\left(\sum_{k=2}^{n-2} knT(n-1,k) + \sum_{k=2}^{n-2}(n-1)T(n-1,k) - \sum_{k=2}^{n-2} 2kT(n-1,k)\right)$$
$$= \mu_{n-1} + \frac{1}{n} - \frac{2\mu_{n-1}}{n}$$

Where the last equality above used equation 3. Therefore in the limit as $n \to \infty$ $\mu_n \to \frac{1}{2}$ as we expected.

4 Shaving random recursive trees

Let T_n be a RRT. We say that the induced subtree T_n^s such that $V(T_n^s) = \{v \in V(T_n) : \deg(v) > 1\}$ is T_n *shaved*. In this section we produce a simple algorithm for deducing the Prüfer sequence of $V(T_n^s)$.

Let $P_n = (a[1], a[2], \ldots, a[n-2])$ be the Prüfer sequence associated with the random recursive tree T_n on n vertices built in the way described in Section 2.5. Recall that we have built P_n in such a way that $a[i]$ is the parent of vertex $i+2$ for $i = 3, 4, \ldots, n$ and that the set $L_n = \mathbb{N}_n / P_n$ corresponds to the leaves of T_n.

We denote the Prüfer sequence corrosponding to T_n^s by

$$P_n^s = (a'[1], a'[2], \ldots, a'[m])$$

where $m = n - |L_n|$. Loosely speaking we will build P_n^s by removing entries from P_n. Let R be the set of removed vertices from P_n. An algorithm which describes this process is as follows:

(i) Let R be the set of removed vertices from P_n and initially $R = \emptyset$.

(ii) For every $l \in L_n$ we set $a[l-2] \in R$.

(iii) Let the quotient $Q = P_n/R$ be the sequence of vertices in P_n and not in R. If we write $Q = (a'[1], a'[2], \ldots, a'[m])$ then $a'[1]$ is the first entry in P_n that is not in R, $a'[2]$ is the second entry in P_n that is not in R and so on. Then $Q = P_n^s$

Remark 1. *The shaved RRT, T_n^s, is* not *itself a random recursive tree since the vertices' labels need not be consecutive integers (or even start at 1), however it is relatively straight forward to produce a RRT isomorphic to T_n^s. Since $|V(T_n^s)| = m$ is finite we can order $V(T_n^s)$ from lowest to highest so let $f : V(T_n^s) \to \mathbb{N}_m$ take the vertex with the lowest label to 1, the second lowest label to 2 and so on. We call the RRT built in the way described the* reduced RRT *and denote it by $\tilde{T}_n^s$ and the corrosponding Prüfer sequence is denoted by $\tilde{P}_n^s = (\tilde{a}'[1], \tilde{a}'[2], \ldots, \tilde{a}'[m])$.*

Question 1. *Does there exists a $(1, j)$-star in T_n^s which permutes $(1, k)$-stars in T_n?*

We have not yet built the appropriate tree to answer this question. We need to classify the leaves of T_n^s by the relevant $(1, k)$-star that has been shaved off. Let m(i) denote the multiplcity of each $i \in P_n$ and colour each $i \in R$ by $m(i)$. So the leaves, $\tilde{L}_n^s$, of $\tilde{T}_n^s$ can be partitioned into classes $C_1, C_2, \ldots$ according to their colour.

Now we can define the *k-reduced random recursive forest*, $\tilde{T}_{n,k}^s$ which is built as follows:

(i) The set of leaves of $\tilde{T}_{n,k}^s$ are defined to be $\tilde{L}_{n,k}^s := C_k$.

(ii) Remove every element $\tilde{a}_n^s[i]$ such that $i - 2 \notin C_k$.

We can associate the Prüfer sequence $\tilde{P}_{n,k}^s$ with any k-reduced random recursive forest $\tilde{T}_{n,k}^s$.

Example 4.1.

References

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