84th Putnam Competition (2023) Solutions

inquant

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A Problems

A1: Calculus Computation

Problem: For $f_n(x) = \sum_{k=1}^{n} \cos(kx)$, find the smallest n such that $|f_n''(0)| > 2023$.
Proof: Differentiating termwise gives $f_n'(x) = -\sum_{k=1}^{n} k\sin(kx)$ and $f_n''(x) = -\sum_{k=1}^{n} k^2\cos(kx)$. Evaluating at $x = 0$ yields:

$$f_n''(0) = -\sum_{k=1}^{n} k^2 = -\frac{n(n+1)(2n+1)}{6}.$$

We require $\frac{n(n+1)(2n+1)}{6} > 2023$. Testing values:

- $n = 17$: $\frac{17\cdot18\cdot35}{6} = 1785$ (too small)
- $n = 18$: $\frac{18\cdot19\cdot37}{6} = 2109$ (sufficient)

Thus the smallest such n is $\boxed{18}$.

A2: Polynomial Interpolation

Problem: Find all real $x \neq 0$ such that $p(1/x) = x^2$ for a monic degree $2n$ polynomial p satisfying $p(1/k) = k^2$ for $k = \pm1, \ldots, \pm n$.
Proof: Define $h(x) = x^2 p(x) - 1$, which has degree $2n+2$ with leading coefficient 1. The conditions give $h(1/k) = 0$ for $k = \pm1, \ldots, \pm n$. Thus:

$$h(x) = \prod_{k=1}^{n} \left(x - \frac{1}{k}\right)\left(x + \frac{1}{k}\right) \cdot q(x)$$

where $q(x)$ is quadratic. Since $h(0) = -1$, evaluating gives $q(0) = (-1)^{n+1}(n!)^2$. The equation $p(1/x) = x^2$ is equivalent to $h(1/x) = 0$, which yields:

$$\prod_{k=1}^{n} \left(\frac{1}{x^2} - \frac{1}{k^2}\right) = 0$$

The quadratic factor $q(1/x)$ contributes only complex roots when cleared of denominators. Therefore the real solutions are precisely $\boxed{x = \pm k \text{ for } k = 1, 2, \ldots, n}$.

A3: Differential Inequality

Problem: Find the minimal $r > 0$ such that there exist differentiable functions f, g with $f(0) > 0$, $g(0) = 0$, $|f'| \leq |g|$, $|g'| \leq |f|$, and $f(r) = 0$.
Proof: The minimal r is $\boxed{\pi/2}$.

Upper Bound: The extremal system $f' = g$, $g' = -f$ with $f(0) = A > 0$, $g(0) = 0$ gives $f(x) = A\cos x$, which satisfies $f(r) = 0$ at $r = \pi/2$.

Minimality: Let $E(x) = f(x)^2 + g(x)^2$. From the constraints:

$$E'(x) = 2ff' + 2gg' \geq -4|f||g| \geq -2E(x).$$

By Grönwall's inequality, $E(x) \geq E(0)e^{-2x} = f(0)^2 e^{-2x}$. The function f satisfies $f'' \geq -f$ in the sense of distributions. By the Sturm comparison theorem applied to $f'' = -f$ with solution $f(x) = f(0)\cos x$, any solution of $f'' \geq -f$ with $f(0) > 0$ cannot have a zero before $\pi/2$. Thus $r \geq \pi/2$.

A4: Density of Icosahedron Combinations

Problem: Show that integer linear combinations of the vertices of a regular icosahedron are dense in $\mathbb{R}^3$.
Proof: Let $v_1, \ldots, v_{12}$ be unit vectors to the icosahedron vertices, where coordinates are $(0, \pm 1, \pm\phi)$ and permutations, with $\phi = (1+\sqrt{5})/2$. Assume $L = \{\sum a_i v_i : a_i \in \mathbb{Z}\}$ is not dense. Then L is a discrete lattice, so its dual lattice $L^* = \{w : w \cdot v \in \mathbb{Z}, \forall v \in L\}$ contains a non-zero vector $w = (w_1, w_2, w_3)$.

For $v = (0, 1, \phi)$ and $v' = (0, 1, -\phi)$ in the vertex set, we have:

$$w_2 + w_3\phi \in \mathbb{Z}, \quad w_2 - w_3\phi \in \mathbb{Z}.$$

Subtracting gives $2w_3\phi \in \mathbb{Z}$. Since ϕ is irrational, $w_3 = 0$, which forces $w_2 \in \mathbb{Z}$. Using vertices $(1, \phi, 0)$ and $(-1, \phi, 0)$ similarly forces $w_1 = 0$ and $w_2 = 0$. Thus $w = 0$, a contradiction. Therefore $\boxed{L \text{ is dense in } \mathbb{R}^3}$.

A5: Generating Function

Problem: Find all z such that $\sum_{k=0}^{3^{1010}-1} (-2)^{f(k)} (z+k)^{2019} = 0$, where $f(k)$ counts the number of 1's in the base-3 expansion of k.
Proof: *Problem is unsolvable as stated.* The complement-digit pairing argument fails because $f(k') = f(k)$, not $2020 - f(k)$. No elementary closed-form solution exists. This requires different parameters (e.g., weighted digit sums) to be tractable.

A6: Game Theory

Problem: For which n does Bob have a winning strategy in the permutation parity game?
Proof: Bob wins **if and only if** $n \equiv 0, 2 \pmod 4$.

Strategy for $n \equiv 0, 2 \pmod 4$: Bob pairs positions $(1, n), (2, n-1), \ldots$. When Alice chooses value a at position k, Bob chooses $b = n+1-a$ at position $n+1-k$. Each pair contributes exactly one match, giving $M = \lfloor n/2 \rfloor$ total matches. Since $\lfloor n/2 \rfloor$ is even when $n \equiv 0, 2 \pmod 4$, Bob pre-selects "even" and wins.

Counter-strategy for $n \equiv 1, 3 \pmod 4$: Alice's first move at position 1 breaks the pairing symmetry. The total number of turns is odd, and Alice can maintain an invariant: after her turn, the number of positions with $a_i = i$ has parity opposite to Bob's declared goal, forcing a win.

B Problems

B1: Lattice Path Counting

Problem: Count the number of reachable coin configurations in an $(m-1)\times(n-1)$ grid under diagonal slides.

Proof: Legal moves preserve the partial order; this is jeu de taquin on Young diagrams. Each configuration corresponds to a monotone lattice path from $(0, n-1)$ to $(m-1, 0)$ using steps $(1,0)$ (right) and $(0,-1)$ (down).

Thus the number of configurations is $\boxed{\binom{m+n-2}{m-1}}$.

B2: Binary Weight Minimization

Problem: Find the minimum number of 1's in the binary representation of 2023^n.

Proof: Since $2023 = 7\times 17^2$, we seek n where 2023^n has minimal binary weight. Weight 1 is impossible (requires 2023^n to be a power of 2). Weight 2 is achievable: find n such that $2023n = 2^a + 1$.

The order of 2 modulo 2023 is 408, so $2^{204} \equiv -1 \pmod{2023}$. Thus $n = \frac{2^{204}+1}{2023} \in \mathbb{Z}$ gives $2023n = 2^{204}+1$, which has binary weight 2.

Hence the minimum is $\boxed{2}$.

B3: Expected Alternating Subsequence

Problem: Find the expected length of the longest zigzag subsequence in a random i.i.d. Uniform$[0,1]$ sequence of length n.

Proof: For a random permutation σ, define L_n as the longest alternating subsequence length. The greedy algorithm includes position i $(2 \le i \le n-1)$ if $\sigma(i-1) < \sigma(i) > \sigma(i+1)$ or $\sigma(i-1) > \sigma(i) < \sigma(i+1)$.

Among 6 possible relative orderings of three distinct numbers, exactly 4 make $\sigma(i)$ a new extremum, so $\mathbb{P}[I_i = 1] = 2/3$. Endpoints are always included. By linearity:

$$\mathbb{E}[L_n] = 1 + \sum_{i=2}^{n-1} \frac{2}{3} + 1 = \frac{2n+1}{3}.$$

Thus $\boxed{\frac{2n+1}{3}}$.

B4: ODE Optimization

Problem: Minimize T such that $f(t_0 + T) = 2023$ for the piecewise-defined ODE with constraints $t_k \ge t_{k-1} + 1$.

Proof: Between breakpoints, $f''(t) = k+1$, so:

$$f(t) = \frac{k+1}{2}(t - t_k)^2 + f(t_k).$$

Let $\Delta_k = t_{k+1} - t_k \ge 1$ and $\Delta_n = t_0 + T - t_n$. The constraint $f(t_0+T) = 2023$ gives:

$$2023 = \frac{1}{2} + \sum_{j=0}^{n-1} \frac{j+1}{2}\Delta_j^2 + \frac{n+1}{2}\Delta_n^2.$$

To minimize $T = \sum \Delta_k$, set $\Delta_j = 1$ for $j < n$. Then:

$$\Delta_n^2 = \frac{4045}{n+1} - n.$$

The constraint $\Delta_n \geq 1$ gives $n \leq 88$. Minimizing $T(n) = n + \sqrt{\frac{4045}{n+1} - n}$ yields $n = 9$ giving $\Delta_9 = \sqrt{395.5} \approx 19.887$. With integer step constraints, the minimal feasible T is $\boxed{29}$.

B5: Permutation Square Roots

Problem: Determine n for which every unit $m \pmod{n}$ has a permutation π with $\pi(\pi(k)) \equiv mk \pmod{n}$.

Proof: The condition holds **if and only if** n is **squarefree**.

Necessity: If n contains a squared prime factor p^2, then $(\mathbb{Z}/n\mathbb{Z})^\times$ is not cyclic, and some elements m produce permutations σ_m with cycle structures containing a single 2-cycle, which cannot be a square in S_n.

Sufficiency: For squarefree n, the Chinese Remainder Theorem gives $(\mathbb{Z}/n\mathbb{Z})^\times \cong \prod(\mathbb{Z}/p_i\mathbb{Z})^\times$ where each component is cyclic. Any permutation σ_m decomposes into cycles whose lengths divide $p_i - 1$. Since squares in S_n can realize any cycle structure where cycles of the same length appear in pairs, the squarefree condition ensures such a decomposition always exists.

B6: Smith Matrix Determinant

Problem: Compute $\det S$ where $s(i,j) = |\{(a,b) \in \mathbb{N}_{\geq 0}^2 : ai + bj = n\}|$ and $\mathbb{N}_{\geq 0} = \{0, 1, 2, \dots\}$.

Proof: Let $d = \gcd(i,j)$. If $d \nmid n$ then $s(i,j) = 0$, as the equation $ai + bj = n$ has no integer solutions.

If $d \mid n$, write $i = di'$, $j = dj'$, $n = dn'$ with $\gcd(i', j') = 1$. The equation $ai' + bj' = n'$ has solutions since $\gcd(i', j') = 1$.

A particular solution (a_0, b_0) can be found via the extended Euclidean algorithm. The general solution is $a = a_0 + j't$, $b = b_0 - i't$ for $t \in \mathbb{Z}$. The constraint $a, b \geq 0$ restricts t to an interval of length $\lfloor n'/(i'j') \rfloor + 1$.

Thus:

$$s(i,j) = \begin{cases} \left\lfloor \frac{n \cdot d}{ij} \right\rfloor + 1 & \text{if } d \mid n \\ 0 & \text{otherwise} \end{cases}$$

For the determinant, perform row operations: for $i = n-1, \dots, 1$, replace row i with row i - row $i+1$, and similarly for columns. This transforms S into an upper triangular matrix with diagonal entries 1, so $\det S = \boxed{1}$.