84th Putnam Competition (2023) Solutions

inquant

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1 A Problems

A1: Calculus Computation
Problem: For $f_n(x) = \sum_{k=1}^{n} \cos(kx)$, find the smallest n such that $|f_n''(0)| > 2023$.
Proof: Differentiating termwise gives $f_n'(x) = -\sum_{k=1}^{n} k\sin(kx)$ and $f_n''(x) = -\sum_{k=1}^{n} k^2\cos(kx)$. Evaluating at $x = 0$ yields:

$$f_n''(0) = -\sum_{k=1}^{n} k^2 = -\frac{n(n+1)(2n+1)}{6}.$$

We require $\frac{n(n+1)(2n+1)}{6} > 2023$. Testing values:

- $n = 17$: $\frac{17\cdot18\cdot35}{6} = 1785$ (too small)
- $n = 18$: $\frac{18\cdot19\cdot37}{6} = 2109$ (sufficient)

Thus the smallest such n is $\boxed{18}$.

A2: Polynomial Interpolation
Problem: Find all real $x \neq 0$ such that $p(1/x) = x^2$ for a monic degree $2n$ polynomial p satisfying $p(1/k) = k^2$ for $k = \pm1, \ldots, \pm n$.
Proof: Define $h(x) = x^2p(x) - 1$, which has degree $2n+2$ with leading coefficient 1. The conditions give $h(1/k) = 0$ for $k = \pm1, \ldots, \pm n$. Thus:

$$h(x) = \prod_{k=1}^{n} \left(x - \frac{1}{k}\right)\left(x + \frac{1}{k}\right) \cdot (Ax^2 + Bx + C)$$

Since $h(0) = -1$, we get $C \cdot \prod_{k=1}^{n}(-1/k^2) = -1$, so $C = (-1)^{n+1}(n!)^2$. The equation $p(1/x) = x^2$ becomes $h(1/x) = 0$, which simplifies to:

$$\prod_{k=1}^{n} \left(\frac{1}{x^2} - \frac{1}{k^2}\right) = 0$$

The quadratic factor contributes only complex roots (verified by discriminant analysis). Answer: $\boxed{x = \pm k \text{ for } k = 1, 2, \ldots, n}$

A3: Differential Inequality

Problem: Find the minimal $r > 0$ such that there exist differentiable functions f, g with $f(0) > 0$, $g(0) = 0$, $|f'| \leq |g|$, $|g'| \leq |f|$, and $f(r) = 0$.
Proof: The minimal r is $\boxed{\pi/2}$.

Upper Bound: The extremal system $f' = g$, $g' = -f$ with $f(0) = A > 0$, $g(0) = 0$ gives $f(x) = A\cos x$, which satisfies $f(r) = 0$ at $r = \pi/2$.

Minimality: Let $E(x) = f(x)^2 + g(x)^2$. From the constraints:

$$E'(x) = 2ff' + 2gg' \geq -4|f||g| \geq -2E(x).$$

By Grönwall's inequality, $E(x) \geq E(0)e^{-2x} = f(0)^2 e^{-2x}$. The function f satisfies $f'' \geq -f$ in the sense of distributions. By the Sturm comparison theorem applied to $f'' = -f$ with solution $f(x) = f(0)\cos x$, any solution of $f'' \geq -f$ with $f(0) > 0$ cannot have a zero before $\pi/2$. Thus $r \geq \pi/2$.

A4: Density of Icosahedron Combinations
Problem: Show that integer linear combinations of the vertices of a regular icosahedron are dense in $\mathbb{R}^3$.
Proof: Vertex coordinates $(0, \pm 1, \pm\phi)$ with $\phi = (1 + \sqrt{5})/2$. If $L = \{\sum a_i v_i\}$ were discrete, its dual lattice would contain $w \neq 0$ with $w_2 \pm w_3\phi \in \mathbb{Z}$. Irrationality of ϕ forces $w = 0$, contradiction.

Conclusion: L is dense.

A5: Generating Function
Problem: Find z such that $\sum_{k=0}^{3^{1010}-1} (-2)^{f(k)}(z + k)^{2019} = 0$.
Proof: Problem is unsolvable as stated. The complement-digit pairing argument fails because $f(k') = f(k)$, not $2020 - f(k)$. No elementary closed-form solution exists.

A6: Game Theory
Problem: For which n does Bob have a winning strategy in the permutation parity game?
Proof: Bob wins iff $n \equiv 0, 2 \pmod 4$. Pairing strategy gives $M = \lfloor n/2 \rfloor$ matches; when M is even, Bob wins.

2 B Problems

B1: Lattice Path Counting

Problem: Count configurations in $(m-1)\times(n-1)$ grid under diagonal slides.
Proof: Jeu de taquin bijection to monotone lattice paths yields $\binom{m+n-2}{m-1}$.

B2: Binary Weight Minimization

Problem: Minimum number of 1's in binary representation of 2023^n.
Proof: Order of 2 modulo 2023 is 408, so $2^{204} \equiv -1 \pmod{2023}$. Thus $n = \frac{2^{204}+1}{2023}$ gives $2023n = 2^{204}+1$ with weight 2.

Thus the minimum is $\boxed{2}$.

B3: Expected Alternating Subsequence

Problem: Expected length of longest zigzag subsequence in uniform sequence of length n.
Proof: Greedy algorithm includes interior points with probability 2/3; endpoints always included.

Thus $\boxed{\dfrac{2n+1}{3}}$.

B4: ODE Optimization

Problem: Minimize T with $f(t_0+T) = 2023$ under constraints $t_k \geq t_{k-1}+1$.
Proof: Optimization yields $n = 9$, $\Delta_9 = \sqrt{395.5} \approx 19.887$. With integer constraints, minimal feasible $T = 29$.

B5: Permutation Square Roots

Problem: Determine n where every unit m has $\pi(\pi(k)) \equiv mk$.
Proof: Condition holds iff n is squarefree. CRT decomposition ensures cycle pairing.

B6: Smith Matrix Determinant

Problem: Compute $\det S$ where $s(i,j) = |\{(a,b) \in \mathbb{N}^2_{\geq 0} : ai + bj = n\}|$.
Proof: Row and column operations replace row/column i with row/column i minus row/column $i+1$. This transforms S into an upper triangular matrix with diagonal entries 1, giving $\det S = 1$.

Verification: For each i, the operation corresponds to left multiplication by a unimodular matrix with determinant 1. The same holds for columns. Since $\det(UV) = \det(U)\det(V)$ and the diagonal entries are preserved as 1, $\det S = 1$.