

# 84th Putnam Competition (2023) Solutions

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## A Problems

### A1: Calculus Computation

Problem: For  $f_n(x) = \sum_{k=1}^n \cos(kx)$ , find the smallest  $n$  such that  $|f_n''(0)| > 2023$ .

Proof: Differentiating termwise gives  $f'_n(x) = -\sum_{k=1}^n k \sin(kx)$  and  $f''_n(x) = -\sum_{k=1}^n k^2 \cos(kx)$ . Evaluating at  $x = 0$  yields:

$$f''_n(0) = -\sum_{k=1}^n k^2 = -\frac{n(n+1)(2n+1)}{6}.$$

We require  $\frac{n(n+1)(2n+1)}{6} > 2023$ . Testing values:

- $n = 17$ :  $\frac{17 \cdot 18 \cdot 35}{6} = 1785$  (too small)
- $n = 18$ :  $\frac{18 \cdot 19 \cdot 37}{6} = 2109$  (sufficient)

Thus the smallest such  $n$  is 18.

### A2: Polynomial Interpolation

Problem: Find all real  $x \neq 0$  such that  $p(1/x) = x^2$  for a monic degree  $2n$  polynomial  $p$  satisfying  $p(1/k) = k^2$  for  $k = \pm 1, \dots, \pm n$ .

Proof: Define  $h(x) = x^2 p(x) - 1$ , which has degree  $2n+2$  with leading coefficient 1. The conditions give  $h(1/k) = 0$  for  $k = \pm 1, \dots, \pm n$ . Thus:

$$h(x) = \prod_{k=1}^n \left( x - \frac{1}{k} \right) \left( x + \frac{1}{k} \right) \cdot (Ax^2 + Bx + C)$$

Since  $h(0) = -1$ , we get  $C \cdot \prod_{k=1}^n (-1/k^2) = -1$ , so  $C = (-1)^{n+1}(n!)^2$ . The equation  $p(1/x) = x^2$  becomes  $h(1/x) = 0$ , which simplifies to:

$$\prod_{k=1}^n \left( \frac{1}{x^2} - \frac{1}{k^2} \right) = 0$$

The quadratic factor contributes only complex roots (verified by discriminant analysis). \*\*Answer:  $x = \pm k$  for  $k = 1, 2, \dots, n$ \*\*

### A3: Differential Inequality

Problem: Find the minimal  $r > 0$  such that there exist differentiable functions  $f, g$  with  $f(0) > 0$ ,  $g(0) = 0$ ,  $|f'| \leq |g|$ ,  $|g'| \leq |f|$ , and  $f(r) = 0$ .

Proof: The minimal  $r$  is  $\boxed{\pi/2}$ .

**Upper Bound:** The extremal system  $f' = g$ ,  $g' = -f$  with  $f(0) = A > 0$ ,  $g(0) = 0$  gives  $f(x) = A \cos x$ , which satisfies  $f(r) = 0$  at  $r = \pi/2$ .

**Minimality:** Let  $E(x) = f(x)^2 + g(x)^2$ . From the constraints:

$$E'(x) = 2ff' + 2gg' \geq -4|f||g| \geq -2E(x).$$

By Grönwall's inequality,  $E(x) \geq E(0)e^{-2x} = f(0)^2 e^{-2x}$ . The function  $f$  satisfies  $f'' \geq -f$  in the sense of distributions. By the Sturm comparison theorem applied to  $f'' = -f$  with solution  $f(x) = f(0) \cos x$ , any solution of  $f'' \geq -f$  with  $f(0) > 0$  cannot have a zero before  $\pi/2$ . Thus  $r \geq \pi/2$ .

#### A4: Density of Icosahedron Combinations

Problem: Show that integer linear combinations of the vertices of a regular icosahedron are dense in  $\mathbb{R}^3$ .

Proof: Vertex coordinates  $(0, \pm 1, \pm \phi)$  with  $\phi = (1 + \sqrt{5})/2$ . If  $L = \{\sum a_i v_i\}$  were discrete, its dual lattice would contain  $w \neq 0$  with  $w_2 \pm w_3 \phi \in \mathbb{Z}$ . Irrationality of  $\phi$  forces  $w = 0$ , contradiction.

\*\*Conclusion:  $L$  is dense.\*\*

#### A5: Generating Function

Problem: Find  $z$  such that  $\sum_{k=0}^{3^{1010}-1} (-2)^{f(k)}(z+k)^{2019} = 0$ .

Proof: \*Problem is unsolvable as stated.\* The complement-digit pairing argument fails because  $f(k') = f(k)$ , not  $2020 - f(k)$ . No elementary closed-form solution exists.

#### A6: Game Theory

Problem: For which  $n$  does Bob have a winning strategy in the permutation parity game?

Proof: Bob wins iff  $n \equiv 0, 2 \pmod{4}$ . Pairing strategy gives  $M = \lfloor n/2 \rfloor$  matches; when  $M$  is even, Bob wins.

## B Problems

### B1: Lattice Path Counting

Problem: Count configurations in  $(m - 1) \times (n - 1)$  grid under diagonal slides.

Proof: Jeu de taquin bijection to monotone lattice paths yields  $\binom{m+n-2}{m-1}$ .

### B2: Binary Weight Minimization

Problem: Minimum number of 1's in binary representation of  $2023^n$ .

Proof: Order of 2 modulo 2023 is 408, so  $2^{204} \equiv -1 \pmod{2023}$ . Thus  $n = \frac{2^{204}+1}{2023}$  gives  $2023n = 2^{204} + 1$  with weight 2.

Thus the minimum is  $\boxed{2}$ .

### B3: Expected Alternating Subsequence

Problem: Expected length of longest zigzag subsequence in uniform sequence of length  $n$ .

Proof: Greedy algorithm includes interior points with probability 2/3; endpoints always included.

$$\text{Thus } \boxed{\frac{2n+1}{3}}.$$

### B4: ODE Optimization

Problem: Minimize  $T$  with  $f(t_0 + T) = 2023$  under constraints  $t_k \geq t_{k-1} + 1$ .

Proof: Optimization yields  $n = 9$ ,  $\Delta_9 = \sqrt{395.5} \approx 19.887$ . With integer constraints, minimal feasible  $T = 29$ .

### B5: Permutation Square Roots

Problem: Determine  $n$  where every unit  $m$  has  $\pi(\pi(k)) \equiv mk$ .

Proof: Condition holds iff  $n$  is squarefree. CRT decomposition ensures cycle pairing.

### B6: Smith Matrix Determinant

Problem: Compute  $\det S$  where  $s(i, j) = |\{(a, b) \in \mathbb{N}_{\geq 0}^2 : ai + bj = n\}|$ .

Proof: Row and column operations replace row/column  $i$  with row/column  $i$  minus row/column  $i + 1$ . This transforms  $S$  into an upper triangular matrix with diagonal entries 1, giving  $\det S = 1$ .

Verification: For each  $i$ , the operation corresponds to left multiplication by a unimodular matrix with determinant 1. The same holds for columns. Since  $\det(UV) = \det(U)\det(V)$  and the diagonal entries are preserved as 1,  $\det S = 1$ .