Linear Algebra Basics

Scalars and Vectors

In linear algebra, a **scalar** is a value that can represent size or magnitude. A **vector** is an object that has both magnitude(size) and direction. Vectors are used to represent quantities like force, velocity, or displacement in physics. Algebraically, we represent vectors as an ordered list of values written in a column or a row. For example…

$$\begin{bmatrix} 1 \\ 9 \\ -13 \end{bmatrix} \text{ or } \begin{bmatrix} 20 & 5 & -6 \end{bmatrix}$$

Matrix[1]

A **matrix** is a rectangular array or table of numbers, symbols, or expressions with elements (or entries) arranged in rows and columns, which is used to represent a mathematical object or property. We can consider a matrix as a set of vectors with equal number of components. For example…

$$\begin{bmatrix} 1 & 9 & -13 \\ 20 & 5 & -6 \end{bmatrix}$$

is a matrix with two rows and three columns. This is referred to as a 2x3 matrix.

We can refer to the entire matrix with a symbol such as M and we can refer to an individual element in an array using the matrix symbol with subscripts. For example, if the matrix above were referred to as M, the element at the second row in the third column would be referred to as $m_{2,3}$ and the value of the element is -6.

Note In linear algebra, the row is notated before the column. When referring to a single element, use subscripts to denote the coordinate within the array.

Transposition

The transpose of an $m \times n$ matrix **A** is the $n \times m$ matrix $\boldsymbol{A}^T$ is formed by turning rows into columns and vice versa:

$$(\boldsymbol{A}^T)_{i,j} = \boldsymbol{A}_{j,i}$$

For example:

$$\begin{bmatrix} 1 & 9 & -13 \\ 20 & 5 & -6 \end{bmatrix}^T = \begin{bmatrix} 1 & 20 \\ 9 & 5 \\ -13 & -6 \end{bmatrix}$$

Addition

We can add two matrices together as long as the matrices are of the same dimension. The sum of **A**+**B** of two $m \times n$ matrices is calculated entry wise:

$$(\boldsymbol{A} + \boldsymbol{B})_{i,j} = \boldsymbol{A}_{i,j} + \boldsymbol{B}_{i,j}, 1 \leq i \leq m\,, 1 \leq i \leq m$$

For example, if

$$\boldsymbol{M} = \begin{bmatrix} 1 & 9 & -13 \\ 20 & 5 & -6 \end{bmatrix} \; \boldsymbol{N} = \begin{bmatrix} 14 & 0 & 5 \\ -7 & 2 & -6 \end{bmatrix} \quad \text{then} \quad \boldsymbol{M} + \boldsymbol{N} = \begin{bmatrix} 15 & 9 & -8 \\ 13 & 7 & -12 \end{bmatrix}$$

Scalar Multiplication

We find the product $c\boldsymbol{M}$ of a number c (also called a scalar in this context) and a matrix $\boldsymbol{M}$ by multiplying every entry of $\boldsymbol{M}$ by c.

$$(c\boldsymbol{A})_{i,j} = c \cdot \boldsymbol{A}_{i,j}$$

Example:

$$2\boldsymbol{M} = 2 \cdot \begin{bmatrix} 1 & 9 & -13 \\ 20 & 5 & -6 \end{bmatrix} = \begin{bmatrix} 2 & 18 & -26 \\ 40 & 10 & -12 \end{bmatrix}$$

Matrix Multiplication – Dot Product

The **dot product** of two equal sized matrices will result in a scalar value defined by

$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{A}_1\boldsymbol{B}_1 + \boldsymbol{A}_2\boldsymbol{B}_2 + \cdots + \boldsymbol{A}_n\boldsymbol{B}_n = \sum_{i=1}^{n} \boldsymbol{A}_i\boldsymbol{B}_i$$

Example:

$\boldsymbol{M} = [5 \quad -2 \quad 7]$ $\boldsymbol{N} = [3 \quad 8 \quad -6]$ then $\boldsymbol{M} \cdot \boldsymbol{N} = -43$

Matrix Multiplication – Cross Product

The **cross product** of two matrices is defined if and only if the number of columns of the left matrix is the same as the number of rows of the right matrix. If **A** is an $m \times n$ matrix and **B** is an $n \times p$ matrix, then their matrix product **AB** is the $m \times p$ matrix whose entries are given by the dot product of the corresponding row of **A** and the corresponding column of **B**.

$$[\boldsymbol{AB}]_{i,j} = \boldsymbol{A}_{i,1}\boldsymbol{B}_{1,j} + \boldsymbol{A}_{i,2}\boldsymbol{B}_{2,j} + \cdots + \boldsymbol{A}_{i,n}\boldsymbol{B}_{n,j} = \sum_{r=1}^{n} \boldsymbol{A}_{i,r}\boldsymbol{B}_{r,j}$$

where $1 \leq i \leq m$ and $1 \leq j \leq p$. Figure 1 demonstrates this process.

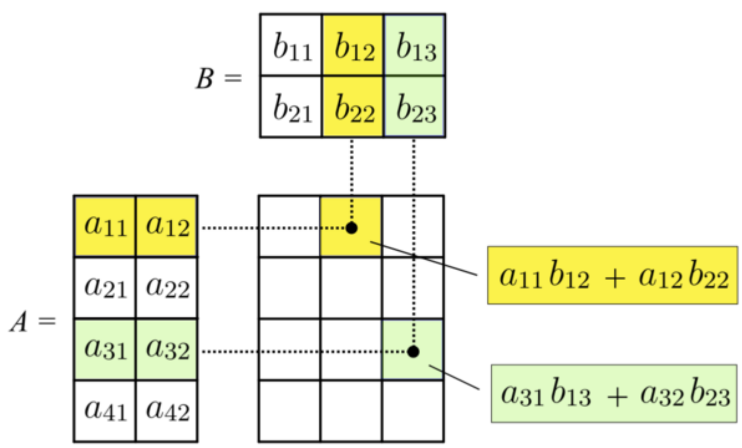


Figure 1. Depiction of matrix product

For example:

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1000 \\ 1 & 100 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 3 & 2340 \\ 0 & 1000 \end{bmatrix}$$

Identity Matrix

The identity matrix is a matrix where when you multiply another matrix to it, you end up with the same matrix.

Example:

$$\begin{bmatrix} 2 & 11 & 4 \\ 8 & 6 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 11 & 4 \\ 8 & 6 & 0 \end{bmatrix}$$

However, the identity matrix to do the opposite direction is different…

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 11 & 4 \\ 8 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 11 & 4 \\ 8 & 6 & 0 \end{bmatrix}$$

Note: Remember that when you are multiplying an $m \times n$ matrix with a $n \times p$ matrix, we get an $m \times p$ matrix. We need an identity matrix that matches the row count of the second if the identity matrix comes first and matches the column count of the first if the identity matrix comes second.

Using Linear Algebra

Scaling

When we do scaling, we are essentially translating each point in our space by a scalar factor away from the origin. This can be done with scalar multiplication.

For example, if we have a point in 2D space defined by vector v and we want to double the distance that point is from the origin, we multiply v by the scalar value of 2.

$$\text{If } \boldsymbol{V} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{then } \boldsymbol{2V} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

This works well assuming we want to scale both components by the same amount. But what if we wanted to scale the x-component and y-component by different scalar values? We can define this with the following equations…

$$x' = S_x x$$
$$y' = S_y y$$

To put this in matrix notation, we consider the 2x2 identity matrix…

$$\boldsymbol{I_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We know that the product of the identity matrix and our vector will be our original vector. To scale, we can substitute the 1's in the identity matrix with our sizing factor.

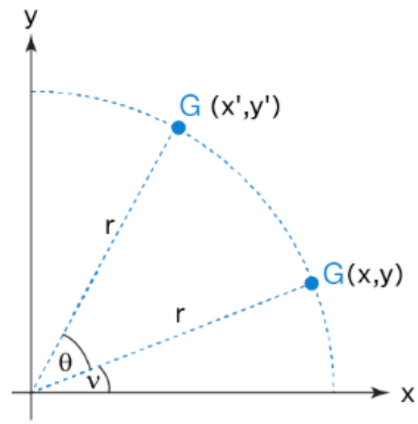
$$\boldsymbol{S} = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$

$$\boldsymbol{SV} = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} S_x x \\ S_y y \end{bmatrix}$$

Rotation

2D Rotation

If we have a point in 2D space, we can look at it like a vector with its origin at location (0,0) offset by an angle **v**. The magnitude of the vector is defined as **r** where we can rotate the point around the origin and allow the magnitude to act like the radius of a circle. The x and y components of the vector along with the magnitude form a right triangle and allow us to use trigonometric calculations.



In polar form, we have

$$x = r\cos v$$
$$y = r\sin v$$

Similarly, expressing (x’,y’) in polar form

$$x' = r\cos(v+\theta)$$
$$y' = r\sin(v+\theta)$$

Using trigonometric identities

$$x' = r(\cos v\cos\theta - \sin v\sin\theta)$$
$$x' = \boldsymbol{r\cos v}\cos\theta - \boldsymbol{r\sin v}\sin\theta$$
$$y' = r(\sin v\cos\theta + \cos v\sin\theta)$$
$$y' = \boldsymbol{r\sin v}\cos\theta + \boldsymbol{r\cos v}\sin\theta$$

If we substitute x and y from above to replace the $\boldsymbol{r\sin v}$ and $\boldsymbol{r\cos v}$, we get

$$x' = x\cos\theta - y\sin\theta$$
$$y' = y\cos\theta + x\sin\theta$$

In linear algebra notation, we can represent these equations

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ y\cos\theta + x\sin\theta \end{bmatrix}$$

To get these results, we would have done the following cross product.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

We can define the matrix $\mathbf{R_1}$ as our rotation matrix

$$\boldsymbol{R_1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Direction of Rotation

We should also note that a positive value of θ will result in counterclockwise rotation while a negative value will result in clockwise rotation. Our single rotation matrix will work for both directions. This makes sense as sine and cosine are periodic functions that are reflections across the y-axis and origin respectively.

Working in 3 Dimensions

For our purposes in 3-dimensional space, we need to account for the z-component even though we are not using it. We can take the rotation matrix $\mathbf{R_1}$ and expand it with quantities from the 3x3 identify matrix to give us our final rotation matrix **R**.

$$\boldsymbol{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can see that our product still handles the rotation and leaves the z-component as 0.

$$\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta + 0 \\ y\cos\theta + x\sin\theta + 0 \\ 0 + 0 + 0 \end{bmatrix}$$

We can do the same process with our scaling matrix.

$$\boldsymbol{S} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can then take the cross product of our rotation and scaling matrices to make a single matrix to handle both operations.

$$\boldsymbol{RS} = \begin{bmatrix} S_x\cos\theta & -S_y\sin\theta & 0 \\ S_x\sin\theta & S_y\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: **RS** is not the same as **SR**.

$$\boldsymbol{SR} = \begin{bmatrix} S_x\cos\theta & -S_x\sin\theta & 0 \\ S_y\sin\theta & S_y\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Translation

In order for us to offset a point, we add translation values to the x and y components.

$$x' = x + T_x$$
$$y' = y + T_y$$

In matrix notation…

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix}$$

We can do this translation using cross products by expanding our identity matrix to include another row and column and adding a value of 1 into our vector.

$$\begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + T_x \\ y + T_y \\ 1 \end{bmatrix}$$

Note: Adding the value of 1 is not arbitrary. We need to make sure the T_x value is added to x after the product is evaluated. Using the value 1 gives us that.

For 3-dimensional space, we expand again to account for a z-component that is always zero.

$$\begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x + T_x \\ y + T_y \\ 0 \\ 1 \end{bmatrix}$$

We can expand our **RS** matrix to 4x4.

$$\boldsymbol{RS} = \begin{bmatrix} S_x \cos\theta & -S_y \sin\theta & 0 & 0 \\ S_x \sin\theta & S_y \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

And finally, combine all three matrices into a single transformation matrix.

$$\boldsymbol{TRS} = \begin{bmatrix} S_x \cos\theta & -S_y \sin\theta & 0 & T_x \\ S_x \sin\theta & S_y \cos\theta & 0 & T_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$