EMTH403

Mathematical Foundation for Computer Science

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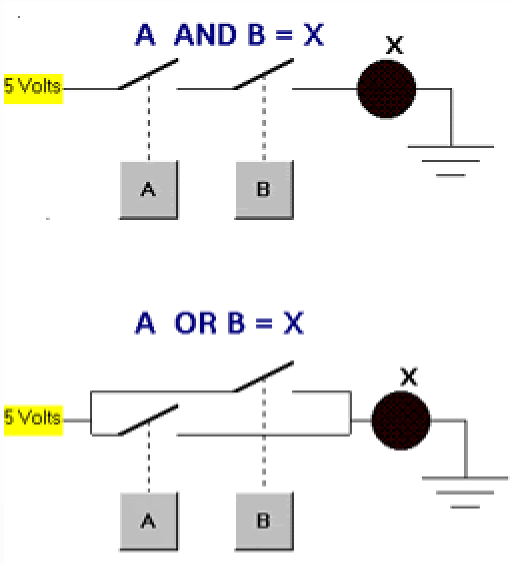
Lecture Outcomes

After this lecture, you will be able to

- understand what are Boolean Functions
- understand what are Boolean Expressions
- understand how to write Boolean Expressions into a logical equivalence.

Boolean Algebra

The circuits in **computers** and other **electronic devices** have **inputs,** each of which is either a 0 or a 1 and produce **outputs** that are also 0s and 1s.



- Circuits can be constructed using any basic element that has two different states.
- Such elements include switches that can be in either the on or the off position and optical devices that can be either lit or unlit.

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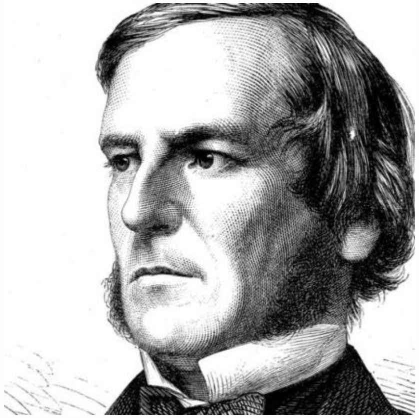
Boolean Algebra

- In 1938 Claude Shannon showed how the basic rules of logic, first given by **George Boole** in 1854 in his ***The Laws of Thought***, could be used to design circuits.



- These rules form the basis for Boolean algebra.
- Here we discuss the basic properties of Boolean algebra.

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Boolean Algebra

The operation of a circuit is **defined by a Boolean function** that specifies the value of an output for each set of inputs.

The first step in constructing a **circuit** is to represent its **Boolean function** by an expression built up using the basic operations of Boolean algebra.

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Boolean Functions – Introduction

Boolean algebra provides the operations and the rules for working with the set **{0, 1}**.

Electronic and optical switches can be studied using this set and the rules of Boolean algebra.

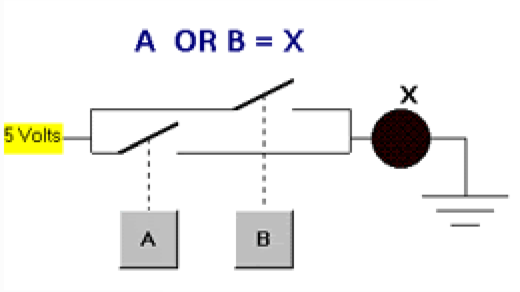
The **three operations** in Boolean algebra that will be used most are complementation, the Boolean sum, and the Boolean product.

Boolean Functions - Introduction

The **complement** of an element, denoted with a bar, is defined by $\bar{0} = 1$ and $\bar{1} = 0$.

The **Boolean sum**, denoted by + or by OR, has the following values:

1 + 1 = 1, 1 + 0 = 1, 0 + 1 = 1, 0 + 0 = 0.



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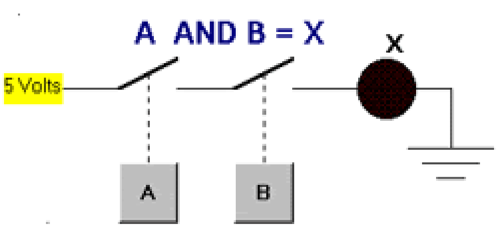
Boolean Functions – Introduction

The **Boolean product**, denoted by · or by AND, has the following values:



$1 \cdot 1 = 1$, $1 \cdot 0 = 0$, $0 \cdot 1 = 0$, $0 \cdot 0 = 0$.

Boolean Functions – Introduction Example 1

Ques:- Find the value of $1 \cdot \overline{0}$

Ans:- $1. \overline{0}$

$= 1.1$

$= 1$

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Boolean Functions – Introduction Example 2

Ques:- Find the value of $1 + \overline{1}$

Ans:- $1 + \overline{1}$

$= 1 + 0$

$= 1$

Boolean Functions – Introduction Example 3

Ques:- Find the value of $0 \cdot \overline{0}$

Sol:- $0 \cdot \overline{0}$

$= 0 \cdot 1$

$= 0$

Boolean Functions – Introduction Example 4

Ques:- Find the value of $\overline{(1+0)}$

Sol:- $\overline{(1+0)}$

$= \overline{(1)}$

$= 0$

Boolean Functions – Introduction Example 5

Ques:- Find the value of $1 \cdot 0 + (\overline{0 + 1})$.

Sol:- $1 \cdot 0 + (\overline{0 + 1})$

$= 0 + (\overline{1})$

$= 0 + 0$

$= 0.$

Boolean Functions – Introduction
Example 1

Illustrates the translation from Boolean algebra to propositional logic.

Ques:- Translate $1 \cdot 0 + (\overline{1+0}) = 0$, the equality found in Example 1, into a logical equivalence.

Ans:- We obtain

$(T \wedge F) \vee \neg(T \vee F) \equiv F$.

Boolean Functions – Introduction – Example 1

Illustrates the translation from propositional logic to Boolean algebra.

Ques:- Translate the logical equivalence $(T \wedge T) \vee \neg F \equiv T$ into an identity in Boolean algebra.

Ans:- We obtain

$(1 \cdot 1) + \overline{0} = 1.$

Boolean Functions

Let $B = \{0, 1\}$. Then $B^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in B$ for $\{1 \leq i \leq n\}$ is the set of all possible n-tuples of 0s and 1s.

The variable x is called a Boolean variable if it assumes values only from B, that is, if its only possible values are 0 and 1.

A function from B^n to B is called a **Boolean function of degree n.**

Boolean Functions

The function $F(x, y) = x\bar{y}$ from the set of ordered pairs of Boolean variables to the set {0, 1} is a Boolean function of degree 2 with F(1, 1) = 0, F(1, 0) = 1, F(0, 1) = 0, and F(0, 0) = 0.

Displayed are these values of F in Table 1 below.

TABLE 1

x	y	$F(x, y)$
1	1	0
1	0	1
0	1	0
0	0	0

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Boolean Expressions

Boolean functions can be represented using expressions made up from variables and Boolean operations.

The Boolean expressions in the variables $x_1, x_2, \ldots, x_n$ are defined recursively as $0, 1, x_1, x_2, \ldots, x_n$ are Boolean expressions;

Each Boolean expression represents a Boolean function.

Boolean Expressions

Ques:- Find the values of the Boolean function represented by F(x, y, z) = $xy + \bar{z}$.

Ans:- The values of this function are as in Table below.

x	y	z	xy	$\bar{z}$	$F(x, y, z) = xy + \bar{z}$
1	1	1	1	0	1
1	1	0	1	1	1
1	0	1	0	0	0
1	0	0	0	1	1
0	1	1	0	0	0
0	1	0	0	1	1
0	0	1	0	0	0
0	0	0	0	1	1

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Boolean Expressions

Show that the distributive law $x(y + z) = xy + xz$ is valid.

The verification of this identity is shown in Table below. The identity holds because the last two columns of the table agree.

Boolean Expressions

Show that the distributive law x(y + z) = xy + xz is valid.

x	y	z	$y+z$	xy	xz	$x(y+z)$	$xy+xz$
1	1	1	1	1	1	1	1
1	1	0	1	1	0	1	1
1	0	1	1	0	1	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Boolean Expressions

Similarly, we can find the validity of the following identities.

Identity	*Name*
$\overline{\overline{x}} = x$	Law of the double complement
$x + x = x$ $x \cdot x = x$	Idempotent laws
$x + 0 = x$ $x \cdot 1 = x$	Identity laws
$x + 1 = 1$ $x \cdot 0 = 0$	Domination laws
$x + y = y + x$	Commutative laws

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Boolean Expressions

Similarly, we can find the validity of the following identities.

$x + (y + z) = (x + y) + z$ $x(yz) = (xy)z$	Associative laws
$x + yz = (x + y)(x + z)$ $x(y + z) = xy + xz$	Distributive laws
$\overline{(xy)} = \overline{x} + \overline{y}$ $\overline{(x + y)} = \overline{x}\,\overline{y}$	De Morgan's laws
$x + xy = x$ $x(x + y) = x$	Absorption laws
$x + \overline{x} = 1$	Unit property
$x\overline{x} = 0$	Zero property

Boolean Expressions – Example 1

Translate the distributive law $x + yz = (x + y)(x + z)$ in into a logical equivalence.

Here we will change the Boolean variables x, y, and z into the propositional variables p, q, and r.

Next, we change each Boolean sum into a disjunction and each Boolean product into a conjunction.

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

Boolean Expressions – Example 2

We can obtain De Morgan's laws for propositions when you transform De Morgan's laws for Boolean algebra in into logical equivalences.

1. $\overline{xy} = \bar{x} + \bar{y}$

Ans:- $\neg(p \wedge q) \equiv \neg p \vee \neg q$

2. $\overline{x+y} = \bar{x}\,\bar{y}$

Ans:- $\neg(p \vee q) \equiv \neg p \wedge \neg q$

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That's all for now…