EMTH403

Mathematical Foundation for Computer Science

Nitin K. Mishra (Ph.D.)

Associate Professor

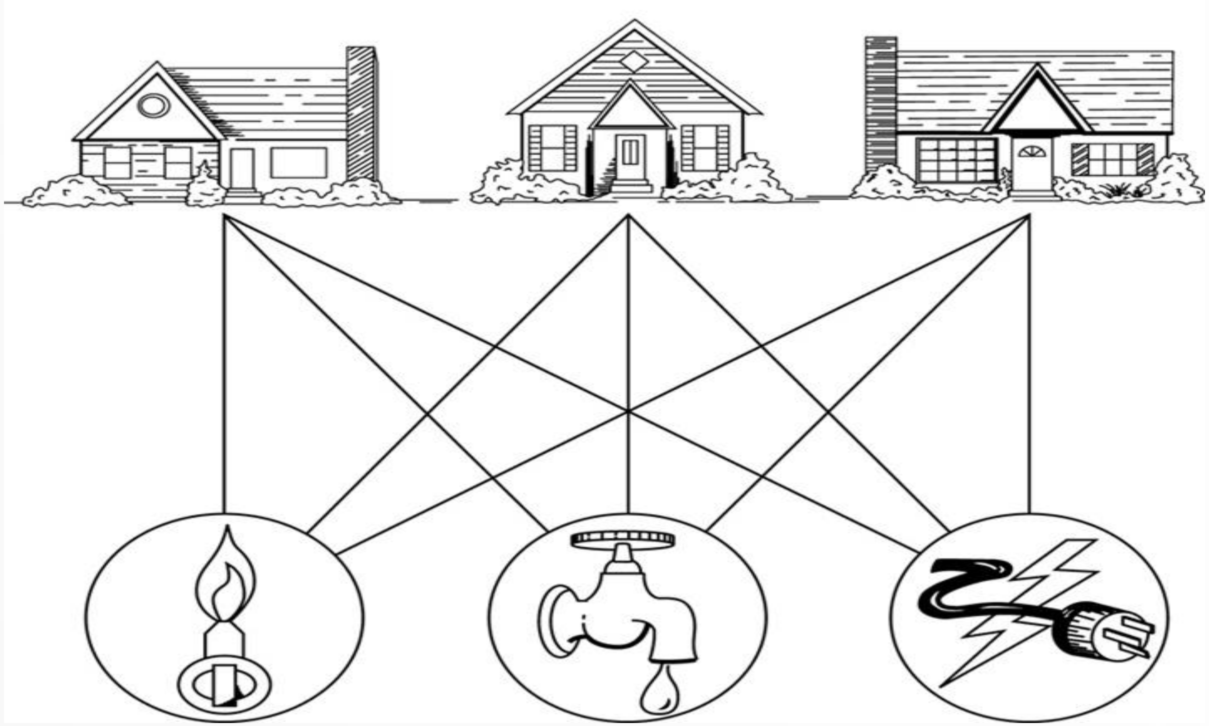
Lecture Outcomes

After this lecture, you will be able to

- Understand what are Planar Graphs
- Understand how $K_{3,3}$ is a non-planar Graph.

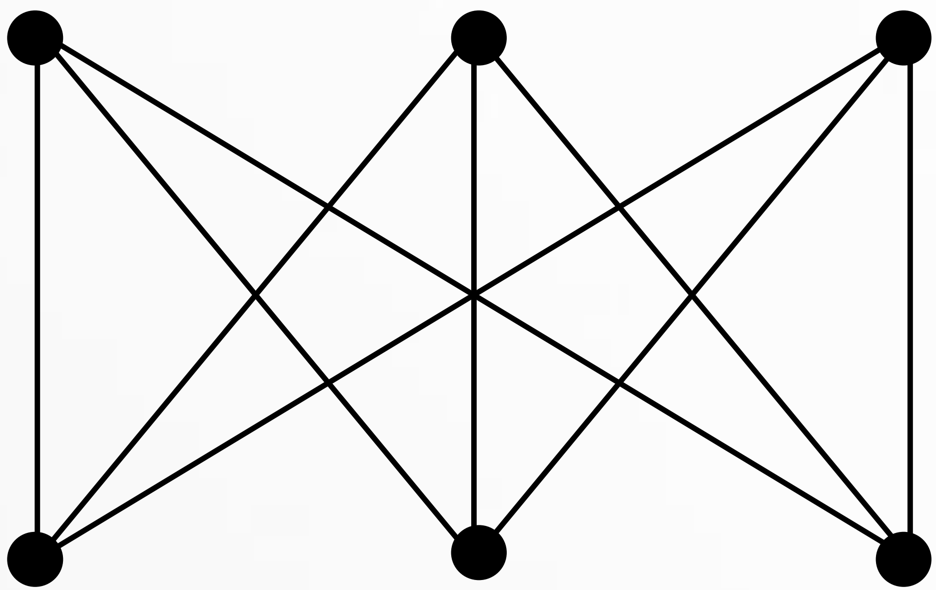
Planar Graphs
The House-and-Utilities Problem

Is it possible to join the three houses to the three utilities in such a way that none of the connections cross?



Planar Graphs

Phrased another way, this question is equivalent to: Given the complete bipartite graph $K_{3,3}$, can $K_{3,3}$ be drawn in the plane so that no two of its edges cross?



$K_{3,3}$

Planar Graphs

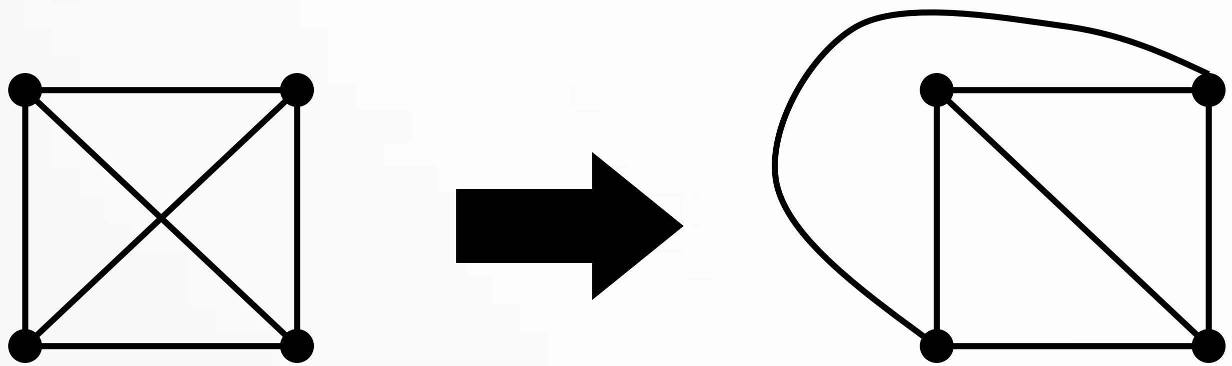
A graph is called planar if it can be drawn in the plane without any edges crossing.

A crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint.

Such a drawing is called a planar representation of the graph.

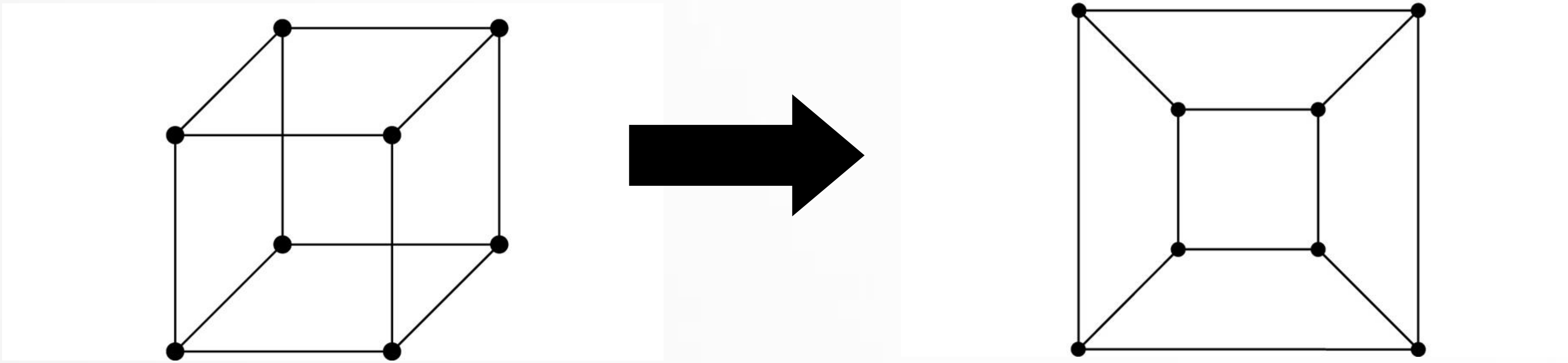
Planar Graphs

A graph may be planar even if it is usually drawn with crossings, since it may be possible to draw it in another way without crossings.



Planar Graphs

A graph may be planar even if it represents a 3-dimensional object.

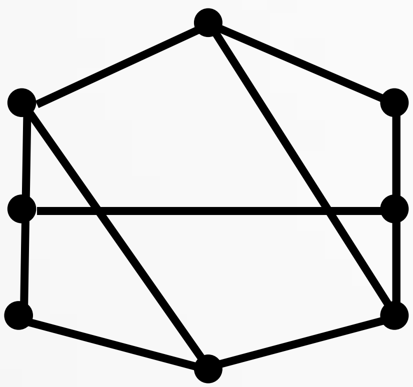


Content Source: **Discrete Mathematics** and **Its Applications**. Seventh Edition. **Kenneth H. Rosen. Monmouth University. Prepared by Jerrold W. Grossman, Oakland University.** (and formerly AT&T Laboratories)

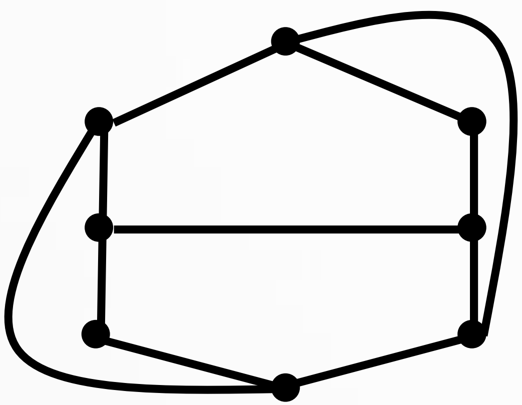
Planar Graphs

Definition:

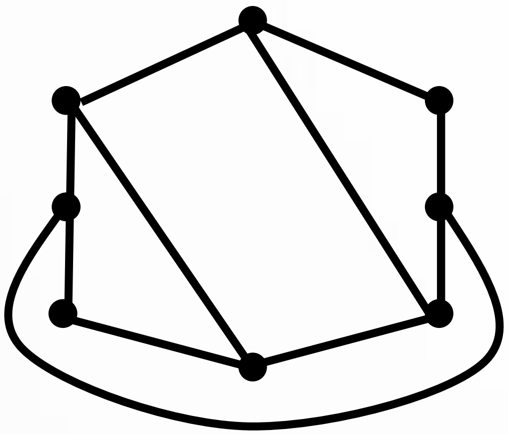
A planar graph G that is drawn in the plane so that no two edges intersect (that is, G is embedded in the plane) is called a plane graph.



(a) planar,
not a plane graph



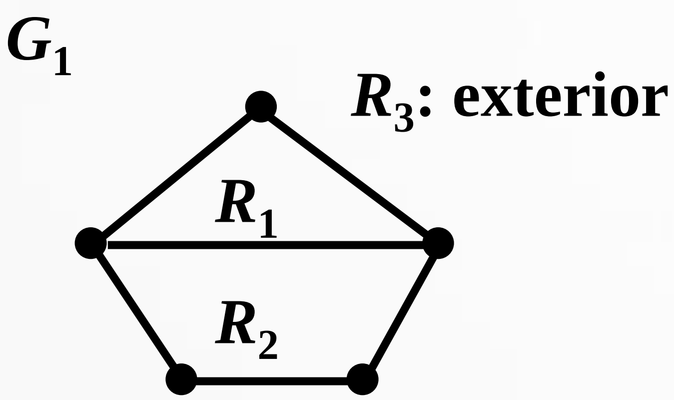
(b) a plane
graph



(c) another
plane graph

Planar Graphs

Note. A given planar graph can give rise to several different plane graph.

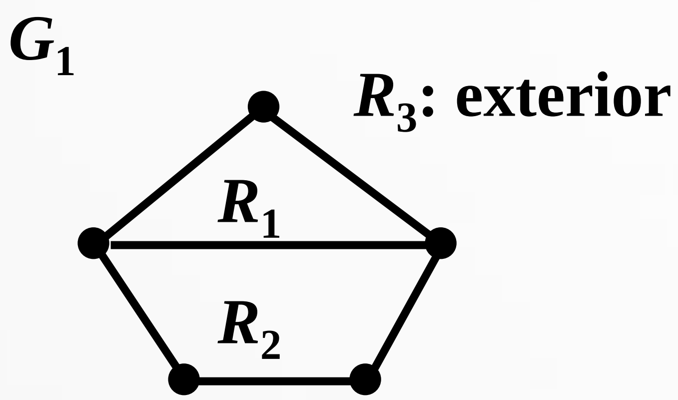


G_1 has 3 regions.

Planar Graphs

Definition:

Let G be a plane graph. The connected pieces of the plane that remain when the vertices and edges of G are removed are called the regions of G.



G_1 has 3 regions.

Planar Graphs

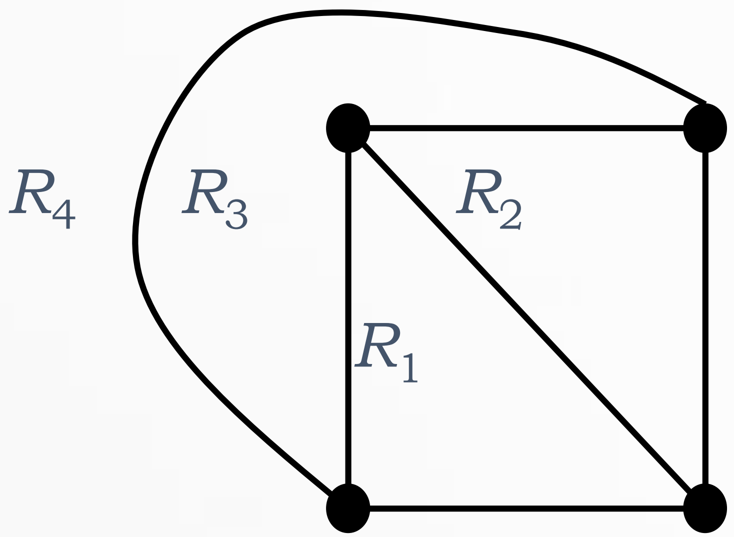
We can prove that a particular graph is planar by showing how it can be drawn without any crossings.

However, not all graphs are planar.

It may be difficult to show that a graph is nonplanar. We would have to show that there is *no way* to draw the graph without any edges crossing.

Regions

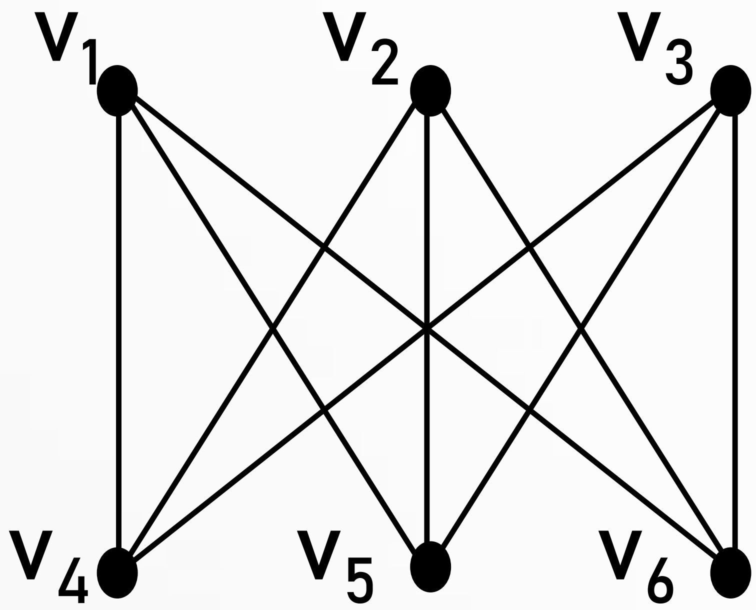
Euler showed that all planar representations of a graph split the plane into the same number of *regions*, including an unbounded region.



Content Source: **Discrete Mathematics** and **Its Applications**. Seventh Edition. **Kenneth H. Rosen. Monmouth University. Prepared by Jerrold W. Grossman, Oakland University.** (and formerly AT&T Laboratories)

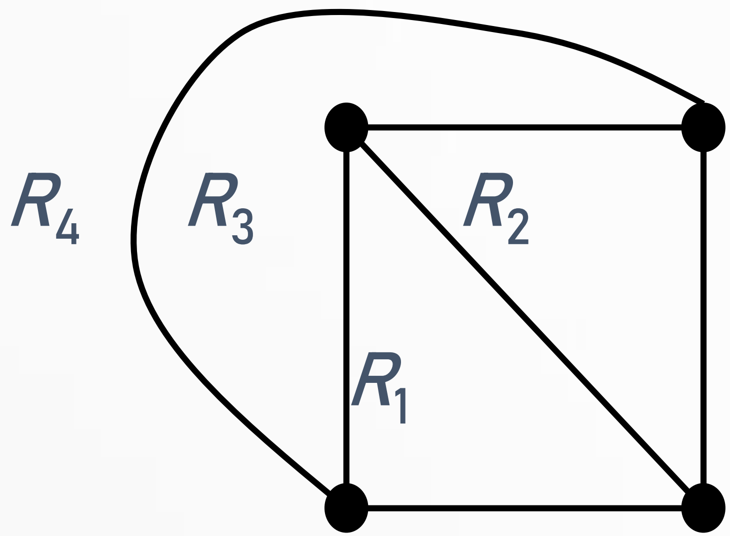
Planar Graphs

In any planar representation of $K_{3,3}$, vertex v_1 must be connected to both v_4 and v_5, and v_2 also must be connected to both v_4 and v_5.



Regions

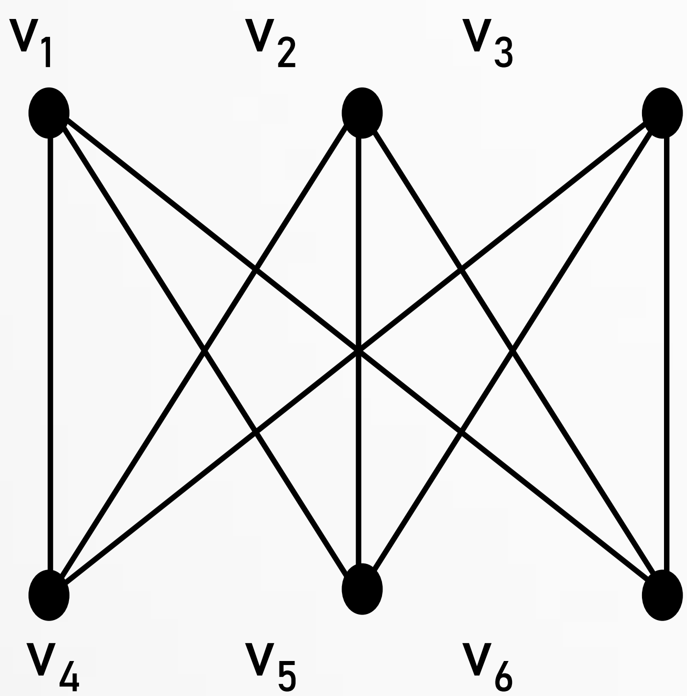
Euler showed that all planar representations of a graph split the plane into the same number of *regions*, including an unbounded region.



Content Source: **Discrete Mathematics** and **Its Applications**. Seventh Edition. **Kenneth H. Rosen. Monmouth University. Prepared by Jerrold W. Grossman, Oakland University.** (and formerly AT&T Laboratories)

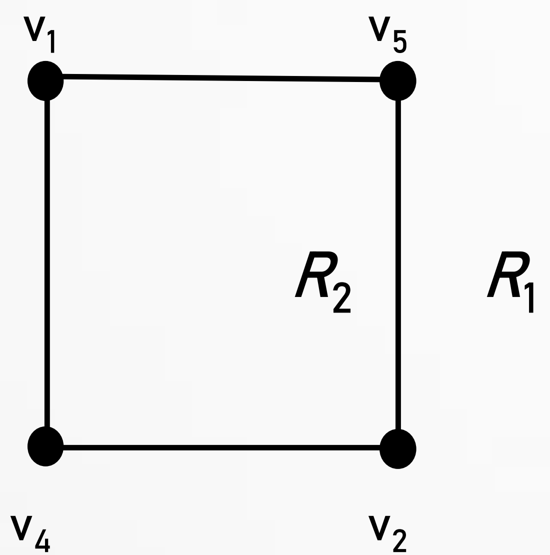
Regions

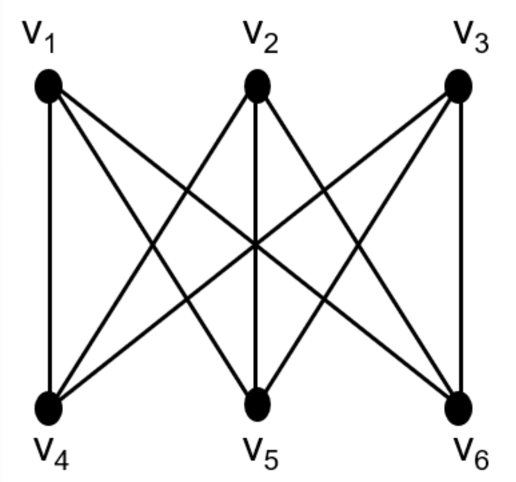
In any planar representation of $K_{3,3}$, vertex v_1 must be connected to both v_4 and v_5, and v_2 also must be connected to both v_4 and v_5.



Regions

The four edges $\{v_1, v_4\}$, $\{v_4, v_2\}$, $\{v_2, v_5\}$, $\{v_5, v_1\}$ form a closed curve that splits the plane into two regions, R_1 and R_2.

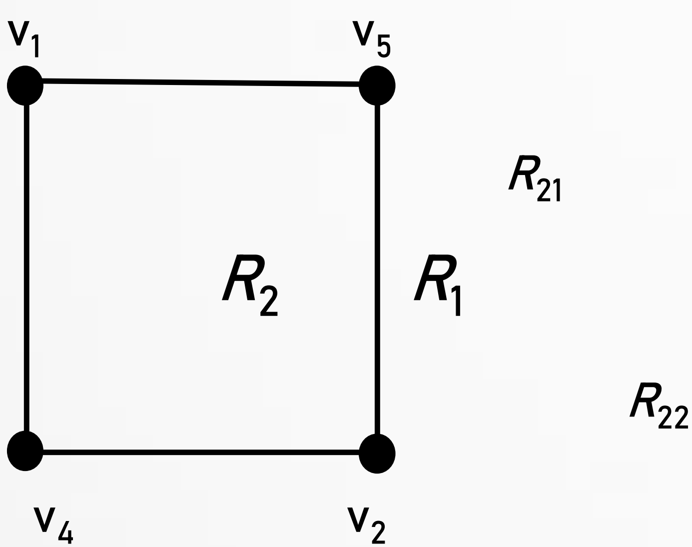


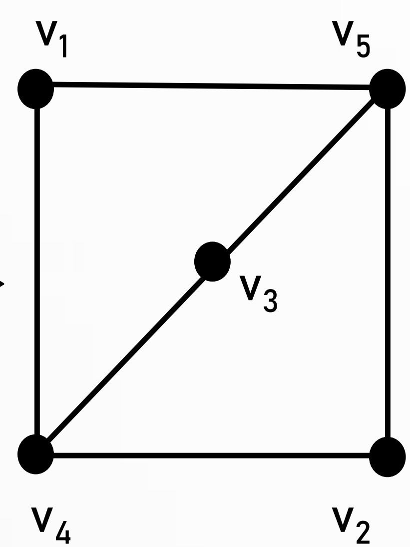


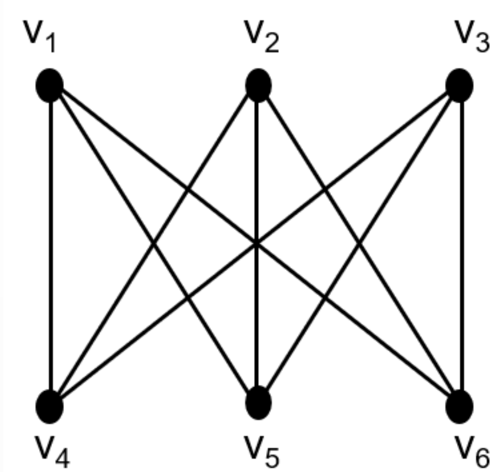
Content Source: Discrete Mathematics and Its Applications. Seventh Edition. Kenneth H. Rosen. Monmouth University. Prepared by Jerrold W. Grossman, Oakland University. (and formerly AT&T Laboratories)

Regions

- Next, we note that v_3 must be in either R_1 or R_2.
- Assume v_3 is in R_2. Then the edges $\{v_3, v_4\}$ and $\{v_4, v_5\}$ separate R_2 into two subregions, R_{21} and R_{22}.





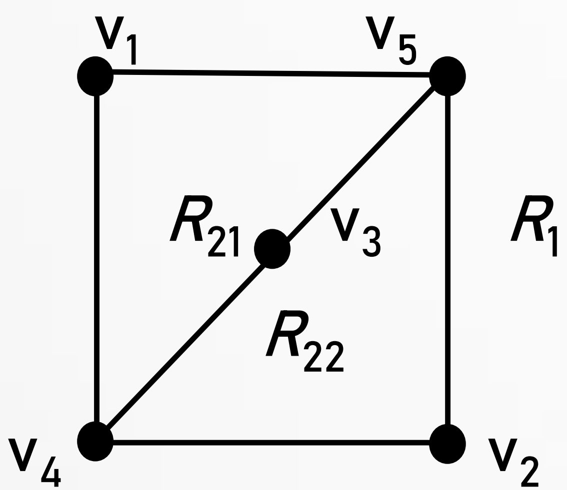


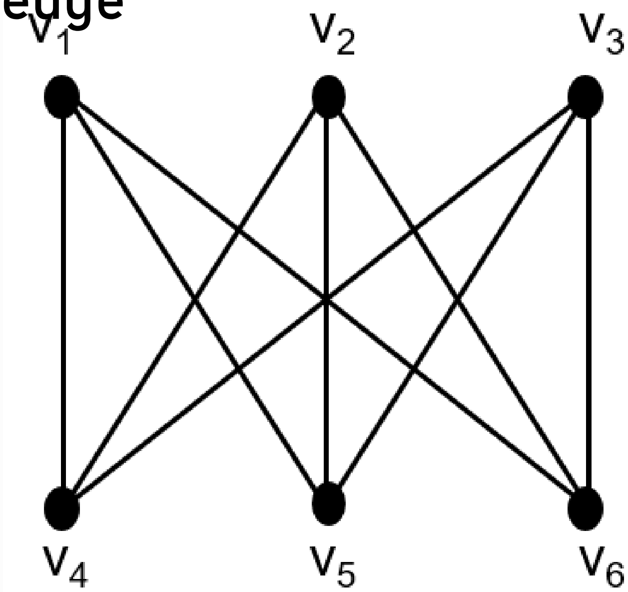
Content Source: Discrete Mathematics and Its Applications. Seventh Edition. Kenneth H. Rosen. Monmouth University. Prepared by Jerrold W. Grossman, Oakland University. (and formerly AT&T Laboratories)

Regions

Now there is no way to place vertex v_6 without forcing a crossing:

- If v_6 is in R_1 then $\{v_6, v_3\}$ must cross an edge
- If v_6 is in R_{21} then $\{v_6, v_2\}$ must cross an edge
- If v_6 is in R_{22} then $\{v_6, v_1\}$ must cross an edge

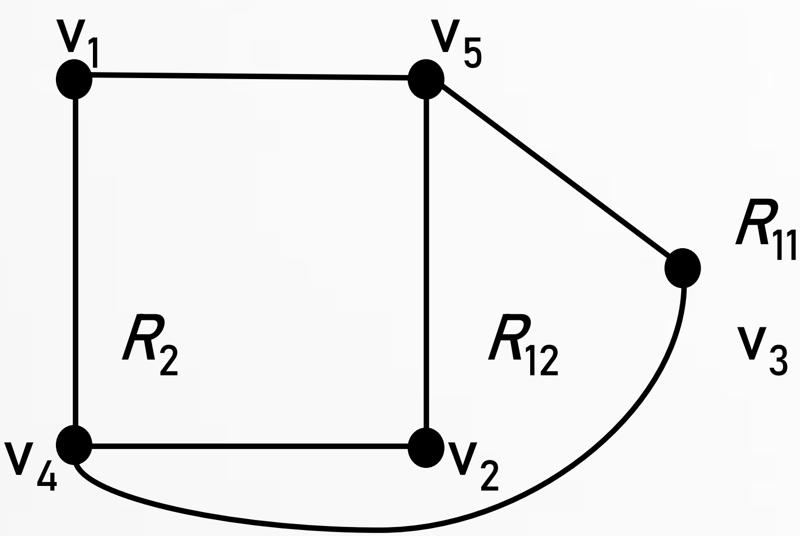


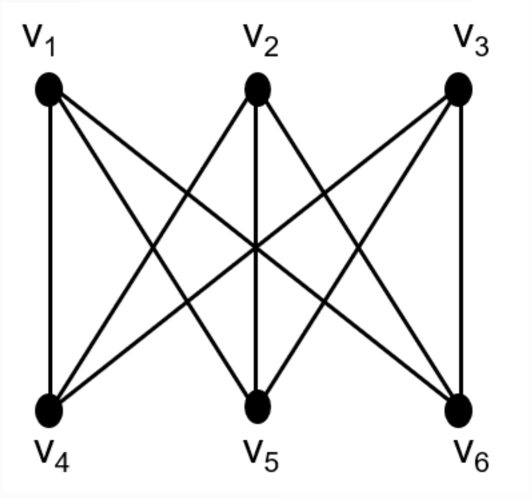


Content Source: Discrete Mathematics and Its Applications. Seventh Edition. Kenneth H. Rosen. Monmouth University. Prepared by Jerrold W. Grossman, Oakland University. (and formerly AT&T Laboratories)

Regions

Alternatively, assume v_3 is in R_1. Then the edges $\{v_3, v_4\}$ and $\{v_4, v_5\}$ separate R_1 into two subregions, R_{11} and R_{12}.





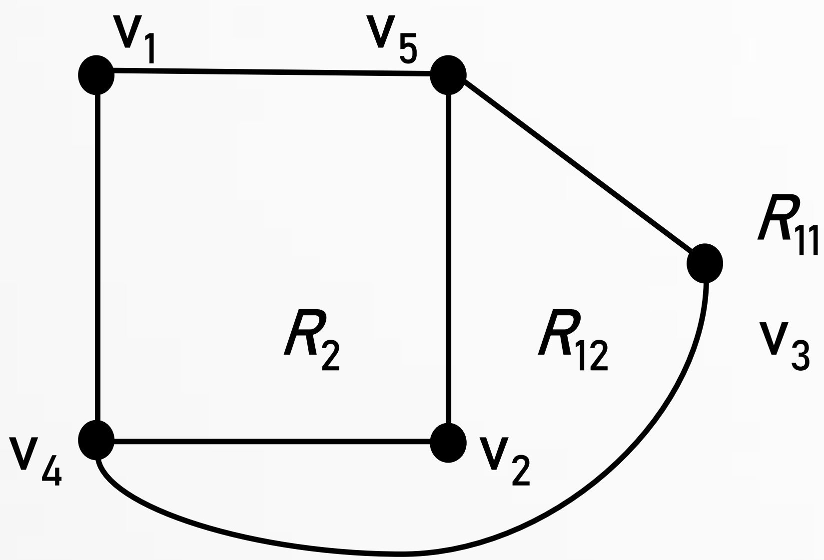
Regions

Now there is no way to place vertex v_6 without forcing a crossing:

If v_6 is in R_2 then $\{v_6, v_3\}$ must cross an edge

If v_6 is in R_{11} then $\{v_6, v_2\}$ must cross an edge

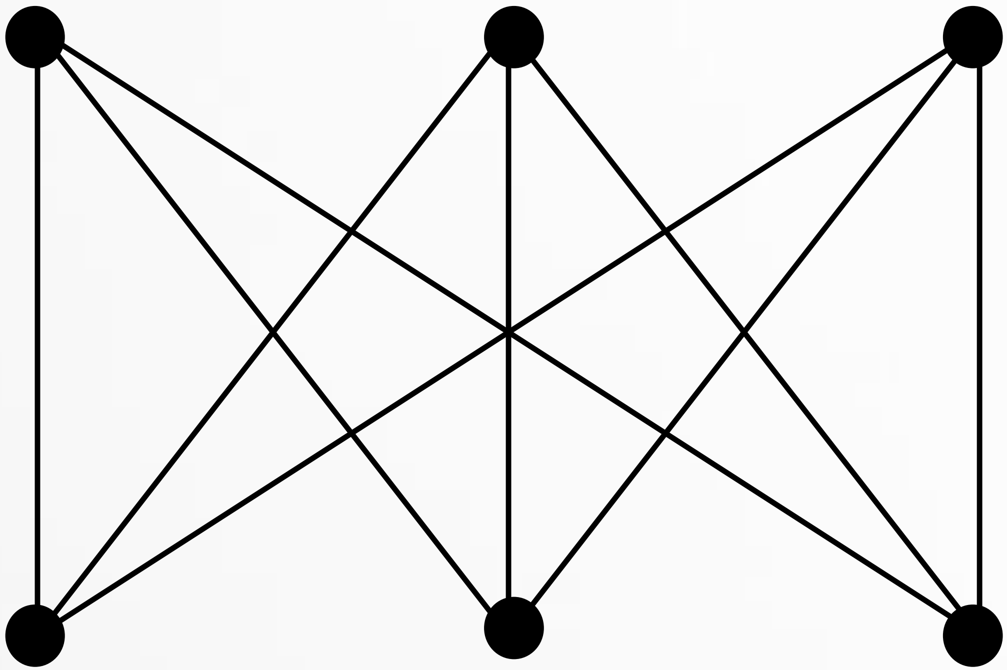
If v_6 is in R_{12} then $\{v_6, v_1\}$ must cross an edge



Content Source: Discrete Mathematics and Its Applications. Seventh Edition. Kenneth H. Rosen. Monmouth University. Prepared by Jerrold W. Grossman, Oakland University. (and formerly AT&T Laboratories)

Planar Graphs

Consequently, the graph $K_{3,3}$ must be nonplanar.



$K_{3,3}$

Content Source: Discrete Mathematics and Its Applications. Seventh Edition. Kenneth H. Rosen. Monmouth University. Prepared by Jerrold W. Grossman, Oakland University. (and formerly AT&T Laboratories)

Theorem 1 (Kuratowski, 1930)

A graph is planar if and only if it does not contain a subgraph that is a K_5 or $K_{3,3}$ configuration.

That's all for now…