EMTH403

Mathematical Foundation for Computer Science

Nitin K. Mishra (Ph.D.)

Associate Professor

Lecture Outcomes

After this lecture, you will be able to

- understand what is adjacency matrix
- understand what is graph isomorphism

Adjacency Matrix

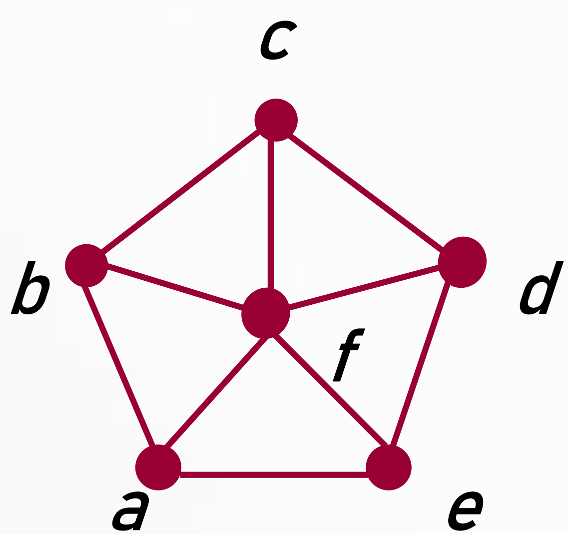
A simple graph $G = (V, E)$ with n vertices can be represented by its adjacency matrix, A, where entry a_{ij} in row i and column j is

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge in G,} \\ 0 & \text{otherwise.} \end{cases}$$

Finding the adjacency matrix

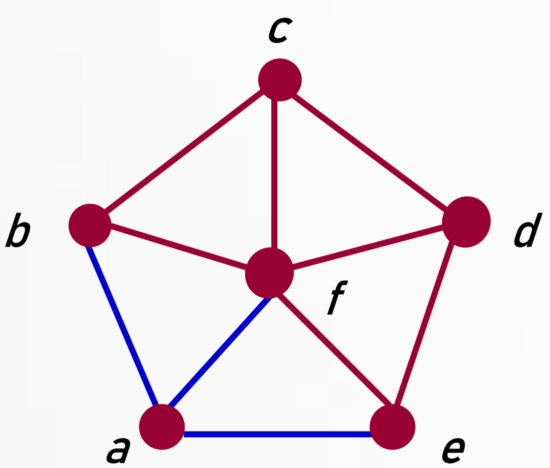
This graph has 6 vertices a, b, c, d, e, f.

We can arrange them in that order.



W_5

Finding the adjacency matrix



W_5

$$
\begin{array}{c c}
 & \text{TO} \\
\text{FROM} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} & \left[\begin{array}{cccccc} 0 & 1 & 0 & 0 & 1 & 1 \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{array}\right]
\end{array}
$$

There are edges from a to b, from a to e, and from a to f

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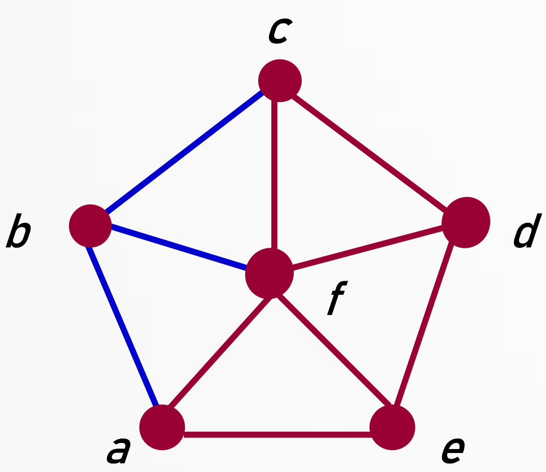
Finding the adjacency matrix



W_5

TO

FROM

$$\begin{array}{cc} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \end{array}$$

There are edges from b to a, from b to c, and from b to f

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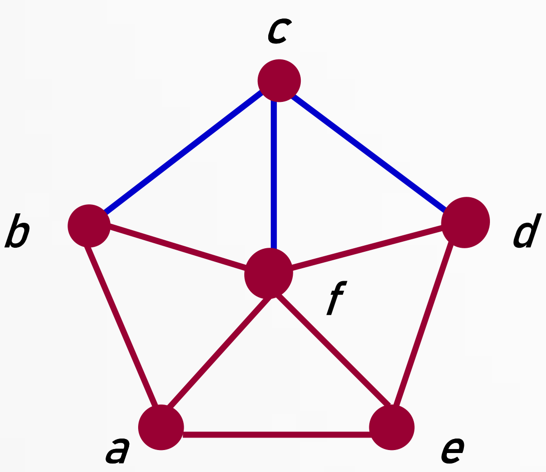
Finding the adjacency matrix



W_5

TO

FROM

$$\begin{array}{cc} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \end{array}$$

There are edges from c to b, from c to d, and from c to f

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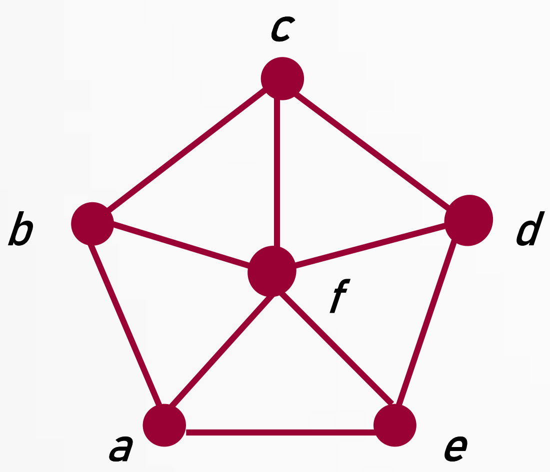
Finding the adjacency matrix



W_5

TO

FROM

$$\begin{array}{cc} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} a\\ b\\ c\\ d\\ e\\ f \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1\\ 1 & 0 & 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 1 & 0 & 1\\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \end{array}$$

Complete the adjacency matrix . . .

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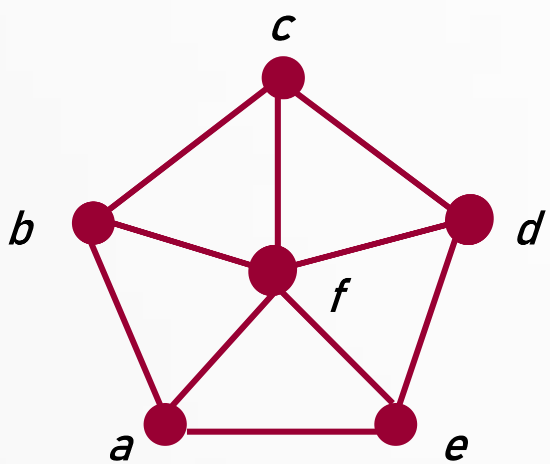
Finding the adjacency matrix



W_5

TO

FROM

$$\begin{array}{cc} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Notice that this matrix is symmetric.

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Graph Isomorphism

Formal definition:

- Simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ are *isomorphic* iff $\exists$ a bijection $f: V_1 \rightarrow V_2$ such that $\forall\ a,b \in V_1$, a and b are adjacent in G_1 iff $f(a)$ and $f(b)$ are adjacent in G_2.
- f is the "renaming" function between the two node sets that makes the two graphs identical.
- This definition can be extended to other types of graphs.

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Finding the adjacency matrix

Necessary but not *sufficient* conditions for $G_1=(V_1, E_1)$ to be isomorphic to $G_2=(V_2, E_2)$:

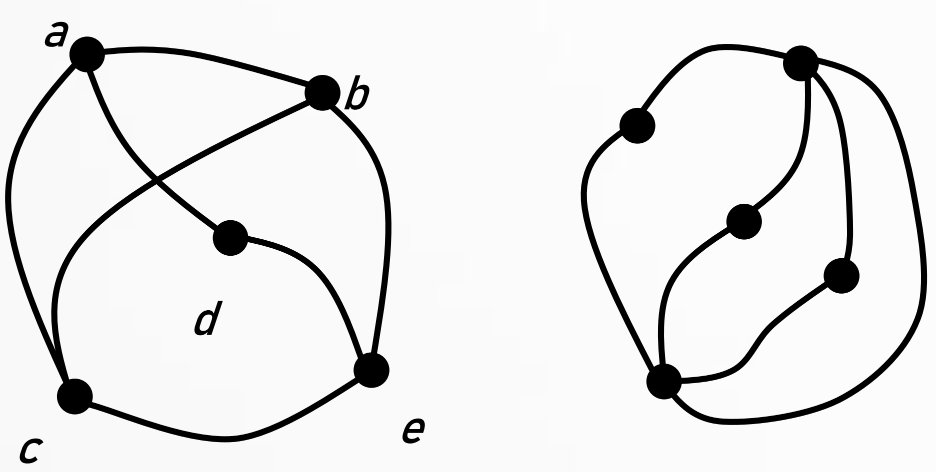
- $|V1|=|V2|$ and $|E1|=|E2|$.
- The number of vertices with degree n is the same in both graphs.
- For every proper subgraph g of one graph, there is a proper subgraph of the other graph that is isomorphic to g.

Graph Isomorphism

Definition:- Two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are said to be isomorphic if there exists a bijection1 $\alpha : V_1 \rightarrow V_2$ such that the edge $(\alpha(a), \alpha(b)) \in E_2$ if and only if $(a, b) \in E_1$.

Graph Isomorphism

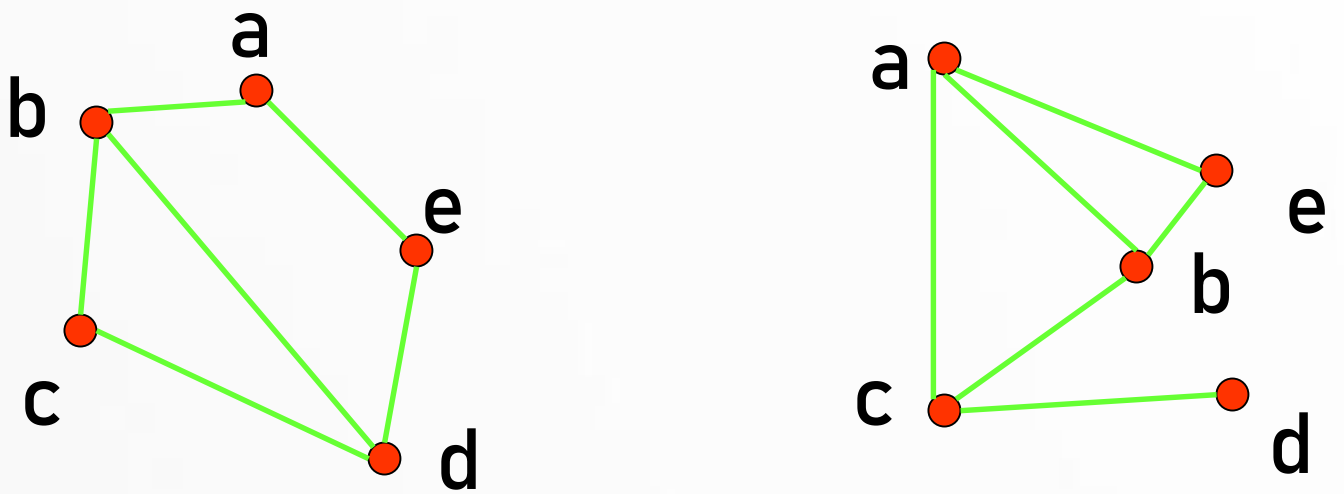
If isomorphic, label the 2nd graph to show the isomorphism, else identify difference.



- *Same No of vertices*
- *Same No of edges*
- *Different No of vertices of degree 2! (1 vs 3)*

Graph Isomorphism

Example II: How about these two graphs?



Solution: No, they are not isomorphic, because they differ in the degrees of their vertices.

Vertex d in right graph is of degree one, but there is no such vertex in the left graph.

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Graph Isomorphism

From a visual standpoint, G1 and G2 are isomorphic if they can be arranged in such a way that their displays are identical (of course without changing adjacency).

Unfortunately, for two simple graphs, each with n vertices, there are n! possible isomorphisms that we have to check in order to show that these graphs are isomorphic.

However, showing that two graphs are not isomorphic can be easy.

Graph Isomorphism

For this purpose we can check invariants, that is, properties that two isomorphic simple graphs must both have.

For example, they must have

- the same number of vertices,
- the same number of edges, and
- the same degrees of vertices.

Note that two graphs that differ in any of these invariants are not isomorphic, but two graphs that match in all of them are not necessarily isomorphic.

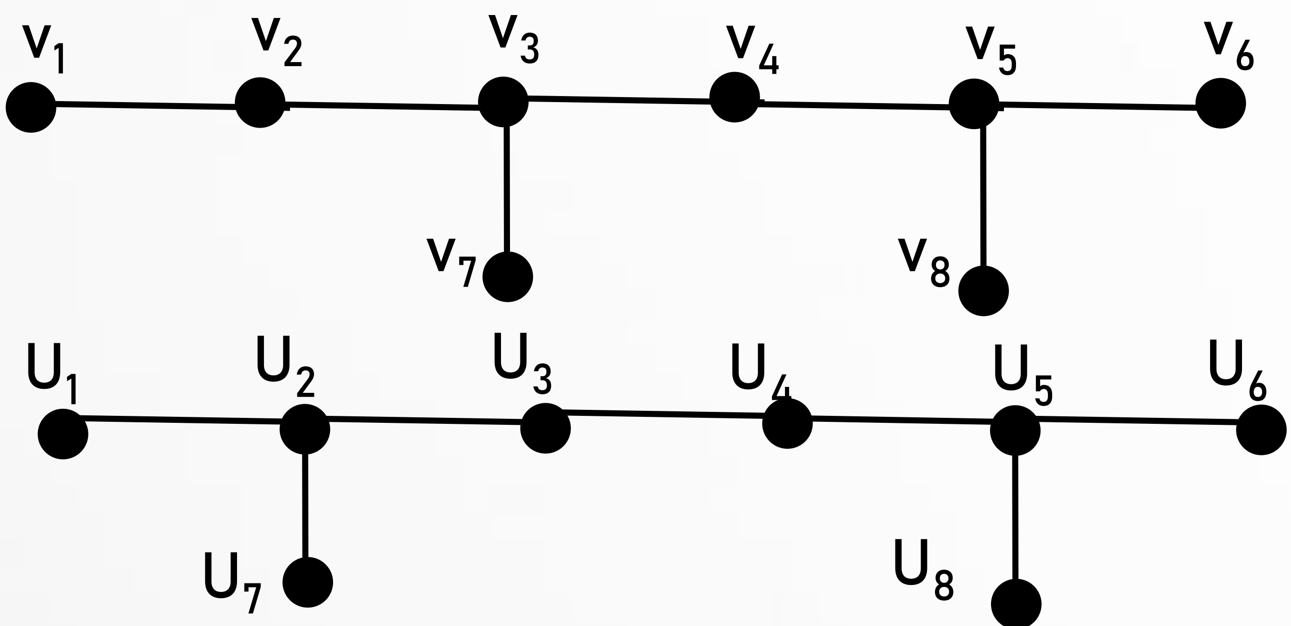
Graph Isomorphism

Determine if the following two graphs G_1 and G_2 are isomorphic:

These graphs are not isomorphic.

In the first graph the vertices of degree 3 are adjacent to a common vertex.

This is not true of the second graph.



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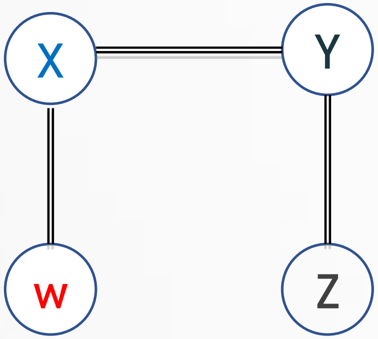
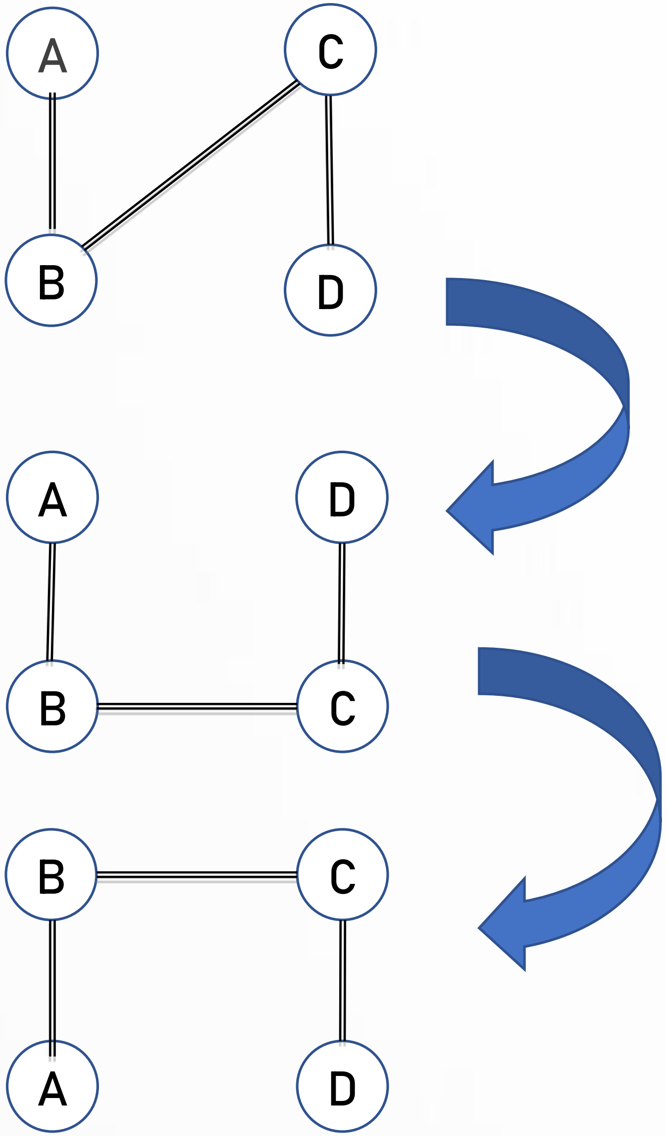
Adjacency Matrix



is isomorphic to



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That's all for now...