



# EMTH403

## Mathematical Foundation for Computer Science

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# Lecture Outcomes



After this lecture, you will be able to

- understand what are Planar Graphs
- understand what is Euler's Formulae and how  $K_{3,3}$  is a non-planar Graph.

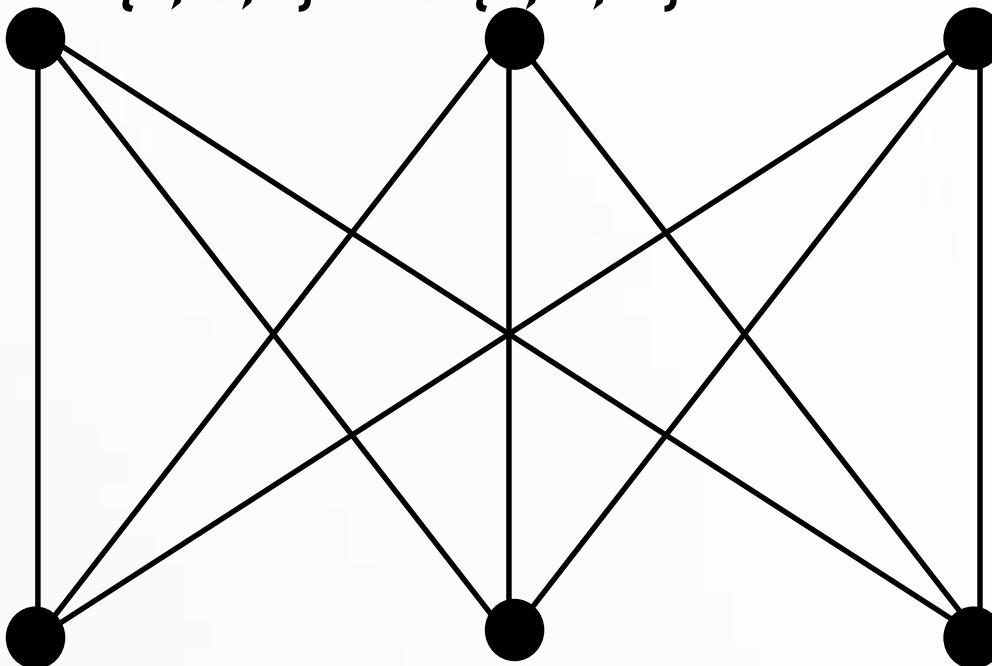
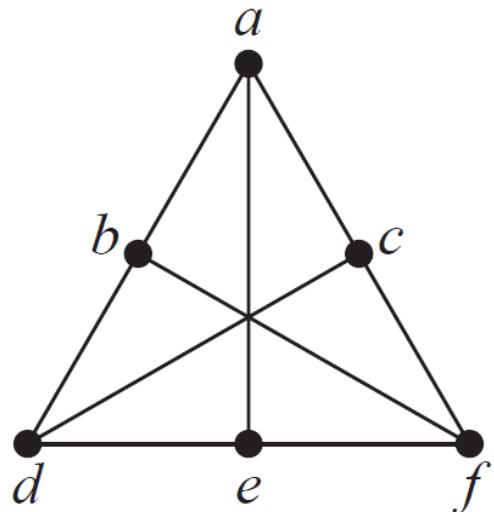
# Theorem 1 (Kuratowski, 1930)

A graph is planar if and only if it does not contain a subgraph that is a  $K_5$  or  $K_{3,3}$  configuration.

# Planar Graphs

Determine whether the given graph is planar. If so, draw it so that no edges cross.

- This is  $K_{3,3}$ , with parts  $\{a, d, f\}$  and  $\{b, c, e\}$ . Therefore it

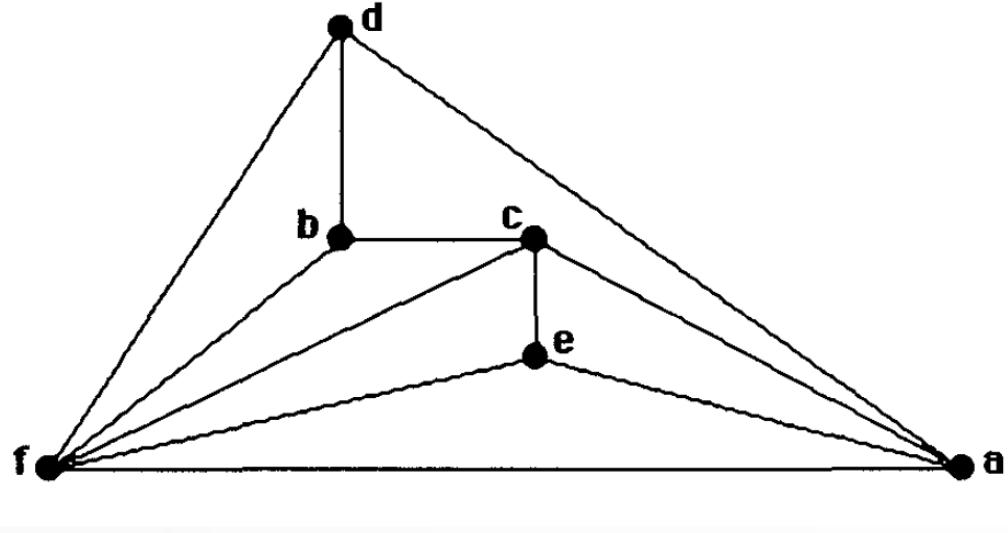
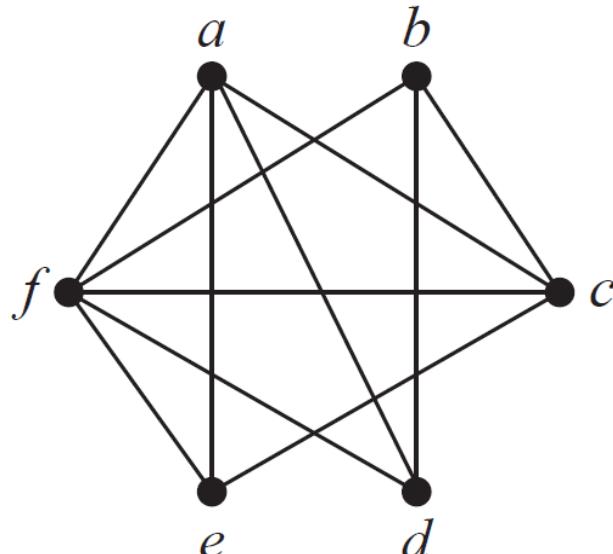


$K_{3,3}$

# Planar Graphs

Ques:- Determine whether the given graph is planar. If so, draw it so that no edges cross.

Sol:- This graph can be untangled if we play with it long enough. The following picture gives a planar representation



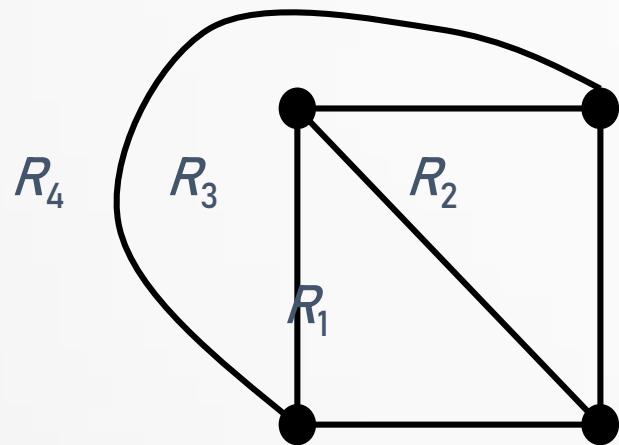
# Euler's Formula

- Euler devised a formula for expressing the relationship between the number of vertices, edges, and regions of a planar graph.
- These may help us determine if a graph can be planar or not.

# Euler's Formula

Let  $G$  be a connected planar simple graph with  $e$  edges and  $v$  vertices. Let  $r$  be the number of regions in a planar representation of  $G$ . Then  $r = e - v + 2$ .

Or  $2 = v - e + r$ .



# of edges,  $e = 6$

# of vertices,  $v = 4$

# of regions,  $r = e - v + 2 = 4$

# Euler's Formula

Euler characteristic (simple form):

$2 = \text{number of vertices} - \text{number of edges} + \text{number of faces}$

Or in short-hand,

$$2 = |V| - |E| + |F|$$

where  $V = \text{set of vertices}$

$E = \text{set of edges}$

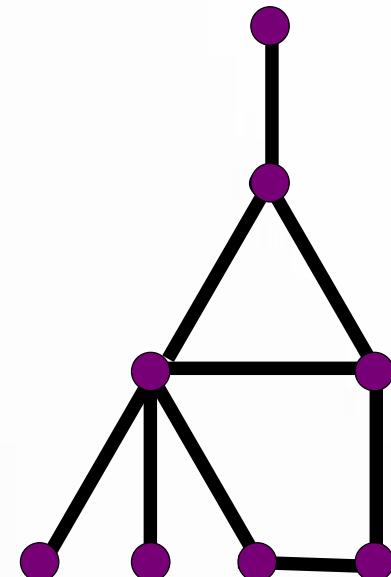
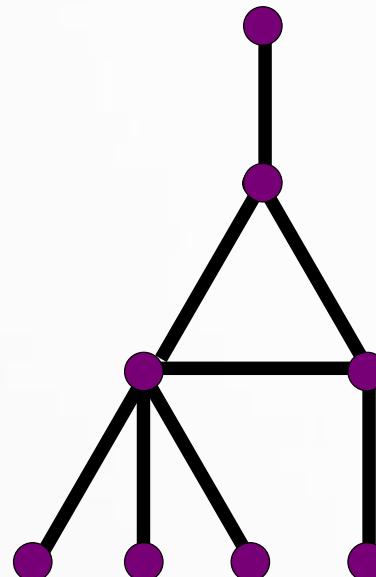
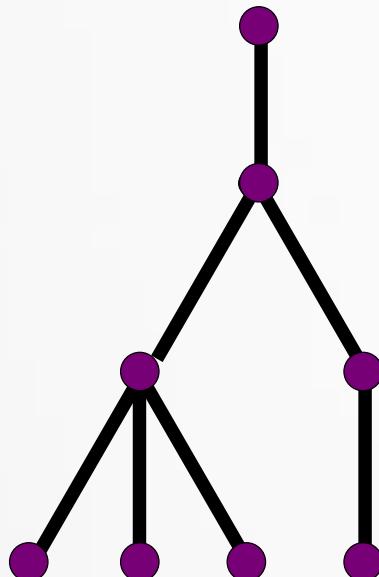
$F = \text{set of faces} = \text{set of regions}$

For a planar connected graph  $|V| - |E| + |F| = 2$

# Euler's Formula

Def<sup>n</sup>: A tree is a connected graph that does not contain a cycle.

A forest is a graph whose components are trees.



**Lemma 2.1:** Any tree with  $n$  vertices has  $n-1$  edges.

# Euler's Formula

$$2 = |V| - |E| + |F|$$



$$\chi = 1 - 0 + 1 = 2$$

# Euler's Formula

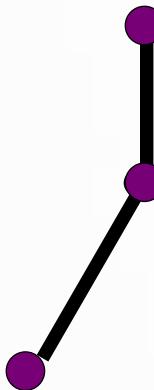
$$2 = |V| - |E| + |F|$$



$$x = 2 - 1 + 1 = 2$$

# Euler's Formula

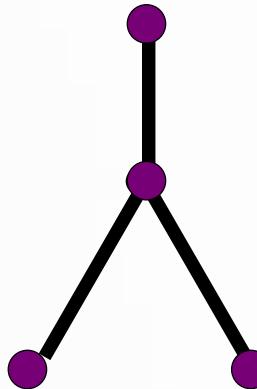
$$2 = |V| - |E| + |F|$$



$$x = 3 - 2 + 1 = 2$$

# Euler's Formula

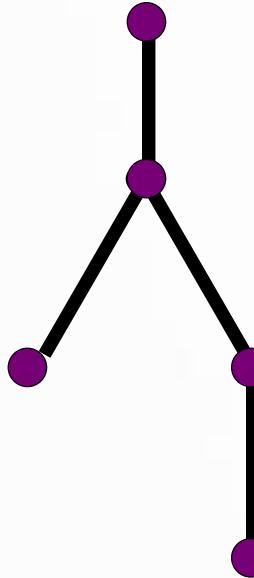
$$2 = |V| - |E| + |F|$$



$$\chi = 4 - 3 + 1 = 2$$

# Euler's Formula

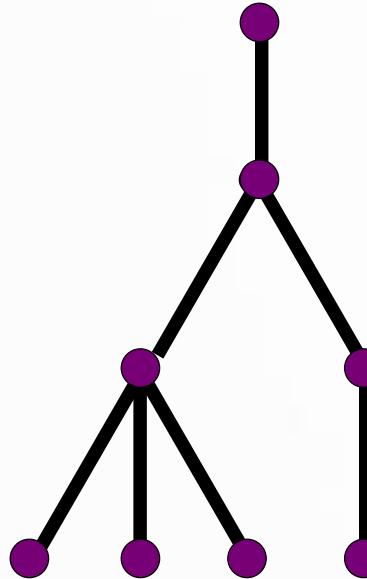
$$2 = |V| - |E| + |F|$$



$$x = 5 - 4 + 1 = 2$$

# Euler's Formula

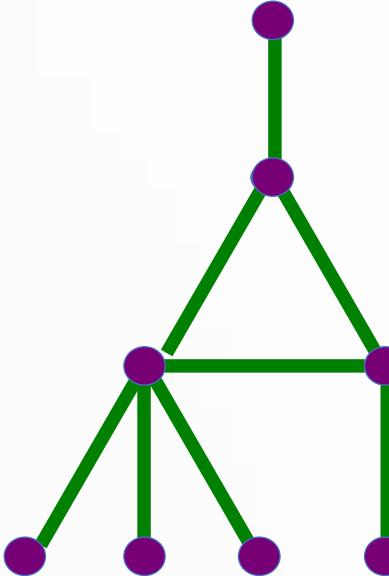
$$2 = |V| - |E| + |F|$$



$$x = 8 - 7 + 1 = 2$$

# Euler's Formula

$$2 = |V| - |E| + |F|$$

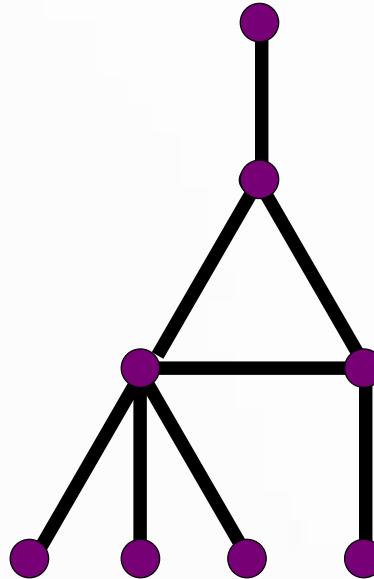


$$x = 8 - 8 + 2 = 2$$

Not a tree.

# Euler's Formula

$$2 = |V| - |E| + |F|$$

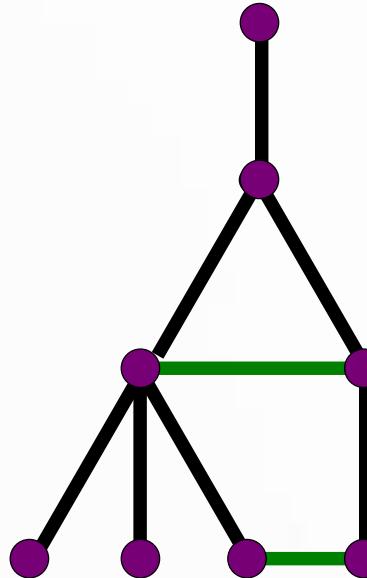


$$\begin{aligned}x &= 8 - 8 + 2 = \\&2\end{aligned}$$

Not a tree.

# Euler's Formula

$$2 = |V| - |E| + |F|$$



$$x = 8 - 9 + 3 = 2$$

Not a tree.

# Euler's Formula

Ques:- Suppose that a connected planar graph has six vertices, each of degree four. Into how many regions is the plane divided by a planar representation of this graph?

Sol:- By Euler's formula  $r = e - v + 2$ . Given  $v = 6$ .

Given that each vertex has degree 4,

Then sum of the degrees is  $6 \times 4 = 24$ .

By the handshaking theorem there are  $e = 24/2 = 12$  edges

Solving  $r = 12 - 6 + 2 = 6 + 2$ ,  $r = 8$ .

Answer: c(8).

# Euler's Formula

Ques:- How many regions would be in a plane graph with 10 vertices each of degree 3?

Sol:- By Euler's formula  $r = e - v + 2$ . Given  $v = 10$ .

Given that each vertex has degree 4,

Then sum of the degrees is  $10 \times 3 = 30$ .

By the handshaking theorem there are  $e = 30/2 = 15$  edges

Solving  $r = 15 - 10 + 2 = 5 + 2$ ,  $r = 7$ .

Answer: c(7).

That's all for now...