EMTH403

Mathematical Foundation for Computer Science

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Lecture Outcomes

After this lecture, you will be able to

- understand what are partial ordering
- understand what are comparable and incomparable elements
- understand what are total, well and lexicographic ordering

Partial Orderings

A relation R on a set S is called a partial ordering or partial order if it is reflexive, antisymmetric, and transitive.

A set S together with a partial ordering R is called a partially ordered set, or poset, and is denoted by (S,R). Members of S are called elements of the poset.

Partial Orderings – Example 1

Ques:- Show that the "greater than or equal" relation ($\geq$) is a partial ordering on the set of integers.

Ans:- Because $a \geq a$ for every integer a, $\geq$ is reflexive. If $a \geq b$ and $b \geq a$, then $a = b$.

Hence, $\geq$ is antisymmetric. Finally, $\geq$ is transitive because $a \geq b$ and $b \geq c$ imply that $a \geq c$.

It follows that $\geq$ is a partial ordering on the set of integers and $(Z, \geq)$ is a poset.

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Partial Orderings – Example 2

Ques:- Show that the inclusion relation ⊆ is a partial ordering on the power set of a set S.

Ans:- Because A ⊆ A whenever A is a subset of S, ⊆ is reflexive.

It is antisymmetric because A ⊆ B and B ⊆ A imply that A = B.

Finally, ⊆ is transitive, because A ⊆ B and B ⊆ C imply that A ⊆ C.

Hence, ⊆ is a partial ordering on P(S), and (P (S),⊆) is a poset.

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Partial Orderings – Example 3

Ques:- Is the following relation on {0, 1, 2, 3} a partial orderings? Determine the properties of a partial ordering that the others lack. {(0, 0), (1, 1), (2, 2), (3, 3)}

Ans:- This is just the equality relation on {0, 1, 2, 3}; more generally, the equality relation on any set satisfies all three conditions and is therefore a partial ordering.

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Partial Orderings – Example 4

Ques:- Is the following relation on {0, 1, 2, 3} a partial orderings? Determine the properties of a partial ordering that the others lack. {(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}.

Ans:- This is not a partial ordering, because although the relation is reflexive, it is not antisymmetric (we have 2 R 3 and 3 R 2, but $2 \neq 3$), and not transitive (3 R 2 and 2 R 0, but 3 is not related to 0).

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Partial Orderings – Example 5

Ques:- Is the following relation on {0, 1, 2, 3} a partial orderings? Determine the properties of a partial ordering that the others lack. {(0, 0), (1, 1), (1, 2), (2, 2), (3, 3)}

Ans:- This is a partial ordering, because it is clearly reflexive; is antisymmetric (we just need to note that (1, 2) is the only pair in the relation with unequal components); and is transitive (for the same reason).

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Partial Orderings – Example 6

Ques:- Is the following relation on {0, 1, 2, 3} a partial orderings? Determine the properties of a partial ordering that the others lack. {(0, 0), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)}

Ans:- This is a partial ordering because it is the "less than or equal to" relation on {1, 2, 3} together with the isolated point 0.

Partial Orderings – Example 7

Ques:- Is the following relation on {0, 1, 2, 3} a partial orderings? Determine the properties of a partial ordering that the others lack. {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}

Ans:- This is not a partial ordering. The relation is clearly reflexive, but it is not antisymmetric (0 R 1 and 1 R 0, but $0 \neq 1$) and not transitive (2 R 0 and 0R1, but 2 is not related to 1).

Partial Orderings – Example 8

Ques:- Let R be the relation on the set of people such that xRy if x and y are people and x is older than y. Show that R is not a partial ordering.

Ans:- R is not reflexive, because no person is older than himself or herself.

That is, $x \not R x$ for all people x. It follows that R is not a partial ordering.

Comparable

The elements a and b of a poset $(S, \preccurlyeq)$ are called **comparable** if either $a \preccurlyeq b$ or $b \preccurlyeq a$.

When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called **incomparable**

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Comparable – Example 1

Ques:- In the poset $(Z^+, |)$, are the integers 3 and 9 comparable?

Ans:- The integers 3 and 9 are comparable, because $3 \mid 9$.

Comparable – Example 2

Ques:- In the poset $(Z^+, |)$, are 5 and 7 comparable?

Ans:-

The integers 5 and 7 are incomparable, because $5 \nmid 7$ and $7 \nmid 5$.

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Totally ordered set

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a totally ordered or linearly ordered set, and $\preccurlyeq$ is called a total order or a linear order.

A totally ordered set is also called a chain.

Totally ordered set - Example

Example 1:- The poset $(Z,\leq)$ is totally ordered, because $a \leq b$ or $b \leq a$ whenever a and b are integers.

Example 2:- The poset $(Z^+, |)$ is not totally ordered because it contains elements that are incomparable, such as 5 and 7.

Well-ordered set

$(S, \preccurlyeq)$ is a well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering, and every nonempty subset of S has a least element.

Well-ordered set – Examples

Example 1:- The set of ordered pairs of positive integers, $Z^+ \times Z^+$, with $(a_1, a_2) \preccurlyeq (b_1, b_2)$ if $a_1 < b_1$, or if $a_1 = b_1$ and $a_2 \leq b_2$ (the lexicographic ordering), is a well-ordered set.

Example 2:- The set Z, with the usual ≤ ordering, is not well-ordered because the set of negative integers, which is a subset of Z, has no least element.

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Lexicographic Order - Example

The words in a dictionary are listed in alphabetic, or lexicographic, order, which is based on the ordering of the letters in the alphabet.

Consider the set of strings of lowercase English letters.

discreet $\prec$ discrete,

because these strings differ first in the seventh position, and e $\prec$ t .

Lexicographic Order - Example

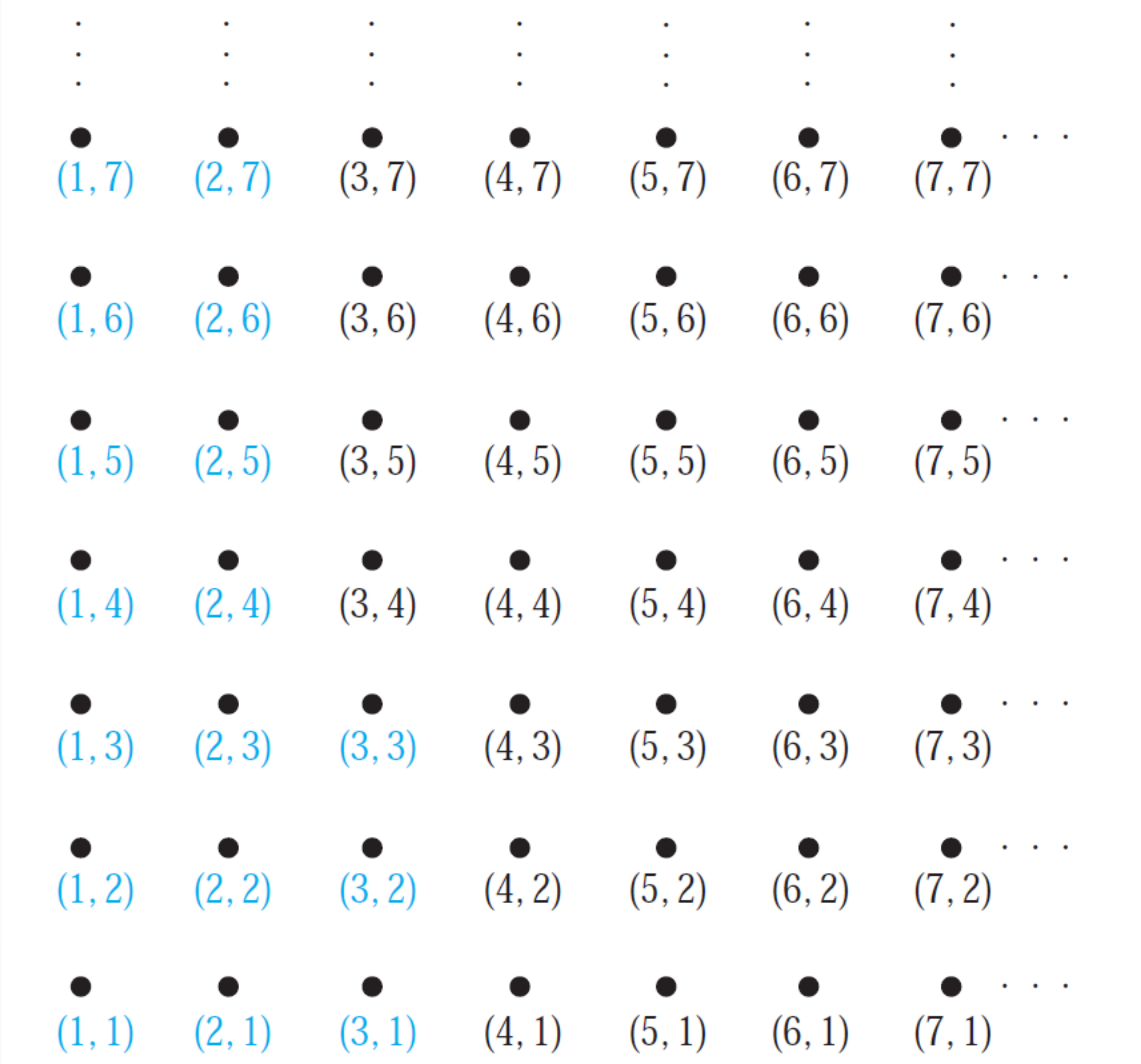
Also,

discreet $\prec$ discreetness, because the first eight letters agree, but the second string is longer.

Furthermore, discrete $\prec$ discretion, because discrete $\prec$ discreti.

Lexicographic Order

The Ordered Pairs Less Than *(3, 4)* in Lexicographic Order.



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Lexicographic Order - Examples

Ques:- Find the lexicographic ordering of these n-tuples:

a) (1, 1, 2), (1, 2, 1)

Ans:- (1, 1, 2) < (1, 2, 1).

b) (0, 1, 2, 3), (0, 1, 3, 2)

Ans:- (0, 1, 2, 3) < (0, 1, 3, 2).

c) (1, 0, 1, 0, 1), (0, 1, 1, 1, 0)

Ans:- (0, 1, 1, 1, 0) < (1, 0, 1, 0, 1).

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That's all for now…