22057029 Assignment 1 N.
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Asympototic Notation

1) Find out whether θ notation satisfies the reflexive, symmetric & transitive property or not (with proof).

Soln $T(n) = \theta(g(n))$ & $g(n) = \theta(h(n))$

$T(n) \stackrel{?}{=} \theta(h(n))$

$$\lim_{n\to\infty} \frac{T(n)}{g(n)} = c_1 > 0 \qquad \lim_{n\to\infty} \frac{g(n)}{h(n)} = c_2 > 0$$

multiplying ① & ② we get

$$\lim_{n\to\infty} \frac{T(n)}{h(n)} = c_1 c_2 > 0$$

$\therefore$ Transitivity is satisfied by theta notation.

- $T(n) = \theta(T(n))$

$$\lim_{n\to\infty} \frac{T(n)}{T(n)} = 1$$

we know if $\lim \frac{T(n)}{f(n)} = c > 0$, then

$T(n) = \theta(f(n))$

$T(n) = O(f(n))$.

$T(n) = \Omega(f(n))$

$\therefore$ Reflexivity is also satisfied by theta notation.

- $T(n) = \theta(f(n))$, is $f(n) = \theta(T(n))$ = ?

If $\lim_{n\to\infty} \frac{T(n)}{f(n)} = c > 0$ $\therefore f(n) = \theta(T(n))$

Symmetricity is satisfied by theta notation.

$$\lim_{n\to\infty} \frac{f(n)}{T(n)} = \frac{1}{c} > 0$$

2. Find out whether Ω-notation satisfies the reflexive, symmetric & transitive property or not (with proof).

Soln. • If $T(n) = \Omega(g(n))$, $g(n) = \Omega(h(n))$

$$\lim_{n\to\infty} \frac{T(n)}{g(n)} = \infty \text{ or } c_1 > 0 \quad \ldots (1)$$

$$\lim_{n\to\infty} \frac{g(n)}{h(n)} = \infty \text{ or } c_2 > 0 \quad \ldots (2)$$

multiplying (1) & (2) we get...

$$\lim_{n\to\infty} \frac{T(n)}{h(n)} = \infty \text{ or } c_1 c_2 > 0$$

$\therefore$ Transitivity satisfies Ω-notation.

• If $T(n) = \Omega(T(n))$.

$$\lim_{n\to\infty} \frac{T(n)}{T(n)} = 1, \text{ we know if } \lim_{n\to\infty} \frac{T(n)}{f(n)} = c > 0,$$

then $T(n) = \Omega(f(n))$

$\therefore$ Reflexivity is also satisfied by Ω notation.

• If $T(n) = \Omega(T(n))$, $f(n) = \Omega(T(n))$

↳ if $\lim_{n\to\infty} \dfrac{T(n)}{f(n)} = c > 0$ or ∞

$$\lim_{n\to\infty} \frac{f(n)}{T(n)} = \frac{1}{c} > 0 \text{ or } 0 \quad \cancel{\text{then } f(n) = \Omega(f(n))}$$

•

3) find out whether O-notation satisfies the reflexive, symmetric & transitive property or not (with proof).

Sol. $T(n) = O(g(n))$ & $g(n) = O(h(n))$

$$\lim_{n\to\infty} \frac{T(n)}{g(n)} = 0 \text{ or } c_1 > 0 \quad \lim_{n\to\infty} \frac{g(n)}{h(n)} = 0 \text{ or } c_2 > 0 \quad \ldots (2)$$

multiplying (1) & (2) we get :-

$$\lim_{n\to\infty} \frac{T(n)}{h(n)} = 0 \text{ or } c_1 c_2 > 0$$

↳ So $T(n) = O(h(n))$.

∴ Transitivity is satisfied by O-notation.

$T(n) = O(T(n))$.

$$\lim_{n\to\infty} \frac{T(n)}{T(n)} = 1$$

↳ its $c > 0$ i.e. $1 > 0$ So $T(n) = O(f(n))$

∴ Reflexivity is also satisfied by O-notation.

$T(n) = O(f(n))$

$$\lim_{n\to\infty} \frac{T(n)}{f(n)} = 0 \text{ or } c > 0$$

but if $\frac{T(n)}{f(n)} = 0$ then $\frac{f(n)}{T(n)} = \infty$ (∴ not > 0 or some +ve const)

∴ Symmetricity is not satisfied O-notation

4) Prove that $O(f(n)) \cdot O(g(n)) = O(f(n) \cdot g(n))$

Sol^n We know $h(n) = O(f(n))$

If $\forall$ $c_1 > 0$ & $n_0 > 0$

① ... $h(n) \le c_1 * f(n)$ for all $n \ge n_0$

Similarly $k(n) = O(g(n))$

If $\forall$ $c_2 > 0$ & $n_1 > 0$

② ... $k(n) \le c_2 * g(n)$ for $n \ge n_1$

of the product function $p(n) = h(n) * k(n)$

To proof $\rightarrow$ If c_3 & n_2 such that $|p(n)| \le c_3 |f(n) * g(n)|$

for all $n \ge n_2$

$$|p(n)| = |h(n) * k(n)| \le |h(n)| * |k(n)|$$
$$\le (c_1 * |f(n)|) * (c_2 * |g(n)|)$$

$$\Rightarrow |p(n)| \le \underbrace{c_1 * c_2}_{\text{constant } (c_3)} * |f(n)| * |g(n)|$$

$\therefore p(n) \le c_3 * |f(n)| * |g(n)|$

$\therefore p(n) = O(f(n) * g(n))$, $n \ge \max(n_1, n_2)$.

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5) Prove that $O(f(n) \cdot g(n)) = f(n) \cdot O(g(n))$

Soln :- $O(f(n) * g(n)) \subseteq O(f(n)) * O(g(n))$

Consider two functions $f(n) = n^2$ & $g(n) = n$.

$f(n) = n^2 = O(n^2)$

$g(n) = n = O(n)$

$f(n) * g(n) = n^3 \cong O(n^3)$

$\because f(n) * O(g(n)) = n^2 * O(n) = O(n^3)$

6) Find the Θ-notation of the following functions :-

a) $f(n) = 4n^3 + 5n^2 + 3n + 4$

$4n^3 \leq 4n^3 + 5n^2 + 3n + 4 \leq 4n^3 + 5n^2 + 3n + 4$

$c_1 n^3 \leq 4n^3 + 5n^2 + 3n + 4 \leq c_2 n^3$

$\left.\begin{array}{l} c_1 = 1 \\ c_2 = 2 \end{array}\right] \rightarrow n^3 \leq 4n^3 + 5n^2 + 3n + 4 \leq 2n^3$ X

$4 + 5 + 3 + 4 = 16$

$c_1 = 1$,

$c_2 = 17 \rightarrow n^3 \leq 4n^3 + 5n^2 + 3n + 4 \leq 17n^3$ ✓

$\therefore f(n) = \Theta(n^3)$

b) $f(n) = \frac{1}{3}n^3 - 4n$

$c_1 \leq \frac{1}{3} - \frac{4}{n^2} \leq c_2$

$f(n) = \Theta(n^3)$

$n = 2$
$n = 2$
$n = 3$
$n = 4$ ✓

$n = 5 \rightarrow +ve$

c) $f(n) = 4 * 2^n + 5n^2 + 3n + 4$

Soln $c_1 \leq 4 2^n + 5n^2 + 3n + 4 \leq c_2$

$4 * 2^n \leq 4 * 2^n + 5n^2 + 3n + 4 \leq 16 \times 2^n$

$c_1 = 4 \qquad c_2 = 16$

$f(n) = \theta(2^n)$

7) Find the O-notation & Ω-notation of the following functions.

a) $f(n) = 2n^3 + 5n^2 - 3n + 4$

Ans $f(n) = O(n^3)$

$2n^3 + 5n^2 - 3n + 4 \geq cn^3$

$c = 2$

$2n^3 + 5n^2 - 3n + 4 \geq 2n^3$

for $n = 1$, $8 \geq 2$ true.

$f(n) = \Omega g(n) = \Omega(n^3)$

b) $f(n) = 5n^2 - 8n$

$f(n) \leq c\, g(n)$

$5n^2 - 8n \leq 6n^2$

$c_1 = 6$

$n = 1$, $-3 \leq 6$ ✓

$f(n) = O(n^2)$

$f(n) \geq c \cdot g(n)$

$5n^2 - 8n \geq 5n^2$ ✗

$\geq 4n^2$

$n = 1$, $5n^2 - 8n$

$\geq 5n^2 - 8n^2$

$n \geq 1$, $5n^2 - 8n \geq -3n^2$

$\therefore -3n^2 \leq 0$ (for $n \geq 1$)

$c = 1$, $n_0 = 1$

$0 \le c\, g(n) \le f(n) \quad \forall n \ge n_0$

$f(n) \cancel{\geq}$ is $\Omega(n^2)$

~~for~~

c) $f(n) = \frac{1}{3} n^3 - 4n$

$f(n) \le c \cdot g(n)$

$\frac{1}{3} n^3 - 4n \le c n^3$

$c = 1, n = 1$

$f(n) = O(n^3)$

For Ω-notation :-

$f(n) \ge c \cdot g(n) \ge 0$

$\frac{1}{3} n^3 - 4n \ge \frac{1}{3} n^3 - 4n^3$

$\Rightarrow \frac{1}{3} n^3 - 4n \ge \frac{-11}{3} n^3$

$\Rightarrow \frac{-11}{3} n^3 \le 0$ (for $n \ge 1$)

$c = 1, n_0 = 1 \cdot f(n)$

$= \Omega(n^3)$

or $n^2 \Longleftrightarrow 2 \log n! \ge n \log n$

$\log n! \ge \frac{1}{2} n \log n$

$\therefore \log(n!) = \Omega(n \log n)$

Since $\log(n!) = \Omega(n \log n) = O(n \log n)$

so $\log(n!) = \Theta(n \log n)$

Q) find the asymptotic bound of the following recurrence equation using iteration method / recursion tree method & substitution method.

Soln (a) $T(n) = T(n-2) + 2 \log n$

$T(n-2) = T(n-4) + 2 \log(n-2)$

$T(n-4) = T(n-8) + 2 \log(n-4)$

$$T(n) = T(n-2) + 2\log n$$
$$= T(n-4) + 2\log(n-2) + 2\log n$$
$$= T(n-6) + 2\log(n-4) + 2\log(n-2) + 2\log n$$
$$\vdots$$
$$T(n-2k) + 2\log(n-(2k-2)) + 2(n-(2k-4)) + \dots 2\log n$$

Assuming $T(1) = 1$,

$$n - 2k = 1$$
$$\text{or } k = \frac{n-1}{2}$$

$$= T(1) + 2\log 3 + 2\log 5 + 2\log 7 \dots + 2\log n$$
$$= T(1) + 2(\log(\underbrace{3 \times 5}_{2n+1} \times 7 \times 9 \times \dots \times n))$$
$$= T(1) + 2(\log((2k+1).((2k+1)+2).((2k+1)+4).((2k+1)+6) \dots \log n$$
$$= T(1) + 2(\log(2k+1)^n)$$
$$= 1 + 2n\log(2k+1) = 1 + 2n\log\left(n\left(\frac{n-1}{2}\right)+1\right)$$
$$= O(n\log n)$$

7) $f(n) = 2^n + 6n^2 - 3n$

Soln $f(n) \leq c \cdot g(n)$

$2^n + 6n^2 - 3n \leq c2^n$

$n = 1, \; 5 \leq 2$

$n = 2, \; 4 + 72 - 6 \leq 4$

$n = 10$

$c = 3$ ✓

$2^n + 6n^2 - 3n \geq c2^n$

$c = 1$

$2^n + 6n^2 - 3n \geq 2^n$

for $n = 25 \geq 2$ thus

$2^n + 6n^2 - 3n$ is $\Omega(2^n)$

8. Prove that $\log n! = O(n \log n)$

$\log n! = \log(1 \times 2 \times 3 \times \ldots (n-1) \times n)$

$\log n! = \log 2 + \log 2 + \log 3 + \ldots + \log n$

$\leq \log n \leq \log n \quad \leq \log n \quad \leq \log n$

$\log n! \leq \log n + \log n + \log n + \ldots \log n$

$\log n! \leq n \log n$

Additionally, $\log(n!) = \Omega(n \log n)$

Trick

$(n!)^2 = n! \times n!$

$= (n \times (n-1) \times (n-2) \times \ldots \times 1) \times (1 \times 2 \times 3 \times 4 \times \ldots \times n)$

or $((n \times 1) \times ((n-1) \times 2) \times ((n-2) \times 3) \times \ldots \times (1 \times n)$

$= \prod_{k=1} (n-k+1) \times k$

$$= \prod_{k=1}^{n} \underbrace{(-k^2 + nk + k)}$$

$$f(k) = -k^2 + nk + k$$

$$= \prod_{k=1}^{n} \underbrace{(-k^2 + nk + k)}$$

$$f(k) = -k^2 + nk + k$$

$$(1 \le k \le n)$$

$$f(k)_{min} = -1^2 + n \times 1 + 1 = n \rightarrow (n!)^2 = \prod_{k=1}^{n} f(k)$$

$$\ge \prod_{k=1}^{n} f(k)_{min}$$

$$\text{or } (n!)^2 \ge \prod^{n} n = n^n$$

b) $T(n) = T(n-1) + cn$

$T(n-1) = T(n-2) + c(n-1)$

$T(n-2) = T(n-3) + c(n-2)$

$T(n) = T(n-2) + c(n-1) + cn$

$= T(n-3) + c(n-2) + c(n-1) + cn$

$\vdots \qquad \vdots$

$n - k = 1 \qquad c(n-(k-1)) + c(n-(k-2))$

$\dots + kcn$

$k = n-1$

$= T(1) + c(2) + c(2) + \dots cn$

$= T(1) + c(1 + 2 + 3 + \dots n)$

$= T(1) + c \cdot \frac{n(n+1)}{2}$

$= O(n^2)$

c) $T(n) = T(n-1) + \frac{1}{n}$

$T(n-1) = T(n-2) + \frac{1}{(n-1)}$

$T(n-2) = T(n-3) + \frac{1}{n-2}$

$T(n) = T(n-1) + \frac{1}{n}$

$= T(n-2) + \frac{1}{n-1} + \frac{1}{n}$

$= T(n-3) + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}$

$\vdots$

$T(n-k) + \frac{1}{n-(k-1)} + \frac{1}{n(k-2)} + \dots \frac{1}{n}$

$n - k = 1$

$k = n-1$

$T(1) + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \dots \frac{1}{n}$

$\simeq O(\lg n)$.

d) $T(n) = 2T(n-1) + cn^2$

$T(n-1) = 2T(n-2) + c(n-1)^2$

$T(n-2) = 2T(n-3) + c(n-2)^2$

$T(n) = 2T(n-1) + cn^2$

$= 2(2T(n-2) + c(n-1)^2) + cn^2$

$= 4T(n-2) + 2c(n-1)^2 + cn^2$

$= 4(2T(n-3) + c(n-2)^2) + 2c(n-1)^2 + cn^2$

$= 8T(n-3) + 4c(n-2)^2 + 2c(n-1)^2 + cn^2$

$\vdots$

$= 2^k T(n-k) + 2^{(k-1)} c(n-(k-1))^2 + 2^{(k-2)} c(n-(k-2))^2 + \dots + cn^2$

$n-k=1$

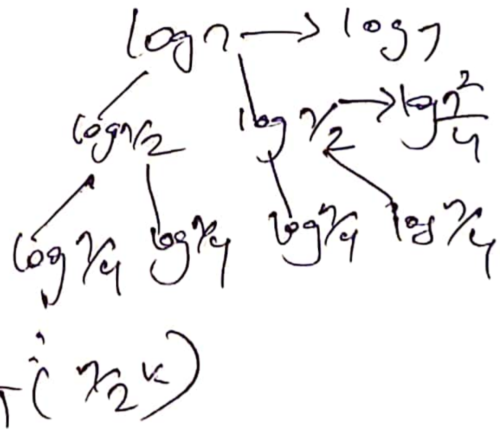
$k=n-1$

$= 2^k + 2^{k-1} c(2)^2 + 2^{k-2}(3)^2 + \dots + cn^2$

$S = 2^k + 2^{k-1} \cdot c2^2 + 2^{k-2} c3^2 + 2^{k-3} c4^2 + \dots cn^2$

$2S = 2^{k+1} + 2^k \cdot c2^2 + 2^{k-1} c3^2 + \dots c$

e) $T(n) = 2T(n/2) + \log n$



$T(n/2) = 2T(n/4) + \log n/2$

$T(n/4) = 2T(n/8) + \log n/4$

$n/2^k = 1$ or $\log_2 n = k$

$= \log n + \log \frac{n^2}{4} + \log \frac{n^4}{256} + \dots$

$= \log \frac{n}{4^0} + \log \frac{n^2}{4^1} + \log n^{4}$

$= 2k \log n - 2k \log 2k$

$= 2 \log n \cdot \log n - 2 \log n \cdot \log(2 \log n)$

$= O(\log n \cdot \log n)$.

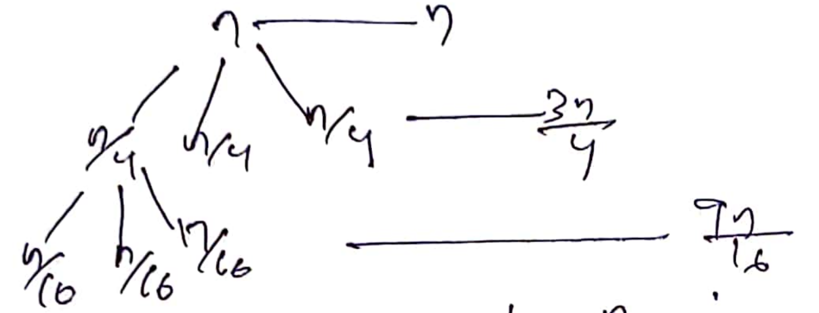
Total $= 2^k + 2\log n \cdot \log n - 2 \log n \cdot \log(2 \log n)$

$= 2^{\log_2 n} + 2 \log n \cdot \log n - 2 \log n \cdot \log(2 \log n)$

$= n + 2 \log n \cdot \log n - 2 \log n \cdot \log(\log n)$

$= O(n)$

f) $T(n) = 3T(n/4) + n$



$\frac{3^k n}{4^k}$

$T\left(\frac{n}{4^k}\right) = T(1)$

$n = 4^k$

$\log_4 n = k$

$3^k = 3^{\log_4 n}$

$= n^{\log_4 3}$

$\text{Total} = n^{\log_4 3} + \frac{3^k n}{4^k}$

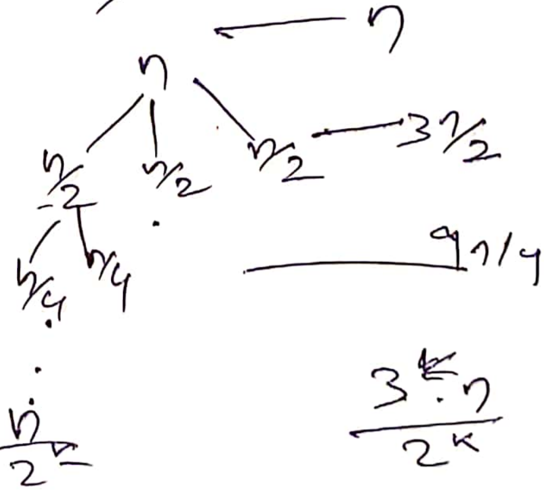
$= n^{\log_4 3} + \frac{n^{\log_4 3} \cdot \cancel{n}}{\cancel{n}}$

$\cong O(n)$

g) $T(n) = 3T(n/2) + n$

$T(n/2) = 3T(n/4) + n/2$

$T(n/4) = 3T(n/8) + n/4$



$\frac{n}{2^k} = 1$

$\Rightarrow n = 2^k$

$\Rightarrow \log_2 n = k$

$3^k = 3^{\log_2 n}$

$= n^{\log_2 3}$

$\text{total} = n^{\log_2 3} + \frac{3^{\log_2 n} \cdot n}{2^{\log_2 n}}$

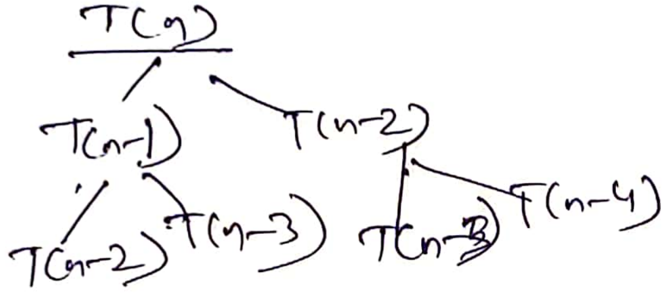
$= n^{\log_2 3} + n^{\log_2 3} = O(n)$

h) $T(n) = T(n-1) + T(n-2) + C \rightarrow$ Assuming $T(1) = C$, $T(2) = 2C$

$T(n-1) = T(n-2) + T(n-3) + C$

$T(n-2) = T(n-3) + T(n-4) + C$

$$T(n) = T(n-1) + T(n-2) + C$$
$$= T(n-2) + T(n-3) + C + T(n-3) + T(n-4) + C$$
$$= T(n-2) + 2T(n-3) + T(n-4) + 3C$$



$$O(2^n)$$

i) $T(n) = 4T(n/2) + n^2$

$$T(n/2) = 4T(n/4) + (n/2)^2$$
$$T(n/4) = 4T(n/8) + (n/4)^2$$

$\frac{n}{2^k} = 1$ $\quad 4^k = 4^{\log_2 n}$

$n = 2^k$ $\quad = n^{\log_2 4}$

$\log_2 n = k$

$$\text{Total} = n^{\log_2 4} + \frac{n^2}{2^k \cdot 2^k} \cdot 4^k$$
$$= n^{\log_2 4} + \frac{n\cdot n}{n\cdot n} \cdot 4^k$$
$$= n^{\log_2 4} + 4^{\log_2 n}$$
$$= n^{\log_2 4} + n^{\log_2 4}$$
$$= n^{\log_2 2^2} + n^{\log_2 2^2}$$
$$= O(n^2)$$

(b) consider a recurrence $T(n) = 2T(n/2) + n$
Show that $T(n) = \Omega(n \log n)$

$T(n) = \Omega(n \log n)$
$T(n) = 2T(n/2) + n$

find constants c & n_0 such that:-
$T(n) \geq c.\, n \log n$ for all $n \geq n_0$.

notice that it is in the form of:-
$T(n) = aT(n/b) + f(n)$, where in this case
$a = 2,\ b = 2,\ f(n) = n$

TC of $f(n) = n$
$n^{\log_b a} = n^{\log_2 2} = n$

$\therefore f(n) = \theta(n^{\log_b a})$
$T(n) = \theta(n^{\log_b a} \cdot \log n)$
$= \theta(n \log n)$.

(ii) 1) Show that $n \log n - 3n + 14 = \Omega(n \log n)$
let $f(n) = n \log n - 3n + 14$, $g(n) = n \log n$
Assuming $f(n) \geq c\, g(n)$ for some constant 'c' & all $n \geq n_0$.
$n \log n - 3n + 14 \geq c.\,(n \log n)$
Subtracting $c.\, n \log n$ from both sides we get:-
$\log n\,(n - cn) \geq 3n - 14$

$= n \cdot (\log n - c) \geq 3n - 14$

Dividing both side by 'n' we get :-

$= \log n - c \geq 3 - \frac{14}{n}$

Now, find a constant 'c' such that $\log n - c$ remains larger than ~~one~~ or equal to $3 - \frac{14}{n}$

let $c = 2$

$\log n - 2 \geq 3 - \frac{14}{n}$

$\log n \geq 5 - \frac{14}{n}$

for all $n \geq 14$ the RHS is +ve & log grows rapidly as 'n' increases.

Thus $n_0 = 14$

for $n \geq 14$ & $c = 2$

$n\log n - 3n + 14 \geq c \cdot n\log n$

for all $n \geq n_0$

Thus $n\log n - 3n + 14 = \Omega(n\log n)$.

12) Show that $\frac{1}{2}n^2 - 7n = \theta(n^2)$

$f(n) = \frac{1}{2}n^2 - 7n$

$f(n) = \frac{1}{2}n^2 - 7n = O(n^2)$ .. ①

$f(n) = \frac{1}{2}n^2 - 7n = \Omega(n^2)$ — ②

Proving ① :- $\frac{1}{2}n^2 - 7n \leq cn^2 \; \forall n \geq n_0$

Dividing both side by n^2 we get :-

$\frac{1}{2} - \frac{7}{n} \leq c$

Since $\frac{7}{n} \to 0$ as n becomes larger,

So $c \geq 1/2$

let $c = 3/4$

$\frac{1}{2} - \frac{7}{n} \leq 3/4$

$\Rightarrow -\frac{7}{n} \leq \frac{1}{4} \to$ this holds true for all $n \to 28$ $\left(\because \frac{7}{28} = \frac{1}{4}\right)$

$\Rightarrow \frac{1}{2}n^2 - 7n \leq \frac{3}{4}n^2$

$\therefore \frac{1}{2}n^2 - 7n = O(n^2)$.

Proving (2) :-

$\frac{1}{2}n^2 - 7n = \Omega(n^2)$

$\frac{1}{2}n^2 - 7n \geq c \cdot n^2 \; \forall n \geq n_0$

Dividing both sides by n^2 :

$\frac{1}{2} - \frac{7}{n} \geq c$

choosing $c = \frac{1}{4}$

$1/2 - \frac{7}{n} \geq \frac{1}{4}$

$\Rightarrow -\frac{7}{n} \geq -\frac{1}{4}$ $\therefore$ $\forall n \geq 28$ Since $\frac{7}{28} = \frac{1}{4}$)

$\Rightarrow \frac{1}{2}n^2 - 7n \geq \frac{1}{4}n^2$

$\Rightarrow \frac{-7}{n} \geq \frac{-1}{4} \quad \because (n \geq 28 \text{ since } \frac{7}{28} = \frac{1}{4})$

$\Rightarrow \frac{1}{2}n^2 - 7n \geq \frac{1}{4}n^2$

Thus, $\frac{1}{2}n^2 - 7n = \Omega(n^2)$

$\therefore$ by proving (1) & (2):—

$\frac{1}{2}n^2 - 7n = \Theta(n^2)$

(13) find the asymptotic bound of the following recurrence equations using master theorem

a) $T(n) = 4T(\frac{n}{2}) + n^2$

$a = 4 \quad f(n) = n^2$

$b = 2 \quad = \Theta(n^2 \log^0 n)$

$k = 2, p = 0$

$\log_2 4 = 2 = k = 2$

$p > -1 \quad \Theta(n^k \log^{p+1} n)$

$= \Theta(n^2 \log n)$

b) $T(n) = 3T(\frac{n}{4}) + n$

$a = 3 \quad f(n) = n$

$b = 4 \quad = \Theta(n^2 \log n) \quad k = 1, p = 0$

$\log_b a = \log_4 3 < k = 1 \quad p \geq 0 \rightarrow \Theta(n^k \log^p n)$

$= \Theta(n)$

c) $T(n) = 2T(n/2) + \sqrt{n}$

$a=2$
$b=2$ $\quad f(n) = \sqrt{n} = (n)^{1/2}$

$\cong \Theta(n^{1/2} \log n)$

$\log_2 2 = 1 > 0.5 \quad k = 1/2, \; P=0$

$\Theta(n^{\log_b a}) \cong \Theta(n)$

d) $T(n) = 4T(n/2) + n^2\sqrt{n}$

$a=4$
$b=2$ $\quad f(n) = n^2\sqrt{n} = n^2 . n^{1/2}$
$= n^{5/2}$

$\log_b a = \log_2 4 \cong \Theta(n^{5/2} \log n)$

$= 2 < 2.5 \; k \quad p \geq 0$

$\hookrightarrow \Theta(n^k \log^p n)$

$\cong \Theta(n^2\sqrt{n})$

e) $T(n) = 2T(n/2) + n^2$

$a=2$
$b=2$ $\quad \log_2 2 = 1 < 2 \qquad f(n) = n^2$

$\cong \Theta(n^k \log^p n)$

$p \geq 0 \qquad \cong \Theta(n^2 \log^p n)$

$\Theta(n^k \log^p n) \qquad k=2, P=0$

$\cong \Theta(n^2)$

f) $T(n) = 16T(n/4) + n^2$

$a=16$
$b=4$ $\quad \Theta(n^k \log^p n)$

$\cong \Theta(n^2 \log^0 n) \to k=2, P=0$

$\log_b a = 2 = k \; p > -1 \quad \Theta(n^k \log^{p+1} n) = \Theta(n^2 \log n)$

g) $T(n) = 2T(n/2) + n^3$

$a=2$, $b=2$, $f(n) = n^3 \simeq \Theta(n^3 \log^0 n)$

$k=3$, $p=0$

$\log_b a = \log_2 2 = 1 < k = 3$

$p \geq 0$

$= \Theta(n^3)$

h) $T(n) = 4T(n/2) + c$

$a=4$, $b=2$, $f(n) = c \simeq \Theta(n^0 \log^0 n)$

$\log_b a = \log_2 4 = 2 > k = 0$

$k=0$, $p=0$

$= \Theta(n^{\log_b a}) = \Theta(n^2)$

14) Find the asymptotic bound of the following recurrence equation using change of variable method.

a) $T(n) = \sqrt{n}\, T(\sqrt{n}) + n$

$n = 2^m$ $\quad n^{1/2} = 2^{m/2}$

$T(2^m) = 2^{m/2} \cdot \underbrace{T(2^{m/2})}_{S(m/2)} + 2^m$

$S(m)$

$S(m) = \frac{m}{2} \cdot S(m/2) + m$

$= m\left(\frac{1}{2} S\left(\frac{m}{2}\right) + 1\right)$

$a = \frac{1}{2}$, $b = 2$, $f(n) = n$

$\log_b a = \log_2 {}^{1/2} = \log_2 1 - \log_2 2 = -1$

Comparing $f(n) = n$

~~nt~~ $n^{\log_b a} = n^{-1}$,

$f(n) = \Theta(n^{\log_b a})$ follows case-2 of masters theorem.

$\cong \Theta(n^{\log_b a} \log n)$

$\cong \Theta(n^{-1} \log n)$ ($\because$ Not sure)

$\cong \Theta\left(\frac{\log n}{n}\right)$

b) $T(n) = 2T(\sqrt{n}) + C$

$n = 2^m \cdot n^{1/2} = 2^{m/2}$ $m = \log n$

$\underbrace{T(2^m)}_{S(m)} = 2\underbrace{T(2^{m/2})}_{S(m/2)} + C$

$S(m) = 2S(m/2) + C$

$a = 2$ $f(n) = C$

$b = 2$ $\cong \Theta(n^0 \log^0 n)$ $k = 0$, $P = 0$

$\log_b a = \log_2 2 = 1 > k = 0$ $\Theta(m^{\log_b a})$

$\cong \Theta(m^1)$

$T(n) = \Theta(\log n)$

c) $T(n) = 4T(n/2) + n$

$n = 2^m$

or $\frac{n}{2} = 2^{m-1}$

$\underbrace{T(2^m)}_{S(m)} = 4\underbrace{T(2^{m-1})}_{S(m-1)} + \underbrace{2^m}_{m}$

$\Rightarrow S(m) = 4S(m-1) + m$

$S(m-1) = 4S(m-2) + m-1$

$S(m-2) = 4S(m-3) + m-2$

$S(m-3) = 4S(m-4) + m-3$

$\vdots$

$S(m) = 4S(m-1) + m$

$= 4(4S(m-2) + m-1) + m$

$= 16S(m-2) + 4(m-1) + m$

$= 16(4S(m-3) + m-2) + 4(m-1) + m$

$= 64S(m-3) + 16(m-2) + 4(m-1) + m$

$= 64(4S(m-4) + m-3) + 16(m-2) + 4(m-1) + m$

$= 256S(m-4) + 64(m-3) + 16(m-2) + 4(m-1) + m$

$\vdots$

$4^k S(m-k) + 4^{k-1}(m-k-1) + 4^{k-2}(m-k-2) + 4^{k-3}(m-1) + m$

$m-k=0$

$k=m$

$S_1 = 4^k + 4^{k-1}(1) + 4^{k-2}(2) + 4^{k-3}(3) + \ldots + 4^{k-k}m$

$4S_1 = 4^{k-1} + 4^{k-2} + 4^{k-3}(2) + \ldots$

$\Theta(n^2)$

d) $T(n) = 7T(n/2) + 3n^2$

$n = 2^m$ | $S(2^m) = 7T(2^{m-1}) + 3 \times 2^{2m}$

$\frac{n}{2} = 2^{m-1}$ | $S(m) = 7T(m-1) + 3 \times 2m$

or $S(m) = 7T(m-1) + 6m$

$\rightarrow S(m-2) = 7S(m-3) + \cancel{6}(m-2)$

$\Rightarrow S(m) = 49(7S(m-3) + 6(m-2))$
$+ 48m - 42$

$\Rightarrow S(m) = 343S(m-3) + 294(m-2) + 48m - 42$

$\Rightarrow S(m) = 343S(m-3) + 294m - 588 + 48m - 42$

$= 343S(m-3) + 342 - 630$

$= 343^k S(n-k) + (6 + 48 + 342 + \cdots 48^k - 42k)$

$\Theta(n^2)$ ~~first found using~~

15) find out the function which is growing faster :-

$a = n^{\sqrt{n}}$ vs $2^{\sqrt{n}\log_2 n}$

taking log on both side

$\log_2 n^{\sqrt{n}}$ .. $\log_2 2^{\sqrt{n}\log_2 n}$

$\sqrt{n}\,\cancel{\log_2 n}$ $\qquad$ $\sqrt{n}\,\cancel{\log_2 n}$

$= 1$ $\qquad$ $= 1$

Both are growing of same

b) $n^{\sqrt{n}}$ vs $n^{\log_2 n}$

Taking log on both side

$\log_2 n^{\sqrt{n}}$ $\quad \log_2 n^{\log_2 n}$

$\sqrt{n}\cancel{\log_2 n}$ $\quad \cancel{\log_2 n}\log_2 n$

$(n)^{1/2}$ $\quad \log_2 n$

taking log on both side:

$\frac{1}{2}\log n > \log \log_2 n$

$n^{\sqrt{n}}$ is growing faster.

c) $n^{\log n}$ vs 2^n

Taking log on both side:—

$\log n \log n = n$

n — grows faster than $\log \log n$

$\therefore$ 2^n grows faster.

d) $\sqrt{\log n}$ vs $\log \log n$

Taking log on both side

$(\log n)^{1/2}$ $\quad \log \log n$

$\frac{1}{2}\log(\log n)$ $\quad \log(\log \log(n))$

$\sqrt{\log n}$ is growing faster.