22057020 Assignment-1 Subject: DAA NAME: AURO SASWAT RAJ

• Asymptotic Notation:-

1. Find out whether θ-notation satisfies the reflexive, symmetric and transitive property or not with proof.

• $T(n) = \theta(g(n))$ and $g(n) = \theta(h(n))$

$T(n) \stackrel{?}{=} \theta(h(n))$

$$\lim_{n \to \infty} \frac{T(n)}{g(n)} = c_1 > 0 \quad \ldots (1)$$

$$\lim_{n \to \infty} \frac{g(n)}{h(n)} = c_2 > 0 \quad \ldots (2)$$

multiplying (1) and (2) we get:-

$$\lim_{n \to \infty} \frac{T(n)}{h(n)} = c_1 c_2 > 0$$

∴ Transitivity is satisfied by Theta notation.

• $T(n) = \theta(T(n))$

$$\lim_{n \to \infty} \frac{T(n)}{T(n)} = 1$$

↳ we know if $\lim_{n \to \infty} \frac{T(n)}{f(n)} = c > 0$, then

$$T(n) = \theta(f(n))$$
$$T(n) = O(f(n))$$
$$T(n) = \Omega(f(n))$$

∴ Reflexivity is also satisfied by Theta notation.

• $T(n) = \theta(f(n))$, is $f(n) = \theta(T(n))$?

↳ if $\lim_{n \to \infty} \frac{T(n)}{f(n)} = c > 0$.

$$\lim_{n \to \infty} \frac{f(n)}{T(n)} = \frac{1}{c} > 0$$

∴ $f(n) = \theta(T(n))$

Symmetricity is satisfied by Theta notation.

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2. Find out whether Ω-notation satisfies the reflexive, symmetric and transitive property or not (with proof)?

- if $T(n) = \Omega(g(n))$, $g(n) = \Omega(h(n))$

$$\lim_{n\to\infty} \frac{T(n)}{g(n)} = \infty \text{ or } c_1 > 0 \quad \dots ①$$

$$\lim_{n\to\infty} \frac{g(n)}{h(n)} = \infty \text{ or } c_2 > 0 \quad \dots ②$$

multiplying ① and ② we get:-

$$\lim_{n\to\infty} \frac{T(n)}{h(n)} = \infty \text{ or } c_1 c_2 > 0$$

$\therefore$ Transitivity satisfies Ω-notation.

- if $T(n) = \Omega(T(n))$

$\lim_{n\to\infty} \frac{T(n)}{T(n)} = 1$, we know if $\lim_{n\to\infty} \frac{T(n)}{f(n)} = c > 0$, then $T(n) = \Omega(f(n))$

$\therefore$ Reflexivity is also satisfied by Ω-notation.

- if $T(n) = \Omega(f(n))$, $f(n) = \Omega(T(n))$

↳ if $\lim_{n\to\infty} \frac{T(n)}{f(n)} = c > 0$ ~~or~~, ∞

$$\lim_{n\to\infty} \frac{f(n)}{T(n)} = \frac{1}{c} > 0 \text{ or } 0$$

so $f(n)$ cannot be said as $\Omega(T(n))$, i.e if $T(n)$ is growing larger than $f(n)$ i.e $\frac{T(n)}{f(n)} = \infty$ the $\frac{f(n)}{T(n)}$ is neither ∞ nor some 'c' > 0

$\therefore$ Symmetricity is not satisfied by Ω-notation

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3. Find out whether O-notation satisfies the reflexive, symmetric, and transitive property or not (with proof).

- $T(n) = O(g(n))$ and $g(n) = O(h(n))$

$\lim_{n \to \infty} \frac{T(n)}{g(n)} = 0$ or $c_1 > 0$ ... ① $\lim_{n \to \infty} \frac{g(n)}{h(n)} = 0$ or $c_2 > 0$ .... ②

multiplying ① and ② we get:-

$\lim_{n \to \infty} \frac{T(n)}{h(n)} = 0$ or $c_1 c_2 > 0$

↳ so $T(n) = O(h(n))$

∴ Transitivity is satisfied by O-notation.

- $T(n) = \theta(T(n))$

$\lim_{n \to \infty} \frac{T(n)}{T(n)} = 1$

↳ its $c > 0$ i.e $1 > 0$ so $T(n) = O(f(n))$

∴ Reflexivity is also satisfied by O-notation.

- $T(n) = O(f(n))$

$\lim_{n \to \infty} \frac{T(n)}{f(n)} = 0$ or $c > 0$

but if $\frac{T(n)}{f(n)} = 0$, then $\frac{f(n)}{T(n)} = \infty$ ($\because$ not > 0 or some +ve const)

∴ Symmetricity is not satisfied by O-notation.

4. Prove that $O(f(n))O(g(n)) = O(f(n) \cdot g(n))$

we know $h(n) = O(f(n))$

if $\forall$ $c_1 > 0$ and $n_0 > 0$

① ... $h(n) \leq c_1 * f(n)$ for all $n \geq n_0$

Similarly $k(n) = O(g(n))$

if $\forall$ $c_2 > 0$ and $n_1 > 0$

② .... $k(n) \leq c_2 * g(n)$ for $n \geq n_1$

let the product function $p(n) = h(n) * k(n)$

To proof $\rightarrow$ $\forall$ c_3 and n_2 such that $|p(n)| \leq c_3 |f(n) * g(n)|$

for all $n \geq n_2$

$|p(n)| = |h(n) * k(n)| \leq |h(n)| * |k(n)| \leq (c_1 * |f(n)|) * (c_2 * |g(n)|)$

$\Rightarrow |p(n)| \leq \underbrace{c_1 * c_2}_{\text{constant } (c_3)} * |f(n)| * |g(n)|$

$\therefore$ $p(n) \leq c_3 * |f(n)| * |g(n)|$

$\therefore$ $p(n) = O(f(n) * g(n))$, $n \geq \max(n_1, n_2)$.

5. Prove that $O(f(n) \cdot g(n)) = f(n) \cdot O(g(n))$

$O(f(n) * g(n)) \subseteq O(f(n)) * O(g(n))$

consider two functions $f(n) = n^2$ and $g(n) = n$.

$f(n) = n^2 = O(n^2)$

$g(n) = n = O(n)$

$f(n) * g(n) = n^3 \cong O(n^3)$

$\because$ $f(n) * O(g(n)) = n^2 * O(n)$

$= O(n^3)$

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5. Find the θ-notation of the following functions:-

(a) $f(n) = 4n^3 + 5n^2 + 3n + 4$

$4n^3 \leq 4n^3 + 5n^2 + 3n + 4 \leq 4n^3 + 5n^2 + 3n + 4$

$c_1 n^3 \leq 4n^3 + 5n^2 + 3n + 4 \leq c_2 n^3$

$\left.\begin{array}{l} c_1 = 1 \\ c_2 = 2 \end{array}\right] \rightarrow n^3 \leq 4n^3 + 5n^2 + 3n + 4 \leq 2n^3$ ✗

$4 + 5 + 3 + 4 = 16$

$c_1 = 1,$

$c_2 = 17 \rightarrow n^3 \leq 4n^3 + 5n^2 + 3n + 4 \leq 17n^3$ ✓

$\therefore f(n) = \theta(n^3)$

(b) $f(n) = \frac{1}{3}n^3 - 4n$

$c_1 \leq \frac{1}{3} - \frac{4}{n^2} \leq c_2$

$n=2$ ✗

$n=2$ ✗

$n=3$ ✗

$n=4$ ✓

$n=5 \rightarrow +ve$

$f(n) = \theta(n^3)$

(c) $f(n) = 4*2^n + 5n^2 + 3n + 4$

$c_1 \leq 4*2^n + 5n^2 + 3n + 4 \leq c_2$

$4*2^n \leq 4*2^n + 5n^2 + 3n + 4 \leq 16 \times 2^n$

$c_1 = 4 \qquad c_2 = 16$

$f(n) = \theta(2^n)$

7) Find the O-notation and Ω-notation of the following functions:

(a) $f(n) = 2n^3 + 5n^2 - 3n + 4$

$f(n) = O(n^3)$ $\quad 2n^3 + 5n^2 - 3n + 4 \geq cn^3$

$c = 2$

$2n^3 + 5n^2 - 3n + 4 \geq 2n^3$

for $n = 1$, $8 \geq 2$ true

$f(n) = \Omega g(n) = \Omega(n^3)$

(b) $f(n) = 5n^2 - 8n$

$f(n) \leq c\,g(n)$

$5n^2 - 8n \leq 6n^2$

$c_1 = 6$

$n = 1$, $-3 \leq 6$ ✓

$f(n) = O(n^2)$

$f(n) \geq c\,g(n)$

$5n^2 - 8n \geq 5n^2 \times$

$\geq 4n^2$

$n = 1$, $5n^2 - 8n \geq 5n^2 - 8n^2$

$\Rightarrow 5n^2 - 8n \geq -3n^2$

$\therefore -3n^2 \leq 0$ (for $n \geq 1$)

$c = 1$, $n_0 = 1$

$0 \leq c\,g(n) \leq f(n)$ $\forall n \geq n_0$

$f(n)$ is $\Omega(n^2)$

(c) $f(n) = \frac{1}{3}n^3 - 4n$

$f(n) \leq c\,g(n)$

$\frac{1}{3}n^3 - 4n \leq cn^3$

$c = 1$, $n = 1$

$f(n) = O(n^3)$

For Ω-notation:-

$f(n) \geq c\,g(n) \geq 0$

$\frac{1}{3}n^3 - 4n \geq \frac{1}{3}n^3 - 4n^3$

$\Rightarrow \frac{1}{3}n^3 - 4n \geq -\frac{11}{3}n^3$

$\Rightarrow -\frac{11}{3}n^3 \leq 0$ (for $n \geq 1$)

$c = 1$, $n_0 = 1$. $f(n) = \Omega(n)$ion

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d) $f(n) = 2^n + 6n^2 - 3n$

$f(n) \leq c\,g(n)$

$2^n + 6n^2 - 3n \leq c\,2^n$ $\hookrightarrow c=3$

$n=1,\ 5 \leq 2$

$n=2,\ 4+72-6 \leq 4$

$n=10,$

$c=3$ ✓

$2^n + 6n^2 - 3n \geq c\,2^n$

$c = 1$

$2^n + 6n^2 - 3n \geq 1 \times 2^n$

for $n=1$ $5 \geq 2$ true

$2^n + 6n^2 - 3n$ is $\Omega(2^n)$

8. prove that $\log n! = O(n \log n)$

$\log n! = \log(1 \times 2 \times 3 \times \cdots (n-1) \times n)$

$\log n! = \log 1 + \log 2 + \log 3 + \cdots + \log n$

$\hookrightarrow \leq \log n$ $\hookrightarrow \leq \log n$ $\hookrightarrow \leq \log n$ $\hookrightarrow \leq \log n$

$\log n! \leq \log n + \log n + \log n + \cdots \log n$

$\log n! \leq n \log n$

- Additionally, $\log(n!) = \Omega(n \log n)$

Trick $\rightarrow (n!)^2 = n! \times n!$

$= (n \times (n-1) \times (n-2) \times \cdots \times 1) \times (1 \times 2 \times 3 \times 4 \times \cdots \times n)$

or $((n \times 1) \times ((n-1) \times 2) \times ((n-2) \times 3) \times \cdots \times (1 \times n)$

$= \prod_{k=1}^{n} (n-k+1) \times k$

$= \prod_{k=1}^{n} \underbrace{(-k^2 + nk + k)}$

$F(k) = -k^2 + nk + k$

$(1 \leq k \leq n)$

$F(k)_{min} = -1^2 + n \times 1 + 1 = n \rightarrow (n!)^2 = \prod_{k=1}^{n} F(k) \geq \prod_{k=1}^{n} F(k)_{min}$

or $(n!)^2 = \prod_{k=1}^{n} n = n^n$

or $n^n \leftrightarrow 2\log n! \geq n\log n$

$\log n! \geq \frac{1}{2} n \log n$

$\therefore \log(n!) = \Omega(n\log n)$

since $\log(n!) = \Omega(n\log n) = O(n\log n)$

so $\log(n!) = \Theta(n\log n)$

(9) Find the asymptotic bound of the following recurrence equation using iteration method, recursion tree method and substitution method.

(a) $T(n) = T(n-2) + 2\log n$

$T(n-2) = T(n-4) + 2\log(n-2)$

$T(n-4) = T(n-8) + 2\log(n-4)$

$T(n) = T(n-2) + 2\log n$

$= T(n-4) + 2\log(n-2) + 2\log n$

$= T(n-8) + 2\log(n-4) + 2\log(n-2) + 2\log n$

$\vdots$

$T(n-2k) + 2\log(n-(2k-2)) + 2(n-(2k-4)) + \cdots 2\log n$

Assuming $T(1) = 1$,

$n - 2k = 1$

or $k = \frac{n-1}{2}$

$= T(1) + 2\log 3 + 2\log 5 + 2\log 7 \cdots + 2\log n$

$= T(1) + 2(\log(\underbrace{3}_{2k+1} \times 5 \times 7 \times 9 \times \cdots \times n))$

$= T(1) + 2(\log((2k+1)\cdot((2k+1)+2)\cdot((2k+1)+4) \cdot ((2k+1)+6) \cdots \log n)$

$= T(1) + 2(\log(2k+1)^n)$

$= 1 + 2n\log(2k+1) = 1 + 2n\log(2(\frac{n-1}{2}) + 1)$

$= O(n\log n)$

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b) $T(n) = T(n-1) + cn$

$T(n-1) = T(n-2) + c(n-1)$

$T(n-2) = T(n-3) + c(n-2)$

$T(n) = T(n-2) + c(n-1) + cn$

$= T(n-3) + c(n-2) + c(n-1) + cn$

$\vdots \qquad \vdots$

$n-k = 1 \qquad c(n-(k-1)) + c(n-(k-2)) \cdots + cn$

$k = n-1$

$= T(1) + c(1) + c(2) + \cdots + cn$

$= T(1) + c(1+2+3+\cdots n)$

$= T(1) + c \cdot \frac{n(n+1)}{2}$

$= O(n^2)$

c) $T(n) = T(n-1) + \frac{1}{n}$

$T(n-1) = T(n-2) + \frac{1}{(n-1)}$

$T(n-2) = T(n-3) + \frac{1}{n-2}$

$T(n) = T(n-1) + \frac{1}{n}$

$= T(n-2) + \frac{1}{n-1} + \frac{1}{n}$

$= T(n-3) + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}$

$\vdots$

$T(n-k) + \frac{1}{n-(k-1)} + \frac{1}{n-(k-2)} + \cdots \frac{1}{n}$

$n-k = 1$

$k = n-1$

$T(1) + \underbrace{\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots + \frac{1}{n}}_{\log n}$

$\simeq O(\log n)$

d) $T(n) = 2T(n-1) + cn^2$

$T(n-1) = 2T(n-2) + c(n-1)^2$

$T(n-2) = 2T(n-3) + c(n-2)^2$

$T(n) = 2T(n-1) + cn^2$

$= 2(2T(n-2) + c(n-1)^2) + cn^2$

$= 4T(n-2) + 2c(n-1)^2 + cn^2$

$= 4(2T(n-3) + c(n-2)^2) + 2c(n-1)^2 + cn^2$

$= 8T(n-3) + 4c(n-2)^2 + 2c(n-1)^2 + cn^2$

$\vdots$

$= 2^k T(n-k) + 2^{(k-1)} c(n-(k-1))^2 + 2^{(k-2)} c(n-(k-2))^2 + \cdots + cn^2$

$n-k=1$

$k=n-1$

$= 2^k + 2^{k-1}c(2)^2 + 2^{k-2}(3)^2 + \cdots\cdots cn^2$

$S = 2^k + 2^{k-1} \cdot c2^2 + 2^{k-2}c3^2 + 2^{k-3}c4^2 + \cdots cn^2$

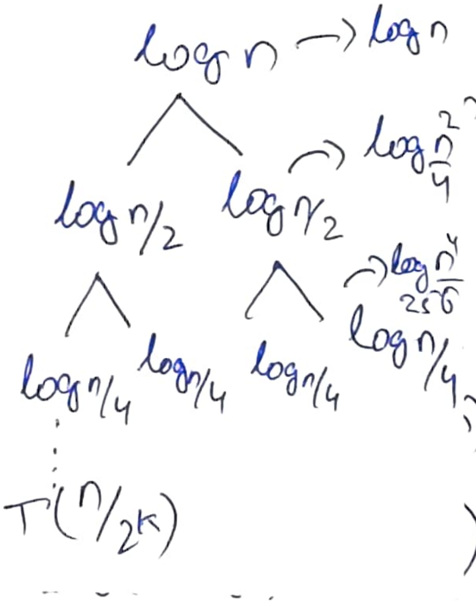
$2S = 2^{k+1} + 2^k \cdot c2^2 + 2^{k-1}c3^2 + \cdots\cdots S$

e) $T(n) = 2T(n/2) + \log n$

$T(n/2) = 2T(n/4) + \log n/2$

$T(n/4) = 2T(n/8) + \log n/4$

$\frac{n}{2^k} = 1$ or $\log_2 n = k$



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$= \log n + \log \frac{n^2}{4} + \log \frac{n^4}{256} + \cdots$

$= \log \frac{n}{4^0} + \log \frac{n^2}{4^1} + \log \frac{n^4}{\cdot}$

$= 2k \log n - 2k \log 2k$

$= 2 \log n \cdot \log n - 2 \log n \cdot \log(2 \log n)$

$\simeq O(\log n \cdot \log n)$

Total $= 2^k + 2\log n \cdot \log n - 2 \log n \cdot \log(2 \log n)$

$= 2^{\log_2 n} + 2 \log n \cdot \log n - 2 \log n \cdot \log (2 \log n)$

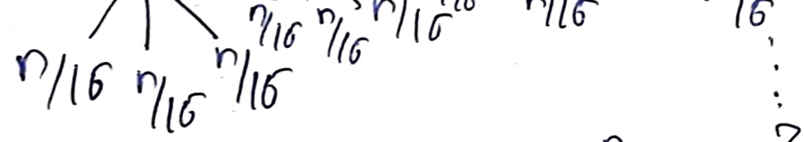
$= n + 2 \log n \cdot \log n - 2 \log n \cdot \log(\log n)$

$\simeq O(n)$

b) $T(n) = 3T(n/4) + n$

$T\left(\frac{n}{4}\right) = 3T\left(\frac{n}{16}\right) + \frac{n}{4}$

$T(n/16) = 3T\left(\frac{n}{64}\right) + \frac{n}{16}$



n — n

$\frac{3n}{4}$

$\frac{9n}{16}$

$\vdots$

$\frac{3^k n}{4^k}$

$\vdots$

$T\left(\frac{n}{4^k}\right) = T(1)$

$n = 4^k$

$\log_4 n = k$

$3^k = 3^{\log_4 n}$

$= n^{\log_4 3}$

Total $= n^{\log_4 3} + \frac{3^k n}{4^k}$

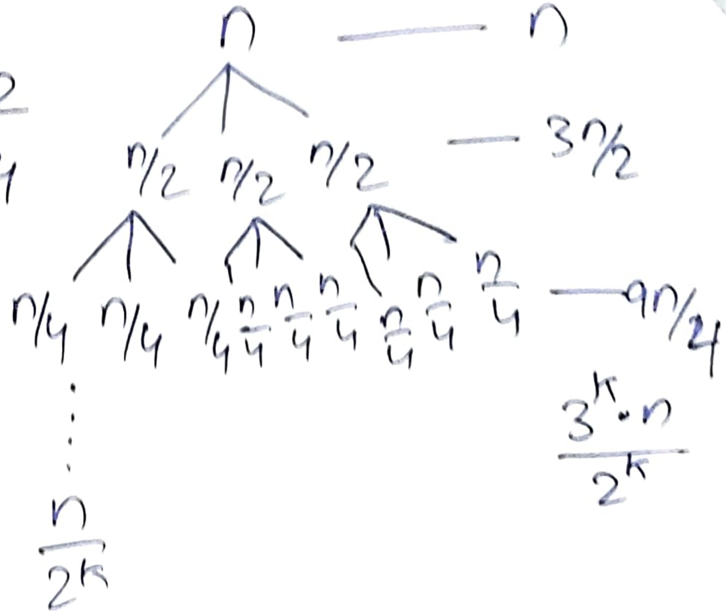
$= n^{\log_4 3} + \frac{n^{\log_4 3} \cdot \cancel{n}}{\cancel{n}}$

$\simeq O(n)$

g) $T(n) = 3T(n/2) + n$

$T(n/2) = 3T(n/4) + n/2$

$T(n/4) = 3T(n/8) + n/4$



$\frac{n}{2^k} = 1$

$\Rightarrow n = 2^k$

$\Rightarrow \log_2 n = k$

$3^k = 3^{\log_2 n}$

$= \cancel{n}$

$= n^{\log_2 3}$

$\text{Total} = n^{\log_2 3} + \frac{3^{\log_2 n} \cdot n}{2^{\log_2 n}}$

$= n^{\log_2 3} + n^{\log_2 3}$

$= O(n)$

(h) $T(n) = T(n-1) + T(n-2) + C \rightarrow$ Assuming $T(1) = C, T(2) = 2C$

$T(n-1) = T(n-2) + T(n-3) + C$

$T(n-2) = T(n-3) + \cancel{0}T(n-4) + C$

$\Rightarrow T(n) = T(n-1) + T(n-2) + C$

$= T(n-2) + T(n-3) + C + T(n-3) + T(n-4) + C + C$

$= T(n-2) + 2T(n-3) + T(n-4) + 3C$

n

$\frac{3n^2}{4}$

$\frac{9n^2}{16}$

$n/4$

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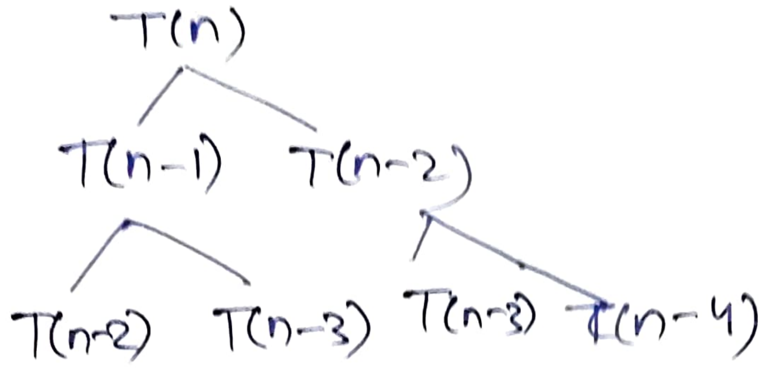
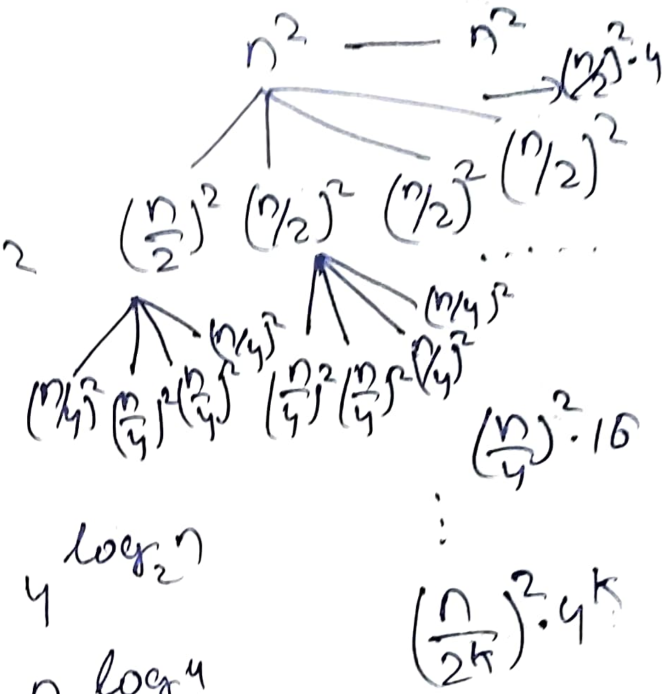


$O(2^n)$

9) $T(n) = 4T(n/2) + n^2$

$T(n/2) = 4T(n/4) + (\frac{n}{2})^2$

$T(n/4) = 4T(n/8) + (n/4)^2$



$\frac{n}{2^k} = 1$

$n = 2^k$

$\log_2 n = k$

$4^k = 4^{\log_2 n}$

$= n^{\log_2 4}$

Total $= n^{\log_2 4} + \frac{n^2}{2^k \cdot 2^k} \cdot 4^k$

$= n^{\log_2 4} + \frac{\cancel{n} \cdot \cancel{n}}{\cancel{n} \cdot \cancel{n}} \cdot 4^k$

$= n^{\log_2 4} + 4^{\log_2 n}$

$= n^{\log_2 4} + n^{\log_2 4}$

$= \cancel{O(n)}\ n^{\log_2 2^2} + n^{\log_2 2^2}$

$= O(n^2)$

10) Consider a recurrence $T(n) = 2T(n/2) + n$
show that $T(n) = \Omega(n \log n)$.

$T(n) = \Omega(n \log n)$

$T(n) = 2T(n/2) + n$,

Find constants c and n_0, such that:-

$T(n) \geq c \cdot n \log n$ for all $n \geq n_0$

Notice that it is in the form of:-

$T(n) = aT(n/b) + f(n)$, where in this case

$a = 2,\ b = 2,\ f(n) = n$

TC of $f(n)$: n

$n^{\log_b a} = n^{\log_2 2} = n$

$\therefore f(n) = \Theta(n^{\log_b a})$

$T(n) = \Theta(n^{\log_b a} * \log n)$

$= \Theta(n \log n)$

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11) Show that $n\log n - 3n + 14 = \Omega(n\log n)$

let $f(n) = n\log n - 3n + 14$, $g(n) = n\log n$

Assuming $f(n) \geq c\, g(n)$ for some constant 'c' and all $n \geq n_0$

$n\log n - 3n + 14 \geq c \cdot n\log n$

Subtracting $c \cdot n\log n$ from both side we get:-

$\log n(n - cn) \geq 3n - 14$

$\Rightarrow n \cdot (\log n - c) \geq 3n - 14$

Dividing both side by 'n' we get:-

$= \log n - c \geq 3 - \frac{14}{n}$

Now, find a constant 'c' such that $\log n - c$ remains larger than or equal to $3 - \frac{14}{n}$

let $c = 2$,

$\log n - 2 \geq 3 - \frac{14}{n}$

$\log n \geq 5 - \frac{14}{n}$

For all $n \geq 14$, the RHS is true and log grows rapidly as 'n' increases.

Thus $n_0 = 14$

For $n \geq 14$ and $c = 2$,

$n\log n - 3n + 14 \geq c \cdot n\log n$

for all $n \geq n_0$.

Thus, $n\log n - 3n + 14 = \Omega(n\log n)$

12) show that $\frac{1}{2}n^2 - 7n = \Theta(n^2)$

$f(n) = \frac{1}{2}n^2 - 7n$

~~f(n) >~~ $f(n) = \frac{1}{2}n^2 - 7n = O(n^2) \ldots ①$

$f(n) = \frac{1}{2}n^2 - 7n = \Omega(n^2) \ldots ②$

Proving ①:-

$\frac{1}{2}n^2 - 7n \leq cn^2 \;\forall n \geq n_0$

Dividing both side by n^2 we get:-

$\frac{1}{2} - \frac{7}{n} \leq c$

since $\frac{7}{n} \to 0$ as n becomes larger,

so $c \geq \frac{1}{2}$

let $c = \frac{3}{4}$

$\frac{1}{2} - \frac{7}{n} \leq 3/4$

$\Rightarrow -\frac{7}{n} \leq \frac{1}{4} \to$ this holds true for all $n \geq 28$ ($\because \frac{7}{28} = \frac{1}{4}$)

$\Rightarrow \frac{1}{2}n^2 - 7n \leq \frac{3}{4}n^2$

$\therefore \frac{1}{2}n^2 - 7n = O(n^2)$

Proving ②:-

$\frac{1}{2}n^2 - 7n = \Omega(n^2)$

$\frac{1}{2}n^2 - 7n \geq c \cdot n^2 \;\forall n \geq n_0$

Dividing both side by n^2:

$\frac{1}{2} - \frac{7}{n} \geq c$

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choosing $c = \frac{1}{4}$

$\frac{1}{2} - \frac{7}{n} \geq \frac{1}{4}$

$\Rightarrow -\frac{7}{n} \geq -\frac{1}{4}$ $\therefore$ ($n \geq 28$ since $\frac{7}{28} = \frac{1}{4}$)

$\Rightarrow \frac{1}{2}n^2 - 7n \geq \frac{1}{4}n^2$

Thus, $\frac{1}{2}n^2 - 7n = \Omega(n^2)$

$\therefore$ by proving ① & ⑫ :-

$\frac{1}{2}n^2 - 7n = \Theta(n^2)$

⑬ Find the asymptotic bound of the following recurrence equations using Master Theorem.

(a) $T(n) = 4T\left(\frac{n}{2}\right) + n^2$

$a = 4$ $\quad f(n) = n^2$

$b = 2$ $\quad \simeq \Theta(n^2 \log^0 n)$

$k = 2, \; p = 0$

$\log_2 4 = 2 = k = 2$ ✓

$p > -1$

$\Theta(n^k \log^{p+1} n)$

$\simeq \Theta(n^2 \log n)$

(b) $T(n) = 3T\left(\frac{n}{4}\right) + n$

$a = 3$ $\quad f(n) = n$

$b = 4$ $\quad = \Theta(n^1 \log^0 n)$

$\log_b a = \log_4 3 < k = 1$ $\quad k = 1, \; p = 0$

$p \geq 0 \rightarrow \Theta(n^k \log^p n)$

$\simeq \Theta(n)$

c) $T(n) = 2T(n/2) + \sqrt{n}$

$a = 2$ $\quad G(n) = \sqrt{n} = (n)^{1/2}$

$b = 2$ $\quad \cong \Theta(n^{1/2} \log^0 n)$

$\log_2^2 = 1 > 0.5$ $\quad k = 1/2, P = 0$

$\Theta(n^{\log_b^a})$

$\cong \Theta(n)$

d) $T(n) = 4T(\frac{n}{2}) + n^2\sqrt{n}$

$a = 4$ $\quad G(n) = n^2\sqrt{n}$

$b = 2$

$= n^2 \cdot n^{1/2}$

$= n^{5/2}$

$\log_b^a = \log_2^4$ $\quad \cong \Theta(n^{5/2} \log^0 n)$

$= 2 < 2.5 = k$

$P \geq 0$

$\hookrightarrow \Theta(n^k \log^P n)$

$\cong \Theta(n^2\sqrt{n})$

e) $T(n) = 2T(\frac{n}{2}) + n^2$

$a = 2$ $\quad \log_2^2 = 1 < 2$ $\quad G(n) = n^2$

$b = 2$ $\quad \cong \Theta(n^k \log^P n)$

$P \geq 0$ $\quad \cong \Theta(n^2 \log^0 n)$

$k = 2, P = 0$

$\Theta(n^k \log^P n)$

$\cong \Theta(n^2)$

G) $T(n) = 16T(n/4) + n^2$

$a = 16$ $\quad \Theta(n^k \log^P n)$

$b = 4$ $\quad \cong \Theta(n^2 \log^0 n) \longrightarrow k = 2, P = 0$

$\log_b^a = 2 = k$

$P > -1$ $\quad \Theta(n^k \log^{P+1} n) = \Theta(n^2 \log n)$

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g) $T(n) = 2T(n/2) + n^3$

$a = 2$ $\qquad$ $f(n) = n^3$

$b = 2$ $\qquad$ $\cong \theta(n^3 \log^0 n)$

$\log_b^a = \log_2 2 = 1 < k = 3$ $\qquad$ $k = 3, P = 0$

$P \geq 0$

$\theta(n^k \log^P n)$

$= \theta(n^3)$

h) $T(n) = 4T(\frac{n}{2}) + C$

$a = 4$ $\qquad$ $f(n) = c$

$b = 2$ $\qquad$ $= \theta(n^0 \log^0 n)$

$\log_b^a = \log_2^4 = 2 > k = 0$ $\qquad$ $k = 0, P = 0$

$\theta(n^{\log_b^a})$

$\cong \theta(n^2)$

14) Find the asymptotic bound of the following recurrence equation using change of variable method.

(a) $T(n) = \sqrt{n}\, T(\sqrt{n}) + n$

$n = 2^m$ $\qquad$ $n^{1/2} = 2^{m/2}$

$$\underbrace{T(2^m)}_{S(m)} = 2^{m/2} \cdot \underbrace{T(2^{m/2})}_{S(m/2)} + 2^m$$

$S(m) = \frac{m}{2} \cdot S(m/2) + m$

$= m\left(\frac{1}{2} S(\frac{m}{2}) + 1\right)$

$a = \frac{1}{2}, b = 2, f(n) = n$

$\log_b^a = \log_2^{1/2} = \log_2^1 - \log_2^2 = -1$

Comparing $f(n) = n$,

$n^{\log_b^a} = n^{-1}$

$f(n) = \Theta(n^{\log_b^a})$ follows case-2 of master theorem

$\cong \Theta(n^{\log_b^a} \log n)$

$\cong \Theta(n^{-1} \log n)$ ($\because$ Not sure)

$\cong \Theta\left(\frac{\log n}{n}\right)$

b) $T(n) = 2T(\sqrt{n}) + C$

$n = 2^m \quad n^{1/2} = 2^{m/2} \quad m = \log n$

$\underbrace{T(2^m)}_{S(m)} = 2\underbrace{T(2^{m/2})}_{S(m/2)} + C$

$S(m) = 2S(m/2) + C$

$a = 2 \quad f(n) = C = \cong \Theta(n^0 \log^0 n)$

$b = 2 \quad k=0, p=0$

$\log_b^a = \log_2^2 = 1 > k = 0$

$\Theta(m^{\log_b^a})$

$\cong \Theta(m^1)$

$\therefore T(n) = \Theta(\log n)$

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c) $T(n) = 4T(\frac{n}{2}) + n$

$n = 2^m$

or $\frac{n}{2} = 2^{m-1}$

$\underbrace{T(2^m)}_{S(m)} = 4T(\underbrace{2^{m-1}}_{S(m-1)}) + \underbrace{2^m}_{m}$

$\Rightarrow S(m) = 4S(m-1) + m$

$S(m-1) = 4S(m-2) + m-1$

$S(m-2) = 4S(m-3) + m-2$

$S(m-3) = 4S(m-4) + m-3$

$S(m) = 4S(m-1) + m$

$= 4(4S(m-2) + m-1) + m$

$= 16S(m-2) + 4(m-1) + m$

$= 16(4S(m-3) + m-2) + 4(m-1) + m$

$= 64S(m-3) + 16(m-2) + 4(m-1) + m$

$= 64(4S(m-4) + m-3) + 16(m-2) + 4(m-1) + m$

$= 256S(m-4) + 64(m-3) + 16(m-2) + 4(m-1) + m$

$\vdots$

$4^k S(m-k) + 4^{k-1}(m-3) + 4^{k-2}(m-2) + 4^{k-3}(m-1) + m$

$m-k=0$

$k=m$

$= \cancel{4^m + 4^{m-1}(m- + 4^{m-2}}$

$= \cancel{4^m + 4^{m-1}(m-3) + 4^{m-2}(m-2) + 4(m-1) \ldots m}$

$S_1 = 4^k + 4^{k-1}(1) + 4^{k-2}(2) + 4^{k-3}(3) \ldots\ldots + 4^{k-k} m$

$4S_1 = 4^{k-1} + 4^{k-2} + 4^{k-3}(2) + \ldots\ldots\ldots$

$\Theta(n^2)$

d) $T(n) = 7T\left(\frac{n}{2}\right) + 3n^2$

$n = 2^m$

$\frac{n}{2} = 2^{m-1}$

$S(2^m) = 7T(2^{m-1}) + 3 \times 2^{2m}$

$S(m) = 7T(m-1) + 3 \times 2m$

or $S(m) = 7T(m-1) + 6m$

$\Rightarrow S(m-2) = 7S(m-3) + 6(m-2)$

$\Rightarrow S(m) = 49(7S(m-3) + 6(m-2)) + 48m - 42$

$\Rightarrow S(m) = 343S(m-3) + 294(m-2) + 48m - 42$

$\Rightarrow S(m) = 343(m-3) + 249m - 588 + 48m - 42$

$= 343S(m-3) + 342m - 630$

$= 343^k S(m-k) + (6 + 48 + 342 + \ldots 48k - 42k)$

$\Theta(n^2)$

$\because$ ~~Not~~ Finally bound using master Theorem

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15) Find out the function which is growing faster:-

a) $n^{\sqrt{n}}$ v/s $2^{\sqrt{n}\log_2 n}$

Taking log on both side

$\log_2 n^{\sqrt{n}}$ $\log_2 2^{\sqrt{n}\log_2 n}$

$\cancel{\sqrt{n}\log_2 n}$ $\cancel{\sqrt{n}\log_2 n}$

$= 1$ $= 1$

Both are growing at same

b) $n^{\sqrt{n}}$ vs $n^{\log_2 n}$

Taking log on both side

$\log_2 n^{\sqrt{n}}$ $\log_2 n^{\log_2 n}$

$\sqrt{n}\,\cancel{\log_2 n}$ $\cancel{\log_2 n}\,\log_2 n$

$(n)^{1/2}$ $\log_2 n$

Taking log on both side:-

$\frac{1}{2}\log n$ $>$ $\log\log_2 n$

$n^{\sqrt{n}}$ is growing faster.

c) $n^{\log n}$ v/s 2^n

Taking log on both side:-

$\log n \log n = n$

n - grows faster than $\log\log n$

$\therefore 2^n$ grows faster.

d) $\sqrt{\log n}$ v/s $\log \cdot \log n$

Taking log on both side

$(\log n)^{1/2}$ $\log \cdot \log n$

$\frac{1}{2} \log(\log n)$ $\log(\log \cdot \log(n))$

$\sqrt{\log n}$ is growing faster.