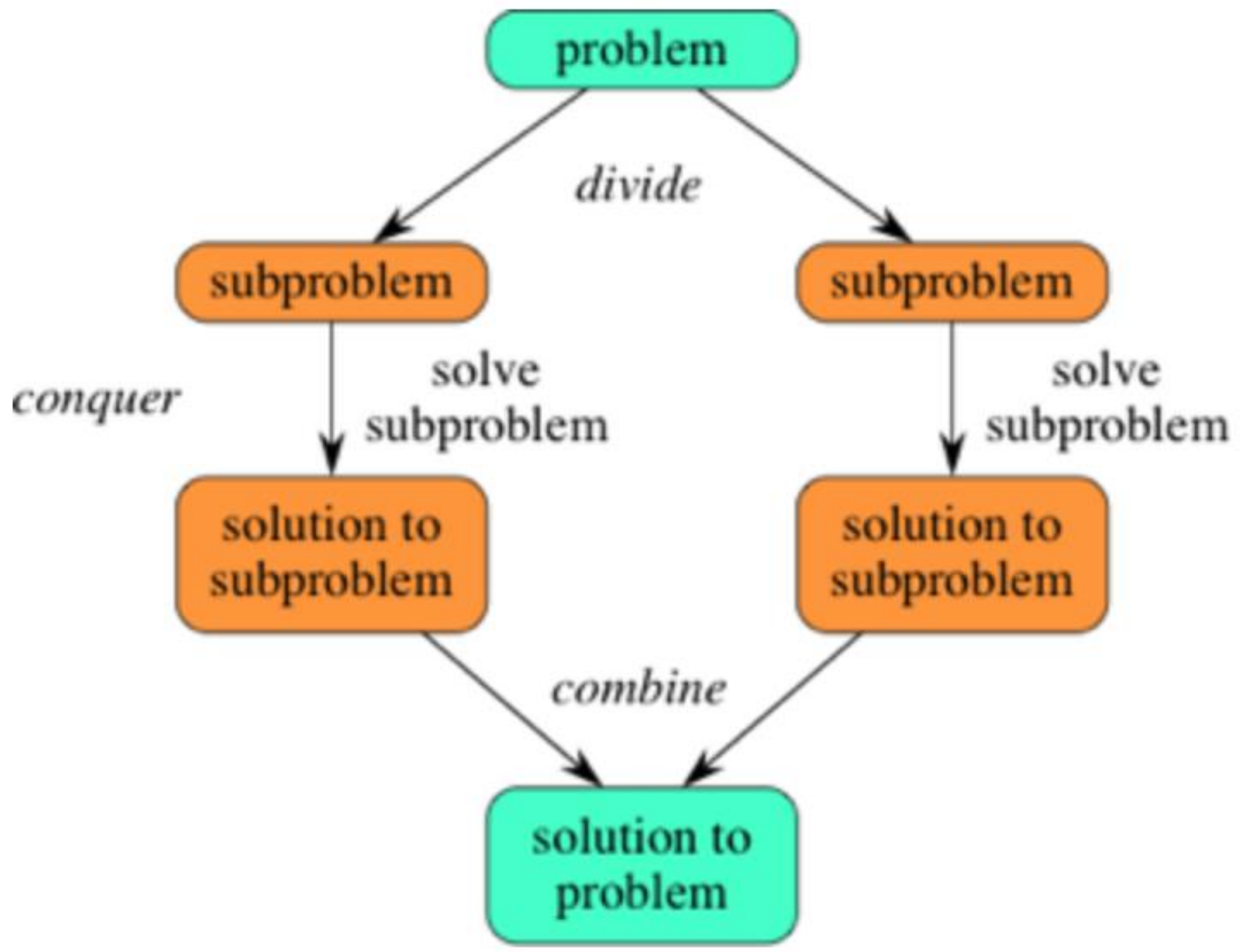
CSE 3318

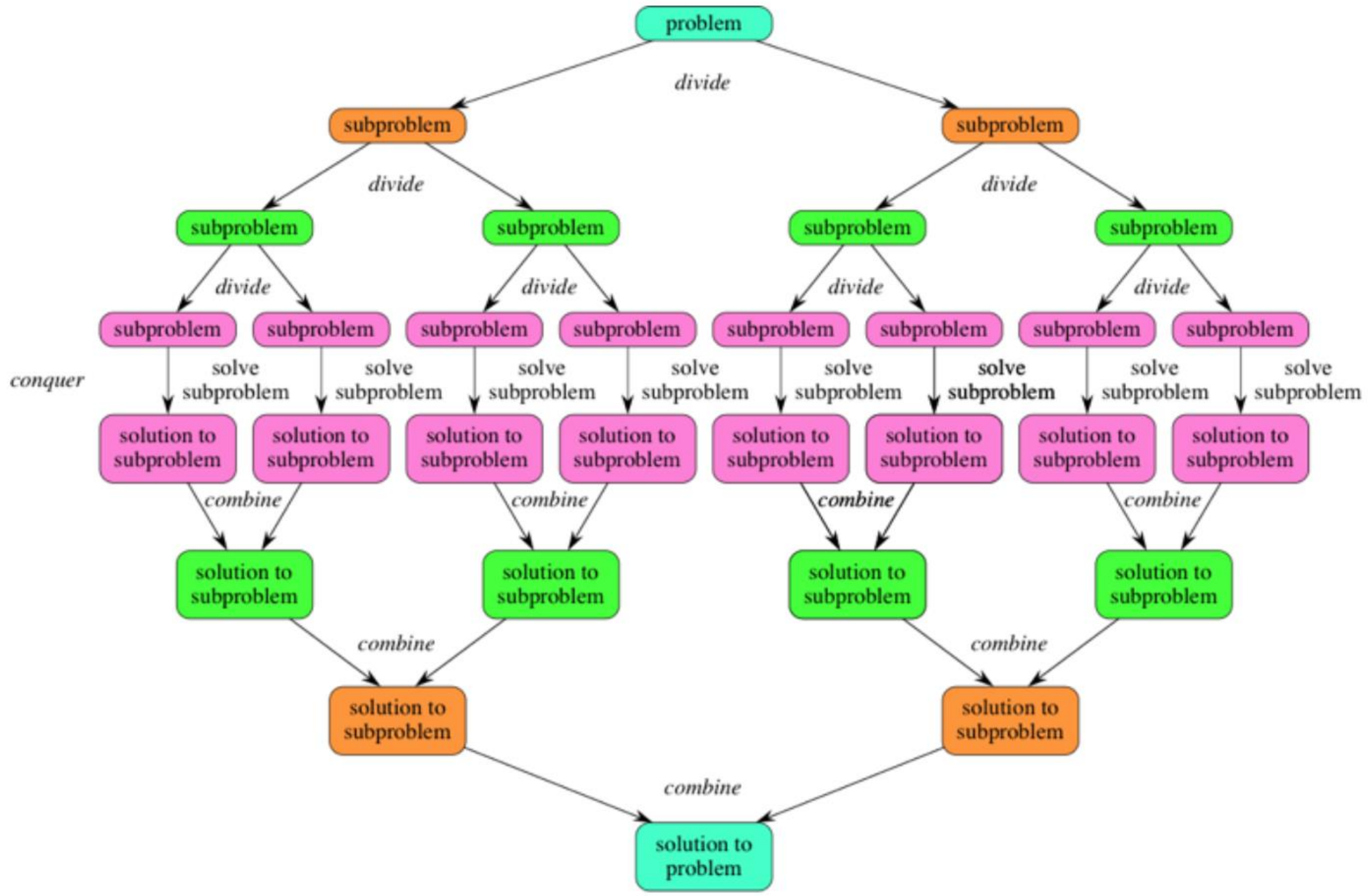
Week of 06/26/2023

Instructor : Donna French

Divide and Conquer



Divide and Conquer



Divide and Conquer
Merge Sort

When using divide-and-conquer to sort, we need to decide what our subproblems are going to look like.

The full problem is to sort an entire array.

The subproblem is to sort a subarray.

Divide and Conquer
Merge Sort

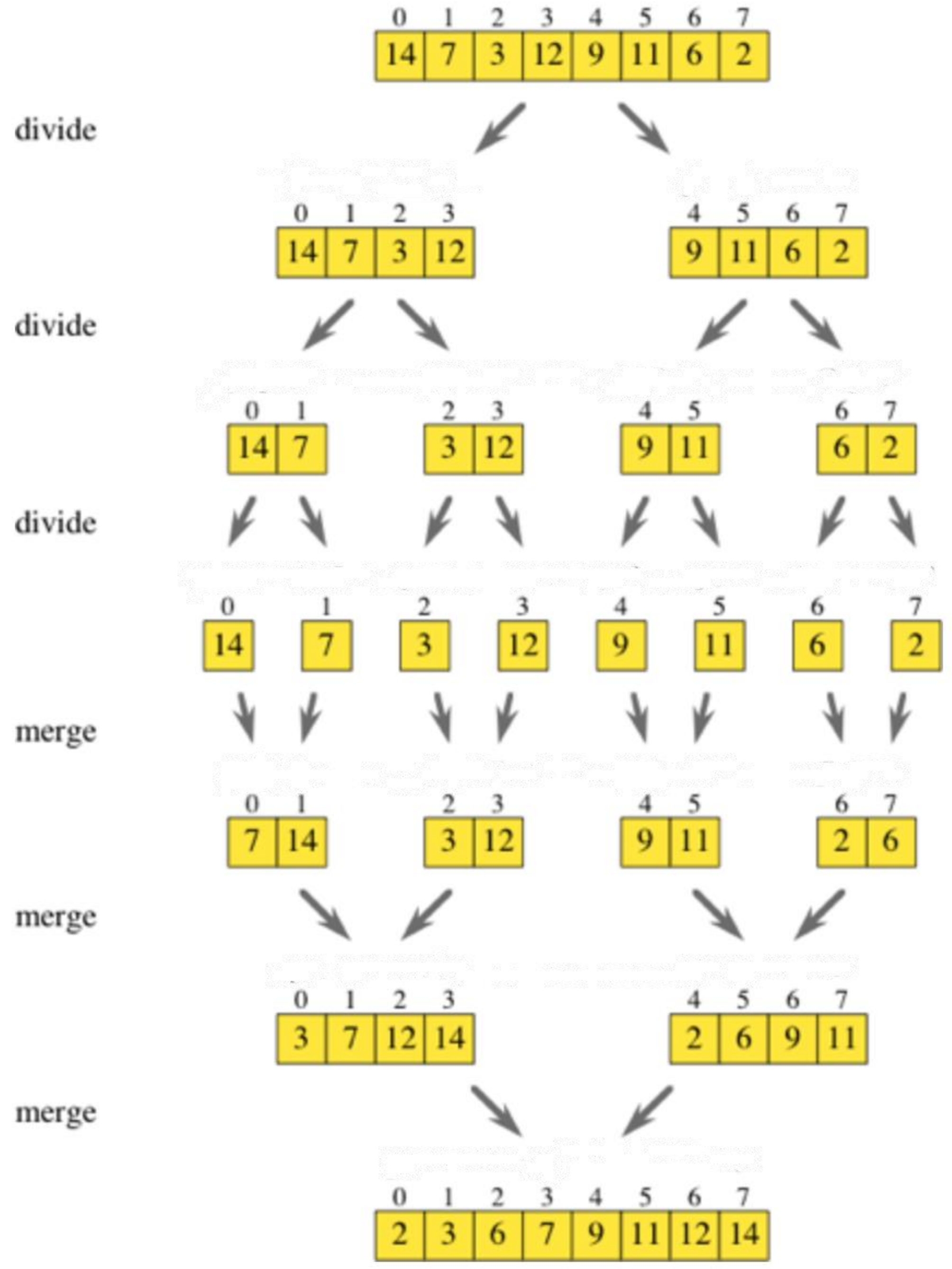
To sort *`A[p .. r]`*

Divide by splitting into two subarrays *`A[p .. q]`* and *`A[q+1 .. r]`* where q is the halfway point of *`A[p .. r]`*

Conquer by recursively sorting the two subarrays *`A[p .. q]`* and *`A[q+1 .. r]`*

Combine by merging the two sorted subarrays *`A[p .. q]`* and *`A[q+1 .. r]`* to produce a single sorted subarray *`A[p .. r].`*

The recursion bottoms out when the subarray has just 1 element, so that it’s trivially sorted – can't divide 1 value.



{14, 7, 3, 12, 9, 11, 6, 2}

{7, 14, 3, 12, 9, 11, 6, 2}

{7, 14, 3, 12, 9, 11, 6, 2}

{3, 7, 12, 14, 9, 11, 6, 2}

{3, 7, 12, 14, 9, 11, 6, 2}

{3, 7, 12, 14, 9, 11, 2, 6}

{3, 7, 12, 14, 2, 6, 9, 11}

{2, 3, 6, 7, 9, 11, 12, 14}

Analysis of Merge Sort

Let's examine the merge function from Merge Sort.

The merge function merges two sorted arrays into a single sorted array.

If the two subarrays have a total of *n* elements, then how many times did we examine each element in order to accomplish the merge?

We have to examine each of the elements in order to merge them together which gives us a merging time of $\Theta(n)$.

Analysis of Merge Sort

Given that the merge function runs in $\Theta(n)$ time when merging n elements, how to demonstrate that Merge Sort runs in $\Theta(n\log_2 n)$ time?

We start by thinking about the three parts of divide-and-conquer and how to account for their running times.

We assume that we are sorting a total of n elements in the entire array.

Analysis of Merge Sort

The divide step takes constant time, regardless of the subarray size.

The divide step just computes the midpoint *q* of the indices *p* and *r*.
(middle = (left+right)/2)

In big-Θ notation, we indicate constant time by Θ(1).

Using 1 with Θ indicates no growth

The conquer step, where we recursively sort two subarrays of approximately $\frac{n}{2}$ elements each, takes some amount of time, but we'll account for that time when we consider the subproblems.

The combine step merges a total of *n* elements, taking Θ(*n*) time.

Analysis of Merge Sort

Let's put the divide and combine steps together.

The $\Theta(1)$ running time for the divide step is a low-order term when compared with the $\Theta(n)$ running time of the combine step. As n increases, the amount of constant time needed for the divide step will be trivial.

So let's eliminate the divide time as a separate consideration and just take the divide and combine steps together as $\Theta(n)$ time.

To make things more concrete, let's say that the divide and combine steps together take cn time for some constant c.

Analysis of Merge Sort

To keep things reasonably simple, let's assume that if $n > 1$, then n is always even.

Why?

So that when we need to think about $\frac{n}{2}$, it's an integer – integer division does not lose anything.

Accounting for the case in which n is odd doesn't change the result in terms of big-Θ notation.

Analysis of Merge Sort

So now we can think of the running time of Merge Sort on an *n*-element subarray as being

the sum of

twice the running time of Merge Sort on an $\frac{n}{2}$-element subarray

(for the conquer step)

plus

cn

(for the divide and combine steps—really for just the merging).

Analysis of Merge Sort

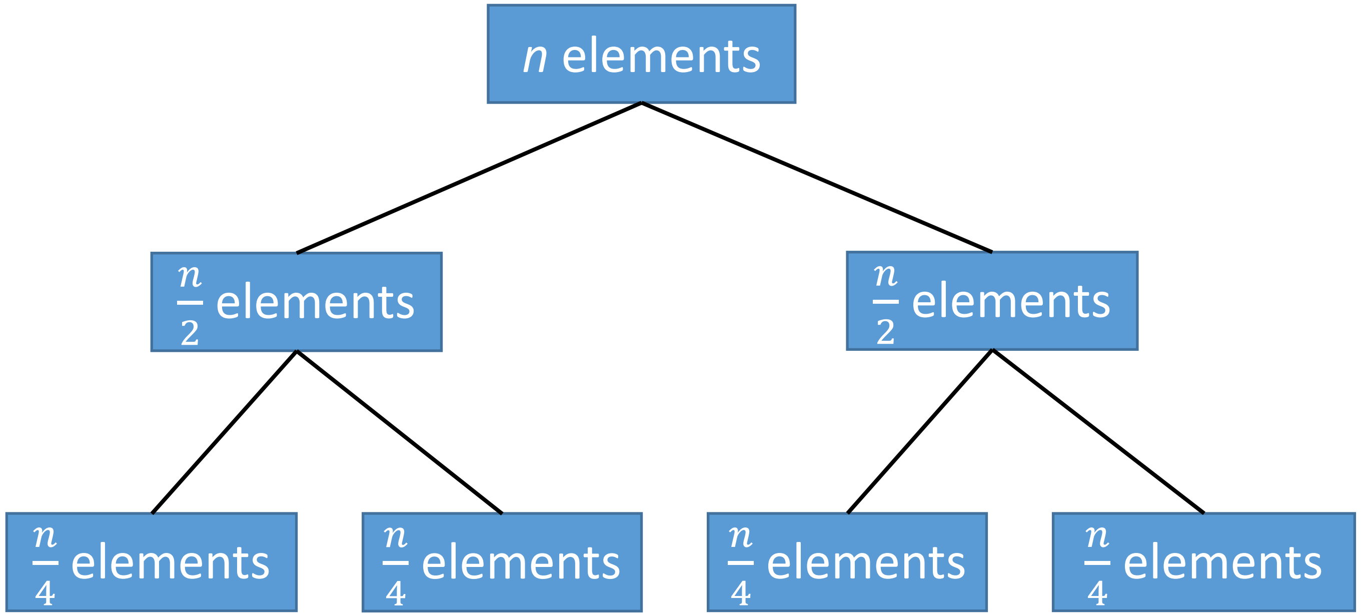
Now we have to figure out the running time of two recursive calls on $\frac{n}{2}$

Each of these two recursive calls takes twice of the running time of Merge Sort on an $\frac{n}{4}$-element subarray (because we have to halve $\frac{n}{2}$) plus $\frac{cn}{2}$ to merge.

We have two subproblems of size $\frac{n}{2}$ and each takes $\frac{cn}{2}$ time to merge.

The total time we spend merging for subproblems of size $\frac{n}{2}$ is 2 * $\frac{cn}{2}$ = *cn*

Analysis of Merge Sort



Total merge time for array of this size

cn

$2 * \frac{cn}{2} = cn$

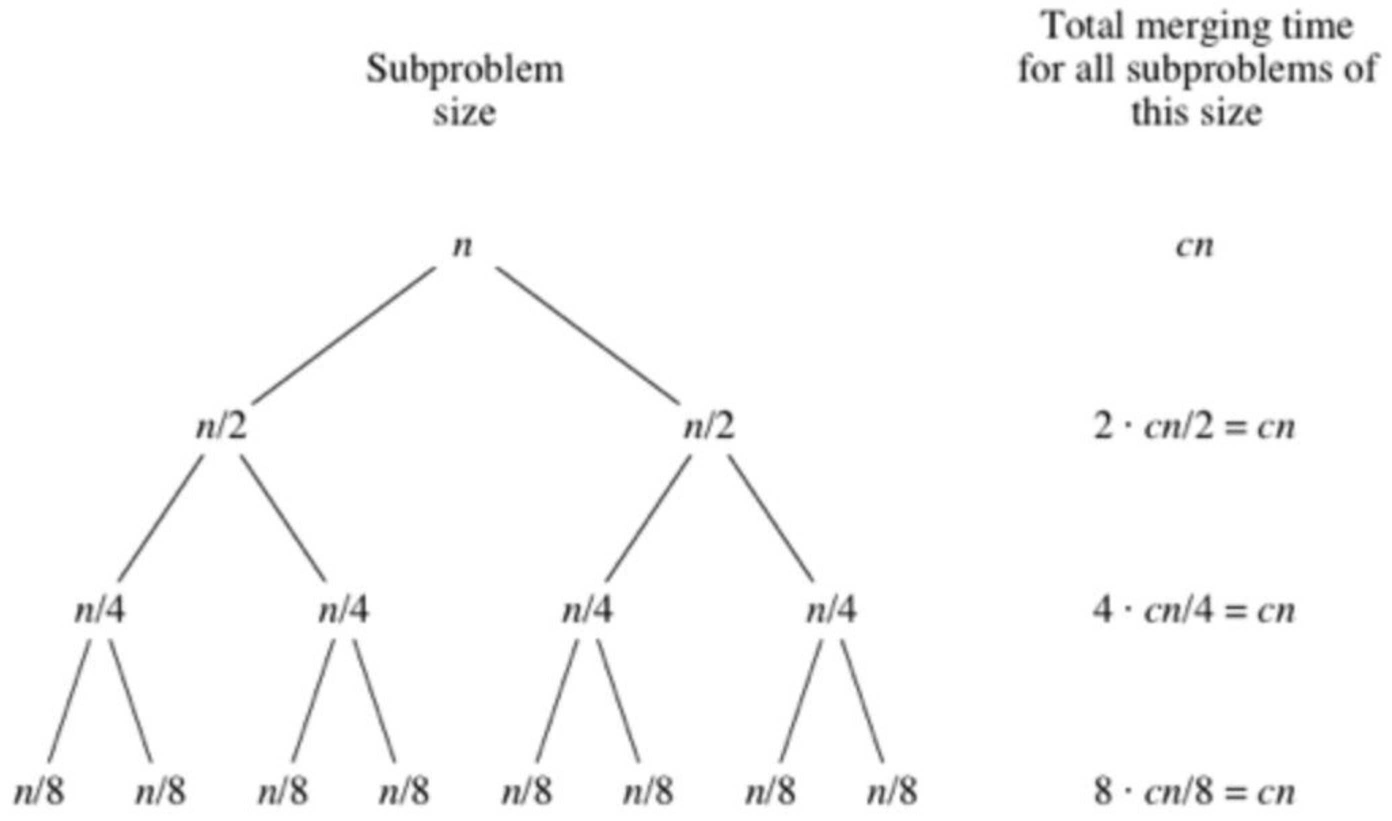
$4 * \frac{cn}{4} = cn$

What would be the merge time for the next level of $\frac{n}{8}$? $8 * \frac{cn}{8} = cn$

Analysis of Merge Sort

As the subproblems/arrays get smaller, the number of subproblems doubles at each "level" of the recursion, but the merging time halves.

The doubling and halving cancel each other out so the total merging time is *cn* at each level of recursion.



Analysis of Merge Sort

Eventually, we get down to subproblems of size 1 (the base case)

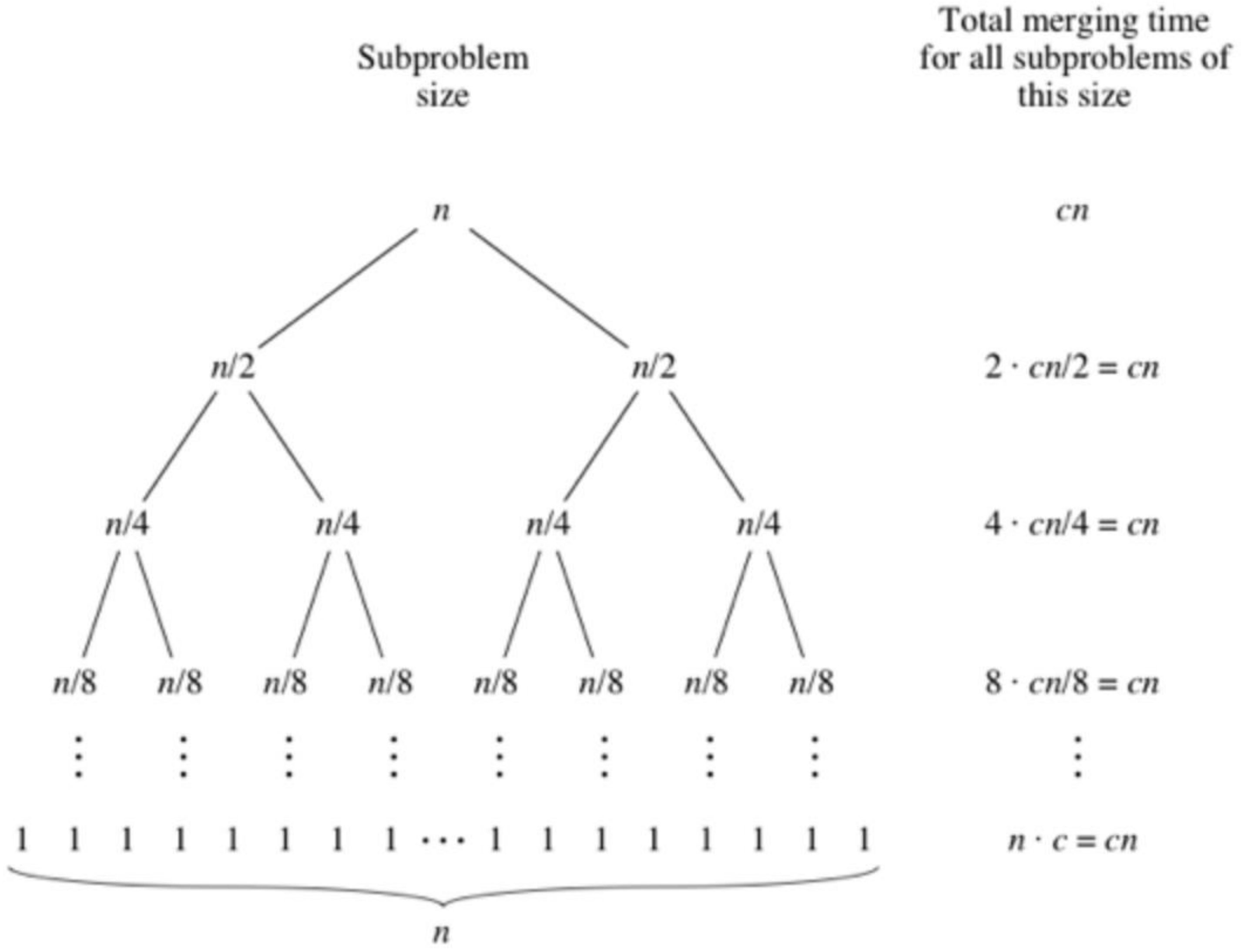
We have to spend Θ(1) time to sort subarrays of size 1

- we have to test whether $p < r$ (left < right)
- this test takes time but that time is constant/the same every time

How many subarrays of size 1 are there?

- Since we started with *n* elements, there must be *n* base cases.
- Since each base case takes Θ(1) time, the base cases take *cn* time

Analysis of Merge Sort



The total time merging in Merge Sort is the sum of the merging times for all the levels.

If there are Z levels in the tree, then the total merging time is

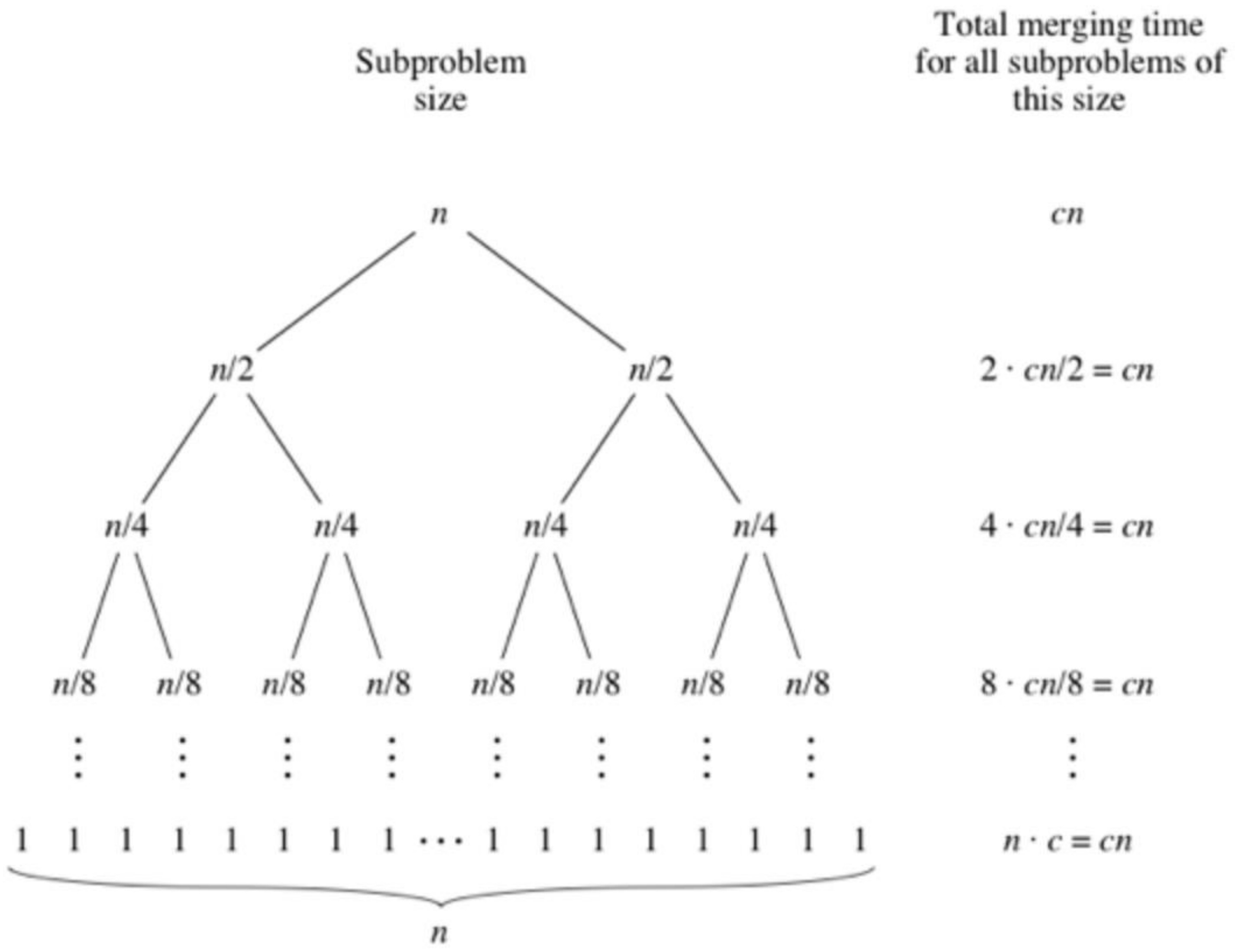
Z * *cn*

So what is Z?

Does this pattern look familiar?

$cn\log_2 n$

Analysis of Merge Sort



n	$\log_2 n$
1	0
2	1
4	2
8	3
16	4
32	5
64	6

Analysis of Merge Sort

n	$\log_2 n$
1	0
2	1
4	2
8	3
16	4
32	5
64	6

You could argue that this runtime should be

$cn\log_2 n + 1$

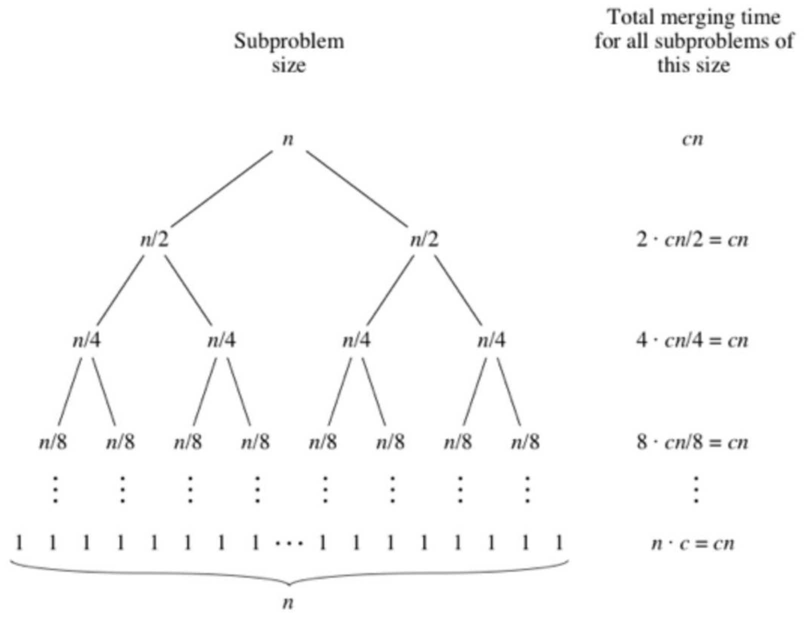
Why + 1?

What happens when we have an array of 1 element?

$cn\log_2 n$ when $n = 1$ is 0

Does it take 0 time to sort an array of 1 element?

You could argue that it takes some time to check for that 1 element.



Analysis of Merge Sort

So we have calculated that the runtime of Merge Sort to be

$cn\log_2 n$ or $cn\log_2 n + 1$

Using Θ notation allows us to discard the constant coefficient *c* and the lower order term (+ 1) if we included it . This gives us a runtime of

$\Theta(n\log_2 n)$

Selection Sort

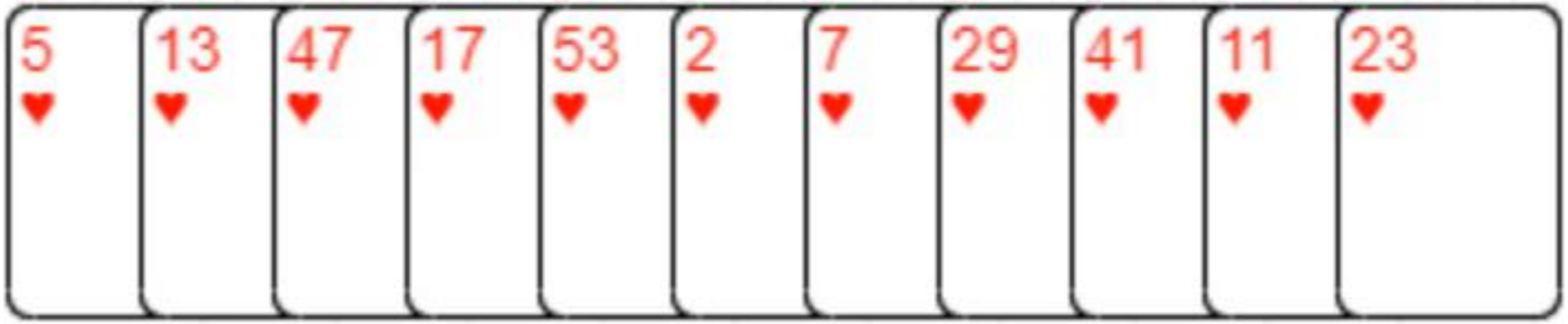
Insertion Sort sorts by taking an item "key" out and inserting it into the correct place (and moved everything else in the array over).

Merge Sort sorts by using divide and conquer with combine(merge) to break the problem down into its smallest pieces and then merge those pieces back together.

Selection Sort is another sort that uses a different technique.

Selection Sort

[redacted]



Selection Sort

1. Find the smallest card. Swap it with the first card.

2. Find the second-smallest card. Swap it with the second card.

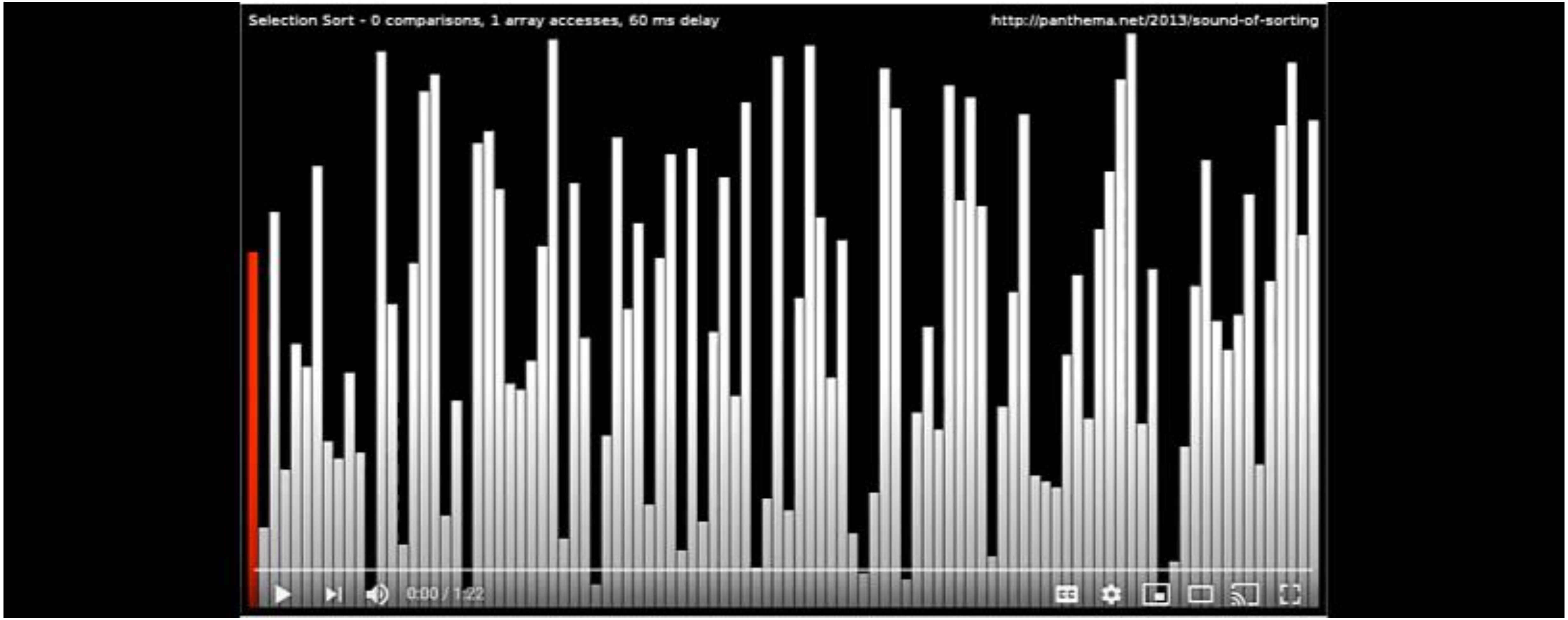
3. Find the third-smallest card. Swap it with the third card.

4. Repeat finding the next-smallest card and swapping it into the correct position until the array is sorted.

This algorithm is called **Selection Sort** because it repeatedly *selects* the next-smallest element and swaps it into place.

Selection Sort

Selection Sort


Selection Sort - 0 comparisons, 1 array accesses, 60 ms delay

http://panthema.net/2013/sound-of-sorting

Selection Sort

What do you think about this algorithm?

What parts of it seem to take the longest?

How do you think it would perform on big arrays?

Keep those questions in mind as we analyze this algorithm.

Selection Sort

Swap

A key step in many sorting algorithms (including selection sort) is swapping the location of two items in an array. Here's a swap function that looks like it might work…

array

index of 1st element to swap

index of 2nd element to swap

```
void swap(int A[], int Swap1, int Swap2)
{
     A[Swap1] =  A[Swap2];
     A[Swap2] =  A[Swap1];
}
```

Selection Sort

```
void swap(int A[], int Swap1, int Swap2)
{
     A[Swap1] =  A[Swap2];
     A[Swap2] =  A[Swap1];
}
```

The first line is OK. Puts the element at `Swap2` into the array element of `Swap1`.

But what happens `A[Swap1]` is assigned to `A[Swap2]`?

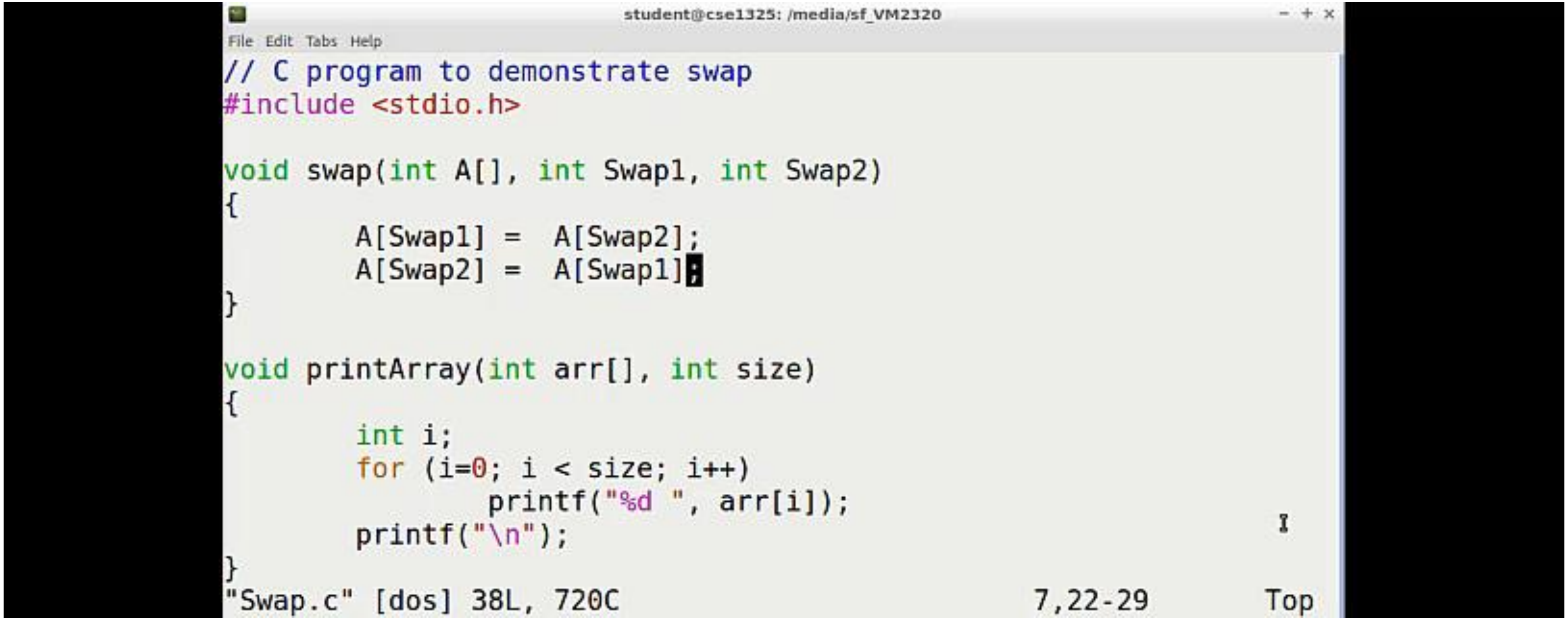
`A[Swap1]` has already been updated with `A[Swap2]`; therefore, it just puts `A[Swap2]` back into `A[Swap2]`.

Selection Sort

```
void swap(int A[], int Swap1, int Swap2)
{
    A[Swap1] =  A[Swap2];
    A[Swap2] =  A[Swap1];
}
```

We need to keep track of A[Swap1] before overwriting it.

Selection Sort



student@cse1325: /media/sf_VM2320

File Edit Tabs Help

```
// C program to demonstrate swap
#include <stdio.h>

void swap(int A[], int Swap1, int Swap2)
{
        A[Swap1] =  A[Swap2];
        A[Swap2] =  A[Swap1];
}

void printArray(int arr[], int size)
{
        int i;
        for (i=0; i < size; i++)
                printf("%d ", arr[i]);
        printf("\n");
}
"Swap.c" [dos] 38L, 720C                          7,22-29       Top
```

Selection Sort

```
student@cse1325:/media/sf_VM2320$
```

Selection Sort

Finding the index of the minimum element in a subarray

One of the steps in Selection Sort is to find the next-smallest element to put into its correct location.

For example, if the array initially has values

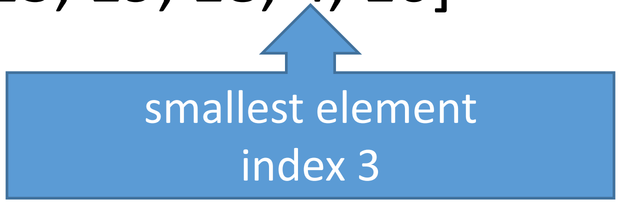
[13, 19, 18, 4, 10]

we first need to find the index of the smallest value in the array.

Selection Sort

Finding the index of the minimum element in a subarray

[13, 19, 18, 4, 10]



4 is the smallest value and its index is 3.

Selection Sort would swap the value at [3] with the value at index [0]

[4, 19, 18, 13, 10]

Now we need to find the index of the second-smallest value to swap into index 1.

Selection Sort

Finding the index of the minimum element in a subarray

We could write code to find the index of the second-smallest value in an array.

It would be more complex than needed - there's a better way.

Notice that since the smallest value has already been swapped into index 0, what we really want is to find the smallest value in the part of the array that starts at index 1.

[13, 19, 18, 4, 10]

[4, 19, 18, 13, 10]

Selection Sort

Finding the index of the minimum element in a subarray

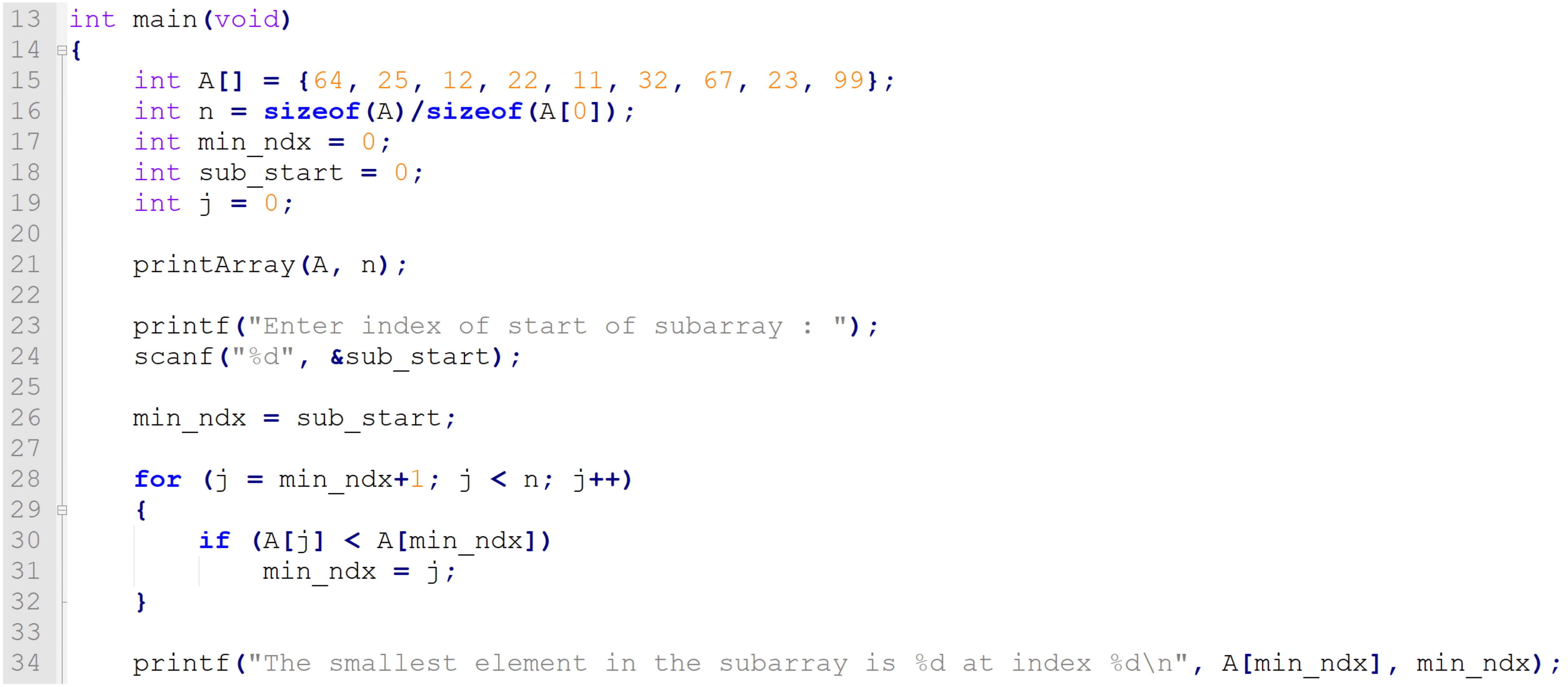
A section of an array is called a **subarray**, so that in this case, we want the index of the smallest value in the subarray that starts at index 1.

[4, 19, 18, 13, 10]

Array element 10 at index 4 is the smallest element in the subarray

The smallest value in the subarray starting at index 1 is 10 at [4] in the original array.

So index 4 is the location of the second-smallest element of the full array.



```c
int main(void)
{
    int A[] = {64, 25, 12, 22, 11, 32, 67, 23, 99};
    int n = sizeof(A)/sizeof(A[0]);
    int min_ndx = 0;
    int sub_start = 0;
    int j = 0;

    printArray(A, n);

    printf("Enter index of start of subarray : ");
    scanf("%d", &sub_start);

    min_ndx = sub_start;

    for (j = min_ndx+1; j < n; j++)
    {
        if (A[j] < A[min_ndx])
            min_ndx = j;
    }

    printf("The smallest element in the subarray is %d at index %d\n", A[min_ndx], min_ndx);
```

```
 0     1      2      3    4       5    6       7    8
64, 25, 12, 22, 91, 32, 67, 23, 99
```

```
Enter index of start of subarray : 0
The smallest element in the subarray is 12 at index 2

Enter index of start of subarray : 1
The smallest element in the subarray is 12 at index 2

Enter index of start of subarray : 3
The smallest element in the subarray is 22 at index 3

Enter index of start of subarray : 4
The smallest element in the subarray is 23 at index 7

Enter index of start of subarray : 8
The smallest element in the subarray is 99 at index 8
```

```
 0    1    2    3    4     5    6    7    8
64,  25,  12,  22,  91,  32,  67,  23,  99
```

```
min_ndx = sub_start;

for (j = min_ndx+1; j < n; j++)
{
   if (A[j] < A[min_ndx])
      min_ndx = j;
}
```

```
min_ndx  j
-------  ---------------
3        min_ndx+1=3+1=4
```

```
Enter index of start of subarray : 3
The smallest element in the subarray is 22 at index 3
```

```
 0    1    2    3   4     5    6    7    8
64,  25,  12,  22,  91,  32,  67,  23,  99
```

```
min_ndx = sub_start;

for (j = min_ndx+1; j < n; j++)
{
   if (A[j] < A[min_ndx])
      min_ndx = j;
}
```

```
min_ndx  j
-------  ---------------
4        min_ndx+1=4+1=5
```

```
Enter index of start of subarray : 4
The smallest element in the subarray is 23 at index 7
```

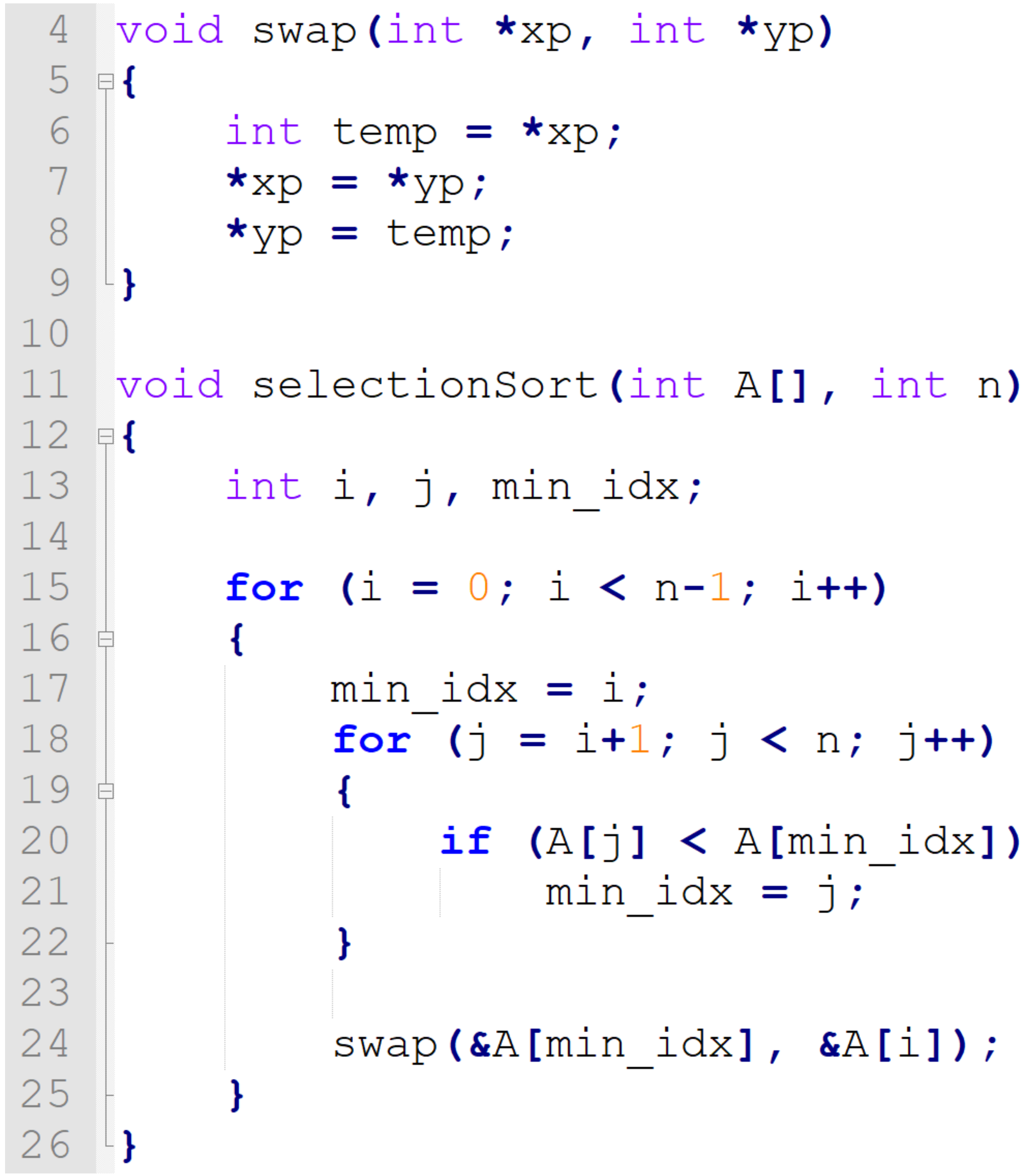
Selection Sort

Going back to our definition of Selection Sort

1. Find the smallest element. Swap it with the first element.
2. Find the second-smallest element. Swap it with the second element.
3. Find the third-smallest element. Swap it with the third card.
4. Repeat finding the next-smallest element and swapping it into the correct position until the array is sorted.

We now have the code to find the smallest, second smallest, third smallest, etc…

We also have the code to swap two array elements.



```c
void swap(int *xp, int *yp)
{
    int temp = *xp;
    *xp = *yp;
    *yp = temp;
}

void selectionSort(int A[], int n)
{
    int i, j, min_idx;

    for (i = 0; i < n-1; i++)
    {
        min_idx = i;
        for (j = i+1; j < n; j++)
        {
            if (A[j] < A[min_idx])
                min_idx = j;
        }

        swap(&A[min_idx], &A[i]);
    }
}
```

Selection Sort

```
void selectionSort(int A[], int n)
{
   int i, j, min_idx;

   for (i = 0; i < n-1; i++)
   {
      min_idx = i;

      for (j = i+1; j < n; j++)
      {
         if (A[j] < A[min_idx])
            min_idx = j;
      }

      swap(&A[min_idx], &A[i]);
   }
}
```

```
{13, 9, 4, 1}

n = 4

 i    min_idx    j
---   -------   ---
 0
```

Selection Sort

```
void selectionSort(int A[], int n)
{
   int i, j, min_idx;

   for (i = 0; i < n-1; i++)
   {
      min_idx = i;

      for (j = i+1; j < n; j++)
      {
         if (A[j] < A[min_idx])
            min_idx = j;
      }

      swap(&A[min_idx], &A[i]);
   }
}
```

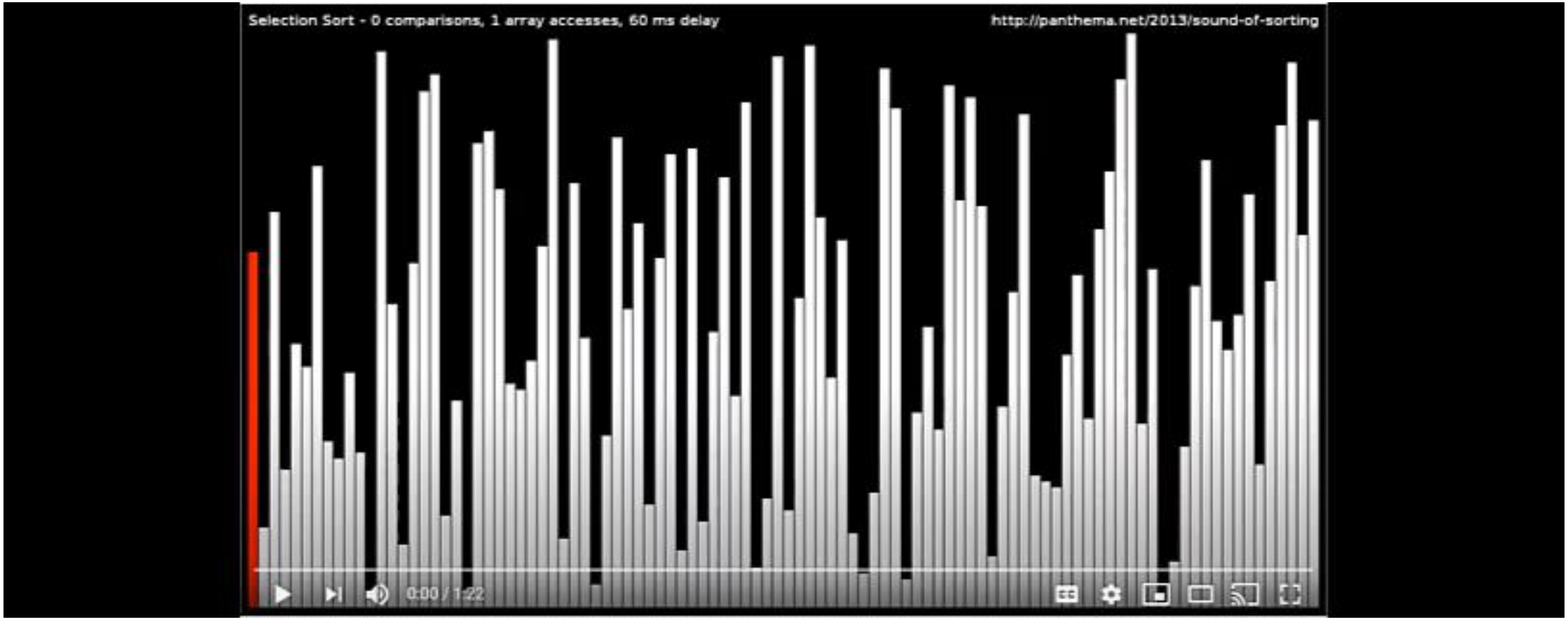
```
{1, 9, 4, 13}

n = 4

 i    min_idx     j
---   -------    ---
 1
```

Selection Sort

```
void selectionSort(int A[], int n)
{
   int i, j, min_idx;

   for (i = 0; i < n-1; i++)
   {
      min_idx = i;

      for (j = i+1; j < n; j++)
      {
         if (A[j] < A[min_idx])
            min_idx = j;
      }

      swap(&A[min_idx], &A[i]);
   }
}
```

```
{1, 4, 9, 13}

n = 4

 i   min_idx   j
---  -------  ---
 2
```

Selection Sort


Selection Sort - 0 comparisons, 1 array accesses, 60 ms delay

http://panthema.net/2013/sound-of-sorting

Analysis of Selection Sort

Selection Sort loops over indices in the array

```
for (i = 0; i < n-1; i++)
```

Let's refer to this loop as the "outer i loop"

For each index, the code loops to find the minimum elements of the subarray

```
for (j = i+1; j < n; j++)
```

Let's refer to this loop as the "inner j loop"

and does a swap.

```
swap(&A[min_idx], &A[i]);
```

If the length of the array is *n*, then there are *n* indices in the array.

Analysis of Selection Sort

How many lines of code are executed by a single call to `swap()`?

```
void swap(int *xp, int *yp)
{
     int temp = *xp;
     *xp = *yp;
     *yp = temp;
}
```

In this implementation, three lines are ALWAYS executed.

We can say that each call to `swap()` takes constant time.

Analysis of Selection Sort

How many lines of code are executed by a single pass through the "outer i loop"?

We also have to account for the "inner j loop".

How many times does this loop execute in a given pass through the "outer i loop"?

It depends on the size of the subarray that it's iterating over. If the subarray is the whole array (as it is on the first step), the loop body runs *n* times.

If the subarray is of size 6, then the loop body runs 6 times.

Analysis of Selection Sort

```
for (i = 0; i < n-1; i++)
{
    min_idx = i;
    for (j = i+1; j < n; j++)
    {
        if (A[j] < A[min_idx])
            min_idx = j;
    }

    swap(&A[min_idx], &A[i]);
}
```

Let's examine an array of size 8.

1st call

When i = 0, the "inner j loop" will run from i+1 to j < n so from 1 to 7.

So we can say that for the first pass through the "outer i loop", the "inner j loop" will run 7 times.

Analysis of Selection Sort

```
for (i = 0; i < n-1; i++)
{
    min_idx = i;
    for (j = i+1; j < n; j++)
    {
        if (A[j] < A[min_idx])
            min_idx = j;
    }

    swap(&A[min_idx], &A[i]);
}
```

Let's examine an array of size 8.

2nd call

When i = 1, the "inner j loop" will run from i+1 to j < n so from 2 to 7.

So we can say that for the first pass through the "outer i loop", the "inner j loop" will run 6 times.

Analysis of Selection Sort

```
for (i = 0; i < n-1; i++)
{
    min_idx = i;
    for (j = i+1; j < n; j++)
    {
        if (A[j] < A[min_idx])
            min_idx = j;
    }

    swap(&A[min_idx], &A[i]);
}
```

Let's examine an array of size 8.

3rd call

When i = 2, the "inner j loop" will run from i+1 to j < n so from 3 to 7.

So we can say that for the first pass through the "outer i loop", the "inner j loop" will run 5 times.

Analysis of Selection Sort

1st call (i = 0) for array of 8 elements	7 times
2nd call (i = 1) for array of 8 elements	6 times
3rd call (i = 2) for array of 8 elements	5 times

Noticing a pattern here?

"outer i loop" goes from 0 to n-1.

When i is 6, "inner j loop" will run from i+1 to j < n so from 7 to 7 < 8.

So "inner j loop" runs once when "outer i loop" is on the last element of the array.

Analysis of Selection Sort

1st call (i = 0) for array of 8 elements	7 times
2nd call (i = 1) for array of 8 elements	6 times
3rd call (i = 2) for array of 8 elements	5 times
4th call (i = 3) for array of 8 elements	4 times
5th call (i = 4) for array of 8 elements	3 times
6th call (i = 5) for array of 8 elements	2 times
7th call (i = 6) for array of 8 elements	1 time
8th call (i = 7) for array of 8 elements	Loop fails

Side Note : Arithmetic Series

How do you compute the sum 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 quickly?

8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 =

(8 + 1) + (7 + 2) + (6 + 3) + (5 + 4) =

9 + 9 + 9 + 9 = 4 * 9 = 36

1. Add the smallest and the largest number.
2. Multiply by the number of pairs.

Side Note : Arithmetic Series

What if the number of integers in the sequence is odd, so that you cannot pair them all up?

It doesn't matter!

Just count the unpaired number in the middle of the sequence as half a pair.

1 + 2 + 3 + 4 + 5

(1 + 5) + (2 + 4) + 3 = 2.5 pairs where each pair has a value of 6.

2.5 * 6 = 15

Side Note : Arithmetic Series

What if the sequence to sum up goes from 1 to *n*?

This an **arithmetic series**.

The sum of the smallest and largest numbers is $n+1$

Because there are *n* numbers total, there are $\frac{n}{2}$ pairs (whether *n* is odd or even).

Therefore, the sum of numbers from 1 to *n* is $(n + 1)(\frac{n}{2})$ which is $\frac{n^2+n}{2}$

Side Note : Arithmetic Series

$$\frac{n^2 + n}{2}$$

8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = (8 + 1) + (7 + 2) + (6 + 3) + (5 + 4) = 9 + 9 + 9 + 9 = 4 * 9 = 36

$$\frac{n^2+n}{2} = \frac{8^2+8}{2} = \frac{72}{2} = 36$$

1 + 2 + 3 + 4 + 5 = (1 + 5) + (2 + 4) + 3 = 2.5 pairs => 2.5 * 6 = 15

$$\frac{n^2+n}{2} = \frac{5^2+5}{2} = \frac{30}{2} = 15$$

Analysis of Selection Sort

The total running time for selection sort has three parts

1. The running time of the "outer i loop"

2. The running time for all the calls to `swap()`.

3. The running time for the "inner j loop"

Analysis of Selection Sort

Parts 1 and 2 are easy

1. The running time of the "outer i loop"

This loop is really just testing and incrementing the loop variable and running the "inner j loop" and calling `swap()`, so it takes constant time for each of the *n* iterations.

Using asymptotic notation, the time for all of these steps is $\Theta(n)$.

2. The running time for all the calls to `swap()`.

We know that there are *n-1* calls to `swap()` and each call takes constant time.

Using asymptotic notation, the time for all calls to `swap()` is $\Theta(n)$.

Analysis of Selection Sort

The running time for the "inner j loop"

Each individual iteration of the loop in "inner j loop" takes constant time. The number of iterations of this loop is *n* in the first call, then *n*−1, then *n*−2 and so on.

We know that this sum, 1 + 2 + … + (*n*−1) + *n* is an arithmetic series and it evaluates to

$$\frac{n^2+n}{2}$$

Therefore, the total time for all calls to "inner j loop" is some constant, c_1, times $\frac{n^2+n}{2}$

Analysis of Selection Sort

Therefore, the total time for all calls to "inner j loop" is some constant, c_1, times $\frac{n^2+n}{2}$

$$c_1\left(\frac{n^2+n}{2}\right) = c_1\left(\frac{1}{2}n^2 + \frac{1}{2}n\right) = \frac{1}{2}c_1(n^2 + n) = \frac{1}{2}c_1n^2 + \frac{1}{2}c_1n$$

In terms of big-Θ notation, we can eliminate c_1 and the factor of $\frac{1}{2}$. We can also eliminate the low-order term of n.

The result is that the running time for all the calls to "inner j loop" is $\Theta(n^2)$.

Analysis of Selection Sort

The total running time for selection sort has three parts

1. The running time of the "outer i loop" $\Theta(n)$
2. The running time for all the calls to `swap().` $\Theta(n)$
3. The running time for the "inner j loop" $\Theta(n^2)$

$\Theta(n) + \Theta(n) + \Theta(n^2) = \Theta(n^2)$

What is big O and big Ω of Selection Sort?

Analysis of Selection Sort

$\Theta(n^2)$

What is big O and big Ω of Selection Sort?

No case is particularly good or particularly bad for selection sort.

The loop in "inner j loop" will always make $\frac{n^2+n}{2}$ iterations regardless of the input.

Selection Sort runs in $\Theta(n^2)$ and $O(n^2)$ and $\Omega(n^2)$.

Analysis of Selection Sort

So what does a runtime of $\Theta(n^2)$ tell us about Selection Sort?

Let's use an example where the constant factor is $\frac{1}{10^6}$ so that selection sort takes approximately $(\frac{1}{10^6})n^2$ seconds to sort n values.

$n = 100$ The running time of selection sort is about $\frac{100^2}{10^6}$ which is $\frac{1}{100}$ seconds. That seems pretty fast.

$n = 1000$ The running time of selection sort is about $\frac{1000^2}{10^6}$ which is 1 second. Not bad??

n grew by a factor of 10 but the runtime of the sort increased by a factor of 100? Hmmm…

What if $n = 1{,}000{,}000$? $\frac{1000000^2}{10^6}$ = 1,000,000 seconds = 11 days and 14 hours.

Increasing the array size by a factor of 1000 increases the running time a million times!

Analysis of Selection Sort

Does the "sortedness" of the array affect the runtime of Selection Sort?

Does it run faster for a sorted array?

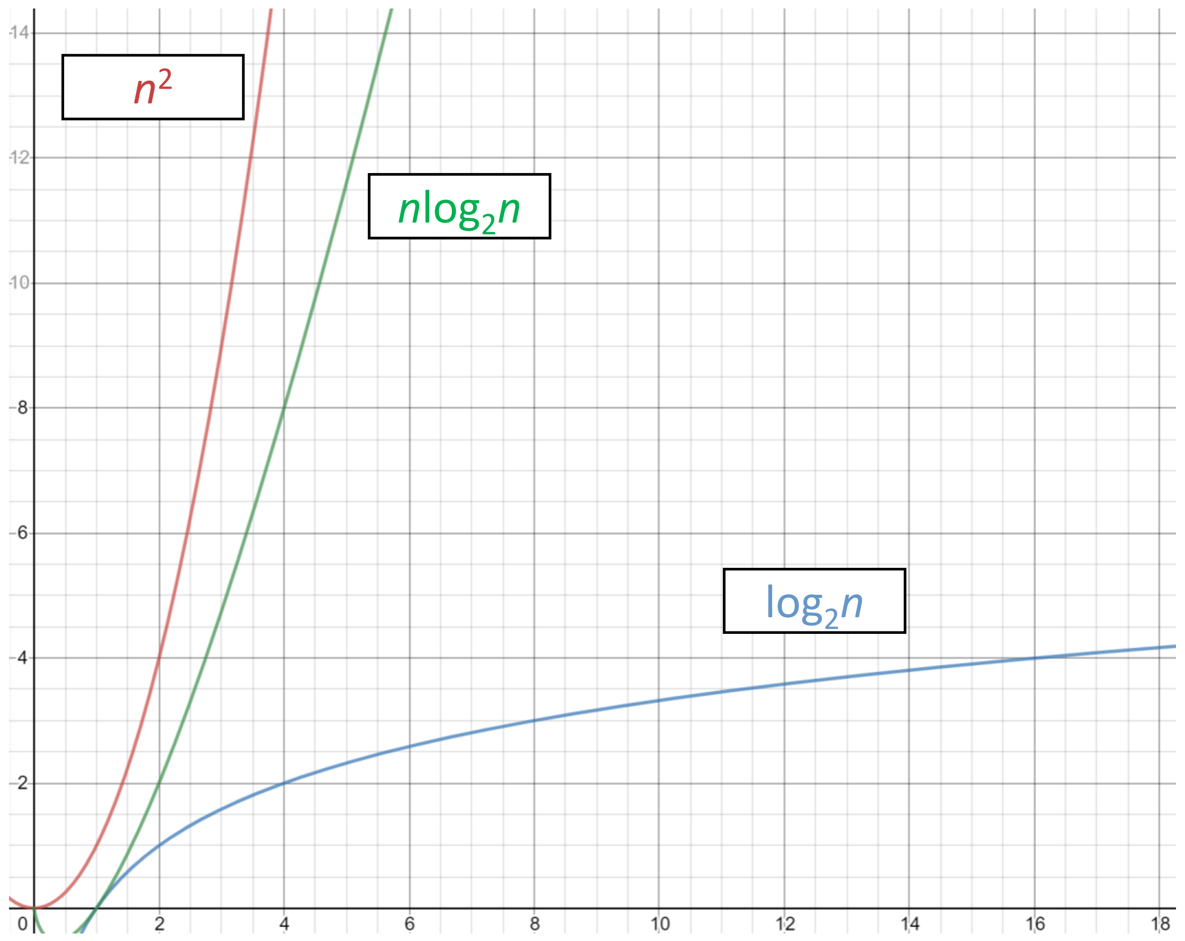
Does it run slower for an array in reverse sorted order?

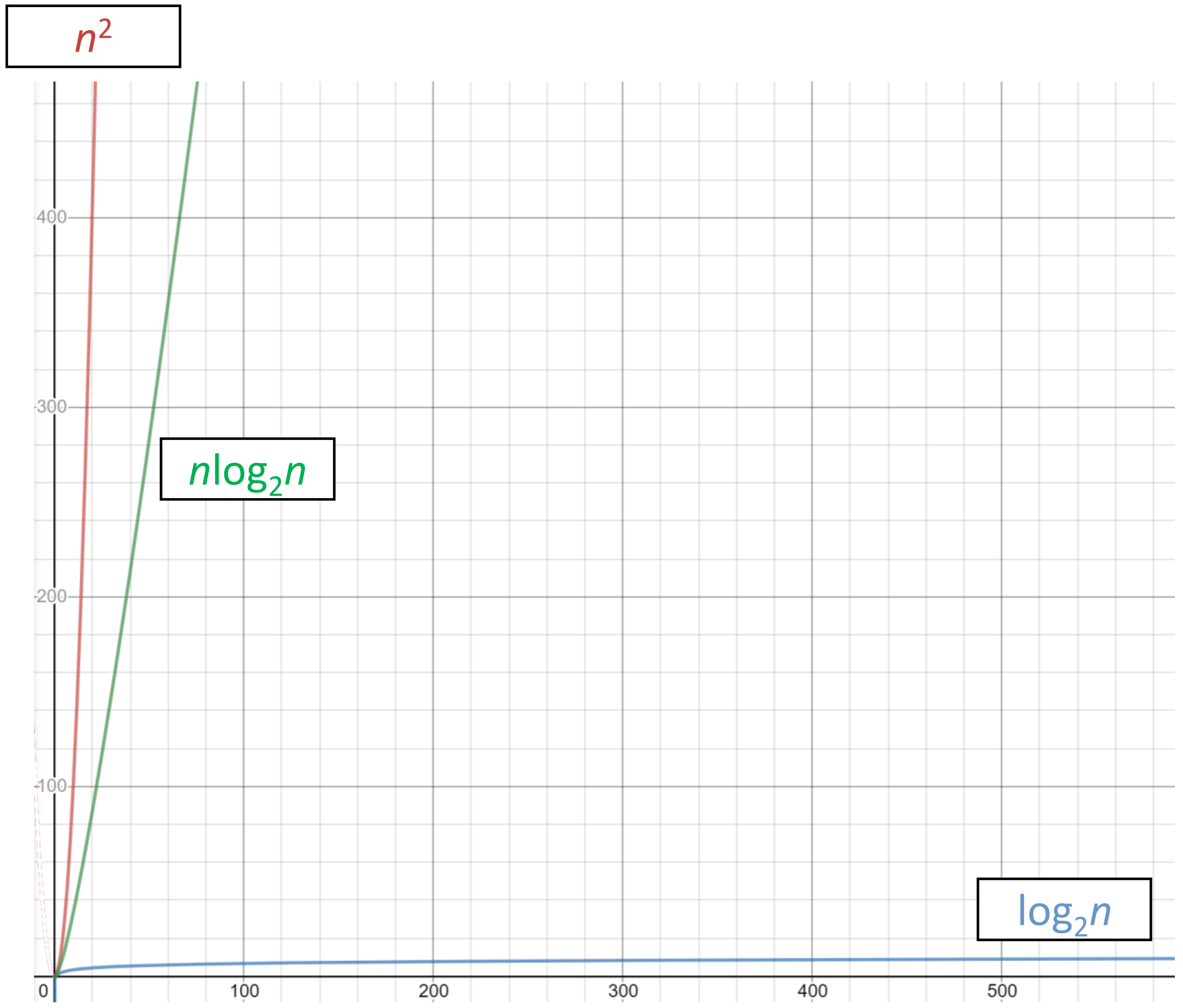
No.

Selection Sort will run through the same number of steps regardless.

Analysis of Selection Sort

$\Theta(n^2)$ is not that great of a run time.





Analysis of Selection Sort

Selection Sort shares many of same benefits of Insertion Sort …

- It performs well on small inputs

    depending on the constant factors involved - it can beat $O(n\log_2 n)$ sorts

- It requires only constant extra space (unlike merge sort)

Selection Sort has some extra benefits

- The code is very simple; therefore, easy to program.

- It only requires $n-1$ swaps (which is better than most sorting algorithms)

- For the same set of elements, it will take the same amount of time regardless of how they are arranged.
    - This can be good for real time applications.

Analysis of Selection Sort

Here are the cons

- $O(n^2)$ is slower than $O(n\log_2 n)$ algorithms (like merge sort) for large inputs.

- Insertion sort, which is also $O(n^2)$, is usually faster than it on small inputs.
  the constant factors would determine which is faster

So what are some situations when you want to use it?

Analysis of Selection Sort

- You need a sort algorithm that is easy to program

- You need a sort algorithm that requires a small amount of code

- You only have a small number of elements to sort, so you feel that it is quick enough and don't want to sacrifice more memory to get more speed.

- Swaps are expensive on your hardware, but you don't want to use a more complicated sort that cuts down on swaps.

- You need the sorting time to be consistent for a given size.

Coding Assignment 3

Make a copy of Coding Assignment 2.

Add a Merge Sort function to your code.

You will run your code on the same files from Coding Assignment 2.

You will graph the results of Insertion Sort and Merge Sort.

Quick Sort

Like Merge Sort, Quick Sort uses divide and conquer and so it's a recursive algorithm also.

The way that Quick Sort uses divide-and-conquer is a little different from how Merge Sort does.

In Merge Sort, the divide step does hardly anything and all the real work happens in the combine step.

Quick Sort is the opposite - all the real work happens in the divide step.

In fact, the combine step in Quick Sort does absolutely nothing.

Quick Sort

Quick Sort has a couple of other differences from Merge Sort.

Quick Sort works in place.

Its worst-case running time is as bad as Selection Sort and Insertion Sort $\Theta\ (n^2)$

But its average-case running time is as good as Merge Sort $\Theta\ (n\log_2 n)$

So why think about quicksort when merge sort is at least as good?

That's because the constant factor hidden in the big-Θ notation for Quick Sort is quite good.

In practice, Quick Sort outperforms Merge Sort and it significantly outperforms Selection Sort and Insertion Sort.

Quick Sort

Quick Sort uses divide-and-conquer.

Just like we did with Merge Sort, think of sorting a subarray

`array[p..r]`

where initially the subarray is `array[0..n-1]`.

Quick Sort

Divide

Choose any element in the subarray `array[p..r]` and call this element the **pivot**.

Rearrange the elements in `array[p..r]` so that all elements in `array[p..r]` that are less than or equal to the **pivot** are to its left and all elements that are greater than the **pivot** are to its right.

This procedure is called **partitioning**.

At this point, it doesn't matter what order the elements to the left/to the right of the pivot are in relation to each other.

We just care that each element is somewhere on the correct side of the pivot.

Quick Sort

As a matter of practice, we'll always choose the rightmost element in the subarray, `array[r]`, as the **pivot**.

So, for example, if the subarray consists of

[9, 7, 5, 11, 12, 2, 14, 3, 10, 6]

then we choose 6 as the pivot.

After **partitioning**, the subarray will look like

[5, 2, 3, 6, 12, 7, 14, 9, 10, 11]

Everything to the left of 6 is less than 6.
Everything to the right of 6 is greater than 6.

Let `q` be the index of where the **pivot** ends up.

Quick Sort

Conquer

Recursively sort the subarrays `array[p..q-1]`

all elements to the left of the pivot, which must be less than or equal to the **pivot**

and `array[q+1..r]`

all elements to the right of the pivot, which must be greater than the **pivot**

Quick Sort

Combine

Do nothing.

Once the conquer step recursively sorts, the sort is complete.

Why?

All elements to the left of the pivot, in `array[p..q-1]`, are less than or equal to the **pivot** and are sorted and all elements to the right of the **pivot**, in `array[q+1..r]`, are greater than the **pivot** and are sorted.

The elements in `array[p..r]` can't help but be sorted!

Quick Sort

Quick Sort is a divide and conquer recursion algorithm. So from `main()`, the quick sort function is called…

```
QuickSort(arr, 0, n-1);
```

with parameters of the array, the start of the array (index 0) and the last index (number of elements in array – 1)

Quick Sort

```
int main(void)
{
    int arr[] = {9, 6, 5, 7};
    int n = sizeof(arr)/sizeof(arr[0]);

    QuickSort(arr, 0, n-1);

    return 0;
}
```

Quick Sort

```
QuickSort(0,3)
{9, 6, 5, 7}

partition(0, 3)
ndx = ?

QuickSort(0,ndx-1)
QuickSort(ndx+1,3)
```

```
void QuickSort(int A[], int low, int high)
{
      if (low < high)
      {
            int ndx = partition(A, low, high);

            QuickSort(A, low, ndx - 1);
            QuickSort(A, ndx + 1, high);
      }
}
```

Note here that the 1st call alters `high` and the 2nd call alters `low`

Quick Sort

```
int partition (int A[], int low, int high)
{
   int i, j = 0;
   int pivot = A[high];

   i = (low - 1);

   for (j = low; j < high; j++)
   {
      if (A[j] < pivot)
      {
         i++;
         swap(&A[i], &A[j]);
      }
   }
   swap(&A[i + 1], &A[high]);

   return (i + 1);
}
```

```
void swap(int *SwapA, int *SwapB)
{
        int temp = *SwapA;
        *SwapA = *SwapB;
        *SwapB = temp;
}
```

```
int partition (int A[], int low, int high)
{
   int i, j = 0;
   int pivot = A[high];

   i = (low - 1);

   for (j = low; j < high; j++)
   {
      if (A[j] < pivot)
      {
         i++;
         swap(&A[i], &A[j]);
      }
   }
   swap(&A[i + 1], &A[high]);

   return (i + 1);
}
```

```
 0   1   2   3
{9,  6,  5,  7}
{6,  5,  7,  9}
```

1st call – `partition(A, 0, 3)`

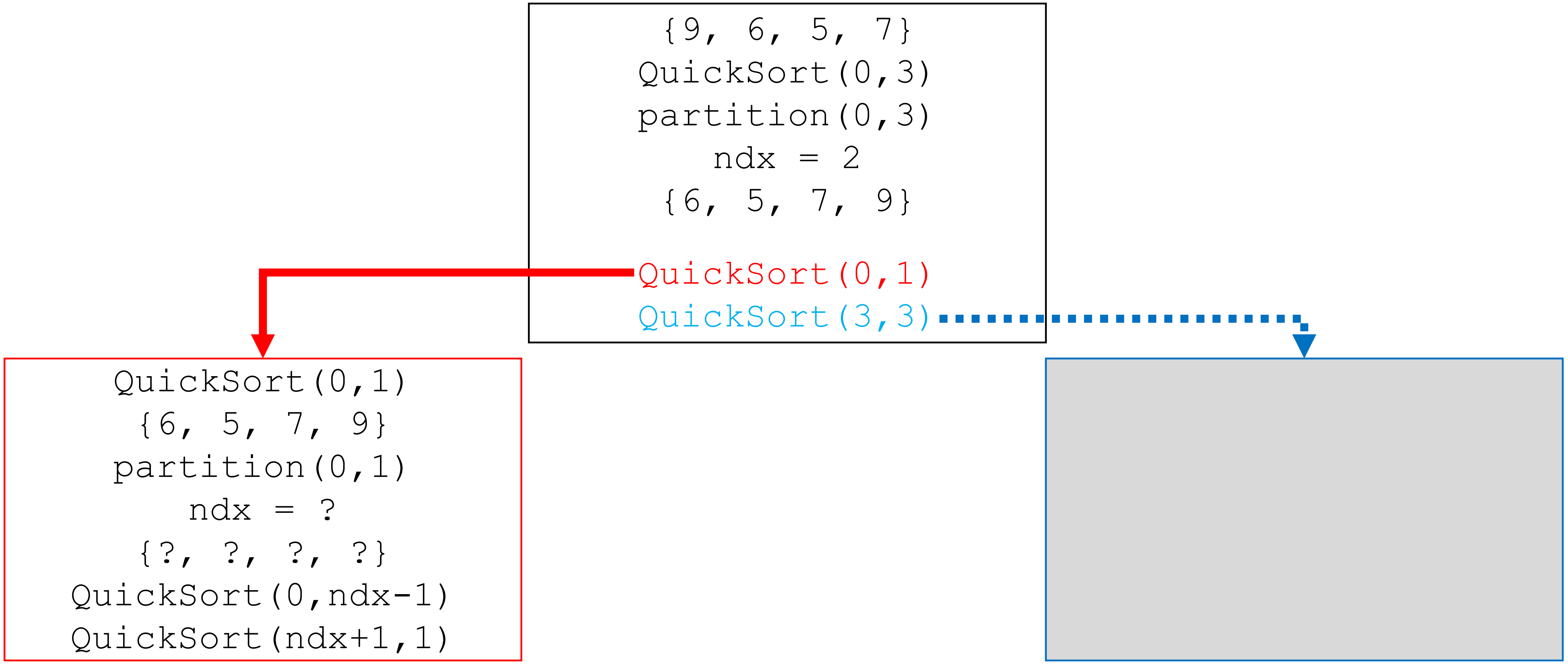
i	j	pivot	low	high

```
{9, 6, 5, 7}
QuickSort(0,3)

partition(0, 3)
ndx = 2
{6, 5, 7, 9}

QuickSort(0,1)
QuickSort(3,3)
```

Quick Sort

```
void QuickSort(int A[], int low, int high)
{
     if (low < high)
     {
          int ndx = partition(A, low, high);

          QuickSort(A, low, ndx - 1);
          QuickSort(A, ndx + 1, high);
     }
}
```

Quick Sort



```
{9, 6, 5, 7}
QuickSort(0,3)
partition(0,3)
ndx = 2
{6, 5, 7, 9}

QuickSort(0,1)
QuickSort(3,3)
```

```
QuickSort(0,1)
{6, 5, 7, 9}
partition(0,1)
ndx = ?
{?, ?, ?, ?}
QuickSort(0,ndx-1)
QuickSort(ndx+1,1)
```

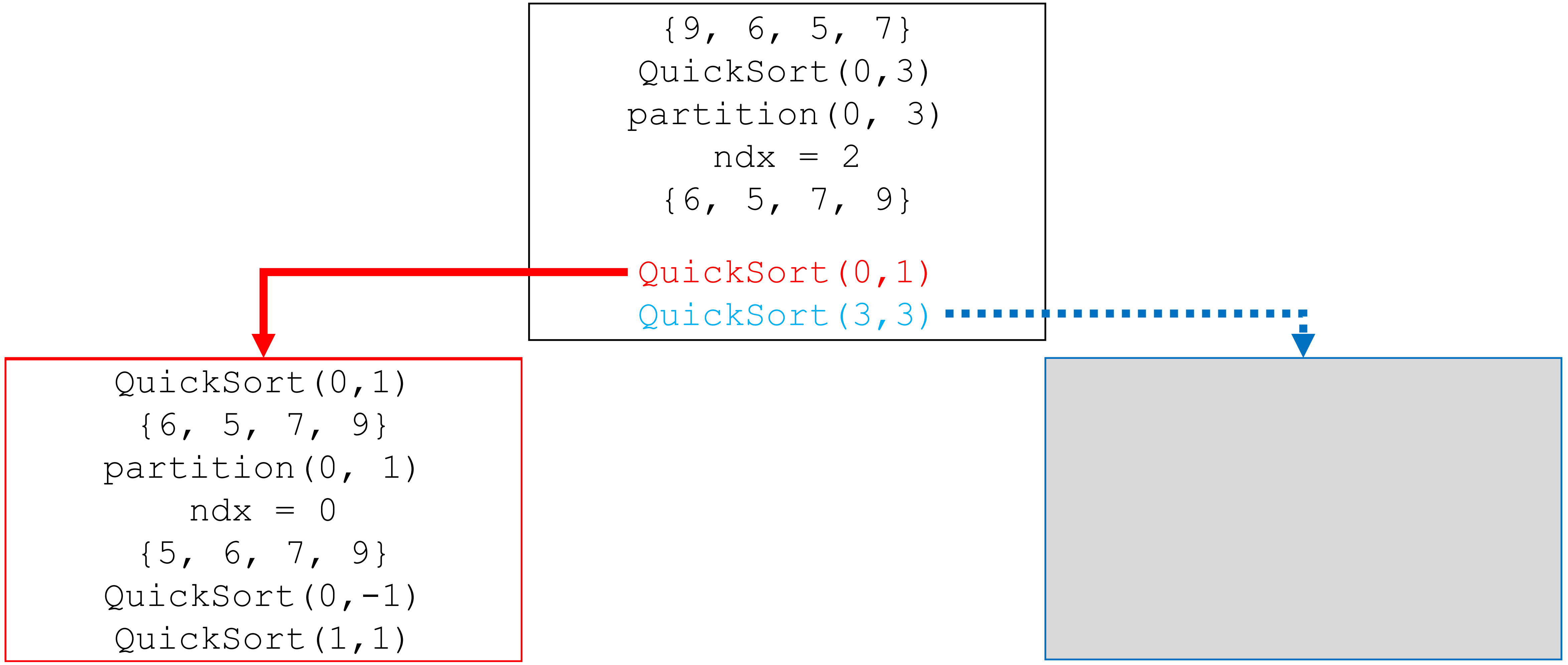
```
int partition (int A[], int low, int high)
{
   int i, j = 0;
   int pivot = A[high];

   i = (low - 1);

   for (j = low; j < high; j++)
   {
      if (A[j] < pivot)
      {
         i++;
         swap(&A[i], &A[j]);
      }
   }
   swap(&A[i + 1], &A[high]);

   return (i + 1);
}
```

```
  0    1    2    3
{6,  5,  7,  9}
{5,  6,  7,  9}
```

2nd call – `partition(A, 0, 1)`

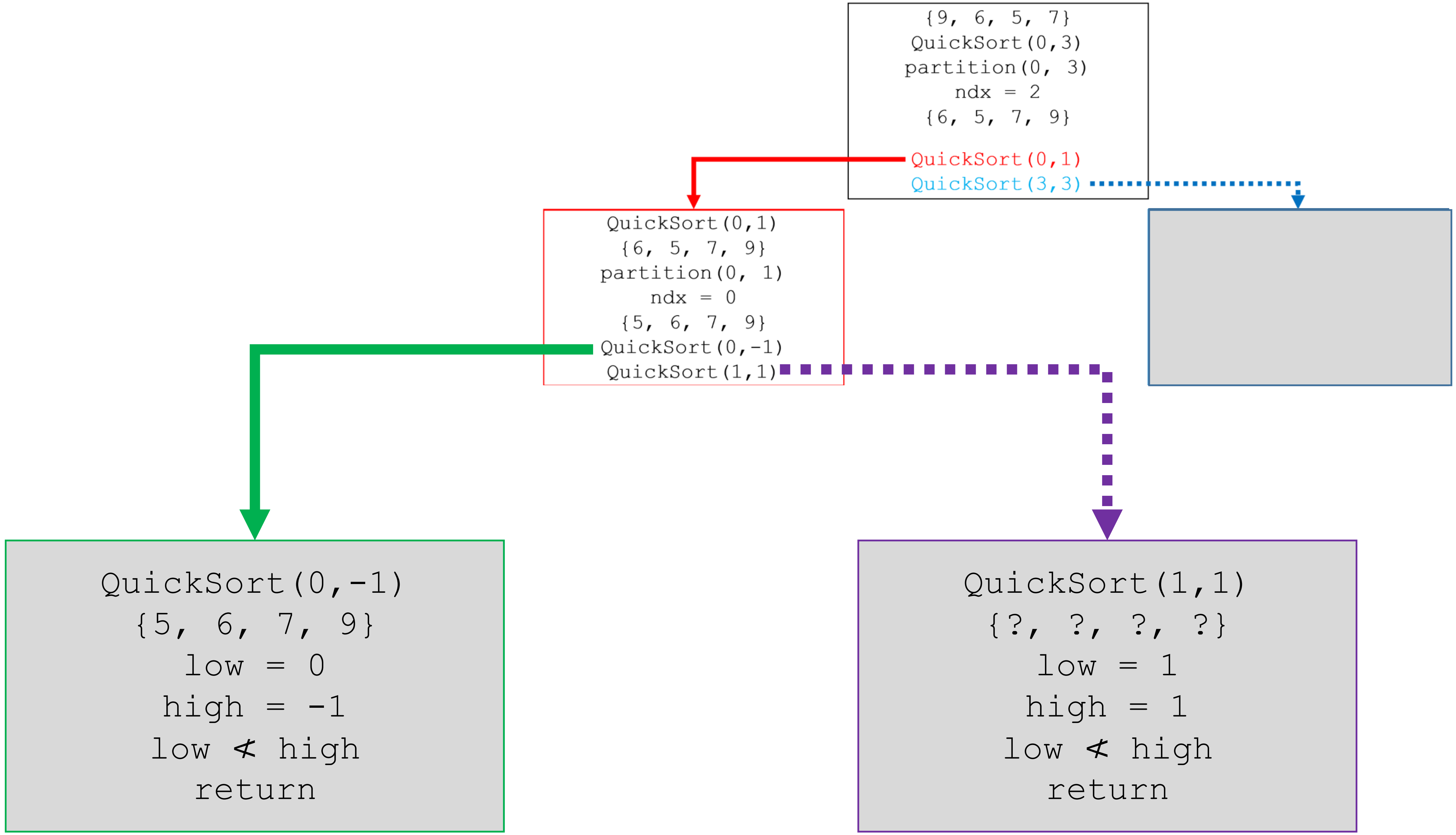
i	j	pivot	low	high

Quick Sort



```
{9, 6, 5, 7}
QuickSort(0,3)
partition(0, 3)
ndx = 2
{6, 5, 7, 9}

QuickSort(0,1)
QuickSort(3,3)
```

```
QuickSort(0,1)
{6, 5, 7, 9}
partition(0, 1)
ndx = 0
{5, 6, 7, 9}
QuickSort(0,-1)
QuickSort(1,1)
```



```
{9, 6, 5, 7}
QuickSort(0,3)
partition(0, 3)
ndx = 2
{6, 5, 7, 9}

QuickSort(0,1)
QuickSort(3,3)
```

```
QuickSort(0,1)
{6, 5, 7, 9}
partition(0, 1)
ndx = 0
{5, 6, 7, 9}
QuickSort(0,-1)
QuickSort(1,1)
```

```
QuickSort(0,-1)
{5, 6, 7, 9}
low = 0
high = -1
low ≮ high
return
```

```
QuickSort(1,1)
{?, ?, ?, ?}
low = 1
high = 1
low ≮ high
return
```

Quick Sort

```
int partition (int A[], int low, int high)
{
  int i, j = 0;
  int pivot = A[high];

  i = (low - 1);

  for (j = low; j < high; j++)
  {
    if (A[j] < pivot)
    {
      i++;
      swap(&A[i], &A[j]);
    }
  }
  swap(&A[i + 1], &A[high]);

  return (i + 1);
}
```

```
{9,6,5,7}     7 is pivot
              9 ≮ 7 => no swap
              6 < 7 => swap 9 and 6
{6,9,5,7}     5 < 7 => swap 9 and 5
{6,5,9,7}     final => swap 9 and 7
{6,5,7,9}     7 was pivot so divide {6,5} and {9}
{6,5}{7}{9}   9 does not process (because low ≮ high)
{6,5}         5 is pivot
              6 ≮ 5 => no swap
              final => swap 6 and 5

{5,6,7,9}
```

Quick Sort

{6,9,7,5}	5 is pivot
	6 ≮ 5 => no swap
	9 ≮ 5 => no swap
	7 ≮ 5 => no swap
	final => swap 6 and 5
{5,9,7,6}	5 was pivot so divide into {} and
{}{5}{9,7,6}	nothing to the left of 5
{9,7,6}	6 is pivot
	9 ≮ 6 => no swap
	7 ≮ 6 => no swap
	final => swap 9 and 6
{6,7,9}	6 was pivot so divide into {} and {7,9}
{}{6}{7,9}	nothing to the left of 6
{7,9}	9 is pivot – see next slide
{5,6,7,9}	

```
int partition (int A[], int low, int high)
{
  int i, j = 0;
  int pivot = A[high];

  i = (low - 1);

  for (j = low; j < high; j++)
  {
    if (A[j] < pivot)
    {
      i++;
      swap(&A[i], &A[j]);
    }
  }
  swap(&A[i + 1], &A[high]);

  return (i + 1);
}
```

```
int partition (int A[], int low, int high)
{
   int i, j = 0;
   int pivot = A[high];

   i = (low - 1);

   for (j = low; j < high; j++)
   {
      if (A[j] < pivot)
      {
         i++;
         swap(&A[i], &A[j]);
      }
   }
   swap(&A[i + 1], &A[high]);

   return (i + 1);
}
```

```
 0   1   2   3
{5,  6,  7,  9}
{5,  6,  7,  9}
```

```
partition(A, 2, 3)
```

i	j	pivot	low	high

Quick Sort

```
{9,7,6,5}         5 is pivot
                  9 ≮ 5 => no swap
                  7 ≮ 5 => no swap
                  6 ≮ 5 => no swap
                  final => swap 9 and 5
{5,7,6,9}         5 was pivot so divide into
                        {} and {7,6,9}
{}{5}{7,6,9}      nothing to the left of 5
{7,6,9}           9 is pivot
```

```
int partition (int A[], int low, int high)
{
  int i, j = 0;
  int pivot = A[high];

  i = (low - 1);

  for (j = low; j < high; j++)
  {
    if (A[j] < pivot)
    {
      i++;
      swap(&A[i], &A[j]);
    }
  }
  swap(&A[i + 1], &A[high]);

  return (i + 1);
}
```

```
int partition (int A[], int low, int high)
{
   int i, j = 0;
   int pivot = A[high];

   i = (low - 1);

   for (j = low; j < high; j++)
   {
      if (A[j] < pivot)
      {
         i++;
         swap(&A[i], &A[j]);
      }
   }
   swap(&A[i + 1], &A[high]);

   return (i + 1);
}
```

0	1	2	3
{5,	7,	6,	9}
{5,	7,	6,	9}

```
partition(A, 1, 3)
```

i	j	pivot	low	high

Quick Sort

{9,7,6,5}	5 is pivot
	9 $\nless$ 5 => no swap
	7 $\nless$ 5 => no swap
	6 $\nless$ 5 => no swap
	final => swap 9 and 5
{5,7,6,9}	5 was pivot so divide into {} and {7,6,9}
{}{5}{7,6,9}	nothing to the left of 5
{7,6,9}	9 is pivot
	7 < 9 => swap 7 and 7
{7,6,9}	6 < 9 => swap 6 and 6
{7,6,9}	final => swap 9 and 9
{7,6,9}	9 was pivot so divide into {7,6} and {9} and {}
{7,6}{9}{}	nothing to the right of 9
{7,6}	6 is pivot
{7,6}	7 $\nless$ 6 => no swap
{5,6,7,9}	final => swap 7 and 6

Quick Sort

`{5,6,7,9}`

`9 is pivot`

```
int partition (int A[], int low, int high)
{
   int i, j = 0;
   int pivot = A[high];

   i = (low - 1);

   for (j = low; j < high; j++)
   {
      if (A[j] < pivot)
      {
         i++;
         swap(&A[i], &A[j]);
      }
   }
   swap(&A[i + 1], &A[high]);

   return (i + 1);
}
```

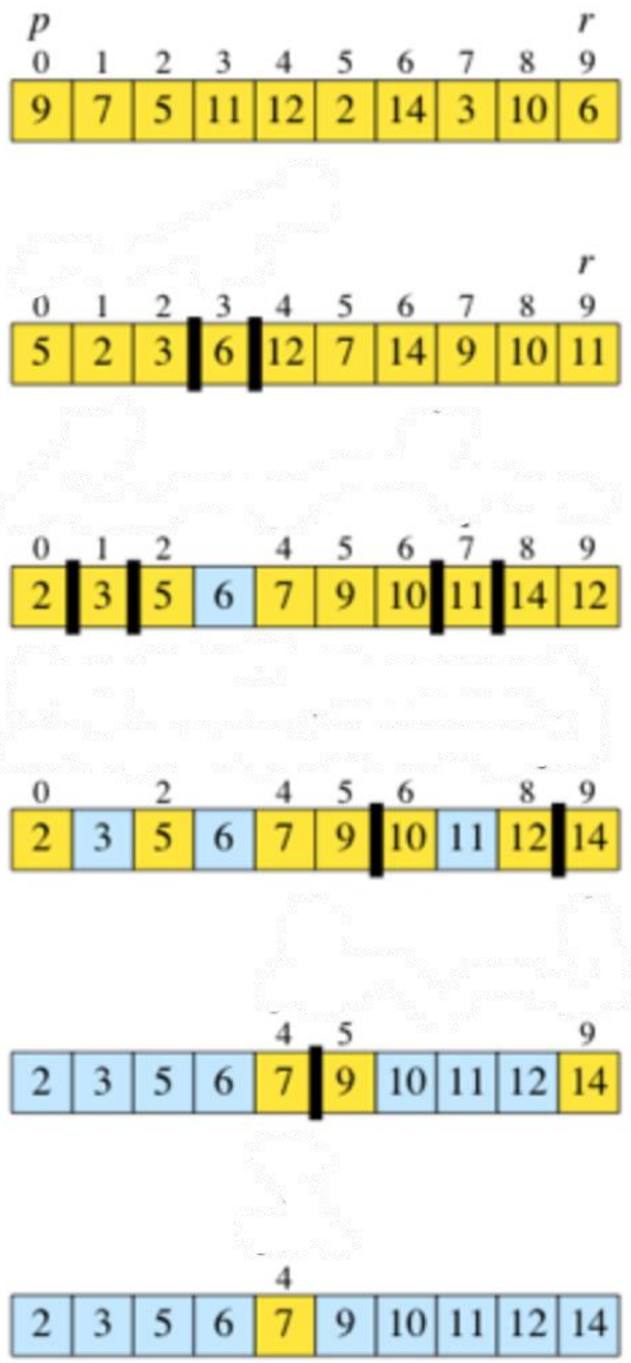
0	1	2	3
{5,	6,	7,	9}
{5,	6,	7,	9}

partition(A, 0, 3)

i	j	pivot	low	high

Quick Sort

{5,6,7,9}	9 is pivot
	5 < 9 => swap 5 and 5
	6 < 9 => swap 6 and 6
	7 < 9 => swap 7 and 7
	final => swap 9 and 9
{5,6,7,9}	9 was pivot so divide into {5,6,7} and {9} and {}
{5,6,7}{9}{}	nothing to the right of 9
{5,6,7}	7 is pivot
	5 < 7 => swap 5 and 5
	6 < 7 => swap 6 and 6
	final => swap 7 and 7
{5,6,7}	7 was pivot so divide into {5,6} and {7} and {}
{5,6}{7}{}	nothing to the right of 7
{5,6}	6 is pivot
	5 < 6 => swap 5 and 5
{5,6,7,9}	final => swap 6 and 6


p r
0 1 2 3 4 5 6 7 8 9
9 7 5 11 12 2 14 3 10 6
r
0 1 2 3 4 5 6 7 8 9
5 2 3 6 12 7 14 9 10 11
0 1 2 4 5 6 7 8 9
2 3 5 6 7 9 10 11 14 12
0 2 4 5 6 8 9
2 3 5 6 7 9 10 11 12 14
4 5 9
2 3 5 6 7 9 10 11 12 14
4
2 3 5 6 7 9 10 11 12 14

Quick Sort

9 7 5 11 12 2 14 3 10 6

5 7 9 11 12 2 14 3 10 6

5 2 9 11 12 7 14 3 10 6

5 2 3 11 12 7 14 9 10 6

5 2 3 6 12 7 14 9 10 11

2 5 3 6 12 7 14 9 10 11

2 3 5 6 7 12 14 9 10 11

2 3 5 6 7 9 14 12 10 11

2 3 5 6 7 9 10 12 14 11

2 3 5 6 7 9 10 11 14 12

2 3 5 6 7 9 10 11 14 12

2 3 5 6 7 9 10 11 14 12

2 3 5 6 7 9 10 11 14 12

2 3 5 6 7 9 10 11 14 12

2 3 5 6 7 9 10 11 14 12

2 3 5 6 7 9 10 11 12 14

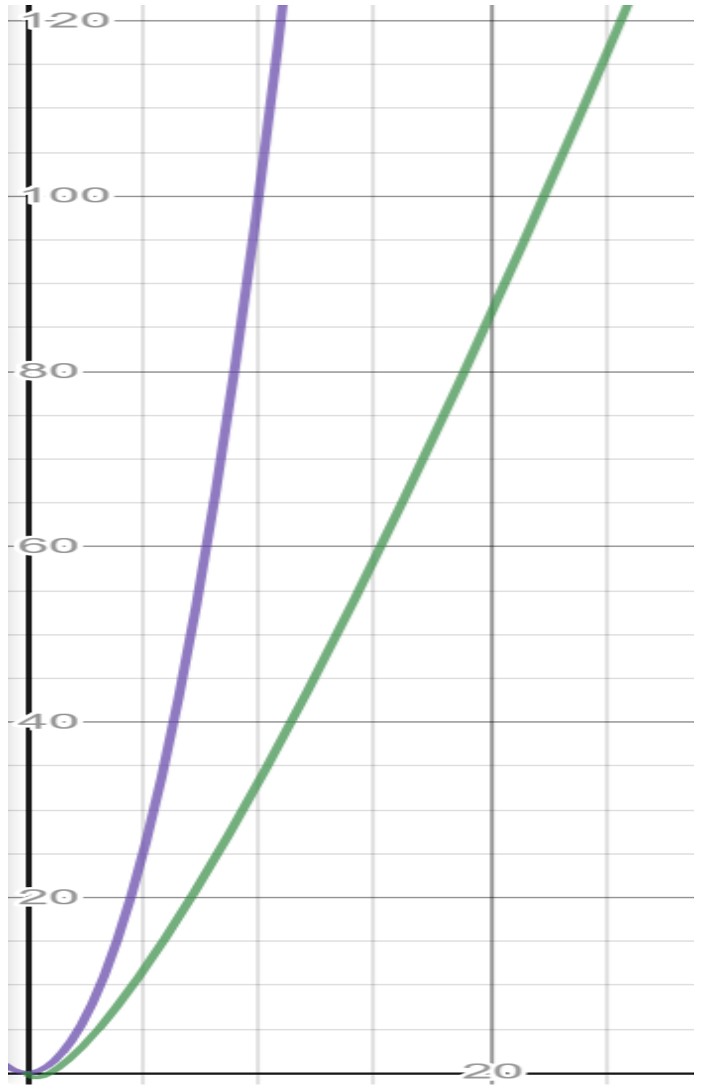
Quick Sort Analysis

Quick Sort's worst case run time is

$\Theta(n^2)$

Quick Sort's best case run time is

$\Theta(n\log_2 n)$



Quick Sort Analysis

So why think about Quick Sort when Merge Sort is at least as good?

Because the constant factor hidden in the big-Θ notation for Quick Sort is quite good.

In practice, Quick Sort outperforms Merge Sort and it significantly outperforms Selection Sort and Insertion Sort.

Quick Sort Analysis

How is it that Quick Sort's worst-case and best-case running times differ?

Let's start by looking at the worst-case running time.

Suppose that we're really unlucky and the partition sizes are really unbalanced.

In particular, suppose that the pivot chosen by the partition function is always either the smallest or the largest element in the *n* element subarray.

Quick Sort Analysis

In particular, suppose that the pivot chosen by the partition function is always either the smallest or the largest element in the *n* element subarray.

Let's start with the case there the pivot is the largest element

{1,2,4,6,7,8,14,18,19}

Pivot would be 19 so all elements are less than pivot so there would be multiple swaps of numbers with themselves but 19 would still remain on the far right.

{1,2,4,6,7,8,14,18} and {}.
{1,2,4,6,7,8,14} and {}.
{1,2,4,6,7,8} and {}

{1,2,4,6,7} and {}
{1,2,4,6} and {}
{1,2,4} and {}
{1,2} and {}

Quick Sort Analysis

As we can see, one of the partitions will contain no elements and the other partition will contain $n - 1$ (all but the pivot) every time.

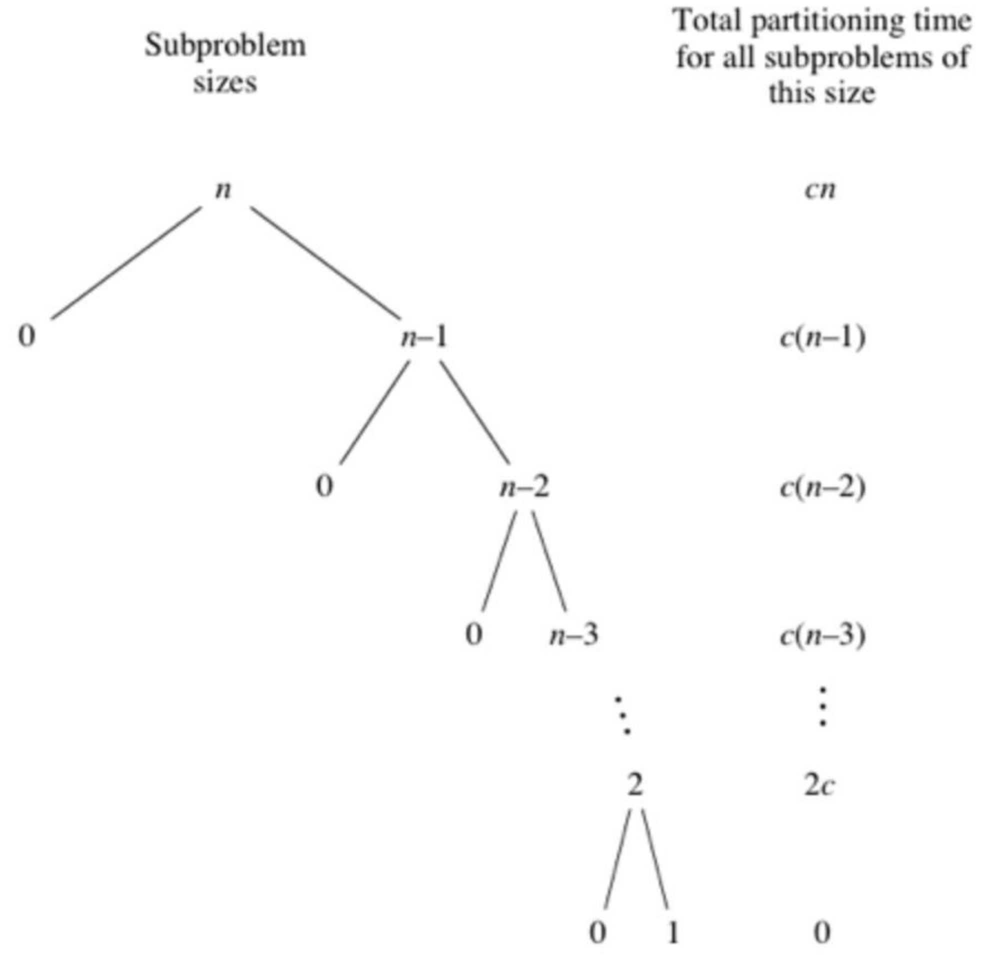
So the recursive calls will be on subarrays of sizes 0 and $n - 1$.

In this situation, divide and conquer with recursion does not help the run time so we don't get the benefit of it either - our runtime suffers.

An array in reverse sorted order would have the same issue.

Quick Sort Analysis

When Quick Sort always has the most unbalanced partitions possible, then the original call takes *cn* time for some constant *c*, the recursive call on *n*−1 elements takes *c*(*n*−1), the recursive call on *n*−2 elements takes *c*(*n*−2) time and so on … until we get to the final 2 elements.



If n is 4

c(4 + (4-1) + (4-2) + (4-3)) =
c(4 + 3 + 2 + 1) = 10

$c(\frac{n(n+1)}{2}) = \frac{4(4+1)}{2} = 10$

Quick Sort Analysis

We can add up the runtime of those partition steps

$cn + c(n-1) + c(n-2) + \cdots + 2c = c(n + (n-1) + (n-2) + \cdots + 2) =$

This pattern indiciates an arithmetic series that we sum with $\frac{n(n+1)}{2}$

Since our pattern ends with 2 instead of 1, we subtract 1

$c(\frac{n(n+1)}{2} + 2 - 1) = c(\frac{1}{2}n^2 + \frac{1}{2}n + 1) = \Theta(n^2)$

Worst case run time

Quick Sort Analysis

Suppose we implement QuickSort so that ChoosePivot always selects the first element of the array. What is the running time of this algorithm on an input array that is already sorted?

- ○ Not enough information to answer this question
- ○ $\Theta(n)$
- ○ $\Theta(n \log n)$
- [✓ $\Theta(n^2)$]

Quick Sort Analysis

What does the best case look like?

Quick Sort's best case occurs when the partitions are as evenly balanced as possible

their sizes either are equal or are within 1 of each other.

Quick Sort Analysis

their sizes are either equal or are within 1 of each other.

if the subarray has an odd number of elements and the pivot is right in the middle after partitioning and each partition has $\frac{n-1}{2}$ elements

if the subarray has an even number of elements and one partition has $\frac{n}{2}$ elements with the other one having $\frac{n}{2} - 1$

Quick Sort Analysis

In either of these cases, each partition has at most $\frac{n}{2}$ elements

The tree of subproblem sizes looks a lot like the tree of subproblem sizes for Merge Sort…

this is where $\log_2 n$ was introduce to the runtime

The partitioning times look like the merging times…

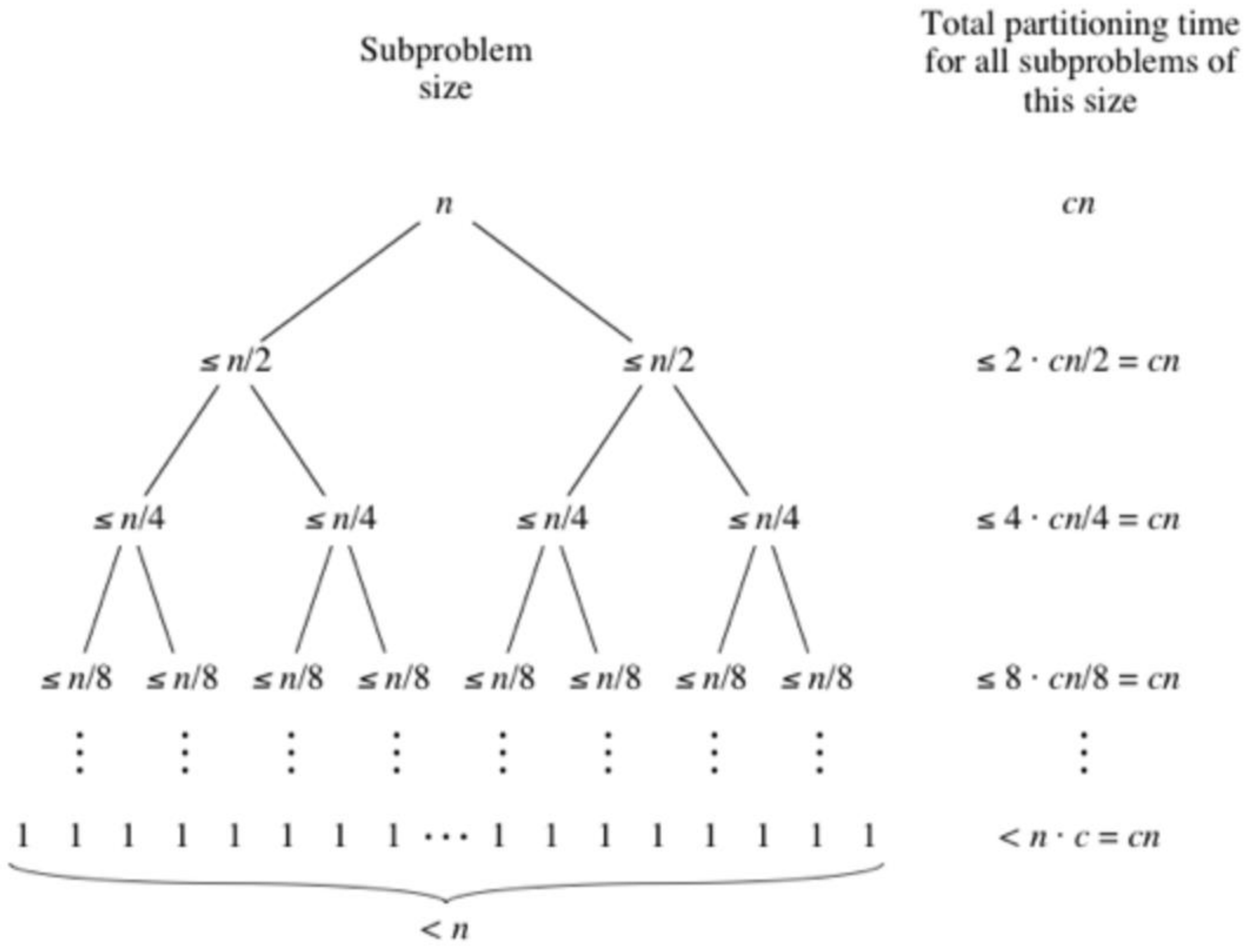
this is where *n* was introduced to the runtime

Quick Sort Analysis

This pattern tell us the same thing it told us about MergeSort.

Partitioning time (*n*) * number of partitions to make ($\log_2 n$)

$O(n\log_2 n)$



Suppose we run QuickSort on some input, and, magically, every recursive call chooses the median element of its subarray as its pivot. What's the running time in this case?

- ○ Not enough information to answer question
- ○ $\Theta(n)$
- [✓ $\Theta(n \log n)$]
- ○ $\Theta(n^2)$

Fix two elements of the input array. How many times can these two elements be compared during the execution of QuickSort ?

- 0
- [✓ 0 or 1]
- 0 or 1 or 2
- Any number in between 0 and n-1

{9, 7, 15, 16, 12}	12 is pivot
{9, 7, 15, 16, 12}	9 < 12 swap 9 with 9
{9, 7, 15, 16, 12}	7 < 12 swap 7 and 7
	15 ≮ 12 so no swap
	16 ≮ 12 so no swap
	final swap of 15 and 12
	recursive call (left and right)
{9, 7} 12, {16, 15}	7 is pivot in left/15 is pivot in right
	9 ≮ 7 so no swap
	final swap of 9 and 7
{7,9} 12 {16,15}	16 ≮ 15 so no swap
	final swap of 16 and 15
{7,9} 12 {15,16}	

Quick Sort Analysis

Worst case is $\Theta(n^2)$ and best case is $\Theta(n\log_2 n)$

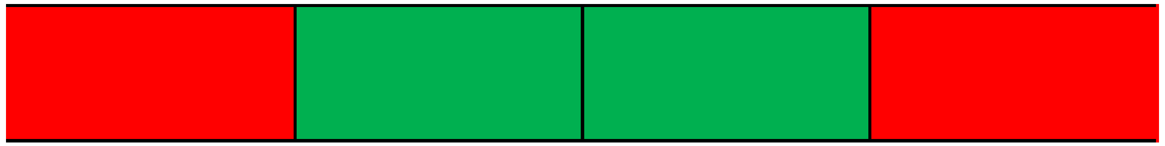
What is an average case?

In the average case, all elements are equally likely to be chosen as the pivot.

Quick Sort Analysis

Worst case is $\Theta(n^2)$ and best case is $\Theta(n\log_2 n)$

After partitioning, we would expect half the time the pivot to end up in the middle two quarters and half the time for it to end up in the outer two quarters.



It can be proven mathematically that this will result in a runtime of $\Theta(n\log_2 n)$ for the average case.

OLQ

- The Quick Sort code from lecture will be provided.
- You will be provided with a copy of this table for reference.

i	j	pivot	low	high

OLQ

You will be given an array to work with (for example)

{9, 7, 5, 12, 11}

Assume that a function to print the entire array is included the line after each call to function `swap()` inside `partition()`.

This call will be in the code provided with the quiz itself and will be in the OLQ review.

OLQ

```
int partition (int A[], int low, int high)
{
   int i, j = 0;
   int pivot = A[high];

   i = (low - 1);

   for (j = low; j < high; j++)
   {
      if (A[j] < pivot)
      {
         i++;
         swap(&A[i], &A[j]);
         printArray(A);
      }
   }
   swap(&A[i + 1], &A[high]);
   printArray(A);

   return (i + 1);
}
```

OLQ

For your given array, you will need to write the output of every call to this function - what does the array look like after each call to function `swap()`?

For example, if your given array is

`{9,7,5,12,11}`

then your answer will be

```
{9,7,5,12,11}
{9,7,5,12,11}
{9,7,5,12,11}
{9,7,5,11,12}
{5,7,9,11,12}
{5,7,9,11,12}
{5,7,9,11,12}
```

OLQ

```
{9, 7, 5, 12, 11}
```

loop swap 9 9	{9,7,5,12,11}
loop swap 7 7	{9,7,5,12,11}
loop swap 5 5	{9,7,5,12,11}
final swap 12 11	{9,7,5,11,12}
final swap 9 5	{5,7,9,11,12}
loop swap 7 7	{5,7,9,11,12}
final swap 9 9	{5,7,9,11,12}

```
i = low -1

for (j = low < high)
{
  if A[j] < pivot
    move i
    swap A[i] A[j]
    print
}
swap A[i+1] with A[high]
print
```

```
i = low -1

for (j = low -> j < high)
{
  if A[j] < pivot
    move i
    swap A[i] A[j]
    print
}
swap A[i+1] with A[high]
print
```

OLQ

{7,12,5,9,11}

loop swap 7 7	{7,12,5,9,11}
loop swap 12 5	{7,5,12,9,11}
loop swap 12 9	{7,5,9,12,11}
final swap 12 11	{7,5,9,11,12}
loop swap 7 7	{7,5,9,11,12}
loop swap 5 5	{7,5,9,11,12}
final swap 9 9	{7,5,9,11,12}
final swap 7 5	{5,7,9,11,12}