Peer Group 3


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Finding eigen values and vectors

$$A = \begin{pmatrix} 4 & 8 & -1 & -2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\begin{pmatrix} 4-\lambda & 8 & -1 & -2 \\ -2 & -9-\lambda & -2 & -4 \\ 0 & 10 & 5-\lambda & -10 \\ -1 & -13 & -14 & -13-\lambda \end{pmatrix} = 0$$

$$(4-\lambda)\begin{pmatrix} -9-\lambda & -2 & -4 \\ 10 & 5-\lambda & -10 \\ -13 & -14 & -13-\lambda \end{pmatrix} - (8)\begin{vmatrix} -2 & -2 & -4 \\ 0 & 5-\lambda & -10 \\ -1 & -14 & -13-\lambda \end{vmatrix} +$$

$$(-1)\begin{pmatrix} -2 & -9-\lambda & -4 \\ 0 & 10 & -10 \\ -1 & -13 & -13-\lambda \end{pmatrix} - (-2)\begin{pmatrix} -2 & -9-\lambda & -2 \\ 0 & 10 & 5-\lambda \\ -1 & -13 & -14 \end{pmatrix}$$

Now let's find each 3x3 determinat separately

*1.

$$\begin{vmatrix} -9-\lambda & -2 & -4 \\ 10 & 5-\lambda & -10 \\ -13 & -14 & -13-\lambda \end{vmatrix} = (-9-\lambda)\begin{vmatrix} 5-\lambda & -10 \\ -14 & -13-\lambda \end{vmatrix} + 2\begin{vmatrix} 10 & -10 \\ -13 & -13-\lambda \end{vmatrix}$$

$$-4\begin{vmatrix} 10 & 5-\lambda \\ -13 & -14 \end{vmatrix}$$

2x2 determinant

$(-9-\lambda)\cdot[(5-\lambda)(-13-\lambda)-(-10)(-14)]+2\cdot[10(-13-\lambda)-(-10)(-13)]$

$-4\cdot[10(-14)-(5-\lambda)(-13)]$

$=(-9-\lambda)\cdot[-65-5\lambda+13\lambda+\lambda^2-140]+2\cdot[-130-10\lambda-130]$

$-4[-140+65-13\lambda]$

$=(-9-\lambda)\cdot[\lambda^2+8\lambda-205]+2\cdot[-260-10\lambda]-4\cdot[-75-13\lambda]$

$=(-9-\lambda)(\lambda^2+8\lambda-205)-520-20\lambda+300+52\lambda$

$=-9\lambda^2-72\lambda+1845-\lambda^3-8\lambda^2+205\lambda-520-20\lambda+300+52\lambda$

$=-\lambda^3-17\lambda^2+165\lambda+1625$

Second 3x3 matrix

$$\begin{vmatrix}-2 & -2 & -4\\ 0 & 5-\lambda & -10\\ 1 & -14 & -13-\lambda\end{vmatrix}=-2\begin{vmatrix}5-\lambda & -10\\ -14 & -13-\lambda\end{vmatrix}+2\begin{vmatrix}0 & -10\\ -1 & -13-\lambda\end{vmatrix}-4\begin{vmatrix}0 & 5-\lambda\\ -1 & -14\end{vmatrix}$$

2x2 determinant

$=-2\cdot[(5-\lambda)(-13-\lambda)-(-10)(-14)]+2[0(-13-\lambda)-(-10)(-1)]-\cancel{4[\ (-14)}$

$4\cdot[0(-14)-(5-\lambda)(-1)]$

$=-2\ [-65-5\lambda+13\lambda+\lambda^2-140]+2\cdot[-10]-4\cdot[5-\lambda]$

$=-2[\lambda^2+8\lambda-205]-20-20+4\lambda$

$=-2\lambda^2-16\lambda+410-20-20+4\lambda$

$=-2\lambda^2-12\lambda+370$

Third 3x3 matrix

$$\begin{vmatrix}-2 & -9-\lambda & -4\\ 0 & 10 & -10\\ -1 & -13 & -13-\lambda\end{vmatrix}=-2\begin{vmatrix}10 & -10\\ -13 & -13-\lambda\end{vmatrix}+(9+\lambda)\begin{vmatrix}0 & -10\\ -1 & -13-\lambda\end{vmatrix}-$$

$$4\begin{vmatrix}0 & 10\\ -1 & -13\end{vmatrix}$$

finding 2x2 determinants

$$= -2\cdot[10(-13-\lambda) - (-10)(-13)] + (9+\lambda)[0\cdot(-13-\lambda) - (-10)(-1)]$$
$$-4[0(-13) - 10(-1)]$$
$$= -2[-130 - 10\lambda - 130] + (9+\lambda)\cdot[-10] - 4[10]$$
$$= -2[-260 - 10\lambda] - 90 - 10\lambda - 40$$
$$= 520 + 20\lambda \cancel{-90-10\lambda-40}$$
$$= 10\lambda + 390$$

fourth 3x3 determinant

$$\begin{vmatrix} -2 & -9-\lambda & -2 \\ 0 & 10 & 5-\lambda \\ -1 & -13 & -14 \end{vmatrix} = -2\begin{vmatrix} 10 & 5-\lambda \\ -13 & -14 \end{vmatrix} + (9+\lambda)\begin{vmatrix} 0 & 5-\lambda \\ -1 & -14 \end{vmatrix} - 2\cdot\begin{vmatrix} 0 & 10 \\ -1 & -13 \end{vmatrix}$$

2x2 determinants

$$= -2[10(-14) - (5-\lambda)(-13)] + (9+\lambda)\cdot[0(-14) - (5-\lambda)(-1)]$$
$$-2[0(-13) - 10(-1)]$$
$$= -2[-140 + 65 - 13\lambda] + (9+\lambda)\cdot[5-\lambda] - 2[10]$$
$$= -2[-75 - 13\lambda] + 45 - 9\lambda + 5\lambda - \lambda^2 - 20$$
$$= 150 + 26\lambda + 45 - 9\lambda + 5\lambda - \lambda^2 - 20$$
$$= \lambda^2 + 22\lambda + 175$$

Bring back all determinants into expansion

$$\det(A-\lambda I) = (4-\lambda)(-\lambda^3 - 17\lambda^2 + 165\lambda + 1625) - 8(-2\lambda^2 - 12\lambda + 370)$$
$$+(-1)(10\lambda + 390) - (-2)(-\lambda^2 + 22\lambda + 175)$$
$$= -(4-\lambda)(\lambda^3 + 17\lambda^2 - 165\lambda - 1625) + 16\lambda^2 + 96\lambda - 2960 - 10\lambda - 390$$
$$-2\lambda^2 + 44\lambda + 350$$
$$= -4\lambda^3 - 68\lambda^2 + 660\lambda + 6500 + \lambda^4 + 17\lambda^3 - 165\lambda^2 - 1625\lambda +$$
$$14\lambda^2 + 130\lambda - 3000$$
$$\det(A-\lambda I) = \lambda^4 + 13\lambda^3 - 219\lambda^2 - 835\lambda + 3500$$

$\det(A - \lambda I) = \lambda_4 + 13\lambda^3 - 219\lambda^2 - 835\lambda + 3500$

Let's use the Newton-Raphson method to find the roots of the polynomial:

$f(\lambda) = \lambda^4 + 13\lambda^3 - 219\lambda^2 - 835\lambda + 3500$

$f'(\lambda) = 4\lambda^3 + 39\lambda^2 - 438\lambda$

Initial Guess: $\lambda = 10$

$f(10) = 10000 + 13000 - 21900 - 8350 + 3500$

$f(10) = -3750$

$f'(10) = 4000 + 3900 - 4380 - 835 = 2685$

$\lambda_{new} = 10 - \frac{-3750}{2685} \approx 11.396$

Next Iteration: $\lambda = 11.396$

$f(11.396) \approx (11.396)^4 + 13(11.396)^3 - 219(11.396)^2 - 835(11.396) + 3500 \approx 276$

$f'(11.396) = 4(11.396)^3 + 39(11.396)^2 - 438(11.396) - 835 \approx 5162$

$\lambda_{new} = 11.396 - \frac{276}{5162} \approx 11.343$

Next Iteration: $\lambda = 11.343$

$f(11.343) \simeq 0$

$\lambda_4 \approx 11.054$

Now, solve the cubic equation:

$$\lambda^4 + 13\lambda^3 - 219\lambda^2 - 835\lambda + 3500 = (\lambda - 11.054)(\lambda^3 + 24.054\lambda^2 + 47.7\lambda - 316.6)$$

$$\lambda^3 + 24.054\lambda^2 + 47.7\lambda - 316.6 = 0$$

Use Newton-Raphson again:

$\lambda_3 \approx 2.675$

$$\lambda^3 + 24.054\lambda^2 + 47.7\lambda - 316.6 = (\lambda - 2.675)(\lambda^2 + 26.729\lambda + 118.3)$$

Now, solve the quadratic equation:

$$\lambda^2 + 26.729\lambda + 118.3 = 0$$

$$\lambda = \frac{-26.729 \pm \sqrt{26.729^2 - 4(118.3)}}{2}$$

$$\lambda = \frac{-26.729 \pm \sqrt{241.4}}{2}$$

Eigen values:

$$\lambda_1 = \frac{-26.729 - 15.54}{2} = -21.125$$

$$\lambda_2 = \frac{-26.729 + 15.54}{2} = -5.604$$

Final Eigen values:

$\lambda_1 = -21.125$,

$\lambda_2 = -5.604$,

$\lambda_3 = 2.675$

$\lambda_4 = 11.054$

For every $\lambda$ we find its own vectors:

$\lambda_1 = -21.125$

$$A - \lambda_1 I = \begin{pmatrix} 25.125 & 8 & -1 & -2 \\ -2 & 12.25 & -2 & -4 \\ 0 & 10 & 26.25 & -10 \\ -1 & -13 & -14 & 8.125 \end{pmatrix}$$

$Av = \lambda v$

$(A - \lambda I) \cdot v = 0$

$$\left(\begin{array}{cccc|c} 25.125 & 8 & -1 & -2 & 0 \\ -2 & 12.125 & -2 & -4 & 0 \\ 0 & 10 & 26.25 & -10 & 0 \\ -1 & -13 & -14 & 8.125 & 0 \end{array}\right) \quad R_1/(25.125) \to R_1$$

$$\left(\begin{array}{cccc|c} 1 & 0.318 & -0.040 & -0.080 & 0 \\ -2 & 12.125 & -2 & -4 & 0 \\ 0 & 10 & 26.25 & -10 & 0 \\ -1 & -13 & -14 & 8.125 & 0 \end{array}\right) \quad R_2 - (-2)R_1 \to R_2$$

$$\left(\begin{array}{cccc|c} 1 & 0.318 & -0.040 & -0.080 & 0 \\ 0 & 12.761 & -2.080 & -4.159 & 0 \\ 0 & 10 & 26.125 & -10 & 0 \\ -1 & -13 & -14 & 8.125 & 0 \end{array}\right) \quad R_4 - (-1)R_1 \to R_4$$

$$\left(\begin{array}{cccc|c} 1 & 0.318 & -0.040 & -0.080 & 0 \\ 0 & 12.761 & -2.080 & -4.159 & 0 \\ 0 & 10 & 26.125 & -10 & 0 \\ 0 & -12.682 & -14.040 & 8.045 & 0 \end{array}\right) \quad R_2/12.761 \to R_2$$

$$\left(\begin{array}{cccc|c} 1 & 0.318 & -0.040 & -0.080 & 0 \\ 0 & 1 & -0.163 & -0.326 & 0 \\ 0 & 10 & 26.125 & -10 & 0 \\ 0 & -12.682 & -14.040 & 8.045 & 0 \end{array}\right) \quad R_3 - 10R_2 \to R_3$$

$$\left(\begin{array}{cccc|c} 1 & 0.318 & -0.040 & -0.080 & 0 \\ 0 & 1 & -0.163 & -0.326 & 0 \\ 0 & 0 & 27.754 & -6.741 & 0 \\ 0 & -12.682 & -14.040 & 8.045 & 0 \end{array}\right) \quad R_4 - -12.682R_2 \to R_4$$

$$\left(\begin{array}{cccc} 1 & 0.318 & -0.040 & -0.080 \\ 0 & 1 & -0.163 & -0.163 \\ 0 & 0 & 27.754 & 27.754 \\ 0 & 0 & -16.106 & -16.106 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0.318 & -0.040 & -0.080 & 0 \\ 0 & 1 & -0.163 & -0.326 & 0 \\ 0 & 0 & 27.754 & -6.741 & 0 \\ 0 & 0 & -16.106 & 3.912 & 0 \end{array}\right) \quad R_3 / 27.754 \to R_3$$

$$\left(\begin{array}{cccc|c} 1 & 0.318 & -0.040 & -0.080 & 0 \\ 0 & 1 & -0.163 & -0.326 & 0 \\ 0 & 0 & 1 & -0.243 & 0 \\ 0 & 0 & -16.106 & 3.912 & 0 \end{array}\right) \quad R_4 - -16.106 \to R_4$$

$$\left(\begin{array}{cccc|c} 1 & 0.318 & -0.040 & -0.080 & 0 \\ 0 & 1 & -0.163 & -0.326 & 0 \\ 0 & 0 & 1 & -0.243 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad R_2 - -0.163R_3 \to R_2$$

$$\left(\begin{array}{cccc|c} 1 & 0.318 & -0.040 & -0.080 & 0 \\ 0 & 1 & 0 & -0.365 & 0 \\ 0 & 0 & 1 & -0.243 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad R_1 - -0.40R_3 \to R_1$$

$$\left(\begin{array}{cccc|c} 1 & 0.318 & 0 & -0.089 & 0 \\ 0 & 1 & 0 & -0.365 & 0 \\ 0 & 0 & 1 & -2.43 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad R_1 - 0.318R_2 \to R_1$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0.027 & 0 \\ 0 & 1 & 0 & -0.365 & 0 \\ 0 & 0 & 1 & -0.243 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 + 0.027\, x_4 = 0 \\ x_2 - 0.365\, x_4 = 0 \\ x_3 - 0.243\, x_4 = 0 \end{cases} \quad (1)$$

$$x_1 = -0.027\, x_4$$
$$x_2 = 0.365\, x_4$$
$$x_3 = 0.243\, x_4$$
$$x_4 = x_4$$

$$V_1 = \begin{pmatrix} -0.027 \\ 0.365 \\ 0.243 \\ 1 \end{pmatrix}$$

$$V_2 = \begin{pmatrix} -16.864 \\ 18.439 \\ -16.446 \\ 1 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} -8.494 \\ 1.428 \\ -1.838 \\ 1 \end{pmatrix}$$

$$V_4 = \begin{pmatrix} -0.071 \\ -0.024 \\ -1.691 \\ 1 \end{pmatrix}$$

Importance of the eigen values in % form

Total sum of absolute values: $|-21.125| + |-5.604| + |2.675| + |11.054| = 40.458$

$$\lambda_1 = \frac{21.125}{40.458} \times 100\% = 52.2\%$$

$$\lambda_2 = \frac{5.604}{40.458} \times 100\% = 13.9\%$$

$$\lambda_3 = \frac{2.675}{40.458} \times 100\% = 6.6\%$$

$$\lambda_4 = \frac{11.054}{40.458} \times 100\% = 27.3\%$$