Higher Order Accurate Derivative and Gradient Calculation in ITK

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Abstract

In this article we describe higher order accurate derivative and gradient image filters for the Insight-Toolkit. These filters are central difference-based numerical derivative approximations that account for additional Taylor series terms and are based on the expressions given by Khan and Ohba.

Contents

1 Introduction

A common way to approximate the first derivative of a function f_k at sample offset k using finite differences is the central difference method.

$$f_0^1 \approx \frac{f_1 - f_{-1}}{2h} \tag{1}$$

Where h is the sampling period.

This expression comes from a Taylor series expansion of the component terms

$$f_1 = f_0 + h f_0^1 + \frac{h^2}{2!} f_0^2 + \cdots + \frac{h^n}{n!} + O(h^{n+1}) \tag{2}$$

Where $O(h^{n+1})$ indicates the series has been truncated after $n+1$ terms.

We also have

$$f_{-1} = f_0 - h f_0^1 + O(h^2) \tag{3}$$

Then we see

$$f_0^1 = \frac{f_1 - f_{-1}}{2h} + O(h^2) \tag{4}$$

Higher order accurate[1] approximations can be made by using additional samples. For instance, a central difference approximation to the first derivative that uses a five point kernel is

$$f_0^1 = \frac{f_{-2} - 8 f_{-1} + 8 f_1 - f_2}{12h} + O(h^4) \tag{5}$$

Khan and Ohba derived closed form expressions for higher order accurate approximations for an arbitrary pth order derivative[1, 2].

For example, the tap-coefficients, d, for a first order derivative

$$d_0 = 0 \tag{6}$$

$$d_k = (-1)^{k+1} \frac{N!^2}{k(N-k)!(N+k)!}, \; k = \pm 1, \pm 2, \ldots, \pm N \tag{7}$$

In this paper we present image derivative and gradient filters for the InsightToolkit (ITK)[3] that implement these expressions.

2 Implementation

The interfaces of `itk::HigherOrderAccurateDerivativeImageFilter` and `itk::HigherOrderAccurateGradientImageFilter` are similar to `itk::DerivativeImageFilter` and `itk::GradientImageFilter`. However, these filters have an additional method, `SetOrderOfAccuracy()`. The approximation will be accurate in Taylor series terms to twice the `OrderOfAccuracy`. They are fast filters implemented with `itk::NeighborhoodIterator`'s. Note that the radius of the `itk::NeighborhoodOperator` will be equal to the `OrderOfAccuracy`. Each order derivative is implemented separately to use recursion during calculation of the factorials[1].

It should be noted that the magnitude of this finite impulse response filter's components falls off dramatically at the ends of the filter with increasing order accuracy. Increasing the order of accuracy beyond four may not be beneficial.

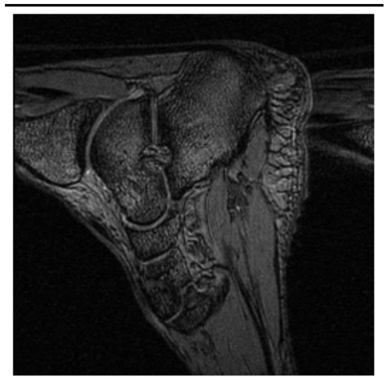
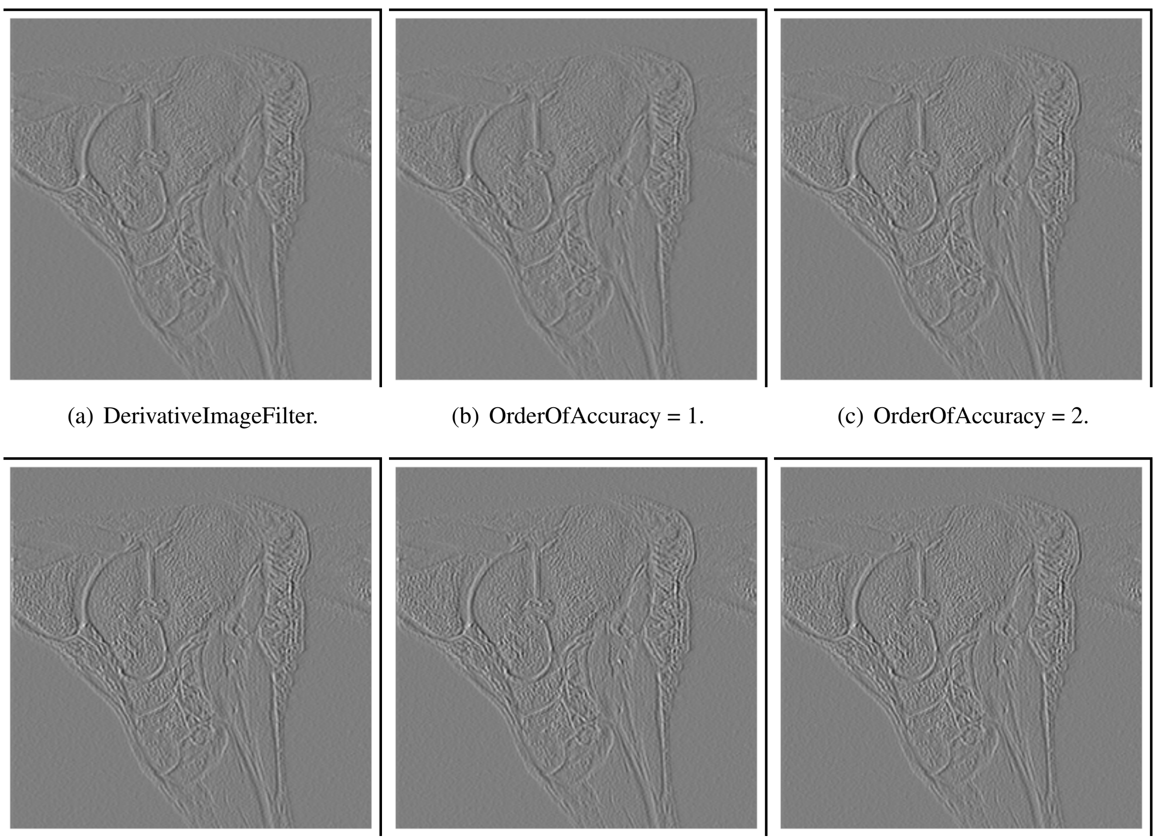


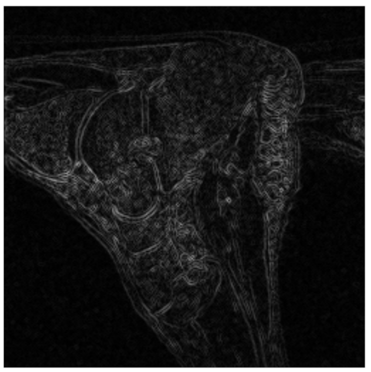
Figure 1: Foot magnetic resonance image (MRI) from which the derivative images in Figure 2 and the gradient images in Figure 3 are generated. This image was taken from the ITK *Testing/Data/Input* directory.



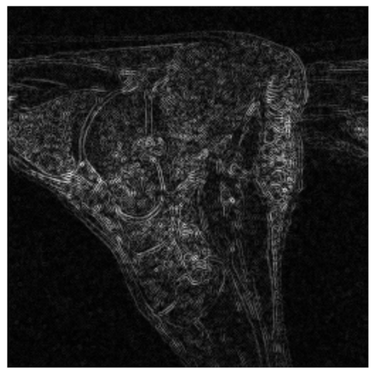
(a) DerivativeImageFilter. (b) OrderOfAccuracy = 1. (c) OrderOfAccuracy = 2.

(d) OrderOfAccuracy = 3. (e) OrderOfAccuracy = 4. (f) OrderOfAccuracy = 5.

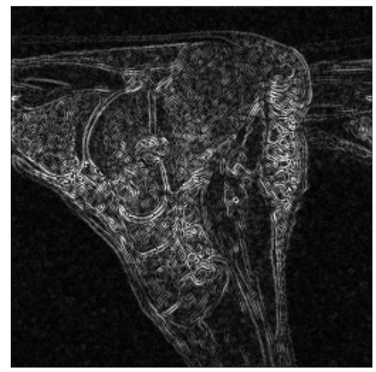
Figure 2: Derivative along Direction 0 in the foot MRI image. The brightness in all the derivative images ranges from -110, black, to 110, white. Figure 2(a) was created with the `DerivativeImageFilter` while Figure 2(b) to 2(f) were created with the `itk::HigherOrderAccurateDerivativeImageFilter` with the `OrderOfAccuracy` set from 1 to 5.



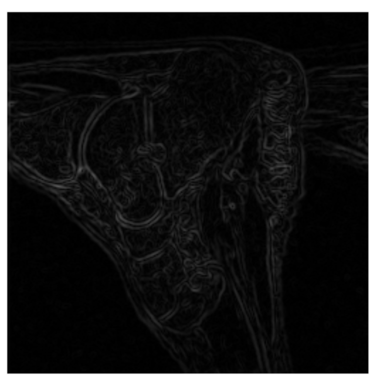
(a) GradientImageFilter.



(b) OrderOfAccuracy = 5.



(c) DoG.



(d) Recursive Gaussian.

Figure 3: Gradient magnitude of Figure 1 with different filters. The brightness in all the gradient images ranges from 0.0, black, to 128.0, white. Figure 3(a) is the `itk::GradientImageFilter`. Figure 3(b) is the `itk::HigherOrderAccurateGradientImageFilter` with order of accuracy set to 5. Figure 3(c) is the `itk::DifferenceOfGaussiansGradientImageFilter` with a width of 1.0. Figure 3(d) is the `itk::GradientRecursiveGaussianImageFilter` with a sigma of 1.0. The spacing in the image is isotropic with a value of 1.0.

3 An Example – Foot MRI Derivative and Gradient Magnitude

Figure 2 and Figure 3 demonstrate the behavior of the filter on an MRI foot image. Note that if the `OrderOfAccuracy` is set to one, `itk::HigherOrderAccurateDerivativeImageFilter` is equivalent to `itk::DerivativeImageFilter` and `itk::HigherOrderAccurateGradientImageFilter` is equivalent to `itk::GradientImageFilter`. The source code to generate these images can be found in the `Testing` subdirectory.

In general, we see the higher order accuracy results in lower noise, higher contrast and higher dynamic range results. Unlike the difference of Gaussians or recursive Gaussian filters, though, it does not remove high frequency content, which is often primarily noise. Choice of the most appropriate filter depends on the application.

4 Future Work

At this time, only the first order derivative is implemented. In the future, higher order derivatives should be implemented.

5 Acknowledgments

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References

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[2] Khan, IR and Ohba, Ryoji. Taylor series based finite difference approximations of higher-degree derivatives. *Journal of Computational and Applied Mathematics*, 154:115 – 124, 2003. URL http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.149.7922\&rep=rep1\&type=pdf. 1

[3] The Insight Segmentation and Registration Toolkit. URL http://www.itk.org/. 1

[1] Here we use the terminology *order of accuracy* to refer to the number of terms used in the Taylor series approximation and *order derivative* to refer to the degree of the derivative.