Using the Fundamental Theorem of Linear Algebra

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March 8, 2017

1 Application

Previously, I went over the Fundamental Theorem of Linear Algebra and the relationships it describes between the different spaces. One of the key ideas of the theorem is orthogonality, which is a very simple but powerful tool when applying linear algebra to real problems. This article will show some ways the theorem and orthogonality can be used to your advantage.

2 A simple example

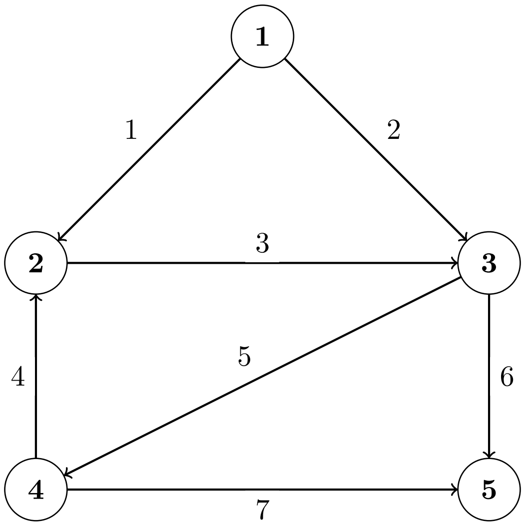
Consider the task of determining if a given column vector $\boldsymbol{b}$ is in the column space of A. One method of figuring this out would be to attempt to solve $\boldsymbol{Ax} = \boldsymbol{b}$ and see if a solution exists. This is a valid approach, but what if you were given multiple $\boldsymbol{b}$'s? You would not want to solve the whole system for each $\boldsymbol{b}$.

With orthogonality, you can easily determine the answer to this question for any $\boldsymbol{b}$. Recall that $\boldsymbol{C(A)} \perp \boldsymbol{N(A^T)}$ by the Fundamental Theorem of Linear Algebra. We know that any vector $\boldsymbol{b}$ that is in $\boldsymbol{C(A)}$ must be orthogonal to any vector in $\boldsymbol{N(A^T)}$. In other words, if $\boldsymbol{b}$ is orthogonal to the basis vectors that make up the left nullspace, then it must be in the column space. All that is required is the complete solution to the left nullspace of A, which is easily obtained via elimination.

The key realization here is that you can make determinations about *all* vectors $\boldsymbol{b}$ through properties of $\boldsymbol{A}$ alone. There is no need to have a particular $\boldsymbol{b}$ to solve for.

3 Circuit design

The previous problem is, perhaps, too much of a toy example to show the importance of orthogonality. A better example is circuit design (and more broadly, network flow). In circuit design, you may wish to take a set of nodes (junctions) with some description of their connectivity and determine voltages to achieve a desired current.



$$\boldsymbol{A} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Above is a circuit with five nodes and seven directed edges which describe how the nodes are connected to each other. $\boldsymbol{A}$ is the adjacency matrix where each row i corresponds to the edge labeled i. The column j represents node j and the entries -1 and 1 in a row indicate the edge begins and ends, respectively, at the corresponding nodes. In the problem $\boldsymbol{Ax} = \boldsymbol{b}$, $\boldsymbol{x}$ represents the voltage at each node and $\boldsymbol{b}$ is the voltage difference across each edge.

$$\boldsymbol{Ax} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \boldsymbol{x_3} \\ \boldsymbol{x_4} \\ \boldsymbol{x_5} \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_1 \\ x_3 - x_2 \\ x_2 - x_4 \\ x_4 - x_3 \\ x_5 - x_3 \\ x_5 - x_4 \end{bmatrix} = \boldsymbol{b}$$

If you wanted to find the set of all voltages which have a zero voltage difference, you simply solve for $\boldsymbol{Ax} = \boldsymbol{0}$, the nullspace! In this matrix, we have a relatively boring nullspace solution of (v, v, v, v, v), which says that if all nodes have the same voltage, no voltage difference is present.

$$\boldsymbol{A^T} = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$\boldsymbol{A^T}$ shows a different view where each row now represents a node and the columns are edges. Before, $\boldsymbol{Ax} = \boldsymbol{b}$ described something about edges (the voltage difference), but $\boldsymbol{A^T y} = \boldsymbol{c}$ is something else:

$$\boldsymbol{A^T y} = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{y_1} \\ \boldsymbol{y_2} \\ \boldsymbol{y_3} \\ \boldsymbol{y_4} \\ \boldsymbol{y_5} \\ \boldsymbol{y_6} \\ \boldsymbol{y_7} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{y_1} - \boldsymbol{y_2} \\ \boldsymbol{y_1} + \boldsymbol{y_4} - \boldsymbol{y_3} \\ \boldsymbol{y_2} + \boldsymbol{y_3} - \boldsymbol{y_5} - \boldsymbol{y_6} \\ \boldsymbol{y_5} - \boldsymbol{y_4} - \boldsymbol{y_7} \\ \boldsymbol{y_6} + \boldsymbol{y_7} \end{bmatrix} = \boldsymbol{c}$$

The vector $\boldsymbol{y}$ represents current along the edges and $\boldsymbol{A^T y}$ is the *net current* (or flow) on each node. What happens when $\boldsymbol{c}$ is the zero vector? It describes *Kirchhoff's current law*, which says that for every node, current in must equal current out (or net current is zero). The left nullspace provides the perfect description of all the currents $\boldsymbol{y}$ that satisfy this law. For this fact to be useful to us, we need to combine it with $\boldsymbol{Ax}$:

$$\boldsymbol{A^T A x} = \boldsymbol{0}$$

This equation ties together the voltages of the nodes to the currents between the nodes and ensures that it satisfies Kirchhoff's current law. [1] It may not be clear why this is necessary, but the Fundamental Theorem of Linear Algebra provides an explanation. $\boldsymbol{Ax}$ produces a vector $\boldsymbol{b}$ that is in the column space, which by the theorem is perpendicular to the left nullspace. Doesn't this mean that all $\boldsymbol{b}$'s satisfy Kirchhoff's current law?

No! The theorem only says $\boldsymbol{y^T b} = \boldsymbol{0}$ if $\boldsymbol{b}$ is in the column space and $\boldsymbol{y}$ is in the left nullspace of A; it says nothing about what $\boldsymbol{A^T y}$ (or $\boldsymbol{A^T A x}$) will be! [2] So, $\boldsymbol{b}$ must be in both the column space and left nullspace of A, which necessitates that $\boldsymbol{x}$ come from the nullspace of A (only the zero vector can be in both the column space and the left nullspace).

References

[1] Gilbert Strang. *Introduction to Linear Algebra, 4th Edition.* Welleslely - Cambridge Press, Wellesley, Massachusetts, 2009.

[1] This is actually a less general form of $\boldsymbol{A^T C A x} = \boldsymbol{f}$, where $\boldsymbol{C}$ is the diagonal "conductance" matrix and $\boldsymbol{f}$ is the external current applied to the circuit.

[2] Try $\boldsymbol{x} = (1, 2, 3, 4, 5)$.