Convergence Rates of Characteristic Functions

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Abstract

This is a small note on the rate of convergence in the quantum central limit theorem. So far it only contains the 0-local and product state case, so in a sense the "classical" case. The more general, "truly quantum" case, will be added later.

1 Setting

We let $\mathcal{X}$ a collection of lattices sites equipped with a distance $d : \mathcal{X} \times \mathcal{X} \to \mathbb{N}$ and consider observables of the form

$$\hat{X} = \sum_{i\in\mathcal{X}} \hat{X}_i \tag{1}$$

with $\hat{X}_i$ hermitian and R-local, i.e., it acts only on sites $\{j \in \mathcal{X} \,|\, d(i,j) \le R\}$. For a given state $\hat{\varrho}$ we write $\langle\cdot\rangle = \mathrm{tr}[\hat{\varrho}\,\cdot]$ and

$$\mu = \langle\hat{X}\rangle, \quad \sigma = \langle(\hat{X}-\mu)^2\rangle. \tag{2}$$

We will be concerned with the characteristic function $\phi : \mathbb{R} \to \mathbb{C}$,

$$\phi(t) = \langle \mathrm{e}^{\mathrm{i}t\hat{X}}\rangle, \tag{3}$$

and its distance to the corresponding Gaussian characteristic function,

$$\Delta(t) = |\phi(t) - \mathrm{e}^{\mathrm{i}\mu t - \sigma^2 t^2/2}|. \tag{4}$$

We will write $|\mathcal{X}| = N$.

2 Product States and 0-local Observables

For 0-local observables and product states we have the following.

Theorem 1 *Let $R = 0$ and $\hat{\varrho} = \otimes_{i\in\mathcal{X}}\hat{\varrho}_i$. If*

$$|t| \sum_{i\in\mathcal{X}} \langle(\hat{X}_i - \langle\hat{X}_i\rangle)^2\rangle^{1/2} \langle(\hat{X}_i-\mu_i)^4\rangle^{1/2} \le \frac{3}{5}\sigma^3 \tag{5}$$

then

$$\Delta(t/\sigma) \le \frac{5}{12}\mathrm{e}^{-t^2/4}|t|^3 \frac{\sum_{i\in\mathcal{X}} \langle(\hat{X}_i - \langle\hat{X}_i\rangle)^2\rangle^{1/2} \langle(\hat{X}_i-\mu_i)^4\rangle^{1/2}}{\sigma^3}. \tag{6}$$

Remark 1: Eq. (6) implies

$$\Delta(t/\sigma) \le \frac{5}{12}\mathrm{e}^{-t^2/4}|t|^3 \frac{\sqrt{\sum_{i\in\mathcal{X}} \langle(\hat{X}_i-\mu_i)^4\rangle}}{\sigma^2}. \tag{7}$$

Remark 2: If $\langle(\hat{X}_i - \langle\hat{X}_i\rangle)^2\rangle = s^2$ and $\langle(\hat{X}_i-\mu_i)^4\rangle^{1/2} = \beta$, one recovers the familiar i.i.d. case,

$$\Delta(t/\sigma) \le \frac{5\beta}{12 s^2}\mathrm{e}^{-t^2/4}|t|^3 \frac{1}{\sqrt{N}} \quad \text{if} \quad |t| \le \frac{3s^2}{5\beta}N^{1/2}. \tag{8}$$

Remark 3: If the $\hat{X}_i$ are bounded, $\|\hat{X}_i\| \le x$ then

$$\Delta(t/\sigma) \le \frac{5}{12} \mathrm{e}^{-t^2/4} |t|^3 \left(\frac{x^2 N}{\sigma^2}\right)^{3/2} \frac{1}{\sqrt{N}} \quad \text{if} \quad |t| \le \frac{3}{5} \left(\frac{\sigma^2}{x^2 N}\right)^{3/2} \sqrt{N}. \tag{9}$$

The proof of Theorem 1 is completely analogous to the classical case, see, e.g., pretty much any book on probability theory. We give it for completeness.

Proof. For $i \in \mathcal{X}$ denote

$$\mu_i = \langle \hat{X}_i \rangle, \quad \sigma_i^2 = \langle (\hat{X}_i - \mu_i)^2 \rangle. \tag{10}$$

As $\hat{\varrho}$ is a product state, we have

$$\sigma^2 = \langle (\hat{X} - \mu)^2 \rangle = \sum_{i,j \in \mathcal{X}} \langle (\hat{X}_i - \mu_i)(\hat{X}_j - \mu_j) \rangle = \sum_{i \in \mathcal{X}} \sigma_i^2 \tag{11}$$

and

$$\Delta(t) = \mathrm{e}^{-\sigma^2 t^2/2} \Big| \mathrm{e}^{\sigma^2 t^2/2} \prod_{i \in \mathcal{X}} \langle \mathrm{e}^{\mathrm{i}t(\hat{X}_i - \mu_i)} \rangle - 1 \Big|. \tag{12}$$

Now,

$$\langle \mathrm{e}^{\mathrm{i}t(\hat{X}_i - \mu_i)} \rangle - 1 = -\int_0^t \mathrm{d}x \int_0^x \mathrm{d}y \, \big\langle (\hat{X}_i - \mu_i) \mathrm{e}^{\mathrm{i}t(\hat{X}_i - \mu_i)} (\hat{X}_i - \mu_i) \big\rangle. \tag{13}$$

By the Cauchy–Schwarz inequality

$$|\langle \hat{A}\hat{B} \rangle|^2 = |\mathrm{tr}[\hat{B}\sqrt{\hat{\varrho}}\sqrt{\hat{\varrho}}\hat{A}]|^2 \le \mathrm{tr}[\hat{B}\hat{\varrho}\hat{B}^\dagger] \mathrm{tr}[\hat{A}^\dagger \hat{\varrho} \hat{A}] = \langle \hat{A}\hat{A}^\dagger \rangle \langle \hat{B}^\dagger \hat{B} \rangle \tag{14}$$

we have

$$|\langle \mathrm{e}^{\mathrm{i}t(\hat{X}_i - \mu_i)} \rangle - 1| \le \sigma_i^2 t^2/2 \le 1/2, \tag{15}$$

where the second inequality is implied by Eq. (5).[1] Hence we may use the principle branch of the logarithm to write

$$\begin{aligned} \Delta(t) &= \mathrm{e}^{-\sigma^2 t^2/2} \Big| \mathrm{e}^{\sigma^2 t^2/2} \mathrm{e}^{\sum_{i \in \mathcal{X}} \log \langle \mathrm{e}^{\mathrm{i}t(\hat{X}_i - \mu_i)} \rangle} - 1 \Big| \\ &= \mathrm{e}^{-\sigma^2 t^2/2} \Big| \exp\Big(\sum_{i \in \mathcal{X}} \big[\sigma_i^2 t^2/2 + \log \langle \mathrm{e}^{\mathrm{i}t(\hat{X}_i - \mu_i)} \rangle \big] \Big) - 1 \Big| \\ &\le \mathrm{e}^{F(t) - \sigma^2 t^2/2} F(t), \end{aligned} \tag{16}$$

where $F(t) = \sum_i F_i(t)$ with

$$\begin{aligned} F_i(t) &= \Big| \frac{\sigma_i^2 t^2}{2} + \log \langle \mathrm{e}^{\mathrm{i}t(\hat{X}_i - \mu_i)} \rangle \Big| \\ &\le \Big| \langle \mathrm{e}^{\mathrm{i}t(\hat{X}_i - \mu_i)} \rangle - 1 + \frac{\sigma_i^2 t^2}{2} \Big| + \Big| \langle \mathrm{e}^{\mathrm{i}t(\hat{X}_i - \mu_i)} \rangle - 1 - \log \langle \mathrm{e}^{\mathrm{i}t(\hat{X}_i - \mu_i)} \rangle \Big| \\ &=: G_i(t) + H_i(t), \end{aligned} \tag{17}$$

where we inserted a zero and used the triangle inequality. For the first term we have

$$G_i(t) = \Big| \int_0^t \mathrm{d}x \int_0^x \mathrm{d}y \int_0^y \mathrm{d}z \, \langle (\hat{X}_i - \mu_i)^2 \mathrm{e}^{\mathrm{i}x(\hat{X}_i - \mu_i)} (\hat{X}_i - \mu_i) \rangle \Big| \tag{18}$$

such that by Eq. (14)

$$\begin{aligned} G_i(t) &\le \int_0^t \mathrm{d}x \int_0^x \mathrm{d}y \int_0^y \mathrm{d}z \, \sqrt{\langle (\hat{X}_i - \mu_i)^4 \rangle \langle (\hat{X}_i - \mu_i)^2 \rangle} \\ &= \sigma_i \langle (\hat{X}_i - \mu_i)^4 \rangle^{1/2} |t|^3/6. \end{aligned} \tag{19}$$

[1] $\sigma_i|t| > 1$ implies the absurdity $\frac{3}{5}\sum_{i \in \mathcal{X}} \langle (\hat{X}_i - \mu_i)^4 \rangle^{1/2} \ge \frac{3}{5}\sigma^2 > \sum_{i \in \mathcal{X}} \langle (\hat{X}_i - \mu_i)^4 \rangle^{1/2}$.

For the second term we use the Mercator series and Eq. (15) to find

$$
\begin{aligned}
H_i(t) &\leq \sum_{n=2}^{\infty} \frac{1}{n} \left| \langle \mathrm{e}^{\mathrm{i}t(\hat{X}_i - \mu_i)} \rangle - 1 \right|^n \leq \sum_{n=2}^{\infty} \frac{1}{n} \left(\frac{\sigma_i^2 t^2}{2} \right)^n \\
&= \frac{\sigma_i^4 t^4}{4} \sum_{n=0}^{\infty} \frac{1}{n+2} \left(\frac{\sigma_i^2 t^2}{2} \right)^n \\
&\leq \frac{\sigma_i^4 t^4}{4} \sum_{n=0}^{\infty} \frac{1}{n+2} 2^{-n} \\
&\leq \frac{\sigma_i^4 t^4}{4} .
\end{aligned} \tag{20}
$$

Hence,

$$
\begin{aligned}
F(t) &\leq \frac{|t|^3}{6} \sum_{i \in \mathcal{X}} \sigma_i \langle (\hat{X}_i - \mu_i)^4 \rangle^{1/2} + \frac{t^4}{4} \sum_{i \in \mathcal{X}} \sigma_i^4 \\
&\leq \frac{|t|^3}{6} \sum_{i \in \mathcal{X}} \sigma_i \langle (\hat{X}_i - \mu_i)^4 \rangle^{1/2} + \frac{t^4}{4} \sum_{i \in \mathcal{X}} \sigma_i^2 \langle (\hat{X}_i - \mu_i)^4 \rangle^{1/2} \\
&\leq \frac{5}{12} |t|^3 \sum_{i \in \mathcal{X}} \sigma_i \langle (\hat{X}_i - \mu_i)^4 \rangle^{1/2} \\
&\leq \sigma^2 \frac{t^2}{4} ,
\end{aligned} \tag{21}
$$

where we used $\sigma_i^2 \leq \langle (\hat{X}_i - \mu_i)^4 \rangle^{1/2}$ (which follows from Eq. (14)) to obtain the second line, Eq. (15) to obtain the third line, and Eq. (5) to obtain the last line. Thus

$$
\Delta(t) \leq \mathrm{e}^{-\sigma^2 t^2/4} F(t) \leq \frac{5}{12} \mathrm{e}^{-\sigma^2 t^2/4} |t|^3 \sum_{i \in \mathcal{X}} \sigma_i \langle (\hat{X}_i - \mu_i)^4 \rangle^{1/2} \tag{22}
$$

and Eq. (6) follows by using the bound in the third line of Eq. (21). $\square$