Notes

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28 January 2026

1 Measure Theory

§1.1 Measure Spaces

Definition 1.1.1. An *algebra* Σ_0 on a set S is a collection of subsets of S such that

- $S \in \Sigma_0$.
- if $F \in \Sigma_0$, then $F^C := S \setminus F \in \Sigma_0$.
- Σ_0 is closed under finite unions.

Notice that this implies that Σ_0 must also be closed under finite intersections.

Definition 1.1.2. A *σ-algebra* Σ is an algebra closed under countably many unions (and thus intersections). Then, the pair (S, Σ) is a *measurable space.*

Likewise with how bases generate topologies, we may *generate* an algebra from a class of subsets:

Definition 1.1.3. Let $\mathcal{C}$ be a class of subsets of S. Then, let the σ-algebra *generated* by $\mathcal{C}$ be the intersection of all σ-algebras on S which superset $\mathcal{C}$.

Example 1.1.4

The $\mathcal{B}(S)$ *Borel σ-algebra* on topological space S, is the σ-algebra generated by the open sets of S.

To turn a measurable space into a measure space, we must assign a measure.

Definition 1.1.5. Let $\mu_0 : \Sigma_0 \to \mathbb{R}_{\geq 0}$. Then, μ_0 is *additive* if for any disjoint sets $F, G \in \Sigma_0$, we have that

$$\mu_0(F \cup G) = \mu_0(F) + \mu_0(G).$$

Likewise, denote μ_0 to be *countably additive* if the additive property applies for countably many added sets.

Definition 1.1.6. A measurable space endowed with a countably additive set function is called a *measure space.*