Notes

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1 Measure Theory

§1.1 Measure Spaces

Definition 1.1.1. An *algebra* Σ_0 on a set S is a collection of subsets of S such that

- $S \in \Sigma_0$.
- if $F \in \Sigma_0$, then $F^C := S \setminus F \in \Sigma_0$.
- Σ_0 is closed under finite unions.

Notice that this implies that Σ_0 must also be closed under finite intersections.

Definition 1.1.2. A *σ-algebra* Σ is an algebra closed under countably many unions (and thus intersections). Then, the pair (S, Σ) is a *measurable space*.

Likewise with how bases generate topologies, we may *generate* an algebra from a class of subsets:

Definition 1.1.3. Let $\mathcal{C}$ be a class of subsets of S. Then, let the σ-algebra *generated* by $\mathcal{C}$ be the intersection of all σ-algebras on S which superset $\mathcal{C}$.

Example 1.1.4

The *Borel σ-algebra* $\mathcal{B}(S)$ on topological space S, is the σ-algebra generated by the open sets of S.

We also denote $\mathcal{B}(\mathbb{R}) = \mathcal{B}$.

To turn a measurable space into a measure space, we must assign a measure.

Definition 1.1.5. Let $\mu_0 : \Sigma_0 \to \mathbb{R}_{\geq 0}$. Then, μ_0 is *additive* if for any disjoint sets $F, G \in \Sigma_0$, we have that

$$\mu_0(F \cup G) = \mu_0(F) + \mu_0(G).$$

Likewise, denote μ_0 to be *countably additive* if the additive property applies for countably many added sets.

Definition 1.1.6. A measurable space endowed with a countably additive set function μ, denoted a *measure*, is called a *measure space*. If $\mu(S) = 1$, then the measure is a *probability measure*.

We have several elementary inequalities which we may be familiar with, namely:

- $\mu(A \cup B) \leq \mu(A) + \mu(B)$.

- $\mu(\bigcup S_i) \leq \sum \mu(S_i)$ for finitely many S_i.

Furthermore, if $\mu(S)$ is finite, then:

- $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$.
- principle of inclusion-exclusion for finitely many S_i.

Proof of the principle of inclusion-exclusion is kind of boring and just induction to be honest.

As a result, we have the following somewhat obvious fact:

Theorem 1.1.7
Let $F_n \in \Sigma$ for $n \in \mathbb{N}$, and let $F_n \subseteq F_{n+1}$ for all n and $\bigcup F_n = F$.

Then, $\mu(F_n)$ approaches $\mu(F)$.

Proof. Let G_n be defined so that $G_1 := F_1$ and $G_n := G_n \setminus G_{n-1}$ for $n \geq 2$. Then,

$$\mu(F_n) = \mu\left(\bigcup_{i \leq n} G_i\right) = \sum_{i \leq n} \mu(G_i),$$

hence

$$\lim_{n \to \infty} \mu(F_n) = \sum \mu(G_n) = \mu(F).$$

$\square$

Similarly, if $F_n \supseteq F_{n+1}$ for all n and $\bigcap F_n = F$, then $\mu(F_n)$ also approaches $\mu(F)$.

Lemma 1.1.8 (Uniqueness of Extension)
Suppose S is a set, and let $\mathcal{I}$ be a π-*system*, or a collection of subsets of S closed under finite intersection.

Let $\Sigma := \sigma(\mathcal{I})$, and suppose μ_1 and μ_2 are measures on (S, Σ) such that $\mu_1(S) = \mu_2(S) < \infty$ and for all $X \in \mathcal{I}$ that

$$\mu_1(X) = \mu_2(X).$$

Then, for all $X \in \Sigma$, we have that

$$\mu_1(X) = \mu_2(X).$$

Theorem 1.1.9 (Carathedory's Extension Theorem)
Let S be a set and Σ_0 an algebra on S. Define $\Sigma := \sigma(\Sigma_0)$. Then, if $\mu_0 : \Sigma_0 \to [0, \infty)$ is a countably additive map, then there exists a corresponding measure μ on Σ such that for all $X \in \Sigma_0$ that

$$\mu(X) = \mu_0(X).$$

If $\mu_0(S)$ is finite, then this extension is unique.

Example 1.1.10 (Lebesgue Measure)
Let $S = (0, 1]$, and for $F \subseteq S$ that $F \in \Sigma_0$ if

$$F = (a_1, b_1] \cup \dots (a_n, b_n]$$

for finite n and $0 \leq a_1 \leq b_1 \leq \dots \leq b_n \leq 1$.

Then, Σ_0 is an algebra on S and can be uniquely extended to the measure space $\mathcal{B}(0, 1]$ with measure μ.

If we additionally denote $\{0\}$ to have length 0, then we have the *Lebesgue measure* on $\mathcal{B}[0, 1]$, which we commonly denote as length.

§1.2 Probability Theory

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the *sample space* of *sample points* $\omega \in \Omega$, and $\mathcal{F}$ is the collection of *events*. Then, we call the *probability* of an event $F \in \mathcal{F}$ occurring to be $\mathbb{P}(F)$.

For example, consider the probability space of infinitely many coin flips. We'd like to show that the number of heads over the number of flips almost surely approaches $\frac{1}{2}$.

Let our sample space be

$$\Omega := \{H, T\}^{\mathbb{N}},$$

so that for each $\omega \in \Omega$, we have that ω is some infinite sequence with members in $\{H, T\}$ as such:

$$\omega = \{\omega_1, \dots\}, \quad \omega_n \in \{H, T\}.$$

Then, we will strategically construct $\mathcal{F}$. Consider the set of all sample points with the nth flip fixed to be heads:

$$F_n = \{\omega \mid \omega_n = H\}.$$

Then, we take

$$\mathcal{F} = \sigma(F_n).$$

§1.3 Random Variables

Definition 1.3.1. A function $f : (X, \Sigma) \to (Y, T)$ is a *measurable function* if for each $F \in T$ that

$$f^{-1}(T) \in \Sigma.$$

If $(Y, T) = \mathcal{B}$, then we will denote our function Σ-*measurable*.

Finally, denote the class of Σ-measurable functions as $m\Sigma$.