Topological Ordering of a List of Randomly-Numbered Elements of a Network

DANIEL J. LASSER, *Missiles and Space Div., Lockheed Aircraft Corp., Sunnyvale, Calif.*

Abstract. A network of directed line segments free of circular elements is assumed. The lines are identified by their terminal nodes and the nodes are assumed to be numbered by a non-topological system. Given a list of these lines in numeric order, a simple technique can be used to create at high speed a list in topological order.

The topological ordering problem, which could arise in any situation requiring the reduction of a network to a list, was encountered in the analysis of PERT networks. The following technique is not the first solution to the problem [1].

PERT (Project Evaluation Review Technique) is a Navy reporting system (the Air Force counterpart is called PEP) in which a major project of developmental work is broken down into discrete quanta of work known as "activities".

If a pair of activities, one of which must precede the other, is depicted as a pair of contiguous lines, then a major developmental project will be represented by a network. Time-sequence analysis requires that this network be reduced to a topologically ordered list.

Some users of the PERT method number the nodes in the network in such a way that a topologically ordered list is generated by numerically ordering a list of the lines. Other organizations, including Lockheed MSD, believe that reduction of networks by computer should permit the nodes to be randomly numbered with respect to topology.

The assumed network (see Fig. 1) consists of directed line segments originating and terminating at intersection points. The segments are named by their limiting points, and although these points are randomly numbered with respect to topology, each point has a unique number. Given a list of these lines in numeric order, the technique consists of maintaining the list in storage in numeric order, while constructing a "threaded list" in topological order.

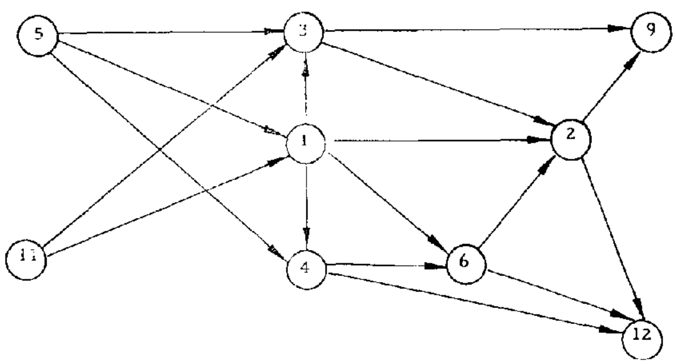

FIG. 1. A small network

Some definitions are necessary. A line segment originates at a predecessor point E_p and terminates at a successor point E_s. In Figure 1, line 5-1 is the direct predecessor of line 1-6 and the indirect predecessor of line 6-2. Conversely, line 6-2 is the direct successor of line 1-6 and the indirect successor of line 5-1. The terms *predecessor* and *sucessor* will be used only in the topological sense. The input list is in order by successor point number within predecessor point number (E_p-E_s).

The rigors of topological order are:

1. All lines having the same predecessor point [each $(\text{set})_p$] will be listed contiguously.
2. Each line will be listed before all of its (topological) successors.

The memory is conceptually divided into three areas:

1. The Numeric Area, containing lines in the numeric order of the input list.
2. The Logical Area, containing lines in topological order as defined above.
3. The Terminal Area, containing in topological order lines none of which have successors in the Logical Area or the Numeric Area.

Initially all lines are in the Numeric Area, which looks like Figure 2. The method is as follows:

1. The physically (not topologically) first $(\text{set})_p$ in the Numeric Area is moved to the Logical Area. (Initially this would be the set 1-2, 1-3, 1-4, 1-6.)
2. The Numeric Area is searched for a $(\text{set})_p$ having an E_p equal to the E_s of the first line in the Logical Area. If such a set (initially 2-9, 2-12) is found, it is moved to the first vacant position in the Logical Area.
3. Taking in turn as a search argument the E_s of every line in the Logical Area, the Numeric and the Logical Areas are searched for $(\text{sets})_p$. If such a $(\text{set})_p$ is found in the Numeric Area, it is moved to the first vacant positions in the Logical Area. If such a $(\text{set})_p$ is found in the Logical Area, but not just

1–2
1–3
1–4
1–6
2–9
2–12
3–2
3–9
4–6
4–12
5–1
5–3
5–4
6–2
6–12
11–1
11–3

FIG. 2. Numeric area at start of routine

before the first vacant position, it is removed from its position, and all lines after it in the Logical Area are "moved up" to close the gap. The found set is now placed in the first vacant position in the Logical Area. If a $(\text{set})_p$ is found in the Logical Area just before the first vacant position it is not moved.

4. The search continues until, its E_s having been used as a search argument, the last line in the Logical Area remains last. At this point no line in the Logical Area has a successor in the Numeric Area.
5. The entire contents of the Logical Area are now moved into the last vacant positions of the Terminal Area. The memory now looks like Figure 3. (Note that 5-1 is now the first activity in the Numeric Area.)
6. Steps 1-5 are repeated until all lines are in the Terminal Area. Each time the contents of the Logical Area are moved to the Terminal Area they are placed in the last vacant positions. Their internal ordering, however, is not changed or reversed. At this point the Terminal Area reads "11-1, 11-3, 5-1, 5-3, 5-4, 1-2, 1-3, 1-4, 1-6, 3-2, 3-9, 4-6, 4-12, 6-2, 6-12, 2-9, 2-12." This constitutes topological order according to the stated rigors.

Although the technique has been described in terms of "areas" and "moves", these terms are used in the conceptual sense only. Physically the memory consists only of the Numeric Area, in which any $(\text{set})_p$ can be found quickly by binary search or its non-existence can be proved.

The Logical Area and the Terminal Area consist of "threaded lists" which are constructed in the Numeric Area by representing each line by three words of storage. The first word contains the E_p, the second the E_s. The third word contains two memory adresses: the address A_p of the preceding member of the threaded list and the address A_s of the following member of the threaded list. To move a line from area to area or within an area consists then of "cutting and splicing" the threads. Physically this means simply changing the contents of the A_s and A_p portions of those members of the list that are next to the cuts or splices. If a circular network (a network in which a line is its own direct or indirect successor) is, through input error, a substantial possibility, it can be isolated by counting the number of times that each line is moved. If there are M lines in a network, moving any line $M+1$ times proves that the line is a member of a circular network. In the IBM 7090, for which the technique was developed, the three words assigned to each line have sufficient capacity to accommodate this move count in addition to E_p, E_s, A_p and A_s.

By use of this technique, a computer of the memory size of the IBM 7090 (32768 36-bit binary words) can reduce a network of 10,000 lines. A sample running time is 30 seconds for 2,000 lines. Networks larger than 10,000 activities can probably be reduced at medium speed by the use of magnetic tape as an extension of addressable storage.

Numeric Area	Logical Area	Terminal Area
5–1		1–2
5–3		1–3
5–4		1–4
11–1		1–6
11–3		3–2
		3–9
		4–6
		4–12
		6–2
		6–12
		2–9
		2–12

FIG. 3. Positions after first execution of steps 1–5

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Eigenvalues of a Symmetric 3 × 3 Matrix

OLIVER K. SMITH, *Space Technology Laboratories, Los Angeles, Calif.*

Recently, in order to find the principal moments of inertia of a large number of rigid bodies, it was necessary to compute the eigenvalues of many real, symmetric 3×3 matrices. The available eigenvalue subroutines seemed rather heavy weapons to turn upon this little problem, so an explicit solution was developed. The resulting expressions are remarkably simple and neat, hence this note.

Let A be a real, symmetric 3×3 matrix, $3m = \operatorname{tr}(A)$, $2q = \det(A - mI)$, and let $6p$ equal the sum of squares of elements of $(A - mI)$. Then, from "Cardano's" trigonometric solution of $\det[(A - mI) - \mu I]$ as a cubic in μ, we find that the eigenvalues of A are

$$m + 2\sqrt{p}\cos\phi$$
$$m - \sqrt{p}\,(\cos\phi + \sqrt{3}\sin\phi)$$
$$m - \sqrt{p}\,(\cos\phi - \sqrt{3}\sin\phi)$$

where

$$\phi = \tfrac{1}{3}\tan^{-1}\frac{\sqrt{p^3 - q^2}}{q},\ 0 \leqq \phi \leqq \pi.$$

Since the eigenvalues are real, p^3 cannot be less than q^2. Of course, if $p^3 = q^2$, as is the case when A has two equal eigenvalues, round-off may cause the occurrence of a small negative value of $p^3 - q^2$. If $p = q = 0$, the argument of the arctangent becomes indeterminate, but correct eigenvalues (all equal to m) result whatever value ϕ is given.

Only moderate accuracy was required in the application for which these results were originally intended. Hence there has been no investigation of such details as the effects of loss of significance in the subtraction $p^3 - q^2$.