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On the comparison of the Spearman and Kendall metrics between linear orders

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Abstract

This paper bears on the comparison of two well-known metrics between linear orders called the *Kendall* and *Spearman metrics* or/and of their normalized versions, respectively, known as the *Kendall tau* and the *Spearman rho*. Using a combinatorial approach based on the partial order intersection of the two compared linear orders, one first proves a relation between these two metrics and a semi-metric, equivalent to the classical Daniels inequality (1948) and to a Guilbaud formula (1980). Then this approach allows to express the difference tau–rho as a simple function of parameters of this same partial order, to compute the maximum value of this difference and to characterize the corresponding pairs of linear orders. Finally, it also leads to discover an ordinal monotonicity property of the Spearman metric. © 1998 Published by Elsevier Science B.V. All rights reserved

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1. Introduction

In several fields (Social Sciences, Statistics, etc.) the problem of comparing two linear orders led to define what has been called *concordance* (or '*correlation*') *coefficients* between two linear orders. The two most used such coefficients are the *Kendall tau* and the *Spearman rho*. Although these coefficients have several alternative definitions, they basically are normalizations, between the values -1 and $+1$, of metrics between linear orders. The first one is a normalization of the *Kendall metric*, i.e. the half of the *symmetric difference distance* between two linear orders, and the second, a normalization of the square of the *Spearman metric*, i.e. the *euclidean distance* between the 'ranks vectors' associated with these linear orders. The comparison of the coefficients tau and rho is an old problem, especially considered by Daniels [2] and Guilbaud [8]. Daniels proved an inequality between tau and rho. Guilbaud introduced

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a third coefficient *sigma* and showed the existence of a linear relation between tau, rho and sigma equivalent with the Daniels inequality. It is not difficult to see that sigma can also be defined as the normalization of a quantity which we call the *Guilbaud metric* (although in fact it is only a semi-metric). The Daniels and Guilbaud proofs rely on euclidean codings of linear orders (especially Guilbaud uses a spectral analysis of the matrices coding all linear orders).

In this paper we use a combinatorial approach based on the partial order $L \cap L'$, intersection of L and L', the two compared linear orders. In Section 2 we give our notation and we recall some basic notions on relations and partial orders. In Section 3 our approach allows to prove a relation between the Spearman, Kendall and Guilbaud metrics, which is equivalent to the Guilbaud relation between rho, tau and sigma, or to the Daniels inequality. In Section 4 this approach allows to obtain a simple expression of $\rho(L,L') - \tau(L,L')$ in function of parameters of $L \cap L'$, and then to determine the extremal values of this difference. We can also give some examples of the situation where tau equals rho (to characterize this situation amounts to solve an apparently difficult combinatorial problem.) Another by-product of this approach, presented in Section 5, is to prove that the Spearman metric between two linear orders L and L' is a decreasing function of $L \cap L'$. Proofs of these results and more bibliographical or historical comments can be found in [11].

2. Notation and recalls

In this section we recall some basic definitions on relations and partial orders. Throughout this paper $S = \{x, y, z, \ldots\}$ denotes a set of n elements with n finite and greater than one.

A binary relation on S is a subset R of S^2. We write indifferently $(x, y) \in R$ or xRy when the two elements x and y of S are related by R, and $(x, y) \notin R$ or $xR^{c}y$ when they are not related. If R and R' are two binary relations on S, the *difference* $R - R'$ is the binary relation defined by $x(R - R')y$ if xRy and $xR'^{c}y$. The *symmetric difference* is the relation $(R - R') \cup (R' - R)$. It is well known that the cardinality of the symmetric difference defines a metric on the set of all binary relations on S called the *symmetric difference distance*.

A *strict partial order* P on S is a *transitive* (for all $x, y, z \in S$, xPy and yPz imply xPz), and *asymmetric* (for all $x, y \in S$, xPy implies $yP^{c}x$) binary relation defined on S. Notice that such a relation is *irreflexive* (for every $x \in S$, $xP^{c}x$). A strict partial order P is a *strict linear order* if it is also *connected* (for all $x, y \in S$, $x \neq y$ and $xP^{c}y$ imply yPx). In all this paper, a strict partial (respectively, linear) order will be simply called a partial (respectively, linear) order. When P is a partial order, the binary relation P^{d}, defined by $xP^{d}y$ if yPx, is a partial order called the *dual order* of P. Two partial orders P and P' on S are of the same *type* if they are isomorphic (i.e. if there exists a bijection f of S into S such that xPy if and only if $f(x)P'f(y)$).

The *covering relation* $\prec_P$ associated with the partial order P is defined by $x \prec_P y$ if xPy and there does not exist z with xPz and zPy (there is no element 'between' x and y). Then we say that x is covered by y or that y covers x. The *Hasse diagram* of a partial order is a planar representation of its covering relation. If L is a linear order on S and x an element of S we set $r_L(x) = 1 + |\{y \in S\colon yLx\}|$ and we call this number the *rank* of x (in L). By ranking the elements of S according to their ranks from 1 to n, we obtain a permutation $x_1x_2\ldots x_i\ldots x_n$ of S, with $r_L(x_i) = i$ for each $i = 1,2,\ldots,n$. We can call this permutation the 'position' permutation associated with L (see Section 6 for the 'rank' permutation associated with L). Conversely, any permutation $x_1x_2\ldots x_i\ldots x_n$ of S defines a linear order L on S (x_iLx_j if $i<j$) and we shall generally use this permutation presentation to give examples of linear orders. So, if for instance $S=\{a,b,c,d\}$, and $L=\{(b,c),(b,a),(b,d),(c,a),(c,d),(a,d)\}$, we write $L=bcad$. Notice that if $x_1x_2\ldots x_i\ldots x_n$ is the permutation associated with the linear order L, the permutation associated with the dual order L^{d} is $x_nx_{n-1}\ldots x_i\ldots x_1$.

When L and L' are two linear orders on S, their intersection $L\cap L'$ is a partial order on S which we call the *partial order associated* with L and L'.

For any other basic notion not introduced here see, for instance, [1] or [6].

3. The Kendall, Spearman and Guilbaud metrics and coefficients

We first define the Kendall and Spearman metrics between linear orders and the associated concordance coefficients.

Let L and L' be two linear orders on S:

— The *Kendall metric* (or *distance*) between L and L' is the half of the symmetric difference distance between L and L'. It is denoted by $d_{\mathrm{K}}(L,L')$. Since (x,y) belongs to $L-L'$ if and only if (y,x) belongs to $L'-L$ one has

$$d_{\mathrm{K}}(L,L')=|\{\{x,y\}\subseteq S\colon xLy \text{ and } yL'x\}|.$$

In other words, $d_{\mathrm{K}}(L,L')$ is the number of pairs $\{x,y\}$ of S on which L and L' 'disagree'.

The maximum distance $d_{\mathrm{K}}(L,L')$ between two linear orders L and L' is $n(n-1)/2$, value obtained if and only if L' is the linear order L^{d} dual of L.

— The *Spearman metric* (or *distance*) between L and L' is the euclidean metric between the two associated 'rank vectors'.

$$d_{\mathrm{S}}(L,L')=\Sigma\{[r(x)-r'(x)]^2,\ x\in S\}^{1/2}.$$

The square of the maximum distance $d_{\mathrm{S}}(L,L')$ between two linear orders L and L' is $n(n+1)(n-1)/3$, value obtained if and only if L' is the dual of L.

The two classical concordance coefficients Kendall tau and Spearman rho between two linear orders are obtained by normalizing the two quantities d_{K} and d_{S}^2 so that they vary between $+1$, obtained if and only if $L=L'$, and -1, obtained if and only if $L' = L^{\mathrm{d}}$. Since the normalization formula for the quantity q is $1 - 2q/\max q$, one gets

$$\tau(L,L') = 1 - 4d_{\mathrm{K}}(L,L')/n(n-1),$$
$$\rho(L,L') = 1 - 6d_{\mathrm{S}}^2(L,L')/(n^3 - n).$$

We introduce now a third quantity d_{G} and its normalization σ, that appear implicitly in [2] and explicitly in [8].

Definitions. Let L and L' be two linear orders on S written as permutations of S. We say that L' is a *circular transformation* of L if $L' = L$ or if L' is obtained from L by a circular permutation. For example, the circular transformations of the linear order *abcd* are *abcd*, *bcda*, *cdab* and *dabc*.

If L is a linear order on S and $\{x,y,z\}$ a subset of S, we denote by $L_{\{x,y,z\}}$ the order restriction of L to $\{x,y,z\}$.

Let L, L' be two linear orders on S and $\{x,y,z\}$ a subset of S. We say that L and L' have a *circular agreement* on $\{x,y,z\}$ if $L'_{\{x,y,z\}}$ is a circular transformation of $L_{\{x,y,z\}}$ (or equivalently if $d_{\mathrm{K}}(L_{\{x,y,z\}}, L'_{\{x,y,z\}})$ is even). We say that L and L' have a *circular disagreement* on $\{x,y,z\}$ if $L'_{\{x,y,z\}}$ is a circular transformation of $L^{\mathrm{d}}_{\{x,y,z\}}$ (or equivalently if $d_{\mathrm{K}}(L_{\{x,y,z\}}, L'_{\{x,y,z\}})$ is odd).

So, if for example the restriction of L to $\{x,y,z\}$ is xyz, L' has a circular agreement with L on $\{x,y,z\}$ if its restriction to $\{x,y,z\}$ is xyz, yzx or zxy, and it has a circular disagreement with L on $\{x,y,z\}$ if its restriction to $\{x,y,z\}$ is yxz, xzy or zyx.

We set

$$A(L,L') = \{\text{subsets } \{x,y,z\} \text{ of } S, \text{ for which } L \text{ and } L' \text{ have a circular agreement}\},$$
$$D(L,L') = \{\text{subsets } \{x,y,z\} \text{ of } S, \text{ for which } L \text{ and } L' \text{ have a circular disagreement}\},$$

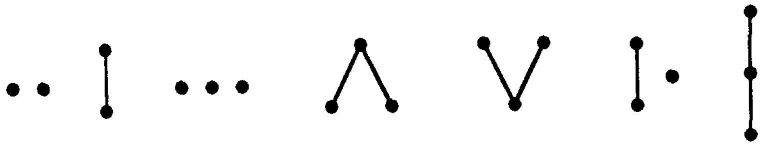
$$a_{\mathrm{G}}(L,L') = |A(L,L')|,$$
$$d_{\mathrm{G}}(L,L') = |D(L,L')|.$$

Then

$$d_{\mathrm{G}}(L,L') + a_{\mathrm{G}}(L,L') = n(n-1)(n-2)/6.$$

Recall that d is a *semi-metric* on a set X if d is a map from the set X^2 of all ordered pairs of X into $\mathbb{R}^+$ which is symmetrical $(d(x,y) = d(y,x))$, satisfies the triangular inequality $(d(x,y) \leqslant (d(x,z) + d(z,y))$ and satisfies $d(x,y) = 0$ if $x = y$.

Proposition 1. *The quantity $d_{\mathrm{G}}(L,L')$ is a semi-metric on the set of all linear orders on S. One has $d_{\mathrm{G}}(L,L') = 0$ if and only if L' is a circular transformation of L and $d_{\mathrm{G}}(L,L') = n(n-1)(n-2)/6$ if and only if L' is a circular transformation of L^{d}.*



I C D_1 D'_2 D''_2 A_2 A_1

Fig. 1.

In the same way that tau and rho have been defined as normalizations of d_{K} and d_{S}^2, we define a normalization of d_{G} called σ:

$$\sigma(L,L') = 1 - 2d_{\mathrm{G}}(L,L')/\max d_{\mathrm{G}} = 1 - 12d_{\mathrm{G}}(L,L')/n(n-1)(n-2).$$

So one has $-1 \leqslant \sigma(L,L') \leqslant +1$ with $\sigma(L,L') = +1$ (respectively -1) if and only if L' is a circular transformation of L (respectively L^{d}).

The coefficient σ and the semi-metric d_{G} will be called (respectively), the *Guilbaud coefficient and metric.*

Remark. Let π be a permutation of S and $L = x_1x_2\dots x_n$ a linear order on S. We denote by πL the linear order defined by the permutation $\pi(x_1)\pi(x_2)\dots\pi(x_n)$. Then for $q \in \{d_{\mathrm{K}}, d_{\mathrm{S}}, d_{\mathrm{G}}, \tau, \rho, \sigma\}$ we have $q(\pi L, \pi L') = q(L,L')$, what means that all these quantities are invariant by a relabelling of S. In particular, if $S = \{1,2,\dots,n\}$ we can always assume that $L = 12\dots n$. We will use this fact in the remark after Theorem 9.

Our next step consists of showing that all the above quantities $q(L,L')$ can be expressed as a function of parameters of the partial order $L \cap L'$. In fact we define the needed parameters for an arbitrary partial order P on S. We consider all the types of partial orders defined on a set with two or three elements. There are seven types represented by Hasse diagrams and named in Fig. 1. The two types D'_2 or D''_2 are dual and we say that a partial order is of type D_2 if it is of type D'_2 or D''_2.

Let P be a partial order defined on S. We say that the subset $\{x,y\}$ of S is of type I (respectively C) if the restriction $P_{\{x,y\}}$ of P to this subset is of type I (respectively C), i.e. if $P_{\{x,y\}}$ is isomorphic with the partial order representing the type I (respectively C) on Fig. 1. Notice that in the first case (respectively second case) one says that x and y are *incomparable* (respectively *comparable*) in P.

We set

$i(P)$ = number of subsets $\{x,y\}$ of S of type I,

$c(P)$ = number of subsets $\{x,y\}$ of S of type C.

We say that the subset $\{x,y,z\}$ of S is of type D'_2 (respectively D''_2) if the restriction $P_{\{x,y,z\}}$ is of type D'_2 (respectively D''_2).

We set

$$d_2'(P) = \text{number of subsets } \{x,y,z\} \text{ of } S \text{ of type } D_2',$$
$$d_2''(P) = \text{number of subsets } \{x,y,z\} \text{ of } S \text{ of type } D_2''.$$

Finally, for $i \in \{1,2\}$, we say that the subset $\{x,y,z\}$ of S is of type D_i (respectively A_i) if the restriction $P_{\{x,y,z\}}$ is of type D_i (respectively A_i). We say that it is of type D (respectively A) if it is of type D_1 or D_2 (respectively, A_2 or A_1).

For $i \in \{1,2\}$, we set

$$d_i(P) = \text{number of subsets } \{x,y,z\} \text{ of } S \text{ of type } D_i,$$
$$a_i(P) = \text{number of subsets } \{x,y,z\} \text{ of } S \text{ of type } A_i,$$
$$d(P) = \text{number of subsets } \{x,y,z\} \text{ of } S \text{ of type } D \ (= d_1(P) + d_2(P)),$$
$$a(P) = \text{number of subsets } \{x,y,z\} \text{ of } S \text{ of type } A \ (= a_1(P) + a_2(P)).$$

So we have

$$c(P) + i(P) = n(n-1)/2,$$
$$d(P)+a(P) = d_1(P)+d_2(P)+a_1(P)+a_2(P) = d_1(P)+d_2'(P)+d_2''(P)+a_1(P)+a_2(P)$$
$$= n(n-1)(n-2)/6,$$
$$(n-2)c(P) = 3a_1(P) + a_2(P) + 2d_2(P).$$

Now it is easy to check the following facts:

Lemma 2. *Let L and L' be two linear orders defined on S, and $L \cap L'$ the associated partial order*:

$$d_K(L,L') = i(L \cap L') = n(n-1)/2 - c(L \cap L'),$$
$$d_G(L,L') = d(L \cap L'),$$
$$a_G(L,L') = a(L \cap L').$$

The following result linking the ranks $r(x)$ and $r'(x)$ in the two linear orders L and L' with the parameters of the partial order $L \cap L'$ is much less obvious. One obtains it by writing $r(x) = 1 + |\{y \in S\colon\ yLx\}| = 1 + |\{y \in S\colon\ yL \cap L'x\}| + |\{z \in S\colon\ zL \cap L'^{\mathrm{d}}x\}|$ and similarly for $r'(x)$.

Proposition 3. *For L, L' two linear orders on S, we have*

$$\begin{aligned}\Sigma\{[r(x)r'(x)],\ x \in S\} &= n^2 + c(L \cap L') + a_2(L \cap L') + d_1(L \cap L') \\ &\quad +2a_1(L \cap L') + 2d_2(L \cap L') \\ &= n(n+1)(n+2)/6 + c(L \cap L') \\ &\quad +a_1(L \cap L') + d_2(L \cap L').\end{aligned}$$

Since $d_S^2(L,L')=\Sigma\{[r(x)-r'(x)]^2, x\in S\}=2\Sigma\{[r(x)^2], x\in S\}-2\Sigma\{[r(x)r'(x)], x\in S\}=2n(n+1)(2n+1)/6-[2n^2+2c+2a_2+2d_1+4a_1+4d_2]$ one obtains after some computations,

Theorem 4. *For L,L' two linear orders on S, we have*

$$d_S^2(L,L')=nd_K(L,L')-d_G(L,L').$$

Remark. This result shows that the Spearman metric $d_S(L,L')$ can also be expressed as a function of the parameters c and d of the partial order $L\cap L'$.

Replacing in Theorem 4, $d_S^2(L,L')$, $d_K(L,L')$ and $d_G(L,L')$ by their expressions functions of $\rho(L,L')$, $\tau(L,L')$ and $\sigma(L,L')$, one gets:

Corollary 5 (Guilbaud [8]). *For L,L' two linear orders on S, we have*

$$\tau(L,L')=[2(1+1/n)\rho(L,L')]/3+[(1-2/n)\sigma(L,L')]/3,$$

or equivalently

$$\sigma(L,L')=(3n/(n-2))\tau(L,L')-(2(n+1)/(n-2))\rho(L,L').$$

Since σ lies between -1 and $+1$, the last formula gives immediately:

Corollary 6 (Daniels [2]). *For L,L' two linear orders on S, we have*

$$-1\leqslant(3n/(n-2))\tau(L,L')-(2(n+1)/(n-2))\rho(L,L')\leqslant+1. \tag{1}$$

We can also deduce the above results several expressions of rho in function of the parameters of $L\cap L'$, for instance $\rho(L,L')=-1+12[c(L\cap L')+a_1(L\cap L')+d_2(L\cap L')]/(n^3-n)=((n+1)[a_1(L\cap L')-d_1(L\cap L')]-(n-1)[a_2(L\cap L')-d_2(L\cap L')])/n(n^2-1)(n-2)$.

Remark. There exists another relation between tau and rho obtained by Durbin and Stuart [5]. It would be interesting to study this relation from an ordinal point of view.

4. The comparison of the Kendall and Spearman metrics

In fact, we compare these metrics by using their normalizations tau and rho which have the advantage to have the same range $[-1,+1]$. A basic result is the following proposition that is obtained by using the expressions of $\tau(L,L')$ and $\rho(L,L')$ in function of the parameters of the partial order $L\cap L'$.

Proposition 7. *For L, L' two linear orders on S, we have*

$$\rho(L,L') - \tau(L,L') = 4[d_2(L \cap L') - a_2(L \cap L')]/(n^3 - n).$$

So, in order to determine the maximum *bias* between ρ and τ, i.e. the extremal values of $\rho - \tau$, we must seek the extremal values of the quantity $d_2(P) - a_2(P)$, where P is a partial order intersection of two linear orders. We conjectured the answer, but it was obtained by resolving a more general graph problem that we first present. Let $G = (V, E)$ be a (undirected) graph; we denote by $d_2(G)$ (respectively, $a_2(G)$) the number of its subgraphs with three vertices and two edges (respectively, with three vertices and one edge). Now, we seek the extremal values of the quantity $d_2(G) - a_2(G)$ when G is an arbitrary graph with a fixed number of vertices. The answer was given by Le Conte de Poly-Barbut [10]. We say that G is a *complete bipartite* graph if there exists a bipartition $V_1 + V_2$ of the set V of its vertices such that xy is an edge of G if and only if $x \in V_1$ and $y \in V_2$. If $|V_1| = p$ and $|V_2| = q$ a such graph is denoted by $K_{p,q}$. We say that G is *a balanced complete bipartite* graph if G is a complete bipartite graph $K_{p,q}$ with $|p - q| \leqslant 1$. We say that G is a *balanced biclique graph* if the complementary graph G^c of G (xy is an edge of G^c if and only if xy is not an edge of G) is a balanced complete bipartite graph.

Lemma 8 (Le Conte de Poly-Barbut [10]). (a) *The quantity $d_2(G)$ is maximum on the set of all graphs with n vertices if and only if G is a balanced complete bipartite graph.*

(b) *The quantity $a_2(G)$ is maximum on the set of all graphs with n vertices if and only if G is a balanced biclique graph.*

(c) *The quantity $|d_2(G) - a_2(G)|$ is maximum on the set of all graphs with n vertices if and only if G is a balanced complete bipartite graph or a balanced biclique graph.*

(d) *The maximum value of $d_2(G)$ on the set of all graphs with n vertices equals the maximum value of $a_2(G)$ and the maximum value of $|d_2(G) - a_2(G)|$. These maximum values are equal to $n^2(n-2)/8$ if n is even and to $(n^2-1)(n-2)/8$ is n is odd.*

In order to apply this result to our original problem we have only to give a definition and do two remarks. If P is a partial order, we define its *comparability* graph $G_C(P) = (S, C(P))$ as the graph whose set of vertices is S and whose edges are the pairs $\{x, y\}$ such that xPy or yPx, (i.e. such that x and y are comparable in P). Noticing that this graph is undirected, the first remark is that one has $d_2(P) = d_2(G_C(P))$ and $a_2(P) = a_2(G_C(P))$. The second is that the balanced complete bipartite graphs or the balanced biclique graphs of the above lemma are comparability graphs of partial orders intersections of two linear orders (this is explicited in the following theorem). Then this lemma also gives the answer to our original question and allows to derive the extremal values of $\rho - \tau$. In Theorem 9 we take for S the set $\{1, 2, 3, \ldots, n\}$ of the first n integers.

Theorem 9. *Let* $S = \{1,2,3,\ldots,n\}$ *and* $L = 12\ldots i\ldots n$. (a) *The quantity* $\rho(L,L') - \tau(L,L')$ *is maximum on the set of all linear orders defined on* S *if and only if* $L' = i\ldots 1n\ldots i+1$, *with* $i = p$ *if* $n = 2p$, *and* $i = p$ *or* $p+1$ *if* $n = 2p+1$.

For n even,

$$\mathrm{Max}(\rho - \tau) = n(n-2)/2(n^2-1)$$

$$(\textit{with } \rho = (n^2+2)/2(n^2-1) \textit{ and } \tau = 1/(n-1)).$$

For n odd,

$$\mathrm{Max}(\rho - \tau) = (n-2)/2n \quad (\textit{with } \rho = 1/2 \textit{ and } \tau = 1/n)$$

(b) *The quantity* $\rho(L,L') - \tau(L,L')$ *is minimum on the set of all linear orders defined on* S *if and only if* $L' = i+1\ldots n1\ldots i$, *with* $i = p$ *if* $n = 2p$, *and* $i = p$ *or* $p+1$ *if* $n = 2p+1$.

For n even,

$$\mathrm{Min}(\rho - \tau) = n(2-n)/2(n^2-1)$$

$$(\textit{with } \rho = -(n^2+2)/2(n^2-1) \textit{ and } \tau = -1/(n-1))$$

For n odd,

$$\mathrm{Min}(\rho - \tau) = (2-n)/2n \quad (\textit{with } \rho = -1/2 \textit{ and } \tau = -1/n).$$

Then we have:

Corollary 10. *For* $n \to +\infty$, $\mathrm{Max}(\rho - \tau) \nearrow +1/2$ (*with* $\rho \searrow +1/2$ *and* $\tau \searrow 0$) *and* $\mathrm{Min}(\rho - \tau) \searrow -1/2$ (*with* $\rho \nearrow -1/2$ *and* $\tau \nearrow 0$).

Remark. According to the remark following Proposition 1, the pairs of linear orders corresponding to the extremal values of $\rho - \tau$ are all the pairs $\{\pi(12\ldots i\ldots n), \pi(i\ldots 1n\ldots i+1)\}$ and $\{\pi(12\ldots i\ldots n), \pi(i+1\ldots n1\ldots i)\}$, π arbitrary permutation of S. These pairs form very particular configurations. The case $\rho - \tau$ maximum corresponds to the situation where L and L' are in complete disagreement within two classes of equal (or almost equal) cardinalities but in complete agreement between these two classes. Dually, in the case $\tau - \rho$ maximum there is a complete agreement within classes and a complete disagreement between the two classes.

From Proposition 7 one has $\tau(L,L') = \rho(L,L')$ if and only if $a_2(L \cap L') = d_2(L \cap L')$. So the problem of characterizing such pairs amounts to the problem of characterizing the partial orders $L \cap L'$ such that $a_2(L \cap L') = d_2(L \cap L')$, or more generally of characterizing the graphs G such that $a_2(G) = d_2(G)$. It seems difficult to find a general characterization of such graphs. Here, we only give four infinite families of examples:

Proposition 11. (1) *Let* $\{a_1, a_2, \ldots, a_p\}$, $\{b_1, b_2, \ldots b_p\}$ *be a partition of* S *into two classes of same cardinality. The two linear orders* $L = a_1 a_2 \cdots a_p b_1 b_2 \cdots b_p$ *and* $L' = b_1 a_1 \cdots b_i a_i \cdots b_p a_p$ *(or* $L' = a_p b_p \cdots a_i b_i \cdots a_1 b_1$*) satisfy* $\tau(L, L') = \rho(L, L') = 0$.

(2) *Let* $\{a_1, a_2, \ldots, a_p\}$, $\{b_1, b_2, \ldots, b_q\}$ *and* $\{c_1, c_2, \ldots, c_r\}$ *be three subsets of* S *forming a partition of* S. *The two linear orders* $L = a_1 a_2 \cdots a_p b_1 b_2 \cdots b_q c_1 c_2, \ldots, c_r$ *and* $L' = c_r c_{r-1} \cdots c_1 a_p a_{p-1} \cdots a_1 b_q b_{q-1} \cdots b_1$ *(or* $L' = b_1 b_2 \cdots b_q a_1 a_2 \cdots a_p c_1 c_2 \cdots c_r$*) satisfy* $\tau(L, L') = \rho(L, L') = 0$ *if and only if* $r = (p + q - 2)/2$.

For example, with $L = 12345678$ the four orders $L' = 51627384, 48372615, 87321654$ and 45612378 satisfy $\tau(L, L') = \rho(L, L')$.

5. An ordinal property of the Spearman distance

We denote by $\delta(L, L')$ an arbitrary metric between two linear orders L and L'. An *ordinal property* of such a metric is a property that depends only on the partial order $L \cap L'$. We write two such properties concerning four arbitrary linear orders L_1, L_2, L_3 and L_4:

P1 $\quad L_1 \cap L_2 = L_3 \cap L_4 \;\Rightarrow\; \delta(L_1, L_2) = \delta(L_3, L_4)$,

P2 $\quad L_1 \cap L_2 \subseteq L_3 \cap L_4 \;\Rightarrow\; \delta(L_1, L_2) \geqslant \delta(L_3, L_4)$.

Property P1 says that the metric δ is a function of the partial order $L_1 \cap L_2$. The Spearman metric satisfies this property since it is a function of the parameters $c(L_1 \cap L_2), a_1(L_1 \cap L_2)$ and $d_2(L_1 \cap L_2)$. The aim of the following developments is to prove that d_S satisfies the stronger property P2, i.e. that it is a decreasing function of the partial order $L_1 \cap L_2$. We shall need results (Lemmas 12 and 13) that are true for any partial order.

Let P be an arbitrary partial order on the set S and $I(P)$ its incomparability relation ($xI(P)y$ if $xP^c y$ and $yP^c x$).

We set:

$$Px = \{y \in S,\ y \neq x\colon\ yPx\},\quad xP = \{y \in S,\ y \neq x\colon\ xPy\},$$

$$I_P(x) = \{y \in S\colon\ xI(P)y\},$$

$$x \sim_P y \quad \text{if } Px = Py \quad \text{and} \quad xP = yP.$$

Notice that if $x \sim_P y$ we have also $I_P(x) = I_P(y)$ and that the relation $\sim_P$ is an equivalence on S. We recall a classical (and easy) result: if P is a partial order and $(x, y) \notin P$, $Q, = P \cup \{(x, y)\}$ is a partial order if and only if $yP \subseteq xP$ and $Px \subseteq Py$.

Let Q be a partial order obtained from the partial order P by the adjunction of a single ordered pair (x, y), thus satisfying the characterization just recalled. By considering the changes in the restrictions of P on the subsets $\{x, y, z\}$, z different from x and y, one gets the following results:

Lemma 12. (1) *Let P and Q be two partial orders on S with $P \subset Q = P \cup \{(x,y)\}$. Then*

$$d(Q) = d(P) - |yP| - |Px| - |I_P(x) \cap I_P(y)| + |Py \cap I_P(x)| + |xP \cap I_P(y)|.$$

Moreover $|d(Q) - d(P)| \leqslant (n-2)$ with
(a) *$d(Q) = d(P) + n - 2$ if and only if x is minimal in P, y is maximal in P and $I_P(x) \cap I_P(y) = \emptyset$.*
(b) *$d(P) = d(Q) + n - 2$ if and only if $x \sim_P y$.*

(2) *Let P and Q be two partial orders on S with $P \subseteq Q$ and $|Q - P| = k$ $(0 \leqslant k \leqslant n(n-1)/2)$. Then $|d(Q) - d(P)| \leqslant k(n-2)$.*

Assuming now that we have four linear orders L_1, L_2, L_3, L_4 with $L_1 \cap L_2 \subseteq L_3 \cap L_4$, we can apply this lemma to the partial orders $L_1 \cap L_2$ and $L_3 \cap L_4$. Since, $d_S^2(L_1,L_2) - d_S^2(L_3,L_4) = nd_K(L_1,L_2) - d_G(L_1,L_2) - nd_K(L_3,L_4) + d_G(L_3,L_4) = n(c(L_3 \cap L_4) - c(L_1 \cap L_2)]) - [d(L_1 \cap L_2) - d(L_3 \cap L_4)]$ one finally obtains:

Proposition 13. *Let L_1, L_2, L_3, L_4 be four linear orders on S with $L_1 \cap L_2 \subseteq L_3 \cap L_4$ and $c(L_3 \cap L_4) - c(L_1 \cap L_2) = k$ $(0 \leqslant k \leqslant n(n-1)/2)$. Then*

$$d_S^2(L_1,L_2) - d_S^2(L_3,L_4) \in [2k, 2k(n-1)].$$

Corollary 14. *The Spearman metric satisfies property P2:*

$$L_1 \cap L_2 \subseteq L_3 \cap L_4 \Rightarrow d_S(L_1,L_2) \geqslant d_S(L_3,L_4).$$

Corollary 15. *Let L_1, L_2, L_3 be three linear orders on S, such that L_3 is obtained from L_2 by the interchange of two elements x, y, with $(x,y) \in L_1$ and y covered by x in L_2. Then*

$$d_S(L_1,L_2) > d_S(L_1,L_3).$$

Remark that if $L_1 \cap L_2 \subset L_3 \cap L_4$ we have $d_S(L_1,L_2) > d_S(L_3,L_4)$ so $d_S(L,L')$ is a strictly decreasing function of $L \cap L'$.

6. Conclusion

We emphasize the fact that since the set $\mathscr{L}_n$ of all linear orders on $S = \{1,2,\ldots,n\}$ is in bijection with the set Σ_n of all permutations of S, to study metrics on $\mathscr{L}_n$ is equivalent to study metrics on Σ_n. In their 1977 paper Diaconis and Graham consider four 'metrics' on Σ_n and prove certain relations between them (other than those we consider here). Especially what they call S corresponds to the square of the Spearman metric on the set $\mathscr{L}_n$ (so it does not always satisfy the triangular inequality), and what they call I corresponds with the Kendall metric. Notice that in their paper the (implicit) association between Σ_n and $\mathscr{L}_n$ is made by means of the 'rank permutation' and not by the 'position permutation' like in our paper. For instance, if $S=\{1,2,3,4\}$, the linear order $L=\{(2,4),(2,1),(2,3),(4,1),(4,3),(1,3)\}$ has been denoted in our paper by the position permutation 2413. The rank permutation associated with this linear order is the permutation $r_L(1)r_L(2)r_L(3)r_L(4)=3142$. Remark that the position and rank permutations are two inverse permutations. Other results and references on metrics between permutations can be found in [3] and [7].

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