Symmetries as Otter-morphisms

WDRP - Polyhedra

Consider four points labeled counterclockwise 1 to 4. We saw that one operation we could perform on these points is $1 \mapsto 2$ and $2 \mapsto 1$. Another might be $2 \mapsto 3$ and $3 \mapsto 2$.

These operations are all *symmetries* of the collection of four points. But what makes it a symmetry? To be a symmetry it's not as simple as asking that reflecting across a line gives back the same picture, as the symmetries above showed. Instead it has to be something called an *Automorphism*. To define an automorphism we need to define some other properties of functions.

Definition 0.1. A function is called **one-to-one** if every output has a unique input that maps to it. This is the same as the horizontal line test for functions.

Example. Recall that the symbol for the integers is $\mathbb{Z}$ and that a function with domain and codomain the integers would be written as $f : \mathbb{Z} \to \mathbb{Z}$

1. Then the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = n+1$ is a one-to-one function. Indeed, for any number n, the unique number mapping to it under f is $n - 1$.

2. Then the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = 2n$ is a one-to-one function. Indeed, for any number $2n$, the unique number mapping to it under f is n.

3. The function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = n^2$ is <u>not</u> one-to-one. Notice that $f(2) = f(-2) = 4$, so the number mapping to 4 is not unique.

4. If <u>*otter*</u> is the collection of otters with significant others, then the function $f : \underline{otter} \to \underline{otter}$ sending each otter to it's significant other is one-to-one since otters are monogamous.

Definition 0.2. A function is called **onto** if everything in the codomain is the output of some input.

Example.

1. The function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = n + 1$ is onto since for any number n, $f(n - 1) = n$.

2. The function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = 2n$ is <u>not</u> onto. There is no integer mapping to 1. If there was, it would have to satisfy $2n = 1$, or equivalently $n = 1/2$ which is not an integer.

3. The function $f : \mathbb{Z} \to \mathbb{N}$ given by $f(n) = n^2$ is <u>not</u> onto. Indeed, something like 5 is not an output of the function. If there was an input mapping to 5 it would have to satisfy $n^2 = 5$, or equivalently $n = \pm 5$, which is not an integer.

4. If <u>*Otter*</u> is the collection of all otters (including those without significant others), the function $f : \underline{otter} \to \underline{Otter}$ sending each otter to it's significant other is not surjective since some otters are too invested in their career to find love.

As we saw from the examples there are some functions which are onto, but not one-to-one and vice versa. We also saw there are some which are both onto and one-to-one, these are given a name.

Definition 0.3. A function $f : D \to C$, which is both onto and one-to-one, is called bijective. If the domain and codomain are the same, then the function is called an automorphism. Like when someone write a book about themselves its called an *auto*biography.

Example.

1. We saw that the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = n + 1$ is a bijection, and in particular, an automorphism. In fact $f(n) = n + a$ for any integer a is an automorphism. Can you think of any others which would be an automorphism?

2. In a perfect world where otters can find love and excel in their respective careers, the function $f : \underline{Otter} \to \underline{Otter}$ would be an automorphism, or more accurately, an Otter-morphism.

In general you should think of a bijection as pairing up the things in the domain and codomain. Bijections are how you show two sets are the same size in math, and lots of other properties. In general, an automorphism should be thought of as a symmetry. And as we'll see in the sequel, the symmetries (automorphisms) of something always form a group.