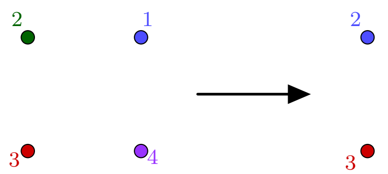
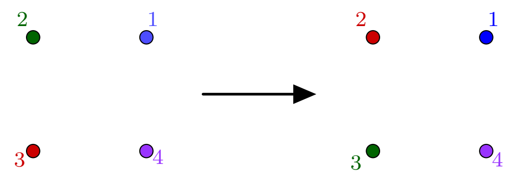
Notation for Symmetries

WDRP - Polyhedra

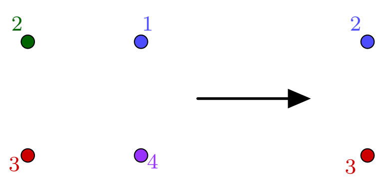
Let's consider the four points labeled counterclockwise 1 to 4.We saw that one symmetry might be We also saw that symmetries are automorphisms from the set of four labeled points to four labeled points $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$. How do we write this symmetry as a function? A natural way is by writing each number above which number it is sent to. Here are some examples

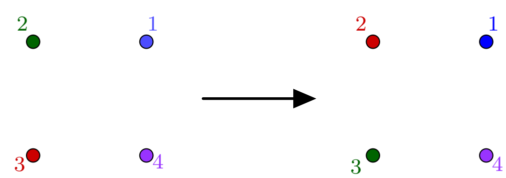




$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \qquad f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

But we will use a more compact notation called **cycle notation**. Here are the same permutations but in cycle notation.





$$f = (1\ 2) \qquad f = (1\ 2\ 3)$$

You read these permutations as "1 goes to 2, 2 goes to 1" and "1 goes to 2, 2 goes to 3, 3 goes to 1" reading from left to right. The right most number always goes to the first number, which is why its called *cycle* notation, kind of like when pac man hits the right side of the screen he re appears on the left side! If a number excluded then it is fixed by the permutation. We may write our cycles as (1 2)(3)(4) but this leads to clutter so we exclude the points that are fixed.

Now that we have notation for permutations, how do we combine them? For example if we want to combine (1 2 3) and (1 3) we write it as (1 2 3)(1 3). Start with ”(1 ” and starting with the cycle on the right, ask where 1 goes. Since its (1 3), 1 goes to 3. Then the cycle on the left says that 3 goes to 1, remember pac-man? So 1 is fixed which means we exclude it and should start again with ”(2 ”. Now, where does 2 go? Starting with the cycle on the right, since there is no 2 we say 2 goes to 2. The cycle on the left says that 2 goes to 3 so now we have ”(2 3 ”. Now we ask where 3 goes, starting with the cycle on the right we get that 3 goes to 1. The cycle on the left says 1 goes to 2, so because we are back to the same number on the left of our cycle we close the parenthesis to get (2 3). Thus, (1 2 3)(1 3) = (2 3). Try doing it the other way (1 3)(1 2 3) and see if it comes out the same way!

We also saw the cyclic group. Although this is a subgroup of the symmetric group, it's easier to think of it as it's own entity and so it has it's own notation. If r is the rotation by the smallest positive degree, so in the square case by $90°$, then the rotation by $180°$ is given by r^2, by $270°$ is r^3 and by $0°$ or equivalently $360°$ by ι. You can write it as r^4, but in general it is the identity element so it gets a special name. We also had the special flip about the line through the vertex labeled 1. We call this element s. As it turns out all the other reflections can be written as s with some power of r. So all the elements in, say D_4, are $\{\iota, r, r^2, r^3, s, sr, sr^2, sr^3\}$. So how do we put these together?

To put these together we need a special rule for how s and r behave together. The rule is $sr^k = r^{-k}s$ where r^{-k} is the inverse of r^k. For example working in D_4, r is the inverse of r^3 since $r \cdot r^3 = r^4 = \iota$. So how does this work? Consider $sr \cdot sr^2$. The inverse of r is r^3 in D_4 so we can write $rs = sr^3$ which means we have the following

$$sr \cdot sr^2 = s(r \cdot s)r^2 = s(sr^3)r^2 = s^2r^5$$

Notice that doing a flip twice takes us back to how we were so $s^2 = \iota$, or s is the inverse of s. Then

$$s^2r^5 = \iota \cdot r^5 = r^5$$

we also know that $r^4 = \iota$ so

$$r^5 = r^4 \cdot r = \iota \cdot r = r$$

Thus, $sr \cdot sr^2 = r$. This works the same way for C_4, except we only need to use r's, not s is involved.