Symmetries as Groups

WDRP - Polyhedra

We saw some examples of symmetries, and we saw how mathematicians think of symmetries as automorphisms of objects. It turns out the collections of symmetries are not JUST automorphisms, but they some meet some other criteria called **axioms** which are like a set of rules. We don't just want to think of ANY automorphisms of something as symmetries, they should somehow match our intuition of what symmetries should be. Without these axioms there are some very weird examples we could come up with which we wouldn't consider symmetries.

note that the elements of a groups don't explicitly have to be the automorphisms of something, its enough to just have a set of "symmetries" and an operation that lets you "add" them.

Definition 0.1. A set with an associative binary operation is a group if the following axioms are met

1. Closure: any two symmetries can be put together to make another automorphism
2. Inverses: Any symmetry has a reverse symmetry
3. Identity: There is a symmetry ι which when put together with any other symmetry, is the same as only doing the other symmetry.

Associativity is really just a technicality, the groups we naturally run into and the symmetries that match our intuition are associative. You see associativity all the time with regular numbers, for example $(1 + 2) + 3 = 1 + (2 + 3)$. You should think of closure as saying that symmetries are compatible with each other, as well as a sort of completeness. Inverses are a natural request. Remember, symmetries are operations which leaves the object(s) the same after moving it around. Naturally, I should be able to put my object(s) back to how they started before I did something to them. This axiom is why it's important we use automorphisms to define symmetries. As for the identity, this is probably more of a technicality. Intuitively, doing nothing to our object(s) should always be a symmetry, in some vacuous sense.

A good definition in math captures your intuitive understanding of how things work while also encompassing interesting properties. Despite this, sometimes a good definition can still include some unintuitive examples.

Example.

1. The integers $\mathbb{Z}$ under addition form a group
2. The rational numbers $\mathbb{Q}$ (fractions) form a group under addition, as well as under multiplication
3. The radian measures of a circle form a group under addition. In some sense this is a "continuous" cyclic group.

What are all of these odd groups symmetries of? I don't really know! It's possible they are symmetries of themselves! One thing to notice is the definition of a group did not limit us to finite things. All of the examples are infinite things, and as we know from calculus, things get weird when we start including infinite things!