Meeting Topic: Down with Determinant!

이번 주 주제는 Sheldon Axler의 “Down with Determinants!"로 행렬식을 쓰지 않고 학부생 선형대수학의 주요한 내용을 살펴봅니다.

세부 내용은 변할 수 있습니다.

1. Linear Algebra Done Right

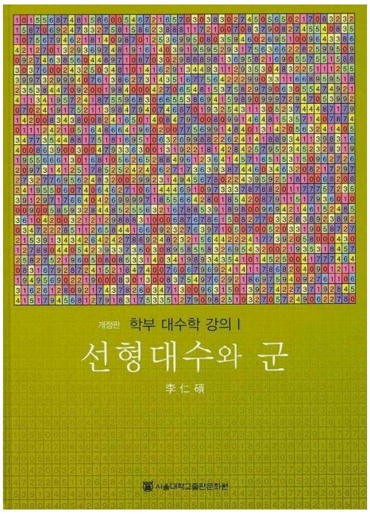
- Springer 버전: https://link.springer.com/content/pdf/10.1007/978-3-031-41026-0.pdf
- 27 January 2024 버전: https://linear.axler.net/LADR4e.pdf
- Errata: https://linear.axler.net/LADRErrata4e.html

2. Introduction

3. Down with Determinants!

- https://www.axler.net/DwD.html
- https://www.maa.org/sites/default/files/pdf/awards/Axler-Ford-1996.pdf

선형대수는 왜 배우나?



- 이원석, 마음의 고향
- 미적분

미적분은 쉽게 말해 곡선을 다루는데 곡선은 굽이 굽이 휘어있어 어렵다. 곡선은 어려우니까 곡선을 가까이서 보아 liearnization하면 선형대수가 된다.

space와 association에서 vector space를 space로 두고 spaces간의 자연스런 변환인 linear map을 다루는 분야가 linear algebra이다.

In 1996 this paper received the Lester R. Ford Award for expository writing from the Mathematical Association of America.

3.1. Introduction

"I find it hard conceive of a situation in which the numerical value of a determinant is needed: Cramer's rule, because of its inefficiency, is completely impractical, while the magnitude of the determinant is an indication of neither the condition of the matrix nor the accuracy of the solution."

https://www.reddit.com/r/math/comments/15ahoci/comment/jtmqn24/?utm_source=share&utm_medium=web2x&context=3

"Thank you for the question. Let me clear up a few misconceptions about my book Linear Algebra Done Right (LADR).

First, determinants are not excluded from LADR. Specifically, 25 of the 331 pages of the third edition of LADR deal with determinants. True, the 25 pages about determinants appear in the last chapter of the book. That was a political decision on my part to emphasize that the important theorems of linear algebra can be proved (with cleaner proofs!) without using determinants. For example, compare the proof in LADR of the crucial theorem that every operator on a finite-dimensional complex vector space has an eigenvalue to the standard determinant-based proof. Sure, it is a matter of taste as to which proof you prefer, but most people think that the nondeterminant proof provides more insight.

Second, of course I know that determinants are useful in many areas of research mathematics and in the change of variables formula in multivariable integration, and I have never hinted otherwise. My point is that determinants are not super-useful in basic linear algebra.

Millions of high school students every year waste time learning to solve systems of linear equations using Cramer's rule. The toy 2-by-2 and 3-by-3 systems that they solve give students the false impression that Cramer's rule is useful in the real world. Probably many thousand large systems of linear equations are solved by computers every day, none of them using Cramer's rule (which should not be used even on small 10-by-10 systems of equations).

Let me mention that the forthcoming fourth edition of LADR has a new clean approach to determinants via alternating multilinear forms (but still in the last chapter). The fourth edition of LADR will appear in late November 2023. The electronic version will be an Open Access book, meaning that I am making the electronic version legally available for free to the world. You can get it at the book's website (https://linear.axler.net/), where two sample chapters are already available (including the chapter on eigenvalues, which gives the nondeterminant proof mentioned above)."

3.2. Eigenvalues and Eigenvectors

"Because a basis-free approach seems more natural, this paper will mostly use the language of linear transformation; readers who prefer the language of matrices should have no trouble making the appropriate translation."

Def The term *linear operator* will mean a linear transformation from a vector space to itself; thus a linear operator corresponds to a square matrix (assuming some choice of basis).

$$L : V \to V$$

Notation used throughout the paper: n denotes a positive integer, V denotes an n-dimensional **complex** vector space, T denotes a linear operator on V, and I denotes the identity operator.

Def A complex number λ is called an *eigenvalue* of T if $T - \lambda I$ is not injective.

$$\det(T - \lambda I)$$

의 해로 eigenvalue를 보통 정의했지만 여기서는 $T - \lambda I$가 injective하지 않는 λ로 eigenvalue를 정의했습니다.

Thm 2.1. *Every linear operator on a finite-dimensional complex vector space has an eigenvalue.*

Proof) $v \in V, v \neq 0$

$$v, Tv, T^2v, T^3v, \cdots, T^nv : \text{linearly dependent}$$

$$a_0v + a_1Tv + \cdots + a_nT^nv = 0$$

By the fundamental theorem of algebra,

$$a_0 + a_1 + \cdots + a_nz^n = a_n(z - r_1)\cdots(z - r_n)$$

$$\begin{aligned} 0 &= (a_0I + a_1T + a_2T^2 + \cdots + a_nT^n)v \\ &= a_n(T - r_1I)(T - r_2I)\cdots(z - r_nI)v \end{aligned}$$

만약 모든 $T - r_iI$가 단사라면 $v = 0$ 이므로 모순이다. 따라서 최소한 하나의 eigenvalue가 존재한다.

Def A vector $v \in V$ is called an *eigenvector* of T if $Tv = \lambda v$ for some eigenvalue λ.

Prop 2.2 non-zero eigenvectors corresponding to distinct eigenvalues of T are linearly independent.

Proof)

$$v_1, \cdots, v_m \text{ : eigenvectors}$$

$$\lambda_1, \cdots, \lambda_m \text{ : corresponding eigenvalues with } \lambda_i \neq \lambda_j$$

$$a_1 v_1 + \cdots + a_m v_m = 0$$

Apply the linear operator

$$(T - \lambda_2 I)(T - \lambda_3 I) \cdots (T - \lambda_m I)$$

to both sides of the equation above, getting

$$\begin{aligned}
(T - \lambda_2 I)(T - \lambda_3 I) v_1 &= (T - \lambda_2 I)(T v_1 - \lambda_3 I v_1) \\
&= (T - \lambda_2 I)(\lambda_1 v_1 - \lambda_3 v_1) \\
&= (T - \lambda_2 I) v_1 (\lambda_1 - \lambda_3) \\
&= v_1 (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)
\end{aligned}$$

$$a_1 (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \cdots (\lambda_1 - \lambda_3) v_1 = 0$$

$$\Rightarrow a_1 = 0$$

같은 방식으로 계속 진행하면 $a_2 = a_3 = \cdots = 0$ 이므로 linearly independent하다. ■

3.3. Generalized Eigenvectors

Unfortunately, the eigenvectors of T need not span V. We will see that the generalized eigenvectors of T always span V.

Def A vector $v \in V$ is called a *generalized eigenvector* of T if

$$(T - \lambda I)^k v = 0$$

for some eigenvalue λ of T and some positive integer k.

Obviously, the set of generalized eigenvectors of T corresponding to an eigenvalue λ is a subspace of V.

The following lemma shows that in the definition of generalized eigenvector, instead of allowing an arbitrary power of $T - \lambda I$ to annihilate v, we could have restricted attention to the n^{th} power, where n equals the dimension of V.

Def ker is an abbreviation for kernel, which is defined as the set of vectors that get mapped to 0.

$$Tv = 0$$

인 모든 v를 $\ker T$이다.

Lemma 3.1 The set of generalized eigenvectors of T corresponding to an eigenvalue λ equals $\ker(T - \lambda I)^n$.

Proof)

$$\begin{aligned} v \in \ker(T - \lambda I)^n &\Rightarrow (T - \lambda I)^n v = 0 \\ &\Rightarrow v \text{ is a generalized eigenvector of } T \end{aligned}$$

$v \in \mathrm{GE}(T, \lambda), v \neq 0$. We need to prove that $(T - \lambda I)^n v = 0$.

$$(T - \lambda I)^k v = 0 \text{ for some positive integer } k$$

We will be done if we show that $k \leq n$.

$$v, (T - \lambda I)v, (T - \lambda I)^2 v, \cdots, (T - \lambda T)^{k-1} v$$

이 linearly independent함을 보이고자 한다.

$$a_0T + a_1(T - \lambda I)v + a_2(T - \lambda I)^2 v + \cdots + a_{k-1}(T - \lambda I)^{k-1} v = 0$$

Applying $(T - \lambda I)^{k-1}$ to both sides of equation above, getting

$$a_0 \underbrace{(T - \lambda I)^{k-1} v}_{v \neq 0} = 0$$

$$\Rightarrow a_0 = 0$$

Now applying $(T - \lambda I)^{k-2}$ to both sides of the equation, getting

$$a_1 (T - \lambda I)^{k-1} v = 0$$

$$\Rightarrow a_1 = 0$$

Continuing in this fashion, we see that $a_j = 0$ as desired.

Prop 3.4 The generalized eigenvectors of T span V. (i.e. $\mathrm{GE}(T) = V$)

Proof) The proof will be by induction on n, the dimension of V. Obviously, the result holds when $n = 1$. (Why?)

Suppose $n > 1$. Let λ be any eigenvalue of T. (Why?) We first show that

$$V = \underbrace{\ker(T - \lambda I)^n}_{V_1} \oplus \underbrace{\mathrm{Im}(T - \lambda I)^n}_{V_2} = V_1 \oplus V_2$$

Suppose $v \in V_1 \cap V_2$.

$$v \in V_1 \Rightarrow (T - \lambda I)^n v = 0$$
$$v \in V_2 \Rightarrow (T - \lambda I)^u = v \text{ for some } u \in V$$

$$(T - \lambda I)^{2n} u = (T - \lambda I)^n v = 0$$

$$\Rightarrow (T - \lambda I)^n = 0 \qquad \because \text{by Lemma 3.1}$$
$$\Rightarrow v = (T - \lambda I)^n u = 0$$

Thus, $V_1 \cap V_2 = \{0\}$.

Because V_1 and V_2 are the kernel and image of a linear operator on V, we have

$$\dim V = \dim V_1 + \dim V_2$$

. This implies $V = V_1 \oplus V_2$.

$$\begin{aligned}
&\lambda \text{ is an eigenvalue of } T \\
&\Rightarrow T - \lambda I : \text{not injective} \\
&\Rightarrow (T - \lambda I)v = 0 \qquad \text{for some } v \neq 0 \\
&\Rightarrow (T - \lambda I)^n v = 0 \\
&\Rightarrow v \in \ker(T - \lambda I)^n \\
&\Rightarrow V_1 \neq \{0\} \\
&\Rightarrow \dim V_1 \geq 1
\end{aligned}$$

Thus $\dim V_2 < n$. Furthermore, because T commutes with $(T - \lambda I)^n$.

$$T(T - \lambda I)^n = (T - \lambda I)^n T$$

We easily see that (recall $V_2 = \mathrm{Im}(T - \lambda I)^n = (T - \lambda I)^n V$)

$$T(T - \lambda I)^n V = (T - \lambda I)^n TV$$

$$TV_2 = (T - \lambda I)^n (TV) \subseteq V_2$$

$$T|_{V_2} : V_2 \to V_2$$

즉 V_2는 invariant subspace of T이다. 따라서 $T|_{V_2}$가 linear operator이므로 선형 대수를 적용할 수 있다.

By our induction hypothesis, V_2 is spaned by the generalized eigenvectors of $T|V_2$, each of which is obviously also a generalized eigenvector of T. So is V_1. Hence this gives the desired result. ■

Def An operator is called *nilpotent* if some power of it equals 0. (i.e. $T^k = 0$ for some k)

Cor If 0 is the only eigenvalue of T, then T is nilpotent.

Proof)

$$V = \ker(T - 0I)^n = \ker T^n$$

$$\forall v \in V, T^v = 0 \Rightarrow T^n = 0 \quad \blacksquare$$

이 때 주의할 점이 n이 T를 annihilate하는 최소값이 아닐 수 있다는 점이다.

Prop 3.8 Non-zero generalized eigenvectors corresponding to distinct eigenvalues of T are linearly independent.

Proof) Suppose that $v_1, \cdots, v_m$ are non-zero generalized eigenvectors of T corresponding to distinct eigenvalues $\lambda_1, \cdots, \lambda_m$.

$$a_1 v_1 + \cdots + a_m v_m = 0$$

Let k be the smallest positive integer such that $(T - \lambda_1 I)^k v = 0$. Apply the linear operator

$$(T - \lambda_1 I)^{k-1} (T - \lambda_2 I)^n \cdots (T - \lambda_m I)^n$$

to both sides, getting

$$a_1 (T - \lambda_1 I)^{k-1} (T - \lambda_2 I) \cdots (T - \lambda_m I)^n v_1 = 0$$

여기서 $(T - \lambda_i I)$와 $\left(T - \lambda_j I\right)$는 commute하고 $v_2, v_3, \ldots$는 generalized eigenvector이므로 전부 날라간다.

$$a_1 (T - \lambda_1 I)^{k-1} (T - \lambda_2 I) \cdots (T - \lambda_m I)^n v_1$$
$$= a_1 (T - \lambda_1 I)^{k-1} \{(T - \lambda_1 I) + (\lambda_1 - \lambda_2) I\}^n \cdots \{(T - \lambda_1 I) + (\lambda_1 - \lambda_m) I\}^n v_1 = 0$$

and then expand each $\left((T - \lambda_1 I) + \left(\lambda_1 - \lambda_j\right) I\right)^n$ using the binomial theorem, we get

$$a_1 (T - \lambda_1 T)^{k-1} \underbrace{(\lambda_1 - \lambda_2)^n \cdots (\lambda_1 - \lambda_m)^n}_{\neq 0} v_1 = 0$$

$$a_1 \underbrace{(T - \lambda_1 T)^{k-1} v_1}_{\neq 0} = 0$$

$$\Rightarrow a_1 = 0$$

In a similar fashion, $a_j = 0$ for each j, as desired. $\blacksquare$

Now we can pull everything together into the following structure theorem.

Thm 3.11 Let $\lambda_1, \cdots, \lambda_m$ be the distinct eigenvalues of T, with $U_1, \cdots, U_m$ denoting the corresponding sets of generalized eigenvectors. Then,

(1) $V = U_1 \oplus \cdots \oplus U_m$ (Primary decomposition theorem)

(2) $T|_{U_j} : U_j \to U_j$ (Invariant subspaces)

(3) Each $(T - \lambda_j I)|_{U_j}$ is nilpotent. ($\to$ Jordan form)

(4) Each $T|_{U_j}$ has only one eigenvalue, namely λ_j.

Proof)

(1) Prop 3.4

(2) Suppose $v \in U_j$. then $(T - \lambda_j I)^k v = 0$ for some positive integer k. Tv 가 다시 U_j에 들어가는지 확인한다.

$$(T - \lambda_j I)^k Tv = T(T - \lambda_j I)^k v = T(0) = 0$$

$$\Rightarrow Tv \in U_j$$

$$\Rightarrow T \mid U_j : U_j \to U_j$$

(3)
$$v \in U_j \iff v \in \ker(T - \lambda_j I)^n$$

$$\iff (T - \lambda_j I)^n v = 0 \text{ for all } v \in U_j$$

$$\iff T - \lambda_j I|_{U_J} \text{ is nilpotent}$$

(4) λ_j, λ : eigenvalue of $T|_{U_j}$, $v \neq 0$: generalized eigenvector of λ_j.

$$Tv = \lambda v$$

$$(T - \lambda_j I)v = Tv - \lambda_j v = \lambda v - \lambda_j v = (\lambda - \lambda_j)v$$

$$\Rightarrow (T - \lambda_j I)^k v = (\lambda - \lambda_j)^k v$$

$$\Rightarrow 0 = (\lambda - \lambda_j)^k v$$

$$\Rightarrow \lambda = \lambda_j$$

3.4. The Minimal Polynomial

Because the space of linear operators on V is finite dimensional, there is a smallest positive integer k such that $I, T, T^2, \cdots, T^k$ are not linearly independent.

$$\{\forall L : V \to V\}$$

$\dim V \times \dim V$

Thus there exists **unique** complex numbers $a_0, \cdots, a_{k-1}$ such that

$$a_0 I + a_1 T + a_2 T^2 + \cdots + a_{k-1} T^{k-1} + T^k = 0$$

Def The polynomial $a_0 + a_1 z + a_2 z^2 + \cdots + a_{k-1} z^{k-1} + z^k$ is called the *minimal polynomial* of T. It is the monic polynomial p of smallest degree such that $p(T) = 0$.

최고차항 계수는 가장 작다는 조건에 의해 0이 아니다. 따라서 일반성을 잃지 않고 최고차항 계수를 1로 둘 수 있다.

가장 작은 k 이므로 만약 다른 계수 b_i가 존재한다면 서로 빼서 $a_i = b_i$ 임을 알 수 있다.

최소다항식을 정의할 때 기저를 고려하지 않았다. 그렇기 때문에 최소다항식은 기저에 무관하고 오로지 T에만 의존한다.

```
format long g

N = 3;

% T = randi([-10, 10], N);
T = [
    1 2 4
    1 2 3
    -1 0 1.2
];

% v must not be 0
v = randi([-10, 10], N, 1);

M = [v];
while size(M, 2) <= rank(M)
    M = [M, T * M(:, end)];
end

ker = null(M);
W = zeros(N);
L = eye(N);
for i = 1:length(ker)
    W = W + ker(i) * L;
    L = T * L;
end

ker = ker / ker(end);
clc
ker
```

Thm 4.1 Let $\lambda_1, ..., \lambda_m$ be the distinct eigenvalues of T, Let U_j denote the set of generalized eigenvectors corresponding to λ_j, and let α_j be the smallest positive integer such that $\left(T - \lambda_j I\right)^{\alpha_j} U_j = \{0\}$. Let

$$p(z) = (z - \lambda_1)^{\alpha_1} (z - \lambda_2)^{\alpha_2} \cdots (z - \lambda_m)^{\alpha_m}$$

Then

(1) p has degree at most $\dim V$.

(2) if q is a polynomial such that $q(T) = 0$, then q is a polynomial multiple of p.

(3) p is the minimal polynomial of T.

Proof)

(1) $V = U_1 \oplus \cdots \oplus U_m$ 이므로 $\sum_j \alpha_j \leq \dim V$

(2) Fix j. The polynomial q has them form

$$q(z) = c(z - r_1)^{p_1} \cdots (z - r_M)^{p_M} \left(z - \lambda_j\right)^p$$

where $c \in \mathbb{C}^*$, the r_k's are complex numbers all different from λ_j, the p_k's positive integer, and p is a non-negative integer.

Suppose $v \in U_j$. then $\left(T - \lambda_j I\right)^p v$ is also U_j.

$$q(T)v = 0v = 0$$

$$c(T - r_1 I)^{p_1} (T - r_2 I)^{p_2} \cdots (T - r_M I)^{p_M} \underbrace{\left(T - \lambda_j I\right)^p v}_{\in U_j} = 0$$

U_j에서 eigenvalue는 λ_j 밖에 없다. $(T - r_i I)\left(T - \lambda_j I\right) = \left(T - \lambda_j I\right)(T - r_i I)$ 이므로 $T - r_i I$는 U_j에서 linear operator이다.

$$\left(T - \lambda_j I\right)^k \underbrace{(T - r_i I)v}_{\in U_j} = (T - r_i I)\left(T - \lambda_j I\right)^k v = 0$$

또한 eigenvalue 정의에 의해 $T - r_k I$는 U_j에서 모두 단사이다.

따라서

$$(T - \lambda_j I)^p v = 0$$

Because v was an arbitary element of U, this implies that $\alpha_j \leq p$. q is a polynomial multiple of $(z - \lambda_j)^{\alpha_j}$. The process works for each j. Thus (2) holds.

(3) Since $V = U_1 \oplus \cdots \oplus U_M$, suppose V is a vector in some U_j. If we commute the terms of $(T - \lambda_1 I)^{\alpha_1} \cdots (T - \lambda_M I)^{\alpha_M}$ so that $(T - \lambda_j I)^{\alpha_j}$ is on the right. we see that $p(T)v = 0$.

We know from (2) that no monic polynomial of lower degree has this property. Thus p must be the minimal polynomial of T, completing the proof. ■

Note that by avoiding determinants, we have been naturally led to the description of the minimal polynmoial in terms of generalized eigenvectors.

3.5. Multiplicity and The Characteristic Polynomial

중복도를 정의한다.

Def The ***multiplicity*** of an eigenvalue λ of T is defined to be the dimension of the set of generalized eigenvectors of T corresponding to λ.

The sum of the multiplicities of all eigenvalues of T euqlas the dimension of V.

$$\det(zI - T) = (-1)^n \det(T - zI)$$

행렬식 정의와는 다르게 generalized eigenvector 정의를 쓰면 T의 기하학적 행동과 어떠한 관련이 있는지 명확하게 드러난다.

Def Let $\lambda_1, \cdots, \lambda_m$ denote the distinct eigenvalues of T, with corresponding multiplicities $\beta_1, \cdots, \beta_m$. The polynomial

$$(z-\lambda_1)^{\beta_1}(z-\lambda_2)^{\beta_2} \cdots (z-\lambda_m)^{\beta_m}$$

is called *the characteristic polynomial* of T.

Clearly, it is a polynomial of degree $\dim V$.

기존 선대에서는 det으로 characteristic polynomial을 먼저 정의하고 eigenvalue의 존재성을 보였지만 여기선 그러한 접근을 뒤집었다.

Thm 5.2 (Cayley-Hamilton Theorem) Let q denote the characteristic polynomial of T. Then, $q(T) = 0$.

Proof)

$U_j : \mathrm{GE}(T, \lambda_j)$

α_j : the smallest positive integer such that $(T - \lambda_j I)^{\alpha_j} U_j = \{0\}$

Lemma 3.1을 증명할 때 $T|_{U_j} : U_j \to U_j$ 이고, $v \in U_j$에 대해

$$v, (T-\lambda_j I)v, (T-\lambda_j)^2 v, \cdots, (T-\lambda_j I)^{\alpha_j - 1} v$$

가 linearly independent 함을 보였으므로 $\alpha_j \le \beta_j$. 위 시퀀스가 U_j를 생성함을 보이지 않았기에 $\alpha_j < \beta_j$일 수도 있음.

Hence the characteristic polynomial is a polynomial multiple of the minimal polynomial. Thus q must annihilate T. ■

3.6. Upper-Triangular Form (Jordan Form)

이 부분은 Linear Algebra Done Right 책 내용을 다룹니다.

Camille Jordan (1838-1922) published a proof of 8.46 (Jordan form) in 1870.

8.42 Example: nilpotent operator with nice matrix

Let T be the operator on $\mathbb{C}^4$ defined by

$$\begin{aligned} T(z_1, z_2, z_3, z_4) &:= (0, z_1, z_2, z_3) \\ T^2(z_1, z_2, z_3, z_3) &= (0, 0, z_1, z_2) \\ T^3(z_1, z_2, z_3, z_3) &= (0, 0, 0, z_1) \\ T^4(z_1, z_2, z_3, z_3) &= (0, 0, 0, 0) \end{aligned}$$

Let $v = (1, 0, 0, 0)$.

$$\begin{cases} T^3v = (0, 0, 0, 1) \\ T^2v = (0, 0, 1, 0) \\ Tv \;\;= (0, 1, 0, 0) \\ v \quad\;= (1, 0, 0, 0) \end{cases}$$

are a basis of $\mathbb{C}^4$. The matrix of T with respect to this basis T^3v, T^2v, Tv, v is

$$T = \left[(T^4v) \;\; (T^3v) \;\; (T^2v) \;\; (Tv) \right] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

만약 $v, Tv, T^2v, \cdots$가 V를 생성한다면 Jordan form은 쉽게 구해진다.

8.44 Definition : Jordan basis

Suppose T is an linear operator $V \to V$ (i.e $T \in \mathcal{L}(V)$). A basis of V is called a *Jordan basis* for T if with respect to to this basis T has a block diagonal matrix.

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_p \end{pmatrix}$$

in which each A_k is an upper-triangular matrix of the form

$$A_k = \begin{pmatrix} \lambda_k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{pmatrix}$$

with eigenvalues λ_k of T.

Most of the work in proving that every operator on a finite-dimensional complex vector space has a Jordan basis occurs in **proving the special case below of nilpotent operator.**

8.45 Theorem: every nilpotent operator has a Jordan basis

Suppose $T \in \mathcal{L}(V)$ is nilpotent. Then there is a basis of V that is Jordan basis for T.

Proof) We will prove this result by induction on $\dim V$.

To get started, note that the desired result holds if $\dim V = 1$. (because in that case, the only nilpotent operator is the (0) operator.)

이 때 0이 eigenvalue이다.

Now assume that $\dim V > 1$ and that the desired result holds on all vector space of smaller dimension. Let m be the smallest positive integer such that $T^m = 0$. thus there exists $u \in V$ such that $T^{m-1}u \neq 0$. Let

$$U = \operatorname{span}(u, Tu, ..., T^{m-1}u)$$

The list $u, Tu, ...T^{m-1}u$ is linearly indepdent (Why?).

If $U = V$, then writing this list in reverse order gives a Jordan basis for T and we are done.

> 이 때문에 위 예제를 다뤘.

thus we can assume that $U \neq V$. Note that U is invariant under T (i.e $TU \subseteq U$).

> 왜냐하면 $v \in U$라면
>
> $$\begin{aligned} Tv &= T(a_0 u + a_1 Tu + \cdots + a_{m-1} T^{m-1}u) \\ &= a_0 Tu + a_1 T^2 + \cdots + a_{m-2} T^{m-1}u \in U \end{aligned}$$

다시 U로 생성되니까 $Tv \in U$이다.

By our induction hypothesis, there is a basis of U that is a Jordan basis for $T|_U$.

Strategy⟩ The strategy of our proof is that we will find a subspace W of V such that W is also invariant under T and $V = U \oplus W$. Again by our induction hypothesis, there will be a Jordan basis of W for $T|_W$. Putting together the Jordan bases for $T|_U$ and $T|_W$, we will have a Jordan basis for T.

Let $\varphi \in V^*$ (i.e. dual space of V, $\varphi : V \to \mathbb{C}$) be such that $\varphi(T^{m-1}u) \neq 0$.

> 즉 $T^m = 0$하면 다 죽으니까 마지막까지 버틴 값을 찾고 싶다는 의미이다.

Let $W = \{v \in V : \varphi(T^k v) = 0 \text{ for each } k = 0, 1, ..., m-1\}$.

U에 있는 원소는 $T^m = 0$ 에서만 죽고 그 전에는 죽지 않으니까 U에 없는 원소를 가져오고 싶다는 의미이다.

Then, W is a subspace of V that is invariant T. ($\because \varphi\big(T^k(Tv)\big) = \varphi\big(T^{k+1}v\big) = 0$ for $k = 0, ..., m-1$, and the case $k = m$ holds because $T^m = 0$)

Suppose $v \in U$ with $v \neq 0$.

$c_0 \neq 0$

$$v = c_0 u + c_1 Tu + \cdots + c_{m-1} T^{m-1} u$$

$$T^{m-1} v = c_0 T^{m-1} u$$

$$\varphi(T^{m-1} v) = c_0 \varphi(T^{m-1} u) \neq 0$$

$$\Rightarrow v \notin W$$

$$\Rightarrow U \cap W = \{0\}$$

아직까진 $\dim V = 10$인데 $\dim U = 6, \dim W = 2$여서 $V \neq U \oplus W$일수도 있다. 이러한 상황이 불가능함을 보인다.

Define $S : V \to \mathbb{C}^m$ by

$$Sv = \big(\varphi(v), \varphi(Tv), ..., \varphi(T^{m-1} v)\big)$$

Thus $\ker S = W$. Hence

Dimension theorem $L : V \to$ an vector space,

$$\dim V = \dim \ker L + \dim \operatorname{Im} L$$

$$\begin{aligned} \dim W &= \dim \ker S \\ &= \dim V - \dim \operatorname{Im} S \\ &\geq \dim V - m \end{aligned}$$

$$\begin{aligned}\dim V \geq \dim U \oplus W &= \dim U + \dim W \\ &\geq m + \dim V - m \\ &= \dim V\end{aligned}$$

Thus $U \oplus W = V$, completing the proof. ■

8.46 Jordan form

There is a basis of V that is Jordan basis for $T \in \mathcal{L}(V)$.

Proof) $\lambda_1, \cdots, \lambda_m$: distinct eigenvalue of T.

$$V = \mathrm{GE}(T, \lambda_1) \oplus \cdots \oplus \mathrm{GE}(T, \lambda_m)$$

where each $T - \lambda_k I|_{\mathrm{GE}(T,\lambda_k)}$ is nilpotent (Why?). $T - \lambda_k I|_{\mathrm{GE}(T,\lambda_k)}$ has the only eigenvalue 0. Some basis of each $\mathrm{GE}(T, \lambda_k)$ is a Jordan basis of each $(T - \lambda_k I)|_{\mathrm{GE}(T,\lambda_k)}$. Put these bases together to get a basis of V that is Jordan basis for T. ■

Jordan-Chevalley decomposition

3.7. The Spectral Theorem

We assume that $\langle \cdot, \cdot \rangle$ is an inner product on V.

Inner product $\langle \cdot, \cdot \rangle : V^2 \to \mathbb{F}$ has the following properties.

1. Positivity: $\langle v, v \rangle \geq 0 \forall v \in V$
2. Definiteness: $\langle v, v \rangle = 0$ iff $v = 0$
3. Linearity in first slot: $\langle cx + y, z \rangle = c\langle x, z \rangle + \langle y, z \rangle$
4. Conjugate symmetry: $\langle a, b \rangle = \overline{\langle b, a \rangle}$

Def The *adjoint of* T is the **unique** linear operator T^* on V such that

$$\langle Tu, v\rangle = \langle u, T^*v\rangle$$

for all $u, v \in V$.

적당한 basis를 잡으면 익숙한 행렬로 표현될테니 $\langle Tu, v\rangle = (Tu)^* v = u^* T^* v = \langle u, T^* v\rangle$로 직관적으로 이해할 수 있다.

$\langle Tu, v\rangle = \langle u, Pv\rangle = \langle u, T^*v\rangle$일 때 V의 orthonormal basis 넣어주다보면 $P = T^*$가 되므로 존재한다면 유일하다.

Def The linear operator T is called *normal* if T commutes with its adjoint; in other words, T is normal if $TT^* = T^*T$.

Def The linear operator T is called *self-adjoint* if $T = T^*$.

Obviously, self-adjoint $\Rightarrow$ normal.

Lemma 7.1 If T is normal, then $\ker T = \ker T^*$.

Proof) $v \in V$.

$$\begin{aligned}\mathbb{R} \ni \langle Tv, Tv\rangle = \langle v, T^*Tv\rangle &= \overline{\langle T^*Tv, v\rangle} \\ = \langle T^*Tv, v\rangle &= \langle TT^*v, v\rangle \\ = \langle T^*v, T^*v\rangle &\end{aligned}$$

Thus $Tv = 0$ iff $T^*v = 0$.

The next proposition shows that the eigenvectors of a normal operator span the domain.

Prop 7.2 Every generalized eigenvector of a normal operator is an eigenvector of the operator.

Proof) Suppose T is normal.

Strategy⟩ Prove that $\ker T^* = \ker T$ for every positive integer k. This will complete the proof of the proposition, because we can replace T by $T - \lambda I$ for arbitrary $\lambda \in \mathbb{C}$.

If T is normal, then $T - \lambda I$ is also normal.

$$(T - \lambda I)^* = T^* - \overline{\lambda} I$$

$(T - \lambda I)$ commutes with $\left(T^* - \overline{\lambda} I\right)$. Thus $T - \lambda I$ is normal.

We prove by induction on k, $\ker T^k = \ker T$. If $k = 1$, we are done.

Suppose that k is a positive integer such that $\ker T^k = \ker T$ holds. Let $v \in \ker T^{k+1}$.

$$T\left(T^k v\right) = T^{k+1} v = 0$$

$$\Rightarrow T^k v \in \ker T$$

And so, by Lemma 7.1 ($\ker T = \ker T^*$), $T^*\left(T^k v\right) = 0$. Thus

$$\langle T^*\left(T^k v\right), T^{k-1} v\rangle = \langle 0, T^{k-1} v\rangle = 0$$

$$\langle T^*\left(T^k v\right), T^{k-1} v\rangle = \langle T^k v, T^k v\rangle = 0$$

$$\Rightarrow T^k v = 0$$

Hence, $v \in \ker T^k$, which implies that $v \in \ker T$ by our induction hypothesis. Thus $\ker T^{k+1} = \ker T$.

Prop 7.4 Eigenvectors of a normal operator corresponding to distinct eigenvalues are orthogonal.

Proof)

$$T : \text{normal}, \quad \alpha \neq \lambda : \text{eigenvalues of } T$$
$$u \quad\; v : \text{corresponding eigenvectors}$$

$$(T - \lambda I)v = 0$$
$$\left(T^* - \overline{\lambda} I\right)v = 0$$

$$\begin{aligned}(\alpha - \lambda)\langle u, v\rangle &= \langle \alpha u, v\rangle - \langle u, \overline{\lambda} v\rangle \\ &= \langle Tu, v\rangle - \langle u, T^* v\rangle \\ &= \langle Tu, v\rangle - \langle Tu, v\rangle \\ &= 0 \\ \Rightarrow \langle u, v\rangle &= 0\end{aligned}$$

Thm 7.5 (Spectral Theorem) There is an orthogonal basis of V consisting of eigenvectors of T if and only if T is normal.

Proof) $(\Rightarrow)$

$\exists$ orthonormal basis of V consisting of eigenvectors of T

$\Rightarrow T$ has a diagonal matrix with respect to that basis

$\Rightarrow T^*$: diagonal matrix (conjugated)

$\Rightarrow T$ and T^* commute

$\Rightarrow T$: normal

$(\Leftarrow)$ Suppose T is normal.

T가 normal하면 GE와 ker가 같으므로

$$\begin{aligned}V &= \mathrm{GE}(T, \lambda_1) \oplus \cdots \oplus \mathrm{GE}(T, \lambda_m) \\ &= \ker(T - \lambda_1 I) \oplus \cdots \oplus \ker(T - \lambda_m I)\end{aligned}$$

각각의 $\ker(T - \lambda_i I)$에서 orthonormal basis를 선택하면 당연히 eigenvector로 이루어져 있고 전부 coproduct해도 orthonormal하고 V 전체를 생성하는 orthornomal basis를 얻는다. ■

$\mathrm{GE}(T, \lambda_i) \neq \ker(T - \lambda_i I)$ 였다면 orthonormal basis는 구성할 수 있을지 몰라도 eigenvector로만 이루어져 있다고 보장할 수 없음

Prop 7.6 Every eigenvalue of a self-adjoint operator is real.

Proof)

$$Tv = \lambda v$$

$$\begin{aligned}\lambda\langle v, v\rangle &= \langle \lambda v, v\rangle = \langle Tv, v\rangle = \langle v, T^* v\rangle \\ &= \langle v, Tv\rangle = \langle v, \lambda v\rangle = \overline{\lambda}\langle v, v\rangle\end{aligned}$$

$$\begin{aligned}&\Rightarrow \lambda = \overline{\lambda} \\ &\Rightarrow \operatorname{Im}\lambda = 0 \\ &\Rightarrow \lambda \in \mathbb{R} \quad \blacksquare\end{aligned}$$

3.8. Getting Real

So far we have been dealing only with complex vector spaces. Each linear operator on a real vector space U can be extended to a linear operator on the complexification of U. Our results about linear operators on complex vector spaces can then be translated to information about linear operator on real vector spaces.

Def Suppose that U is a real vector space. As a set, the *complexification* of U, denoted by U_C, equals U times U. Formally, a typical element of U_C is an ordered pair (u, v), where $u, v \in U$, but we will write this as $u + iv$, for obvious reasons.

We define addition on U_C by

$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2).$$

and define multiplication by complex scalar on U_C:

$$(a + ib)(u + iv) = (au - bv) + i(av + bu)$$

for $a, b \in \mathbb{R}$ and $u, v \in U$. Then, U_C becomes a complex vector space.

We can think of U as a subset of U_C by identifying $u \in U$ with $u + i0$.

Clearly, any basis of the real vector space U is also a basis of the complex vector space U_C. Hence the dimension of U as a real vector space equals the dimension of U_C as a complex vector space.

basis에 i를 곱해도 되기 때문이다.

$v \in U \Rightarrow iv \in U_C$.

For S a linear operator on U, the complexification S_C of S is the linear operator on U_C defined by

$$S_C(u + iv) = Su + iSv$$

for $u, v \in U$. If we choose a basis of U and also think of it as a basis of U_C, then clearly S and S_C have the same matrix with respect to this basis.

For S a linear operator on a real vector space U, the complexification

Note that any real eigenvalue of S_C is also an eigenvalue of S (because if $a \in \mathbb{R}$ and $S_{C(u+iv)} = a(u + iv)$, then $Su = au$ and $Sv = av$).

Non-real eigenvalues of S_C come in pairs. More precisely,

$$(S_C - \lambda I)^j (u + iv) = 0 \Leftrightarrow> \left(S_C - \overline{\lambda} I\right)^j (u - iv) = 0$$

for j a positive integer, $\lambda \in \mathbb{C}$, and $u, v \in U$.

$T \in \mathrm{End}(U_S)$라면 어떤 basis를 선택했을 때

$$T = \begin{pmatrix} & & \\ & \cdots & \\ & & \end{pmatrix} = \begin{pmatrix} & & \\ & \cdots & \\ & & \end{pmatrix} + i \begin{pmatrix} & & \\ & \cdots & \\ & & \end{pmatrix} = T_1 + iT_2$$

여기서 $T_1, T_2 \in \mathrm{End}(U)$이다. basis를 바꾸면 T_1, T_2가 다시 분해되므로 여전히 U의 linear operator 두 개로 분해할 수 있다. 이후 basis를 forgetting 하면 basis에 무관한 linear operator를 얻는다.

그러므로 $\overline{Tv} = \overline{T}\overline{v}$가 된다. 또한 $S_C v = Sv$로 정의되므로 $\overline{S_C} = S_C$가 된다.

In particular, if $\lambda \in \mathbb{C}$ is an eigenvalue of S_C, then so is $\overline{\lambda}$, and the multiplicity of λ is the same as the multiplicity of $\overline{\lambda}$.

Because the sum of the multiplicities of all the eigenvalues of S_C equals the (complex) dimension of U_C, we see that if U_C has odd (complex dimension, then S_C must have a real eigenvalue.

모든 j에 대해 성립하기 때문이다.

Thm 8.2 Every linear operator on an odd-dimensional real vector space has a real eigenvalue.

Once again, a proof without determinants offers more insight into why the result holds than the standard proof using determinants.

The minimal and characteristic polynomials of a linear operator S on a real vector space are defined to be the corresponding polynomials of the complexification S_C. Both these polynomials have real coefficients—this follows from our definitions of minimal and characteristic polynomials.

Minimal polynomial

$$c_0 I + c_1 S + c_2 S^2 + \cdots + c_{n-1} S^{n-1} + S^n = 0$$

$$\overline{c_0} I + \overline{c_1} S + \overline{c_2} S^2 + \cdots + \overline{c_{n-1}} S^{n-1} + S^n = 0$$

$$\Rightarrow c_i = \overline{c_i}$$

$$\Rightarrow c_i \in \mathbb{R}$$

Characteristic polynomial

Eigenvalue λ와 $\overline{\lambda}$가 같은 multiplicity를 가지므로

$$\begin{aligned} p(z) &= \cdots (z - \lambda)^{\beta} \left(z - \overline{\lambda}\right)^{\beta} \cdots \\ &= \cdots \left(z^2 - \left(\lambda + \overline{\lambda}\right) z + \lambda\overline{\lambda}\right)^{\beta} \cdots \end{aligned}$$

항이 characteristic polynomial에 나타난다. 만약 홀수 차원을 가졌나면 나머지 eigenvalue 하나는 반드시 실수가 된다.

We make the complexification U_C into a complex inner product space by defining an inner product on U_C in the obvious way.

$$\langle u_1 + iv_1, u_2 + iv_2 \rangle = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle + i\langle v_1, u_2 \rangle - i\langle u_1, v_2 \rangle$$

Note that any orthonormal basis of the real inner product space U is also an orthonormal basis of the complex inner product space U_C.

If S is a self-adjoint operator on U, then obviously S_C is self-adjoint on U_C.

Let S be a self-adjoint operator on U.

$$\begin{aligned}\langle S(u_1 + iv_1), u_2 + iv_2\rangle &= \langle Su_1 + iSv_1, u_2 + iv_2\rangle \\ &= \langle Su_1, u_2\rangle + \langle Sv_1, v_2\rangle + i\langle Sv_1, u_2\rangle - i\langle Su_1, v_2\rangle \\ &= \langle u_1, Su_2\rangle + \langle v_1, Sv_2\rangle + i\langle v_1, Su_2\rangle - i\langle u_1, Sv_2\rangle \\ &= \langle u_1 + iv_1, Su_2 + iSv_2\rangle \\ &= \langle u_1 + iv_1, S(u_2 + iv_2)\rangle\end{aligned}$$

The next theorem gives the formal statement of the result and the details of the proof.

Thm 8.3 Suppose U is a real inner product space and S is a linear operator on U. Then there is an orthonormal basis of U consisting of eigenvectors of S if and only if S is self-adjoint.

Proof) ($\Rightarrow$) Suppose there is an orthonormal basis of U consisting of eigenvectors of S. With respect to that bais, S has a diagonal matrix. Clearly, S^* with respect to the same basis equals the matrix of S. Thus S is self-adjoint.

($\Leftarrow$) Suppose that S is self-adjoint. S_C is self-ajoint on U_C. Thus there is a basis

$$\{u_1 + iv_1, \cdots, u_n + iv_n\}$$

of U_C consisting of eigenvectors of S_C by the complex Spectral Theorem; here each u_j and v_j is in U. Each eigenvalue of S_C is real and thus u_j and each v_j is an eigenvector of S.

$$S_{C(u+iv)} = \lambda(u + iv)$$

$$Su + iSv = \lambda u + i\lambda v$$

$$Su = \lambda u, \quad Sv = \lambda v$$

For any $v \in U$, v is expressed as a linear combination of the basis

$$v = v + 0i = (a_1 + ib_1)(u_1 + iv_1) + \cdots + (a_n + ib_n)(u_n + iv_n)$$
$$= (a_1u_1 - b_1v_1 + \cdots + a_nu_n - b_nv_n) + i(b_1u_1 + a_1v_1 + \cdots + b_nu_n + a_nv_n)$$

for $\forall a_i, b_i \in \mathbb{R}$. Clearly, $\{u_1, v_1, \cdots, u_n, v_n\}$ spans U. In other words, the eigenvectors of S spans U.

For each eigenvalue of S, choose an orthonormal basis of the corresponding eigenspace in U.

> 정 안되면 고른 eigenvalue에 대응하는 u_i, v_i들을 가져다가 inner product space이니 그람 슈미트쓰면 된다. 그람 슈미트를 써도 기존 basis로 표현되니 eigenvalue에 대응하는 eigenspace를 벗어나지 않는다.

Eigenvectors corresponding to distinct eigenvalues are orthogonal. The union of these bases (one for each eigenvalue) is still orthonormal. The span of this union includes every eigenvector of S. We have just seen that the eigenvectors of S span U, and so we have an orthonormal basis of U consisting of eigenvectors S, as needed. ■

3.9. Determinants

At thisstage we have proved most of the major structure theorems of linear algebra without even defining determinants. In this seciton we will give a simple definition of determinants, whose main reasonable use in undergraduate mathematics is in the change of variables formula for multi-variable integrals.

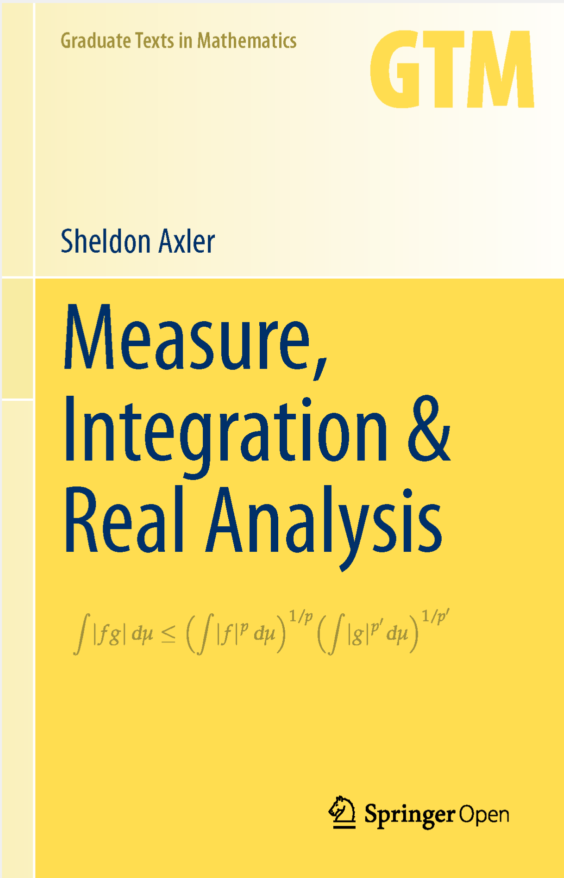
> $\varphi : U \to V$, sending $\mathsf{u} \to \mathsf{v}$
>
> $$\int_V f(\mathsf{v})\nu_V = \int_U f \circ \varphi\,(\mathsf{u}) \det(\varphi'(\mathsf{u}))\nu_U$$
>
> 이 때 φ는 orientation을 바꾸지 않아야 한다. 식으로 쓰니까 어려워보일뿐 사실 d 붙여주고 $\wedge$해주면 된다.

Let's look at some additional motivation for stuyding the product of the eigenvalues.

Suppose we want to know how to make a change of variables in a multivariable integral over some subset of $\mathbb{R}^n$. After linearization, this reduces to the question of how a linear operator S on $\mathbb{R}^n$ changes volumes.

- Linearization 설명, Thurston
- Pushforward

저자의 르베그 측도



- Measure, Integration & Real Analysis https://measure.axler.net/MIRA.pdf
- https://measure.axler.net/

Let's consider the special case where S is self-adjoint. Then there is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of S. A moment's though about the geometry of an orthonormal basis of eigenvectors shows that if E is a subset of $\mathbb{R}^n$, then the volume (whatever that means) of $S(E)$ must equal the volume of E multiplied by the absolute value of the product of the eigenvalues of S.

부피가 orthonormal basis의 합으로 나타내어지고 S가 orthonormal basis 아래서는 단지 scaling 연산 밖에 없으니 eigenvalue만큼 각 basis 길이가 곱해진다. 따라서 모든 eignvalue의 곱이 중요한 object로 나타난다.

At any rate, we see that the product of the eigenvalues seems to be an intersting object.

An arbitrary linear operator on a real vector space need not have any eigenvalues, so we will return to our familiar setting of a linear operator T on a complex vector space V. After getting the basic results on complex vector spaces, we'll deal with real vector spaces by using the notion of complexificatio discussed earlier.

Def The *determinant* of T, denoted $\det T$, is defined to be the product of the eigenvalues of T, counting multiplicity.

This definition would not be possible with the traditional approach to eigenvalues, because that method uses determinants to prove that eigenvalues exist. We already know (by Theorem 3.11 (a)) that T has $\dim V$ eigenvalues, counting multiplicity.

$V = U_1 \oplus \cdots \oplus U_m$이고 multiplicity만큼 eigenvalue가 있다고 보고 있으므로 총 $\dim V$ 만큼 있다.

Thus our simple definition makes sense.

In addition to simplicity, our definition also makes transparent the following result, which is not at all obvious from the standard definition.

Thm 9.1 An operator is invertible if and only if its determinants is non-zero.

Proof) ($\Rightarrow$) T is invertible $\Rightarrow$ $T - 0I$ is injective. Thus T does not have eigenvalue of 0. Its determinants is the product of all eigenvalues of T, each of which is non-zero. Hence $\det T \neq 0$.

($\Leftarrow$) If all eigenvalues are non-zero, then $\det(T)$ is non-zero. This implies that at least one eigenvalue is 0. $T - 0I = T$ is not injective. T is not invertible. ∎

Prop 9.2 The characteristic polynomial of T equals $\det(zI - T)$.

Proof) Let $\lambda_1, \cdots, \lambda_m$ denote the eigenvalues of T, with multiplicities $\beta_1, \cdots, \beta_m$.

Thus for $z \in \mathbb{C}$, the eigenvalues of $zI - T$ are $z - \lambda_1, \cdots, z - \lambda_m$, with multiplicities $\beta_1, \cdots, \beta_m$. (Why?)

Shift property of a eigenvalue

Hence the determinant of $zI - T$ is the product

$$(z - \lambda_1)^{\beta_1} \cdots (z - \lambda_m)^{\beta_m}$$

which equals the characteristic polynomial of T.

Note that determinant is a similarity invariant. In other words, if S is an invertible linear operator on V, then T and $S^{-1}TS$ have the same determinant (because they have the same eigenvalues, counting multiplicity).

$$(S^{-1}TS-\lambda I)^j = (S^{-1}(T-\lambda I)S)^j = S^{-1}(T-\lambda I)^j S$$

이니까 $\mathrm{GE}(T,\lambda)$은 basis만 바뀌어서 $S^{-1}\,\mathrm{GE}(T,\lambda)$에 대응된다.

Fix a basis of V, and for the rest of this section let's identify linear operators on V with matrices with respect to that basis.

How can we find the determinant of T from its matrix, without finding all the eigenvalues? Although getting the answer to that question will be hard, the method used below will show how someone might have discovered the formula for the determinant of a matrix. Even with the derivaiton that follows, determinants are difficult, which is precisely why they should be avoided.

나머지 부분은 Leibniz formula 증명과 T로 선형변환하면 volume이 det 만큼 변한다는 걸 보인다. 그런데 몇 가지 직관적인 가정이 필요하고 이 방식이 괜찮은 지 모르겠다.

det과 순열을 연관시키는 아이디어의 핵심은 Jordan form과 multiplicative이지만 제 생각으로는 det의 characterizing property를 이끌어 내는 편도 나쁘지 않은 것 같다.

Prop 9.5

$$\det(T) = \sum_{\pi} (\operatorname{sgn}\pi) T_{\pi(1),1} \cdots T_{\pi(n),n}$$

Lemma 9.6 Let S be a linear operator on a real inner product space U. Then there exists a linear isometry A on U such that $S = A\sqrt{S^*S}$.

Thm 9.7 Let S be a linear operator on $\mathbb{R}^n$. Then

$$\operatorname{vol} S(E) = |\det S| \operatorname{vol} E$$

for $E \subset \mathbb{R}^n$.

대칭군과 행렬의 basis가 맞물리는 방식이 det의 multiplicative property로 나타난다.