Signals and Systems

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1 Basic concepts

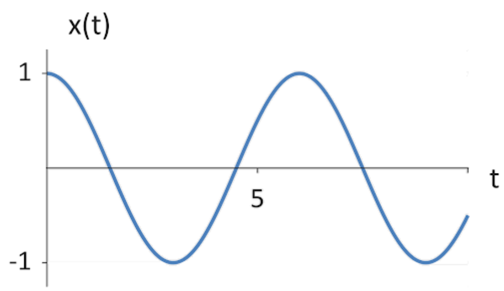
1.1 Signals

Signal is a function that conveys information about a phenomenon.
It is basically a function of an independent variable. For example, $x(t)$ is a signal that varies with time t.
Note that the independent variable can be anything but it is considered to be time unless specified otherwise.

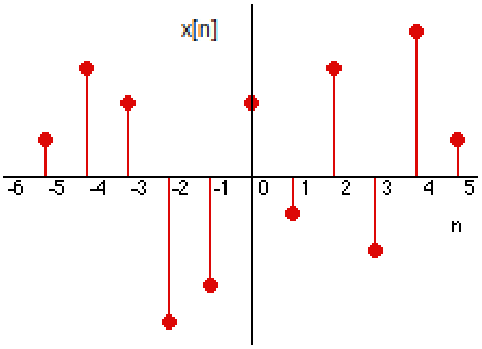
1.2 Classification of Signals

Signals are classified based on different properties.
Based on the nature of independent variable, they are classified as Continuous Time and Discrete Time Signals.

Continuous Time Signal
Signal is defined for every instant of time.



Discrete Time Signal
Signal is defined only for discrete values represented as integers.



Based on the nature of dependent variable, they are classified as Deterministic and Non-deterministic (Random) Signals.

Deterministic Signal
Signal which can be precisely written in form of mathematical equation and one can calculate the exact magnitude of the signal at given point of time.

Non-deterministic / Random Signal
Signal which is not deterministic i.e takes random values and magnitude can't be precisely calculated at any point of time.

1.2.1 Types of Deterministic Signals

- Periodic Signal and Aperiodic Signal
- Real Signal and Complex Signal
- Energy Signal and Power Signal (can be cases where neither satisfy)

Periodic Signal
Signal repeats a pattern after a specific interval called the period.
$x(t) = x(t + T)$ where smallest T is the period.

Aperiodic Signal
Signal that is not periodic.
(more on periodicity later)

Real Signal
Signal that has only real part i.e only one component.

Complex Signal
Signal that has imaginary and real parts i.e two independent components (in-phase and quadrature phase).
$x(t) = re(t) + j\ im(t)$
$Real[x(t)] = re(t)$ and $Imaginary[x(t)] = im(t)$

Energy Signal
Signal which has finite energy and zero power. Meaning, energy of signal can be calculated.

Power Signal
Signal which has infinite energy and finite power. Meaning, power of signal can be calculated.

NENP Signal
Signal which is neither energy nor power signal. Typically, both energy and power are infinite.

Formulae for Energy and Power:
Continuous Time Signal-

$$E_c = \int_{-\infty}^{\infty} |x(t)|^2 dt \qquad \text{Energy}$$

$$P_{apc} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \qquad \text{Aperiodic signal power}$$

$$P_{pc} = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \qquad \text{Periodic signal power}$$

Note that instead of $-T/2$ to $T/2$, the integral can be calculated over any valid period (like from 0 to T).

Discrete Time Signal-

$$E_d = \sum_{-\infty}^{\infty} |x(n)|^2 \qquad \text{Energy}$$

$$P_{apd} = \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-\infty}^{\infty} |x(n)|^2 \qquad \text{Aperiodic signal power}$$

$$P_{pd} = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 \qquad \text{Periodic signal power}$$

Identification of Energy, Power and NENP Signals (Graph method):

Energy Signal

- Finite duration
- Infinite duration with decaying amplitude on both sides

(note that energy signals are absolutely integrable)

NENP Signal

- Finite duration with infinite magnitude at 1 or more points
- Infinite duration with increasing amplitude (one side or both sides)
- Infinite duration with infinite magnitude at 1 or more points

Power Signal

- Infinite duration signal with neither decaying nor increasing amplitudes

(note that periodic signals are either Power or NENP signals)

Energy of standard signals:

- Rectangular pulse of amplitude A and length T; $E = A^2T$
- Triangular pulse of height A and length T; $E = \frac{A^2T}{3}$
- Sinusoidal pulse of amplitude A and length T; $E = \frac{A^2T}{2}$

If $x_1(t)$ and $x_1(t)$ have energies E_1 and E_2, and if $x(t) = x_1(t) + x_2(t)$, then the energy of $x(t)$ is given by,

$$E = E_1 + E_2 + 2Area[x_1(t)\ x_2(t)]$$

1.2.2 Types of Real Signals

Real signals can be classified as Even Signals, Odd Signals or Signals that are neither even not odd (NENO).

Even Signal
Signal which is symmetric about the vertical axis. $x(t) = x(-t)$

Odd Signal
Signal which is symmetric about the origin. $x(t) = -x(-t)$
(note that every odd signal will pass through the origin i.e the amplitude at 0 will be 0)

Any real signal can be expressed as sum of an even signal and an odd signal.
$x(t) = x_e(t)+x_o(t)$ where $x_e(t) = [x(t)+x(-t)]/2$ and $x_o(t) = [x(t)-x(-t)]/2$

1.2.3 Types of Complex Signals

Complex signals can be classified as Conjugate Symmetric Signal or Conjugate Anti-Symmetric (Skew Symmetric) Signal or Signals that are neither.

Conjugate Symmetric Signal
Signal that satisfies : $\bar{x}(t) = x(-t)$

Conjugate Anti-Symmetric (Skew Symmetric) Signal
Signal that satisfies : $\bar{x}(t) = -x(-t)$

Any complex signal can be expressed as sum of a CS signal and a CAS(SS) signal.
$x(t) = x_c(t) + x_s(t)$
where $x_c(t) = [x(t) + \bar{x}(-t)]/2$ and $x_s(t) = [x(t) - \bar{x}(-t)]/2$

Even and Odd for Real Signals correspond to Conjugate-Symmetric and Skew-Symmetric Complex Signals

1.3 Operations on Signals

1. Time Scaling
2. Time Shifting
3. Amplitude Scaling
4. Amplitude Shifting

Time Scaling:
$x(t) \to x(at)$
Compression if $a > 1$ and Expansion if $a < 1$
Note that if $a = -1$, it is called Time Folding which flips the signal w.r.t y axis.

- Will affect period
- Will affect energy
- Will not affect power

Time Shifting:
$x(t) \to x(t + T)$
Left shift if T is positive and Right shift if T is negative

- Will not affect period
- Will not affect energy
- Will not affect power

Amplitude Scaling:
$x(t) \to Ax(t)$
Note that if $A = -1$, it is called Amplitude Folding which flips the signal w.r.t x axis

- Will not affect period
- Will affect energy
- Will affect power

Amplitude Shifting:
$x(t) \to x(t) + k$
This is also called DC shifting

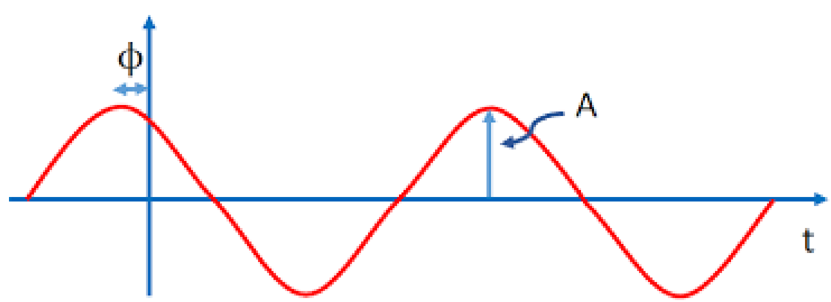
- Will not affect period
- Will affect energy
- Will affect power

Scale(Shift) $\neq$ Shift(Scale)
$x(t) \to Shift \to x(t-T) \to Scale \to x(at-T)$
$x(t) \to Scale \to x(at) \to Shift \to x(a(t-T))$

1.4 Sinusoidal Signals and Periodicity

1.4.1 Continuous Time Sinusoids

A signal of the form $x(t) = Acos(\omega_o t + \phi)$ is a continuous time sinusoidal signal where ω_o is the angular frequency, A is it's amplitude and ϕ is the phase shift.



All continuous time sinusoids are unconditionally periodic with Fundamental Time Period $T_o = 2\pi/\omega_o$.

For a continuous time sinusoid, Time Shift $\leftrightarrow$ Phase Change.
This is because,
$Acos(\omega_o(t+T)+\phi) = Acos(\omega_o t + \omega_o T + \phi) = Acos(\omega_o t + \phi_T)$.

If $\phi = 0$, $x(t)$ is a cosine signal. $\implies$ $x(t) = x(-t)$ i.e it is even signal
If ϕ = -$\pi/2$, $x(t)$ is a sine signal. $\implies$ $x(t) = -x(-t)$ i.e it is odd signal

Basic properties of sine and cosine functions:

1. $sin(\theta + \pi/2) = cos\theta$
2. $cos(\theta + \pi/2) = -sin\theta$
3. $sin(\pi/2 - \theta) = cos\theta$
4. $cos(\pi/2 - \theta) = sin\theta$

Identical CT sinusoids:
Consider 2 CT sinusoidal signals,
$x_1(t) = A_1cos(\omega_1 t + \phi_1)$ and $x_2(t) = A_2cos(\omega_2 t + \phi_2)$.

- If $\omega_1 \neq \omega_2$, then the 2 signals can never be identical
- If $\omega_1 = \omega_2$, then condition for the 2 signals to be identical is, $\omega_1 t + \phi_1 = \omega_2 t + \phi_2 + 2\pi k$, where k is some integer

Period of sum of multiple CT sinusoids (or any periodic signals):

Consider $y(t) = x_1(t) + x_2(t)$ where the periods of $x_1(t)$ and $x_2(t)$ are T_1 and T_2 respectively.

$y(t)$ is only periodic if the ratio of the fundamental frequencies is a rational number i.e $\omega_1/\omega_2 = p/q$

Fundamental frequency, $\omega_o = q\omega_1 = p\omega_2$
Fundamental period, $T_o = 2\pi/\omega_o$

Fundamental frequency of sum of multiple periodic signals assuming the ratios turn out to be rational, will be HCF of all the individual frequencies.

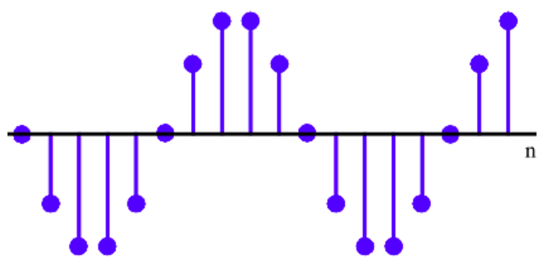
Fundamental time period of sum of multiple periodic signals assuming the ratios turn out to be rational, will be LCM of all the individual time periods.

Note that amplitude and phase can be neglected while calculating the fundamental period or frequency.

- Periodic + Periodic = Periodic or Aperiodic
- Aperiodic + Aperiodic = Aperiodic
- Aperiodic + Periodic = Aperiodic

1.4.2 Discrete Time Sinusoids

A signal of the form $x(n) = Acos(\Omega_o n + \phi)$ is a discrete time sinusoidal signal where Ω_o is the angular frequency, A is it's amplitude and ϕ is the phase shift.



Periodic only if $2\pi/\Omega_o$ is rational number.
$\implies 2\pi m/\Omega_o = N$;
Period N = smallest N such that it is an integer i.e the period has to be an integer.

Time Shift $\implies$ Phase Change : Because $Acos(\Omega_o(n+N)+\phi) = Acos(\Omega_o n+ \Omega_o N + \phi) = Acos(\Omega_o n + \phi_1)$.

Phase Change $\nrightarrow$ Time Shift : Because in $Acos(\Omega_o(n+N)+\phi_1)$, $\phi_1 = \Omega_o N$ is not satisfied for every value of ϕ_1 ; ϕ_1 has to be a multiple of 2π for it to work.

Identical DT sinusoids:
Consider 2 DT sinusoidal signals, $x_1(n) = A_1 cos(\Omega_1 n + \phi_1)$ and $x_2(n) = A_2 cos(\Omega_2 n + \phi_2)$.
Condition for the 2 signals to be identical is :$\Omega_1 n + \phi_1 = \Omega_2 n + \phi_2 + 2\pi k$, where k is some integer.

Hence, it is possible for the 2 signals to be identical even if $\Omega_1 \neq \Omega_2$.

Period of sum of multiple DT sinusoids (or any periodic signals):

Consider $y(n) = x_1(n) + x_2(n)$,
where the periods of $x_1(n)$ and $x_2(n)$ are N_1 and N_2 respectively.

$y(n)$ is unconditionally periodic since the ratio of the fundamental frequencies will certainly be a rational number if both the signals are periodic.

Fundamental frequency, $\Omega_o = q\Omega_1 = p\Omega_2$
Fundamental period, $N = 2\pi/\Omega_o$

Fundamental frequency of sum of multiple periodic signals will be HCF of all the individual frequencies.

Fundamental time period of sum of multiple periodic signals will be LCM of all the individual time periods.

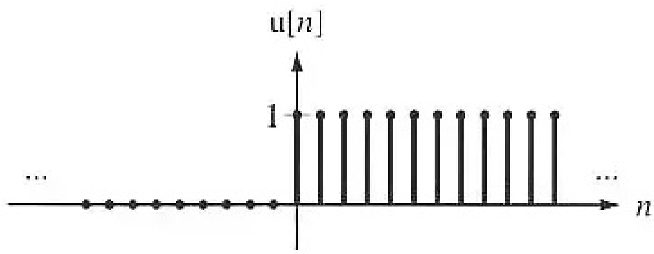
Note that amplitude and phase can be neglected while calculating the fundamental period or frequency.

- Periodic + Periodic = Periodic
- Aperiodic + Aperiodic = Aperiodic
- Aperiodic + Periodic = Aperiodic

1.5 Standard Signals

1.5.1 Discrete Time Unit Step Function

$$u(n) = \begin{cases} 1 & \text{if n} \geq 0 \\ 0 & \text{if n} < 0 \end{cases}$$



1.5.2 Discrete Time Unit Impulse Function

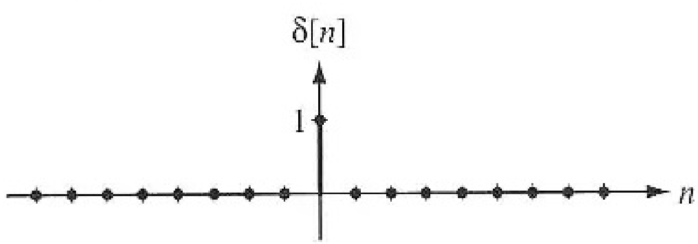
(also called Unit Sample Function)

$$\delta(n) = \begin{cases} 1 & \text{if n} = 0 \\ 0 & \text{if n} \neq 0 \end{cases}$$

$$\rightarrow \delta(n) = u(n) - u(n-1)$$

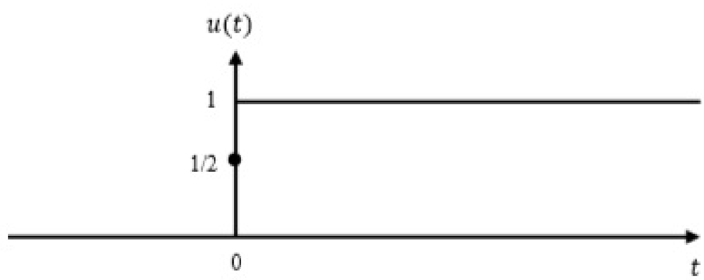
$$\rightarrow u(n) = \sum_{k=0}^{\infty} \delta(n-k)$$

$$\rightarrow u(n) = \sum_{k=-\infty}^{n} \delta(k)$$



1.5.3 Continuous Time Unit Step Function

$$u(t) = \begin{cases} 1 & \text{if t} \geq 0 \\ 0 & \text{if t} < 0 \end{cases}$$



1.5.4 Continuous Time Unit Impulse Function

(also called Dirac Delta Function)

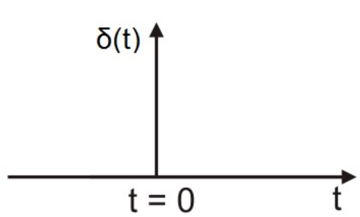
$$\delta(t) = \begin{cases} \infty & \text{if t} = 0 \\ 0 & \text{if t} \neq 0 \end{cases}$$

$$\rightarrow \int_{-\infty}^{\infty} \delta(\tau) d\tau = 1$$

$$\rightarrow \delta(t) = \frac{du(t)}{dt}$$

$$\rightarrow u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

Hence, the Dirac Delta Function has the following properties:
width = 0
height $\rightarrow \infty$
area = 1



Properties of impulse function:

$$\to \delta(t-\tau) = \delta(\tau - t)$$

$$\to \delta(at) = \frac{1}{|a|}\delta(t)$$

$$\to \delta'(at) = \frac{1}{|a|\ a}\delta'(t)$$

$$\to \int_a^b x(t)\ \delta(t-\tau)\ dt = \begin{cases} x(\tau) & if\ a < \tau < b \\ 0 & otherwise \end{cases}$$

$$\to \int_a^b x(t)\ \delta'(t-\tau)\ dt = \begin{cases} -x'(\tau) & if\ a < \tau < b \\ 0 & otherwise \end{cases}$$

$$\to x(t) * \delta(t-\tau) = x(t-\tau)$$

1.6 Systems

System is any physical set of components that takes a signal as input, and produces another signal as output.

$$\xrightarrow{x(t)} \boxed{System} \xrightarrow{y(t)}$$

In the above system, $x(t)$ is the input and $y(t)$ is the output.

1.7 Classification of Systems

1. Based on Linearity: Linear and Non-linear
2. Based on Time Invariance: Time Invariant and Time Variant
3. Based on Memorylessness: Memoryless and Memory
4. Based on Causality: Causal, Non-causal, Anti-causal
5. Based on BIBO Stability: Stable and Unstable
6. Based on Invertibility: Invertible and Non-invertible

1.7.1 Linearity

A system is said to be linear if it follows the principles of Homogeneity and Superposition (Additivity).
Homogeneity: if input signal is amplitude scaled, output should also be amplitude scaled by same factor.
$\implies Ax(t) \rightarrow System \rightarrow Ay(t)$
Additivity: if input is addition of multiple signals, output should be addition of individual outputs.
$\implies x_1(t) + x_2(t) \rightarrow System \rightarrow y_1(t) + y_2(t)$

Therefore, for linearity to be satisfied:
$Ax_1(t) + Bx_2(t) \rightarrow System \rightarrow Ay_1(t) + By_2(t)$

1.7.2 Time Invariance

A system is said to be time invariant if delayed version of input results in delayed version of output (same delay element).
$\implies x(t-T) \rightarrow System \rightarrow y(t-T)$

A system that does not follow this is called Time Varying system.

1.7.3 Memorylessness

A system is said to be memory-less if output depends only on present value of input.
$\implies y(t) = kx(t)$

If output depends on past or future values of input, it has memory.
Such systems are called memory systems or non-memoryless systems.

1.7.4 Causality

A system is said to causal if output depends only on present and past values of input.
$\implies y(t) = kx(t-T) \quad ; T \geq 0$

- Ouput depends only on past values : Purely Causal
- Output depends only on future values : Anti-Causal

- Output depends on future values along with past or present values : Non-Causal

1.7.5 BIBO Stability

A system is said to be "Bounded Input Bounded Output" Stable if for an input which is bounded (i.e does not sum to infinity), the output is also bounded.
$\implies \int x(\tau)d\tau < \infty \implies \int y(\tau)d\tau < \infty$

1.7.6 Invertibility

A system is said to be invertible if it produces distinct output signals for distinct input signals.
Meaning, if there exists such a system which can produce the original input by taking the output as input, then the system is invertible.
$\implies y(t) \rightarrow InverseSystem \rightarrow x(t)$

2 Linear Time-Invariant Systems

LTI System is any system that is both linear and time invariant.

If the system under analysis is an LTI system, then any input signal can be represented as a weighted sum of impulses.
$x(n) = A_0\delta(n) + A_1\delta(n-1) + A_2\delta(n-2) + ... + B_1\delta(n+1) + B_2\delta(n+2) + ...$

Hence, any input signal can be written as value of input at that point multiplied with delta function at the point.
$\implies x(n) = x(0)\delta(n) + x(1)\delta(n-1) + x(2)\delta(n-2) + ... + x(-1)\delta(n+1) + x(-2)\delta(n+2) + ...$

$$\therefore x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

2.0.1 Impulse Response of an System

It is the output of a system if input given is the unit impulse function.
Impulse response is represented by $h(t)$ for CT systems and $h(n)$ for DT systems.
Impulse response can be used to test for time-invariance.
A system is time-invariant if it's impulse response satisfies the following conditions:

- No scaling in time (t or n)
- Amplitude scaling coefficient must be constant
- Any added terms (apart from input) must be constant

Impulse response will play a key role in understanding the response of an LTI system to any arbitrary input.

The frequency domain equivalent of impulse response evaluated at zero initial conditions is called Transfer function (which will be discussed in detail later).

2.1 Convolution

Consider an LTI system of impulse response $h(n)$, with input $x(n)$ and output $y(n)$

$$\xrightarrow{x(n)} \boxed{h(n)} \xrightarrow{y(n)}$$

$y(n) = x(n) * h(n)$ [where * indicates the Convolution operation]
(holds for both CT and DT systems)

Convolution Sum (For Discrete Time) :

$$\boxed{y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)}$$

Length of $y(n)$ = Length of $x(n)$ + Length of $h(n)$ - 1

Convolution Integral (For Continuous Time) :

$$\boxed{y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

Length of $y(t)$ = Length of $x(t)$ + Length of $h(t)$

Lower limit of y = Lower limit of x + Lower limit of h
Upper limit of y = Upper limit of x + Upper limit of h

Properties of Convolution:

- Commutative Law - x(n) * h(n) = h(n) * x(n)
- Associative Law - x(n) * [y(n) * z(n)] = [x(n) * y(n)] * z(n)
- Distributive Law - x(n) * [y(n) + z(n)] = x(n) * y(n) + x(n) * z(n)

(holds for both CT and DT systems)

Cascade connection of 2 systems $\implies$ effective impulse response will be convolution of individual impulse responses.

Rectangular pulse convolution:

- Convolution of two rectangular pulses of amplitude A and equal length T will give triangular pulse of height A^2T and length $2T$.

- Convolution of two rectangular pulses of amplitudes A_1 and A_2 are unequal lengths T_1 and T_2 ($T_2 > T_1$) will give trapezoidal pulse of peak height $A_1 A_2 min(T_1, T_2)$ and length $T_1 + T_2$.
  The trapezoidal pulse will be at it's peak height from T_1 to T_2, measured from the starting point.

2.2 Properties of LTI Systems

Memorylessness Impulse response $h(t) = k\delta(t)$

Causality Impulse response $h(t) = 0 \;\; \forall \;\; t < 0$

BIBO Stability Impulse response must be absolutely integrable.
$\int h(\tau)d\tau < \infty$

Invertibility Impulse response of inverse system is represented as $h^{-1}(t)$
Cascade of $h(t)$ and $h^{-1}(t)$ will result in equivalent system of $\delta(t)$.
$\implies h(t) * h^{-1}(t) = \delta(t)$

Zero Input Response If input $x(t) = 0$, then output $y(t) = 0$.

3 Fourier Analysis

3.1 Continuous Time Fourier Series

Any periodic signal can be expressed as a sum of scaled complex exponentials of the various harmonics of the fundamental frequency.
In linear algebra terms, the complex exponential signals form a basis for the subspace of all periodic signals.

Hence, a linear combination of complex exponential signals of all harmonics of the fundamental frequency will be able to represent any periodic signal.

Exponential Form:

Synthesis equation

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} C_k \; e^{jk\omega_o t}}$$

(this equation is to express the periodic signal $x(t)$ as a linear combination of complex exponents or Fourier series coefficients)

Analysis equation

$$\boxed{C_k = \frac{1}{T_o} \int_{T_o} x(t) \; e^{-jk\omega_o t}}$$

(this equation is to find the value of the scaling factor of each complex exponent in the Fourier series expansion)

Trigonometric Form:

Synthesis equation

$$\boxed{x(t) = a_0 + \sum_{n=1}^{\infty} (a_n cos(\omega_o n t) + b_n sin(\omega_o n t))}$$

Analysis equations

$$\boxed{a_0 = \frac{1}{T_o}\int_{T_o} x(t)dt}$$

$$\boxed{a_n = \frac{2}{T_o}\int_{T_o} x(t)cos(\omega_o nt)dt}$$

$$\boxed{b_n = \frac{2}{T_o}\int_{T_o} x(t)sin(\omega_o nt)dt}$$

T_o is the time period of the periodic signal $x(t)$.
ω_o is the fundamental (angular) frequency of the periodic signal $x(t)$.

Examples-

1) Anti-symmetric periodic square wave (odd signal)
Fourier Series coefficients:

- Odd sequence ($a_0 = 0$)
- Purely imaginary
- Even coefficients are 0

$\implies$ In trigonometric expansion, only sine terms exist

2) Symmetric periodic square wave (even signal)
Fourier Series coefficients:

- Even sequence ($a_0 \neq 0$)
- Purely real
- Even coefficients are 0

$\implies$ In trigonometric expansion, only cosine terms and constant term exist

Point 3 in both the above cases is because the square wave follows **half-wave symmetry** i.e $x(t) = -x(t + \frac{T_o}{2})$.

Half-wave symmetry $\implies$ even coefficients (harmonics) are 0, only odd coefficients (harmonics) exist.

Convergence of the Fourier series-
The Fourier series expansion actually properly represents the required signal if the **Dirichlet conditions** are satisfied.

3.1.1 Dirichlet Conditions

The Fourier series will converge to the actual value of the signal at every point except at the discontinuities if-

- The signal is absolutely integrable in a period
- The signal has finite number of maxima and minima in a period
- The signal has finite discontinuities in a period

Fourier series expansion will converge to average of end points at a finite discontinuity (eg:- square wave).
Fourier series expansion will not converge at infinite discontinuity (eg:- tan function).

3.1.2 Properties of CTFS

Consider periodic signals $x(t)$ with period T_1 and $y(t)$ with period T_2 and their respective exponential CTFS coefficients being C_k and D_k. ($\omega_o = 2\pi/T_1$)

1. Linearity: $Ax(t) + By(t) \rightarrow AC_k + BD_k$
2. Time Shifting: $x(t-\tau) \rightarrow C_k e^{-jk\omega_o\tau}$
3. Frequency Shifting: $e^{jM\omega_o t}x(t) \rightarrow C_{k-M}$
4. Conjugate: $\bar{x}(t) \rightarrow \bar{C}_{-k}$
5. Time Reversal: $x(-t) \rightarrow C_{-k}$
6. Time Scaling: $x(at) \rightarrow C_k$
   $x(at)$ is periodic with period $\frac{T}{a}$, only if a is positive and real

7. Periodic Convolution: $x(t) * y(t) \rightarrow TC_kD_k$
   where $T = LCM(T_1, T_2)$
8. Multiplication: $x(t)y(t) \rightarrow \sum C_l D_{k-l}$
9. Differentiation: $\frac{dx(t)}{dt} \rightarrow jk\omega_o C_k$
10. Integration: $\int_{-\infty}^{t} x(t)dt \rightarrow \frac{C_k}{jk\omega_o}$
11. Conjugate Symmetry for real signals:
    - $C_k = \bar{C}_{-k}$
    - $Re[C_k] = Re[C_{-k}]$
    - $Im[C_k] = -Im[C_{-k}]$
    - $|C_k| = |C_{-k}|$
    - $\angle C_k = -\angle C_{-k}$
12. If $x(t)$ is real and even, C_k is real and even
13. If $x(t)$ is real and odd, C_k is purely imaginary and odd
14. Even-Odd decomposition:
    $x_e(t) \rightarrow Re[C_k]$
    $x_o(t) \rightarrow jIm[C_k]$

Parseval's Theorem
The energy of the signal in remains the same, whether computed from time domain signal or from the Fourier series coefficients.

$$\boxed{\frac{1}{T_o}\int_{T_o} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |C_k|^2 = a_o^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2}$$

3.2 Continuous Time Fourier Transform

The concept of Fourier series can be expanded to aperiodic signals to give the Fourier transform.
An aperiodic signal is simply considered to be a periodic signal with infinite time period. By using this assumption, the Fourier transform is derived.
Fourier transform gives the frequency domain representation of the time domain signal.

Fourier Transform

$$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt}$$

Inverse Fourier Transform

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega}$$

It can be observed that duality exists between the two relations.

Dirichlet conditions hold for convergence of Fourier transform as well.

Note that the Fourier transform is simply the envelope of the Fourier series coefficients of the periodic version of the given aperiodic signal.

This means, the Fourier series coefficients can be obtained by taking samples from the Fourier transform of the aperiodic version of the given periodic signal.

$$\implies C_k = \frac{X(\omega)}{T_o} \quad at \quad \omega = k\omega_o$$

Fourier Transform for periodic signals
If $x(t) = e^{j\omega_o t}$, then $X(\omega) = 2\pi\delta(\omega - \omega_o)$.
Since any periodic signal can be expressed as linear combination of complex exponents, it's Fourier transform will be linear combination of shifted impulses.

$$X(\omega) = \sum_{k=-\infty}^{\infty} C_k \; 2\pi \; \delta(\omega - \omega_o)$$

3.2.1 Properties of CTFT

Consider periodic signals $x(t)$ and $y(t)$ and their respective exponential CTFTs be $X(\omega)$ and $Y(\omega)$.

1. Linearity: $Ax(t) + By(t) \to AX(\omega) + BY(\omega)$
2. Time Shifting: $x(t - \tau) \to X(\omega)e^{-j\omega\tau}$
3. Frequency Shifting: $e^{jM\omega t}x(t) \to X(\omega - M)$
4. Conjugate: $\bar{x}(t) \to \bar{X}(-\omega)$
5. Time Reversal: $x(-t) \to X(-\omega)$
6. Time Scaling: $x(at) \to \frac{1}{|a|}X(\frac{\omega}{a})$
7. Convolution: $x(t) * y(t) \to X(\omega)Y(\omega)$
8. Multiplication: $x(t)y(t) \to \frac{X(\omega)*Y(\omega)}{2\pi}$
9. Differentiation in time: $\frac{dx(t)}{dt} \to j\omega X(\omega)$
10. Integration in time: $\int_{-\infty}^{t} x(t)dt \to \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$
11. Differentiation in frequency: $tx(t) \to j\frac{dX(\omega)}{d\omega}$
12. Conjugate Symmetry for real signals:
    - $X(\omega) = \bar{X}(-\omega)$
    - $Re[X(\omega)] = Re[X(-\omega)]$
    - $Im[X(\omega)] = -Im[X(-\omega)]$
    - $|X(\omega)| = |X(-\omega)|$
    - $\angle X(\omega) = -\angle X(-\omega)$
13. If $x(t)$ is real and even, $X(\omega)$ is real and even
14. If $x(t)$ is real and odd, $X(\omega)$ is purely imaginary and odd
15. Even-Odd decomposition:
    $x_e(t) \to Re[X(\omega)]$
    $x_o(t) \to jIm[X(\omega)]$

Parseval's Theorem
The energy of the signal in remains the same, whether computed from time domain signal or from the frequency domain signal.

$$\boxed{\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega}$$

Duality Property
$x(t) \rightarrow CTFT \rightarrow X(\omega)$

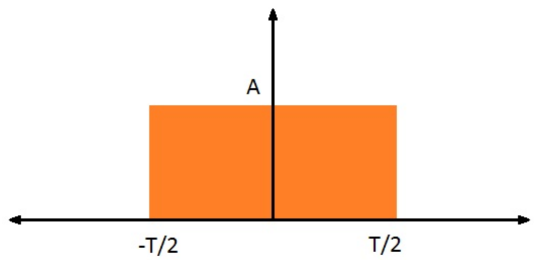
$$\Longrightarrow \boxed{X(t) \rightarrow CTFT \rightarrow 2\pi x(-\omega)}$$

3.2.2 Standard CTFT and CTFS Results

Signal	FT	FS (if periodic)
$\delta(t)$	1	-
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	-
$sgn(t)$	$\frac{2}{j\omega}$	-
$\delta(t-t_o)$	$e^{-j\omega t_o}$	-
$e^{-at}u(t)$	$\frac{1}{a+j\omega}$	-
$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$	-
$te^{-at}u(t)$	$\frac{1}{(a+j\omega)^2}$	-
$e^{j\omega_o t}$	$2\pi\delta(\omega-\omega_o)$	$a_k = \begin{cases} 1 & \text{if k=1} \\ 0 & \text{if k}\neq\text{1} \end{cases}$
$cos(\omega_o t)$	$\pi[\delta(\omega-\omega_o)+\delta(\omega+\omega_o)]$	$a_k = \begin{cases} \frac{1}{2} & \text{if k= 1 or -1} \\ 0 & \text{otherwise} \end{cases}$
$sin(\omega_o t)$	$\frac{\pi}{j}[\delta(\omega-\omega_o)-\delta(\omega+\omega_o)]$	$a_k = \begin{cases} \frac{1}{2j} & \text{if k= 1} \\ \frac{-1}{2j} & \text{if k=-1} \\ 0 & \text{otherwise} \end{cases}$
1	$2\pi\delta(\omega)$	$a_k = \begin{cases} 1 & \text{if k=0} \\ 0 & \text{if k}\neq\text{0} \end{cases}$

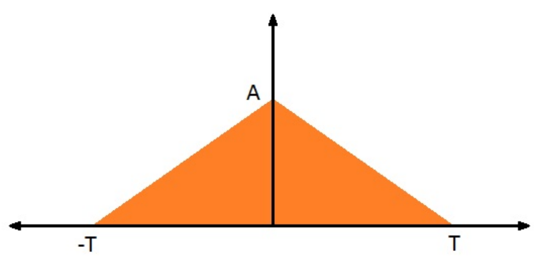
Rectangular/Pulse Signal

$$A \text{ rect}\left(\frac{t}{T}\right) = \begin{cases} A & \text{if } \frac{-T}{2} < t < \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$



Triangular Signal

$$A \text{ tri}\left(\frac{t}{T}\right) = \begin{cases} A - \frac{A}{T}t & \text{if } 0 < t < T \\ A + \frac{A}{T}t & \text{if } -T < t < 0 \\ 0 & \text{otherwise} \end{cases}$$



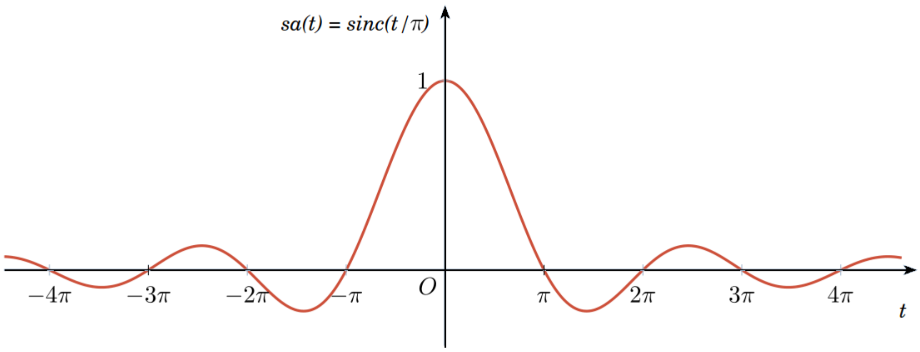
Sinc Function

$$\text{sinc}(at) = \frac{\sin(a\pi t)}{a\pi t}$$

Sample Function

$$\text{sa}(at) = \frac{\sin(at)}{at}$$

$$\implies \text{sinc}(at) = \text{sa}(a\pi t)$$



$\rightarrow \boxed{A \text{ rect}\left(\frac{t}{T}\right) \leftrightarrow AT \text{ sa}\left(\frac{\omega T}{2}\right)}$

$\rightarrow \boxed{A \text{ tri}\left(\frac{t}{T}\right) \leftrightarrow AT \text{ sa}^2\left(\frac{\omega T}{2}\right)}$

$\rightarrow \boxed{\text{sa}(at) \leftrightarrow \frac{\pi}{a} \text{ rect}\left(\frac{\omega}{2a}\right)}$

$\rightarrow \boxed{\text{sa}^2(at) \leftrightarrow \frac{\pi}{a} \text{ tri}\left(\frac{\omega}{2a}\right)}$

(The initial two relations are obtained using Fourier Transform and properties, the last two relations are obtained using duality property and the fact that all the signals here are even)

Gaussian Function

$$\boxed{x(t) = e^{-at^2} \leftrightarrow X(\omega) = \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}}}$$

- Area $= \sqrt{\frac{\pi}{a}}$
- Energy $= \sqrt{\frac{\pi}{2a}}$

Normalized Gaussian Function

$$x(t) = e^{-\pi t^2} \leftrightarrow X(\omega) = e^{-\frac{\omega^2}{4\pi}}$$

Fourier Transform of Gaussian Function will result in Gaussian Function.

3.3 Discrete Time Fourier Series

A discrete complex exponential is inherently periodic, meaning the value of e repeats itself after every N i.e $e^{j\Omega_o kn} = e^{j\Omega_o k(n+N)}$.
This means, to represent a periodic DT signal as a linear combination of complex exponentials, only N complex exponentials are sufficient.

Synthesis equation

$$\boxed{x(n) = \sum_{k=0}^{N-1} C_k e^{jk\Omega_o n}}$$

Analysis equation

$$\boxed{C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-jk\Omega_o n}}$$

It can be observed that duality exists between the two relations.

The differences to notice in DTFS w.r.t CTFS are-

- The DTFS coefficients are periodic as opposed to CTFS coefficients which are not periodic
- DTFS does not have convergence issues, hence Dirichlet Conditions (or any such equivalent) does not exist

3.3.1 Properties of DTFS

Consider periodic signals $x(n)$ with period N_1 and $y(n)$ with period N_2 and their respective exponential DTFS coefficients being C_k and D_k. ($\Omega_o = 2\pi/N_1$)

1. Linearity: $Ax(n) + By(n) \rightarrow AC_k + BD_k$
2. Time Shifting: $x(n-P) \rightarrow C_k e^{-jk\Omega_o P}$
3. Frequency Shifting: $e^{jM\Omega_o n} x(n) \rightarrow C_{k-M}$

4. Conjugate: $\bar{x}(n) \rightarrow \bar{C}_{-k}$

5. Time Reversal: $x(-n) \rightarrow C_{-k}$

6. Time Scaling: $x_m(n) \rightarrow C_k$
   $x_m(n) = x(n/m)$ only if n is multiple of m, else it is 0.

7. Periodic Convolution: $x(n) * y(n) \rightarrow NC_kD_k$
   where $N = LCM(N_1, N_2)$

8. Multiplication: $x(n)y(n) \rightarrow \sum_N C_lD_{k-l}$

9. First Difference: $x(n) - x(n-1) \rightarrow (1 - e^{-jk\Omega_o})C_k$

10. Running Sum: $\sum_{-\infty}^{n} x(n) \rightarrow \frac{C_k}{(1-e^{-jk\Omega_o})}$

11. Conjugate Symmetry for real signals:
    - $C_k = \bar{C}_{-k}$
    - $Re[C_k] = Re[C_{-k}]$
    - $Im[C_k] = -Im[C_{-k}]$
    - $|C_k| = |C_{-k}|$
    - $\angle C_k = -\angle C_{-k}$

12. If $x(n)$ is real and even, C_k is real and even

13. If $x(n)$ is real and odd, C_k is purely imaginary and odd

14. Even-Odd decomposition:
    $x_e(n) \rightarrow Re[C_k]$
    $x_o(n) \rightarrow jIm[C_k]$

Parseval's Theorem

$$\boxed{\frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2 = \sum_{k=0}^{N-1} |C_k|^2}$$

Duality Property
$x(n) \rightarrow \text{DTFS} \rightarrow C_k \implies C_n \rightarrow \text{DTFS} \rightarrow \frac{x(-k)}{N}$

3.4 Discrete Time Fourier Transform

The concept of representing an aperiodic signal as a periodic signal with infinite period will work in DT signals as well.
DTFT will be the envelope of DTFS coefficients, which will be periodic.

Fourier Transform

$$\boxed{X(\Omega) = \sum_{-\infty}^{\infty} x(n)e^{-j\Omega n}}$$

Inverse Fourier Transform

$$\boxed{x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega}$$

3.4.1 Properties of DTFT

Consider periodic signals $x(n)$ and $y(n)$ and their respective exponential DTFTs be $X(\Omega)$ and $Y(\Omega)$.

1. Linearity: $Ax(n) + By(n) \to AX(\Omega) + BY(\Omega)$
2. Time Shifting: $x(n - P) \to X(\Omega)e^{-jk\Omega P}$
3. Frequency Shifting: $e^{jM\Omega n}x(n) \to X(\Omega - M)$
4. Conjugate: $\bar{x}(n) \to \bar{X}(-\Omega)$
5. Time Reversal: $x(-n) \to X(-\Omega)$
6. Time Scaling: $x_{(k)}(n) \to X(k\Omega)$
   $x_{(k)}(n) = x(n/k)$ only if n is multiple of k, else it is 0.
7. Convolution: $x(n) * y(n) \to X(\Omega)Y(\Omega)$
8. Multiplication: $x(n)y(n) \to \frac{X(\Omega)*Y(\Omega)}{2\pi}$
9. First Difference: $x(n) - x(n-1) \to (1 - e^{-j\Omega})X(\Omega)$

10. Running Sum: $\sum_{-\infty}^{n} x(n) \rightarrow \frac{X(\Omega)}{(1-e^{-j\Omega})} + \pi X(0) \sum_k \delta(\Omega - 2\pi k)$

11. Differentiation in frequency: $nx(t) \rightarrow j\frac{dX(\Omega)}{d\Omega}$

12. Conjugate Symmetry for real signals:

    - $X(\Omega) = \bar{X}(-\Omega)$
    - $Re[X(\Omega)] = Re[X(-\Omega)]$
    - $Im[X(\Omega)] = -Im[X(-\Omega)]$
    - $|X(\Omega)| = |X(-\Omega)|$
    - $\angle X(\Omega) = -\angle X(-\Omega)$

13. If $x(n)$ is real and even, $X(\Omega)$ is real and even

14. If $x(n)$ is real and odd, $X(\Omega)$ is purely imaginary and odd

15. Even-Odd decomposition:
    $x_e(n) \rightarrow Re[X(\Omega)]$
    $x_o(n) \rightarrow jIm[X(\Omega)]$

Parseval's Theorem

$$\boxed{\sum_{-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\Omega)|^2 d\Omega}$$

Note that duality exists between CTFS and DTFT.

Since the Fourier Analysis gives frequency domain representation of time domain signals, it is useful to note the following.

Time Domain	Frequency Domain
Periodic	Discrete
Aperiodic	Continuous
Discrete	Periodic
Continuous	Aperiodic

4 Sampling

The process of converting a continuous time signal to a discrete time signal by taking amplitudes of the continuous time signal only at specific instances.

This is equivalent to multiplying a continuous time signal with a train of impulses spaced at discrete intervals.

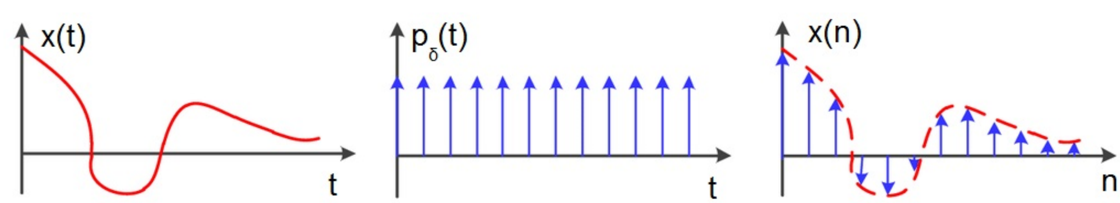
CT signal : $x(t)$
Train of impulses : $p_\delta(t) = \sum \delta(t - kT_s)$, where T_s is sampling period

$\implies x(t) \sum \delta(t - kT_s) = x(kT_s) = x(n)$ where $n = kT_s$

In the frequency domain, the signal is $X(\omega)$. Assume it is band-limited to w i.e the maximum frequency component is w. FT of the impulse train is given by $\omega_s \sum \delta(\omega - k\omega_s)$ where ω_s is the sampling frequency.

Using multiplication property of CTFT, the frequency domain representation of the output will be $X(\omega) * \omega_s \sum \delta(\omega - k\omega_s) = \omega_s \sum X(\omega - k\omega_s)$.

This will be replication of the original spectrum at every integral multiple of the sampling frequency (and amplitude scaled by sampling frequency factor). Hence, by using a low pass filter, the exact version of the original CT signal can be obtained back.



4.1 Sampling Theorem

Any CT signal can be sampled to obtain DT signal and properly reconstructed back to it's original form if the samples are taken such that the sampling frequency is at least twice as the maximum frequency component present in the CT signal.

$$\boxed{\omega_s \geq 2w} \quad (f_s \geq 4\pi w)$$

$2w$ is called the **Nyquist rate**, ω_N .
Note that Nyquist sampling (sampling at Nyquist rate) does not always guarantee proper reconstruction.

Meaning, ideally the sampling rate should be higher than the Nyquist frequency.

- $\omega_s > 2w$ - Over-sampling
- $\omega_s < 2w$ - Under-sampling

Aliasing (consequence of under-sampling)
If a signal is sampled at a frequency less than Nyquist frequency, then there will be a loss of high frequency components of the original signal.
This means the original signal can not be reconstructed from the under-sampled signal.

4.1.1 Sampling and Reconstruction of sinusoidal signals

$x(t) = cos(\omega t) = cos(2\pi f t)$
Sampling frequency, ω_s, meaning $f_s = 2\frac{\pi}{\omega_s}$

Substitute $t = n/f_s$.
$\implies x(n) = cos(\frac{\omega n}{f_s}) = cos(\frac{2\pi f n}{f_s})$

Discrete Frequency, $\Omega = \frac{2\pi f}{f_s}$
Note that Ω is periodic with 2π, meaning it lies between -π to π.

Hence, if $|\frac{2\pi f}{f_s}| > \pi$, it means aliasing has occurred.

The obtained value should be modified using periodicity property of Ω.
Once the discrete frequency is normalized to a value between -π to π (if necessary), re-substitute $n = tf_s$ to get back the recovered CT signal.

5 Frequency Domain Analysis of LTI systems

Since the Fourier Transform of any time domain signal will give it's frequency domain representation, the Fourier Transform is used to analyse an LTI system from frequency domain using it's impulse response.

Consider an LTI system with impulse response $h(t)$ with initial conditions being zero.
The Fourier Transform of $h(t)$, given by $H(\omega)$ is it's frequency response.

5.1 Eigen Function of LTI Systems

Eigen Function is the function which when input to a system will result in output of scaled version of the same function.
The scaling factor is called **Eigen Value**.

For LTI systems, the Eigen Function is $\phi(t) = e^{j\omega_o t}$.

If h(t) is the impulse response of an LTI system and input $x(t) = e^{j\omega_o t}$, then output $y(t) = H(\omega_o)e^{j\omega_o t}$.
Here, $H(\omega_o)$ is the Fourier Transform of h(t) evaluated at $\omega = \omega_o$.

5.2 Distortionless LTI Systems

Magnitude Distortion:
The output of an LTI system has different magnitudes for different frequency components of the input.

Phase Distortion:
The output of an LTI system has different delays for different frequency components of the input.

A system which does not provide magnitude and phase distortion is called **Distortionless system**.
$y(t) = kx(t - \tau)$
$\implies H(\omega) = ke^{-j\omega\tau}$
$|H(\omega)| = k \; ; \qquad \angle H(\omega) = -\omega\tau$

Therefore, a distortionless LTI system will give constant magnitude and linear phase (phase varies linearly with respect to frequency).

Phase Delay
The phase delay gives the time delay in seconds experienced by each sinusoidal component of the input signal.
It is defined as the negative of frequency response of a system divided by the angular frequency.

For a distortionless LTI system,

$$-\frac{\angle H(\omega)}{\omega} = \tau$$

Group Delay
The group delay specifies the delay experienced by a narrow-band group of sinusoidal components which have frequencies within a narrow frequency interval about ω_o.
It is defined as the negative of first derivative of frequency response of a system with respect to the angular frequency.

For a distortionless LTI system,

$$-\frac{d(\angle H(\omega))}{d\omega} = \tau$$

Hence, for a distortionless LTI system, group delay will be equal to phase delay.

The group delay will be constant only if the phase varies linearly with respect to frequency.
It is important for practical systems to have constant group delay because if group delay itself is varying, it means different frequencies in the input will experience different delays.

Note that if the input signal has a time varying envelope, such as $x(t) = a(t)\sin(\omega t)$, then the LTI system (of impulse response $h(t)$) response to such an input would be $y(t) = |H(\omega)|a(t-\tau_g)\sin(\omega(t-\tau_p))$ where τ_g is the group delay and τ_p is the phase delay.

6 Laplace Transform

The Laplace Transform is a more generalized version of the Continuous Time Fourier Transform.

For an LTI system with impulse response $h(t)$, an input of $e^{j\omega t}$ will result in an output of $H(\omega)e^{j\omega t}$ where $H(\omega)$ is CTFT of $h(t)$.

Similarly, if a more generalized complex exponential, with both real and imaginary parts is used i.e e^{st} is input where where $s = \sigma + j\omega$, then output will be $H(s)e^{st}$ where $H(s)$ is the Laplace Transform of $h(t)$.
Note that $Re[s] = \sigma$

For a continuous time signal $x(t)$, the Laplace Transform is defined by $X(s)$.

$$\boxed{X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt}$$

Note that the Laplace Transform of a signal may converge even when the CTFT does not converge.
This is because $x(t)e^{st} = x(t)e^{\sigma t}e^{j\omega t}$
$\implies L[x(t)] = F[x(t)e^{\sigma t}]$

Hence, even if $x(t)$ is not absolutely integrable, Laplace Transform will exist for a certain range of values of σ (such that $x(t)e^{\sigma t}$ will be absolutely integrable).
This range of values of σ under which the Laplace Transform is defined is called as **Region of Convergence (ROC)**.

If $x(t)e^{\sigma t}$ is absolutely integrable for no values of σ i.e there is no ROC, then Laplace Transform for that $x(t)$ does not exist.

It can also be noted that, for any absolutely integrable x(t), $X(\omega) = X(s)$ if $\sigma = 0$.

- Existence of Fourier Transform implies existence of Laplace Transform
- Existence of Laplace Transform does not imply existence of Fourier Transform

Important thing to notice is that Laplace Transform alone does not give 1 to 1 mapping between the two domains (unlike Fourier Transform which does).

Meaning, two signals in time domain can have the exact same Laplace Transform expression, but what will differ is the ROC.
Laplace Transform expression + ROC together will give 1 to 1 mapping between time domain and Laplace domain.

If $X(s)$ can be expressed as ratio of 2 polynomials, i.e $X(s) = N(s)/D(s)$, then it is rational.
Values of s for $N(s) = 0 \implies$ Zeros of $X(s)$ [where $X(s)$ equals 0]
Values of s for $D(s) = 0 \implies$ Poles of $X(s)$ [where $X(s)$ goes to ∞]

6.0.1 Analysis of ROC in the s-plane

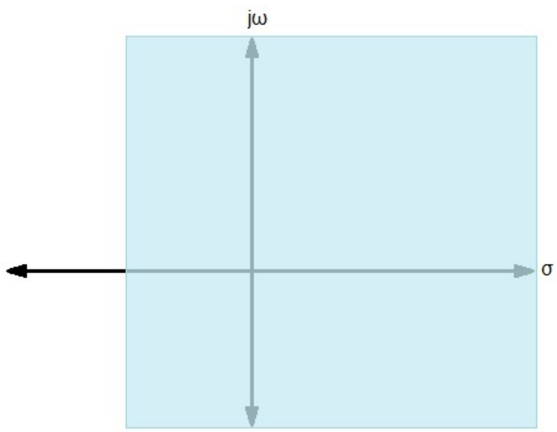
[σ is real axis, $j\omega$ is imaginary axis]

- ROC will be a single strip (finite or infitely extending) parallel to the $j\omega$ axis
- ROC will never contain poles
- If ROC includes the $j\omega$ axis, it implies that the signal is absolutely integrable by itself, hence Fourier Transform exists.
  (this is because $j\omega$ axis being present in ROC means $x(t)e^{\sigma t}$ converges for $\sigma = 0$, meaning $x(t)$ itself is absolutely integrable)
- For a finite duration signal $x(t)$, ROC is the entire s-plane, except maybe 0, ∞, -∞.
  (this is because since $x(t)$ is defined only for a finite duration, irrespective of the value of σ, $x(t)e^{\sigma t}$ will always be absolutely integrable)
- For a right sided signal $x(t)$, if $X(s)$ is rational, then ROC is the region to the right of the right-most pole.
- For a left sided signal $x(t)$, if $X(s)$ is rational, then ROC is the region to the left of the left-most pole
- For an infinite duration signal $x(t)$, if $X(s)$ is rational, then ROC exists between 2 poles. There can be cases where ROC does not exist too.
  (the reasons for the last 3 become trivial upon graphical analysis)

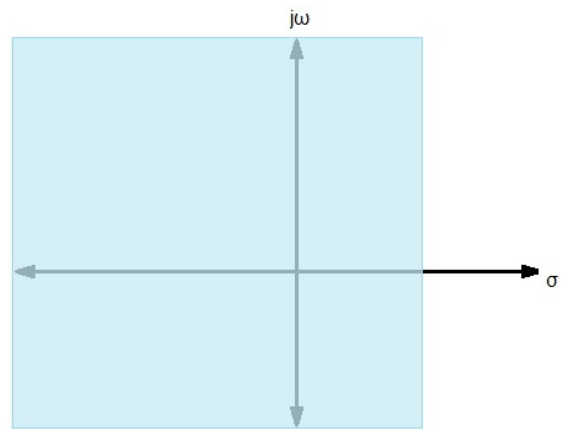
6.0.2 Stability and Causality using properties of ROC

A system with impulse response $h(t)$ can be analysed for stability and causality using ROC of $H(s)$.

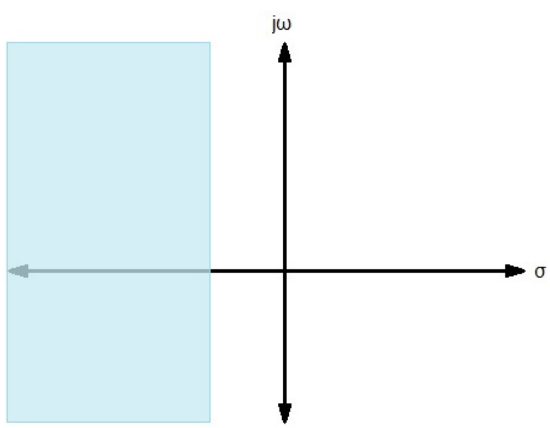
The ROC of $H(s)$ can take one of the following 5 forms.



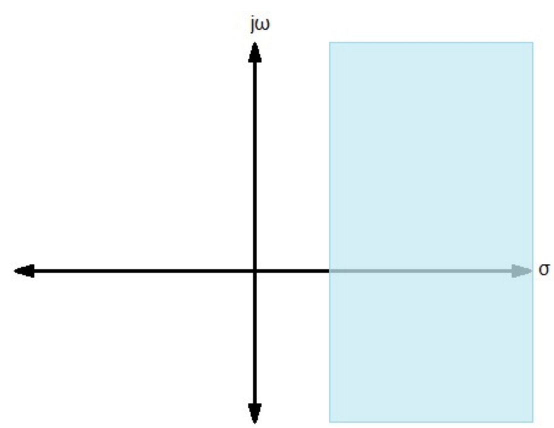
Stable + Causal



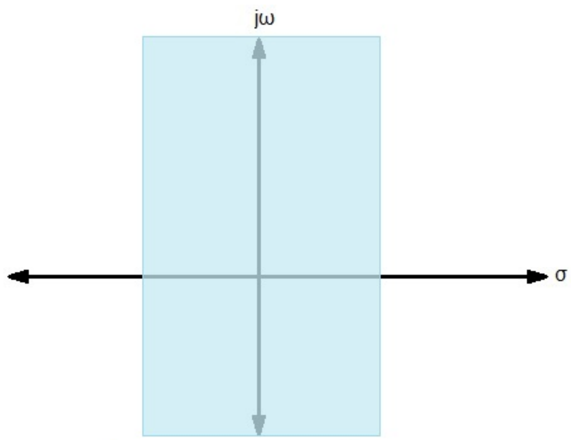
Stable + Anti-causal



Unstable + Causal



Unstable + Anti-causal



Stable + Non-causal (double sided)

Double sided infinitely extending sequences with increasing amplitudes on both sides have no Laplace transform.

Note that the ROCs containing $\sigma = 0$ line i.e the $j\omega$ axis represent stable systems.

6.0.3 Properties of Laplace Transform

Consider CT signals $x(t)$ and $y(t)$ with their respective Laplace Transforms being $X(s)$ and $Y(s)$.

- Linearity: $Ax(t) + By(t) \rightarrow AX(s) + BY(s)$
  (only if ROCs of $X(s)$ and $Y(s)$ have intersection)
- Time Shifting: $x(t-\tau) \rightarrow X(s)e^{-s\tau}$
- Frequency Shifting: $e^{St}x(t) \rightarrow X(s-S)$
- Conjugate: $\bar{x}(t) \rightarrow \bar{X}(\bar{s})$
- Time Reversal: $x(-t) \rightarrow X(-s)$
- Time Scaling: $x(at) \rightarrow \frac{1}{|a|}X(\frac{s}{a})$
- Convolution: $x(t) * y(t) \rightarrow X(s)Y(s)$
- Differentiation in time: $\frac{dx(t)}{dt} \rightarrow sX(s) - x(0)$

- Integration in time: $\int_{-\infty}^{t} x(t)dt \to \frac{X(s)}{s}$
- Differentiation in frequency: $tx(t) \to -\frac{dX(s)}{ds}$
- Integration in frequency: $\frac{x(t)}{t} \to \int_{s}^{\infty} X(s)ds$
- Higher order differentiation:
  - $\frac{dx^n(t)}{dt^n} \to s^n X(s) - s^{n-1}x(0) - s^{n-2}x'(0) - ...sx^{(n-1)}(0) - x^{(n)}(0)$
  - $t^n x(t) \to (-1)^n \frac{d^n X(s)}{ds^n}$

Initial Value Theorem
If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulse at $t = 0$, then

$$\boxed{x(0^+) = \lim_{s\to\infty} sX(s)}$$

Final Value Theorem
If $x(t) = 0$ for $t < 0$ and $x(t)$ is finite as $t \to \infty$, then

$$\boxed{\lim_{x\to\infty} x(t) = \lim_{s\to 0} sX(s)}$$

Note that for FVT to be applicable, $X(s)$ must have no poles in the RHS of s-plane.

Laplace Transform of Periodic Signal
$x(t)$ is a periodic signal with period T, let $x_1(t)$ represent one period of $x(t)$

$$\boxed{X(s) = \frac{X_1(s)}{1 - e^{-sT}}}$$

where $X_1(s)$ is LT of $x_1(t)$

6.0.4 Transfer Function

If the impulse response of an LTI System is $h(t)$, then it's Laplace Transform $H(s)$ evaluated at 0 initial conditions is called the Transfer Function of the system.
$y(t) = x(t) * h(t) \implies Y(s) = X(s)H(s)$
$\therefore H(s) = \frac{Y(s)}{X(s)}$

Transfer function is the ratio of LT of output to LT of input (at 0 initial conditions).

The transfer function representation is generally used for analysis of LTI systems and their response.

This transfer function also is extremely important in control system design. There are various techniques such as block diagrams and signal flow graphs which are made just to find the transfer function of a system efficiently (which will be discussed in control theory).

6.0.5 Standard LT pairs

Signal	Transform	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$\sigma > 0$
$-u(-t)$	$\frac{1}{s}$	$\sigma < 0$
$\delta(t-\tau)$	$e^{-s\tau}$	All s except $\sigma = \infty$ or $-\infty$
$u(t) * u(t) * ... * u(t)[n\ times]$	$\frac{1}{s^n}$	$\sigma > 0$
$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\sigma > 0$
$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\sigma > -\alpha$
$-e^{-\alpha t}u(-t)$	$\frac{1}{1+\alpha}$	$\sigma < -\alpha$
$cos(\omega_o t)u(t)$	$\frac{s}{s^2+\omega_o^2}$	$\sigma > 0$
$sin(\omega_o t)u(t)$	$\frac{\omega_o}{s^2+\omega_o^2}$	$\sigma > 0$
$e^{-\alpha t}cos(\omega_o t)u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_o^2}$	$\sigma > -\alpha$
$e^{-\alpha t}sin(\omega_o t)u(t)$	$\frac{\omega_o}{(s+\alpha)^2+\omega_o^2}$	$\sigma > -\alpha$

Inverse Laplace Transform is best found by splitting given LT expression into partial fractions and reverse mapping using standard pairs.

7 Z Transform

The Z-Transform is a more generalized version of the Discrete Time Fourier Transform.
For an LTI system with impulse response $h(n)$, an input of $e^{j\Omega n}$ will give output $H(\Omega)e^{j\Omega n}$ where $H(\Omega)$ is DTFT of $h(n)$.

A more generalized complex exponential with both real and imaginary parts may be defined as $z^n = re^{j\Omega n}$ where r is magnitude and Ω is phase.

Hence, for input z^n, the output will be $H(z)z^n$ where $H(z)$ is the Z-Transform of $h(n)$.
Note that $|z| = r$

For a discrete time signal $x(n)$, the Z-Transform is defined by $X(z)$ where

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}}$$

Note that the Z-Transform of a signal may exist even when the DTFT does not exist.
This is because x(n) $z^n = x(n)re^{j\Omega n}$
$\implies Z[x(n)] = F[x(n)r^{-n}]$

Hence, even if $x(n)$ is not absolutely integrable, Z-Transform can exist for a certain range of values of r (such that $x(n)r^{-n}$ will be absolutely summable). This range of values of $|z|$ under which the Z-Transform is defined is called as Region of Convergence (ROC).

If $x(n)r^{-n}$ is absolutely summable for no values of $|z|$ i.e there is no ROC, then Z-Transform for that $x(n)$ does not exist.
It can also be noted that, for any absolutely summable $x(n)$, $X(\Omega) = X(z)$ if $|z| = 1$.

- Existence of Fourier Transform implies existence of Z-Transform
- Existence of Z-Transform does not imply existence of Fourier Transform

Important thing to notice is that Z-Transform alone does not give 1 to 1 mapping between the two domains (unlike Fourier Transform which does).

Meaning, two signals in time domain can have the exact same Z-Transform expression, but what will differ is the ROC.
Z-Transform expression + ROC together will give 1 to 1 mapping between time domain and Z-domain.
If $X(z)$ can be expressed as ratio of 2 polynomials, i.e $X(z) = N(z)/D(z)$, then it is rational.
Values of z for $N(z) = 0 \rightarrow$ Zeros of $X(z)$ [where $X(z)$ equals 0]
Values of z for $D(z) = 0 \rightarrow$ Poles of $X(z)$ [where $X(z)$ goes to ∞]

7.0.1 Analysis of ROC in the z-plane

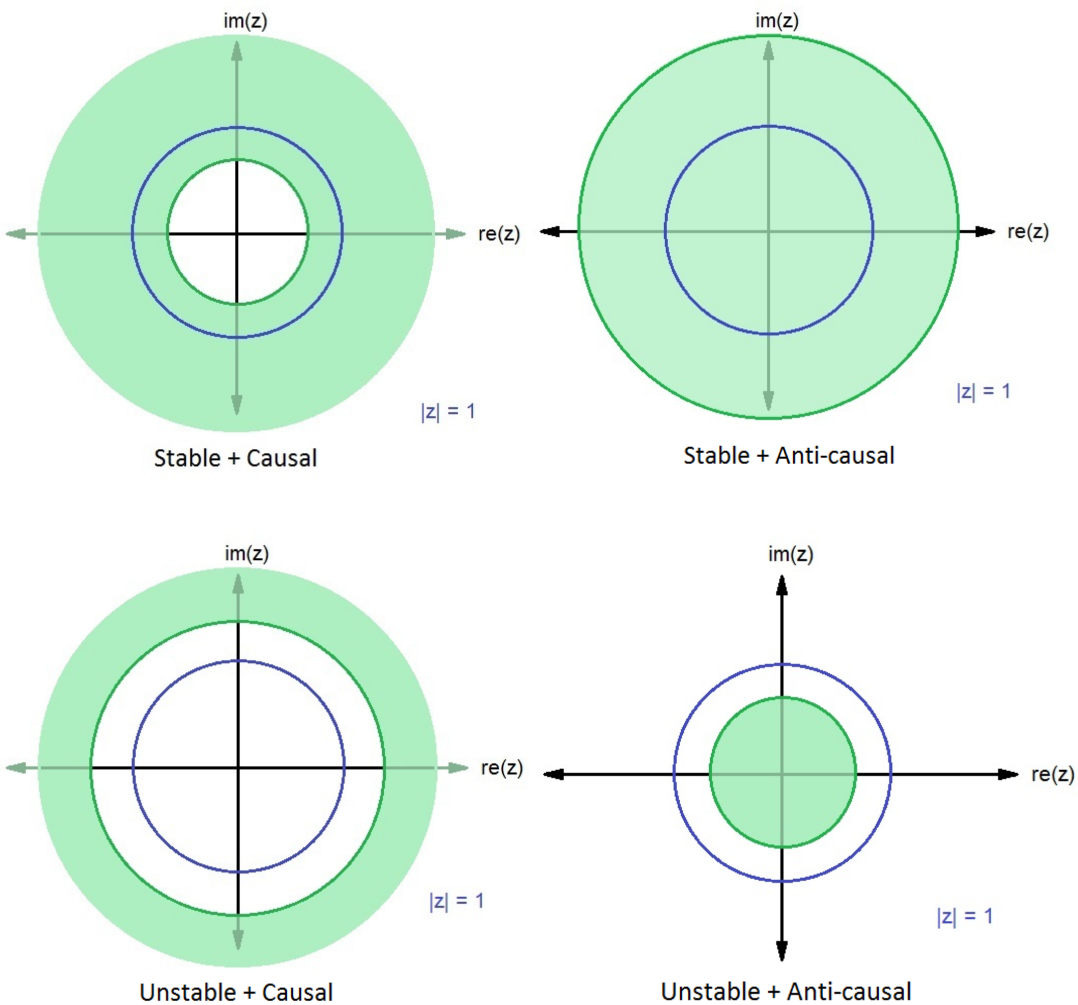
($Re[z]$ is real axis, $Im[z]$ is imaginary axis)

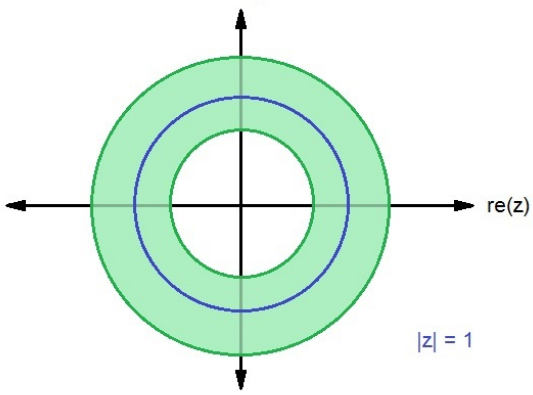
- ROC will be a single ring (finite or infinitely extending) concentric to the unit circle ($|z| = 1$).
- ROC will never contain poles.
- If ROC includes the unit circle, it implies that the signal is absolutely summable by itself, hence Fourier Transform exists.
  (this is because unit circle being present in ROC means $x(n)\ r^{-n}$ converges for $r = 1$, meaning $x(n)$ itself is absolutely summable)
- For a finite duration signal $x(n)$, ROC is the entire z-plane, except maybe $|z| = 0, 1, \infty$.
  (this is because since $x(n)$ is defined only for a finite duration, irrespective of the value of r, $x(n)r$ will always be absolutely summable)
- For a right sided signal $x(n)$, if $X(z)$ is rational, then ROC is the region outside the outermost pole.
- For a left sided signal $x(n)$, if $X(z)$ is rational, then ROC is the region inside the innermost pole
- For an infinite duration signal $x(n)$, if $X(z)$ is rational, then ROC exists between 2 poles as a finite ring. There can be cases where ROC does not exist too.
  (the reasons for the last 3 become trivial upon graphical analysis)

7.0.2 Stability and Causality using properties of ROC

A system with impulse response $h(n)$ can be analysed for stability and causality using ROC of $H(z)$.

The ROC of H(z) can take one of the following 5 forms.





Double sided infinitely extending sequences with increasing amplitudes on both sides have no Z-transform.

Note that the ROCs containing $|z| = 1$ i.e the unit circle represent stable systems.

7.0.3 Properties of Z-Transform

Consider DT signals $x(n)$ and $y(n)$ with their respective Z-Transforms being $X(z)$ and $Y(z)$.

- Linearity: $Ax(n) + By(n) \rightarrow AX(z) + BY(z)$
  (only if ROCs of X(z) and Y(z) have intersection)
- Time Shifting: $x(n - K) \rightarrow X(z)z^{-K}$
- Scaling in z-domain: $a^n x(n) \rightarrow X(\frac{z}{a})$
- Conjugate: $\bar{x}(n) \rightarrow \bar{X}(\bar{z})$
- Time Reversal: $x(-n) \rightarrow X(z^{-1})$
- Time Expansion: $x_{(k)}(n) \rightarrow X(z^k)$
  $x_{(k)}(n) = x(r)$ if $n = rk$ where r is an integer, 0 otherwise
- Convolution: $x(n) * y(n) \rightarrow X(z)Y(z)$
- First Difference: $x(n) - x(n-1) \rightarrow (1 - z^{-1})X(z)$

- Accumulation: $\sum_{-\infty}^{n} x(n) \rightarrow \frac{X(z)}{1-z^{-1}}$
- Differentiation in z-domain:
  - $nx(n) \rightarrow -z\frac{dX(z)}{dz}$
  - $n^2x(n) \rightarrow z^2\frac{d^2X(z)}{dz^2} + z\frac{dX(z)}{dz}$

Initial Value Theorem
If $x(n) = 0$ for $n < 0$, then

$$\boxed{x(0) = \lim_{z\to\infty} X(z)}$$

Final Value Theorem
If $x(n) = 0$ for $n < 0$ and x(n) is finite as $n \rightarrow \infty$, then

$$\boxed{\lim_{x\to\infty} x(n) = \lim_{z\to 1}(1 - z^{-1})X(z)}$$

Note that for FVT to be applicable, $X(z)$ must have no poles outside the unit circle.

7.0.4 Transfer Function

If the impulse response of an LTI System is $h(n)$, then it's Z-Transform $H(z)$ evaluated at 0 initial conditions is called the Transfer Function of the system.
$y(n) = x(n) * h(n) \implies Y(z) = X(z)H(z)$
$H(z) = \frac{Y(z)}{X(z)}$
Transfer function is the ratio of ZT of output to ZT of input (at 0 initial conditions).

The transfer function representation is generally used for analysis of LTI systems and their response.

This transfer function is extensively used in filter design and analysis as well. Different forms of filter realizations are done using Z-transform of the impulse response of the filter.

7.0.5 Standard ZT Pairs

Signal	Transform	ROC
$\delta(n)$	1	All z
$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u(-n-1)$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
$\delta(n-m)$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
$\alpha^n u(n)$	$\frac{1}{1-\alpha z^{-1}}$	$\|z\| > \|\alpha\|$
$-\alpha^n u(-n-1)$	$\frac{1}{1-\alpha z^{-1}}$	$\|z\| < \|\alpha\|$
$n\alpha^n u(n)$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\|z\| > \|\alpha\|$
$-n\alpha^n u(-n-1)$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\|z\| < \|\alpha\|$
$cos(\omega_o n)nu(n)$	$\frac{1-cos\omega_o z^{-1}}{1-2cos\omega_o z^{-1}+z^{-2}}$	$\|z\| > 1$
$sin(\omega_o n)u(n)$	$\frac{sin\omega_o z^{-1}}{1-2cos\omega_o z^{-1}+z^{-2}}$	$\|z\| > 1$
$r^n cos(\omega_o n)u(n)$	$\frac{1-rcos\omega_o z^{-1}}{1-2rcos\omega_o z^{-1}+r^2 z^{-2}}$	$\|z\| > r$
$r^n sin(\omega_o n)u(n)$	$\frac{rsin\omega_o z^{-1}}{1-2rcos\omega_o z^{-1}+r^2 z^{-2}}$	$\|z\| > r$

Inverse Z-Transform is best found by splitting given ZT expression into partial fractions and reverse mapping using standard pairs.

8 Filters, Design and Analysis

Filters are circuits/systems that amplify or pass a certain range of frequencies while attenuating or stopping the other frequencies.

Analysis of filters is done by using Fourier Transform of the impulse response of the circuits/systems, since that gives frequency domain representation.
Output $y(t)$ of a filter with impulse response $h(t)$ and input signal $x(t)$ is given by $y(t) = x(t) * h(t)$.
By taking Fourier Transform on both sides and using convolution property, $Y(\omega) = X(\omega)H(\omega)$.

Hence the frequency response for a given input signal and filter can be found by multiplying the Fourier Transforms of input signal and impulse response of filter, which will give a clear idea on which frequencies are being amplified and which are being attenuated.

Note that since frequency response is a complex valued function, it will have both magnitude and phase. $\implies H(\omega) = |H(\omega)|\angle H(\omega)$

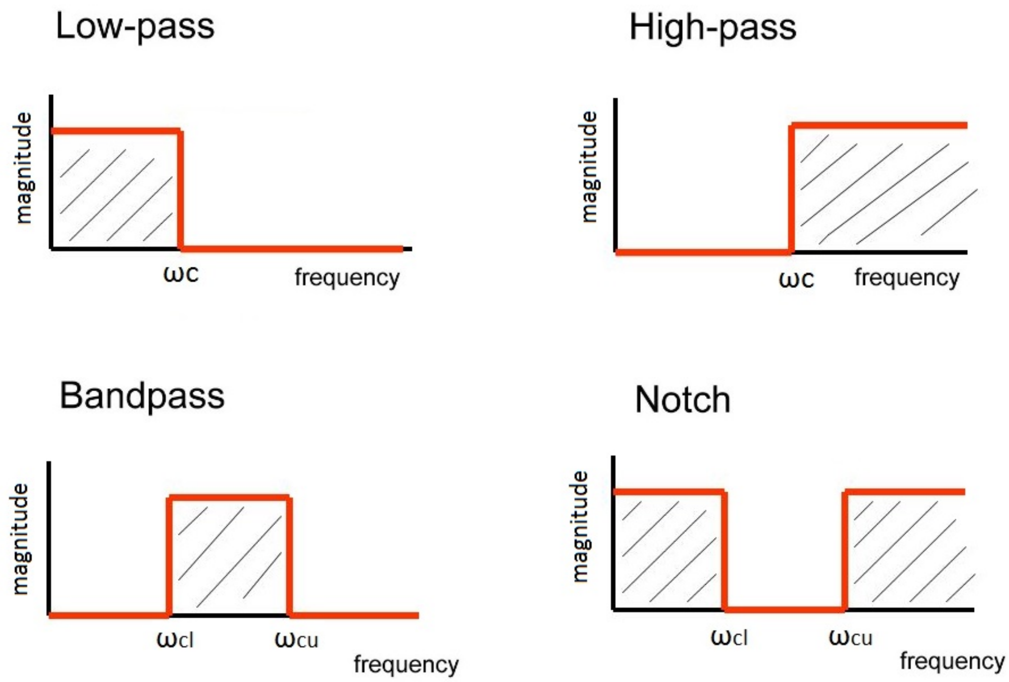
The pass band of a filter is defined as the frequency component range that the filter passes. Hence, the magnitude response in the pass band will typically be 1 or some finite value.
The stop band of a filter is defined as the frequency component range that the filter does not pass. Hence, the magnitude response in the pass band will typically be close to 0.
The frequencies at which the transition from one band to another occurs is called the cut-off frequency.

8.1 Ideal Filters

Sharp/abrupt transition from pass band to stop band (or vice versa).

Limitations/ Disadvantages

- Impulse response is infinitely extending, hence it can't be made causal by delaying
- Difficult to obtain sharp transition band practically



Note that in CT filters, the magnitude and phase plots are plotted against continuous frequency axis that extends to infinity on both sides.

- Low Frequency: $\omega = 0$
- High Frequency: $\omega \to \infty \; or \; -\infty$
- Mid Frequency: ω lies somewhere in between

However in DT filters, the magnitude and phase plots are plotted against discrete frequency axis, so the response repeats after every 2π.

- Low Frequency: $\Omega = 0$
- High Frequency: $\Omega = \pi \; or \; -\pi$
- Mid Frequency: Ω lies somewhere in between
  (exactly mid would be $\pi/2$ or -$\pi/2$)

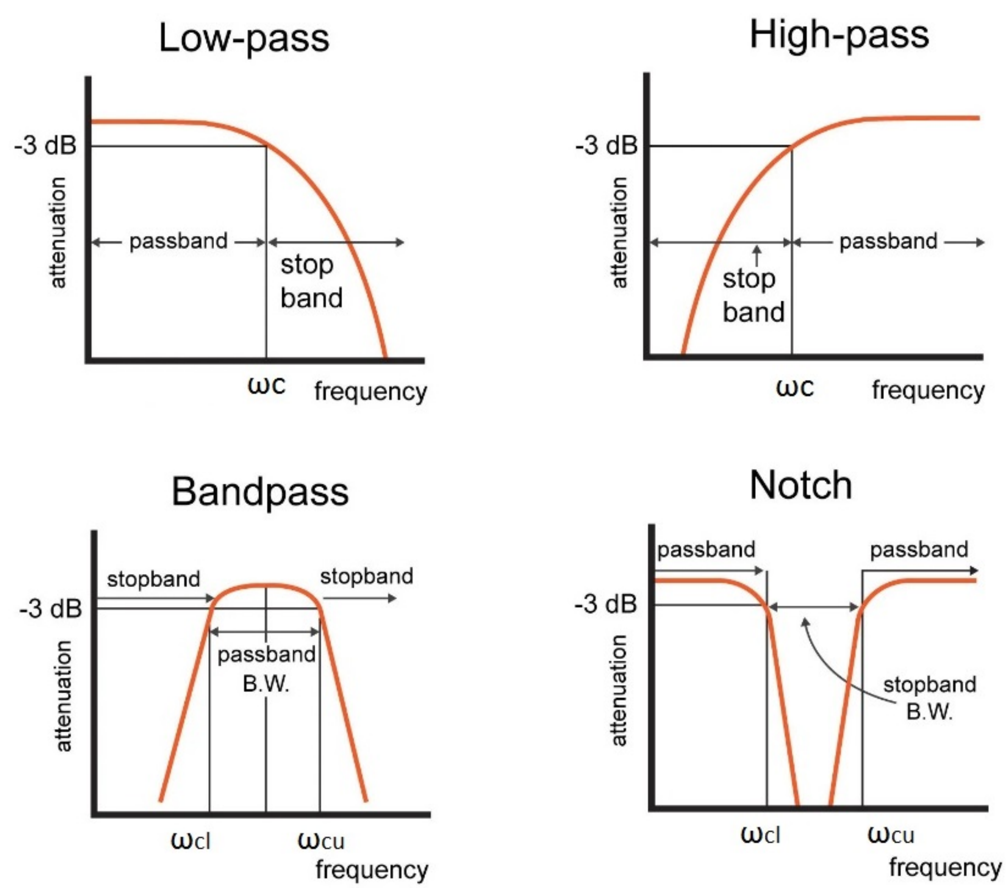
8.2 Practical Filters

Smoother transition from pass band to stop band (or vice versa)

Advantages

- Impulse response is not infinitely extending, hence can be made causal by delaying
- Easier to design using hardware practically

Design and Analysis of these filters is done using Differential or Difference equations in the time domain or using the Transfer function or Frequency response in the frequency domain.



- Pass band refers to all the frequencies which are passed with attenuation lesser than 3 dB.

- Stop band refers to all the frequencies which are passed with attenuation greater than 3 dB.

- The frequencies at which the attenuation is exactly 3 dB are the cut-off frequencies.

3 dB is chosen because it corresponds to where the amplitude is $\frac{1}{\sqrt{2}}$ (or 0.707) times the maximum amplitude.

8.2.1 Moving Average Filter or Finite Impulse Response (FIR) Filter

- Output depends on input samples only.

- Output at an instant is equal to average of surrounding values of input (specified number).

General difference equation of FIR Filter,

$$y(n) = \sum_{k=-M}^{N} a_k x(n-k)$$

Example, 3 point moving average filter
Difference equation:

$$y(n) = \frac{1}{3}[x(n-1) + x(n) + x(n+1)]$$

Transfer function:

$$H(z) = \frac{1}{3}[z + 1 + z^{-1}]$$

(note that the transfer function has only finite number of z terms)

8.2.2 Recursive Filter or Infinite Impulse Response (IIR) Filter

- Output depends on input samples as well as past output samples.
- Feedback concept is used.

General difference equation of IIR Filter,

$$\sum_{k=0}^{M} b_k y(n-k) = \sum_{k=0}^{N} a_k x(n-k)$$

Example (1st order IIR Filter)
Difference equation:

$$y(n) - y(n-1) = x(n)$$

Transfer function:

$$H(z) = \frac{1}{1 - z^{-1}}$$

(note that the transfer function when converted to proper form will have infinite number of z terms)

It is important to observe that FIR filter transfer function will consist only of zeros, whereas an IIR filter transfer function will consist of both poles and zeros.

8.3 Mapping Continuous Time Filters to Discrete Time Filters

Since discrete time processing is more efficient, it is useful to map continuous time filters to discrete time filters.

The main thing to consider while performing this mapping is that the $j\omega$ *axis* ($\sigma = 0$) in s-plane should map to the unit circle ($|z| = 1$) in the z-plane, to ensure stability.

8.3.1 Backward Difference method

This is a simple and intuitive method where the derivatives in the continuous time equation are replaced by backward differences to get the discrete equivalent.

$$\frac{dy_c(t)}{dt} \rightarrow \frac{y_d(n) - y_d(n-1)}{T}$$

$$\implies sY_c(s) \rightarrow (1 - z^{-1})Y_d(z)$$

Though simple, the backward difference method does not give desired results in most cases because here $j\omega axis$ ($\sigma = 0$) in s-plane maps to a smaller circle of radius 0.5 and center (0,0.5) within the unit circle ($|z| = 1$) in the z-plane, which does ensure stability but the characteristics may be lost.

8.3.2 Impulse Invariance method

A better way to perform the mapping is to sample the impulse response of the CT system to obtain the DT version of the same impulse response. This is called Impulse Invariance.
$h_d(n) = h_c(t)|_{t=nT}$

If the continuous time filter is represented as sum of partial fractions as,

$$\boxed{H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}}$$

The corresponding impulse response will be,

$$h_c(t) = \sum_{k=1}^{N} A_k e^{s_k t} u(t)$$

Now by performing sampling at t = nT to obtain discrete time impulse response,

$$h_d(n) = \sum_{k=1}^{N} A_k e^{s_k nT} u(n)$$

The Z-transform of the above discrete time impulse response gives the DT mapping for the CT filter.

$$\boxed{H_d(z) = \sum_{k=1}^{N} \frac{A_k}{1 - e^{s_k T} z^{-1}}}$$

$\therefore$ Pole at $s = s_k$ will map to Pole at $z = e^{s_k T}$ and the coefficients A_k are preserved.

Since this involves sampling the impulse response, there is a chance that aliasing of higher frequencies might occur, which is a disadvantage of this method.

8.3.3 Bi-linear Transformation method

If in case the input signal is not band-limited, aliasing will become intolerable, which is why Impulse Invariance can't be used.

The bi-linear transformation is derived by converting the differential equation governing the CT filter to integral equation and using the Trapezoidal rule to approximate it to a difference equation which will map it to a difference equation.

In the bi-linear transformation process, the mapping is done as follows.

$$\boxed{s = \frac{2}{T}(\frac{1 - z^{-1}}{1 + z^{-1}})}$$

$$\boxed{z = \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s}}$$

Here the relation between CT frequency and DT frequency is non-linear.
$\omega = \frac{2}{T} tan(\frac{\Omega}{2})$

This non-linear mapping can cause some rapid or drastic changes in the mapping characteristics, but the bi-linear transformation completely avoids aliasing.

8.4 Digital Filter Realizations

Once the Transfer function of a digital filter is obtained, it has to be realized using standard structures.
Though a system can be realized in many different ways and all will work on paper, different structures will have different properties, which can be used according to practical needs.

Each structure will consist of the following elements; amplification factors, delays, summing points and take-off points.

Consider a standard example each for FIR and IIR types of filters.
FIR Filter Transfer Function:

$$H_f(z) = a_0 + a_1 z^{-1} + a_2 z^{-2}$$

IIR Filter Transfer Function:

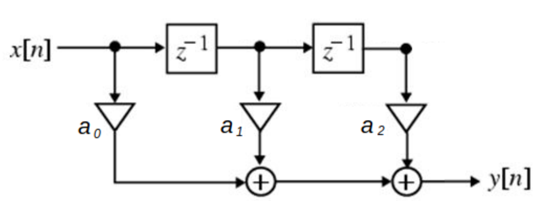
$$H_i(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

8.4.1 Direct Form I

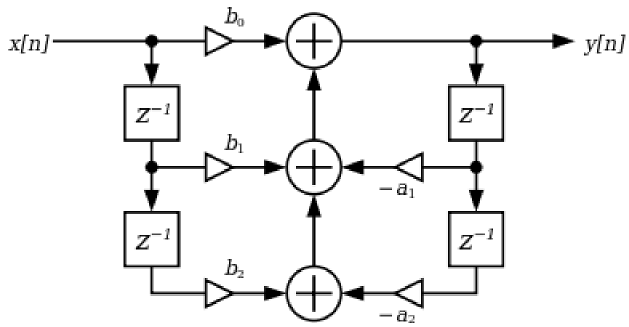
The most basic form of realization obtained directly from the transfer function of the filter.

Simple technique for IIR would be to put coefficients of zeros on the left, negative of coefficients of poles on the right and add them up, with appropriate delays.

FIR Filter:



IIR Filter:

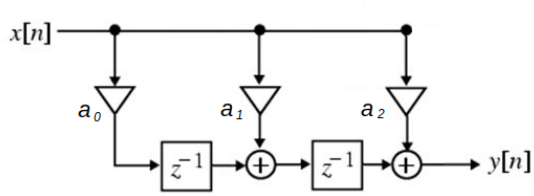


8.4.2 Direct Form II

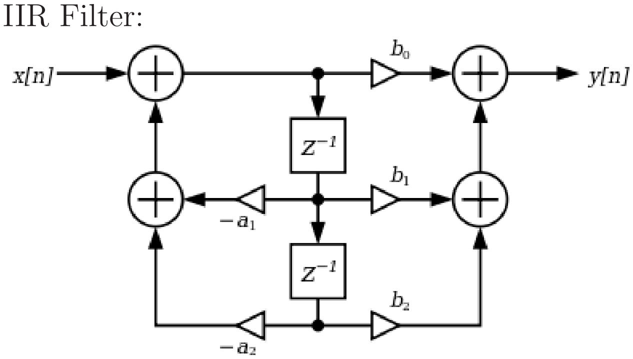
This is the transposition of the first form. Transposition is obtained by swapping the directions and take-off points with summing points.

The change in structure won't have any major impact in FIR case, but in IIR case, Direct Form II combined the delay elements as common for both poles and zeroes, hence reducing the total number of delay elements needed.

FIR Filter:



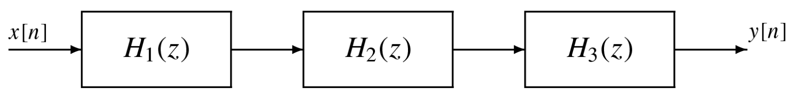
IIR Filter:



8.4.3 Cascade Structure

The cascade structure is obtained by expressing the transfer function as a product of two or more simpler transfer functions, realizing the individual simpler filters using Direct Form II and then cascading them in order to obtain the required filter.

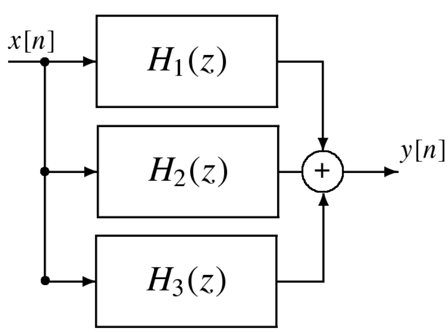
$H(z) = H_1(z)\ H_2(z)\ H_3(z)$



8.4.4 Parallel Structure

The parallel structure is obtained by expressing the transfer function as a sum of two or more simpler transfer functions (typically using Partial Fractions), realizing the individual simpler filters using Direct Form II and the adding them up (with common input and output) to obtain the required filter.

$H(z) = H_1(z) + H_2(z) + H_3(z)$



9 Basics of Digital Signal Processing

The application of digital signal processing allows for many advantages over analog processing in applications, such as error detection and correction in transmission as well as data compression.
Also, digital data occupies less memory and takes lesser computational power when compared to analog data.

9.1 Discrete Fourier Transform

Discrete Fourier transform (DFT) converts a finite length discrete time signal into a same-length sequence of equally-spaced samples of the discrete-time Fourier transform (DTFT).
Meaning, DFT is the sampled version of DTFT.

Let $x(n)$ be a finite length sequence of length N. Since it is not periodic, the DTFT of $x(n)$ i.e $X(\Omega)$ will be a continuous signal.

However, the finite length sequence can be converter to periodic sequence by simply repeating the same sequence after every N intervals.

$$\implies x((n))_N = \sum_{r=-\infty}^{\infty} x(n+rN)$$

The above sequence $x((n))_N$ is periodic with period N and is read as "$x(n \text{ modulo } N)$".
The original finite length sequence $x(n)$ can be obtained back from $x((n))_N$ by simply extracting a single period by multiplying it with a rectangular function of length N.
The above interpretation reveals that a finite length sequence and a periodic sequence are essentially the same, meaning both have N distinct values to define them.

The Discrete Fourier Series coefficients can be calculated for $x((n))_N$, which will give the Discrete Fourier Transform of $x(n)$.

$x((n))_N$ can be expressed as a sum of harmonically related complex exponentials. It can be noted that k also is periodic with N, because as k is varied from 0 to $N-1$, it gives all the complex exponentials necessary to compute the DFS coefficients.

$$x((n))_N = \frac{1}{N}\sum_{k=0}^{N-1} X_s(k)e^{\frac{j2\pi kn}{N}}$$

Hence, the DFS coefficients are calculated as

$$X_s(k) = \sum_{n=0}^{N-1} x(n)e^{-\frac{j2\pi kn}{N}}$$

Hence, $X_s(k)$ is periodic with period N.
Now the same DFS coefficients, if taken only for values of k from 0 to $N-1$, it will give the DFT of $x(n)$.

The Discrete Fourier Transform can more easily be obtained by sampling the Discrete Time Fourier Transform $X(\Omega)$ in order to discretize the sequence in the frequency domain.

$X(\Omega)$ is sampled N times at $\Omega = k\Omega_o$ where $\Omega_o = \frac{2\pi}{N}$ in order to get $X(k)$, which is called the **N-point DFT** of $x(n)$.

For convenience, $e^{-j2\pi N}$ is denoted using W_N.

Discrete Fourier Transform

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n)W_N^{nk}} \quad k = 0, 1 ... N-1$$

Inverse Discrete Fourier Transform

$$\boxed{X(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-nk}} \quad n = 0, 1 ... N-1$$

9.1.1 Properties of DFT

If $x(n)$ and $y(n)$ are two finite length sequences of length N, and their respective DFTs are $X(k)$ and $Y(k)$,

1. Linearity: $Ax(n) + By(n) \rightarrow AX(k) + BY(k)$
2. Time Shifting: $x((n+m))_N \rightarrow X(k)W_N^{-km} = X(k)e^{j2\pi km/N}$
3. Time Folding: $x((-n))_N = x(N-n) \rightarrow X((-k))_N = X(N-k)$
4. Frequency Shifting: $W_N^{kl}x(n) \rightarrow X((k+l))_N$
5. Symmetry: If $x(n)$ is real valued sequence, $X(k) = \bar{X}((N-k))_N$
   - $Re[X(k)] = Re[X((N-k))_N]$
   - $Im[X(k)] = -Im[X((N-k))_N]$
6. Circular Convolution: $x(n) \circledast y(n) \rightarrow X(k)Y(k)$
   (more on this later)

Properties of W_N

- Periodicity : $W_N^{n(k+N)} = W_N^n k$
- Unit Magnitude : $|W_N^{nk}| = 1$

9.1.2 Circular Convolution

It can be observed from the properties of DFT that multiplying the DFTs of 2 sequences and taking IDFT will not give linear convolution of the 2 finite length sequences, but will result in their circular convolution.
Since DFT is calculated for N points, the IDFT of the product of the 2 DFT sequences will also give an N point sequence. However, the linear convolution of 2 finite length sequences of length N will be of length $2N-1$.

In order to implement digital systems using DFT, it is necessary to obtain the linear convolution of the 2 sequences. Hence it is important understand what circular convolution does and how to modify it to obtain the linear convolution.

If $x_1(n)$ and $x_2(n)$ are two finite length sequences of length N, then their circular convolution is given by,

$$\boxed{y_c(n) = \sum_{m=0}^{N-1} x_1(m)x_2((n-m))_N}$$

It is known that the linear convolution of the same two sequences is,

$$y_l(n) = \sum_{r=-\infty}^{\infty} x_1(r)x_2(n-r)$$

$$\implies y_c(n) = \sum_{r=-\infty}^{\infty} y_l(n+rN) \quad n = 0, 1...N-1$$

This means, the circular convolution of the two sequences is repeated linear convolution.

Since the linearly convolution of 2 sequences of length N will be $2N-1$, the extra elements obtained are added back to the first N elements in a periodic manner in the circular convolution to get a sequence of exactly N elements.

$\implies$ Sum of all elements in the linear convolution of 2 sequences is same as sum of all elements in the circular convolution of the 2 sequences.

It is not possible to obtain the linear convolution sequence from the circular convolution since the original elements are already aliased. However, this problem has an easy fix.

It is possible to make sure that the obtained circular convolution sequence is same as the linear convolution sequence by adding extra zeros at the end of the two finite length sequences, such that both the finite length sequences are now of length $2N-1$ instead of N (with all extra zeros).

This way, the circular convolution will result in a sequence of $2N-1$ elements which will be same as the linear convolution sequence since all extra elements are 0 and there will virtually be no periodic addition.

9.2 Fast Fourier Transform

Fast Fourier Transform (FFT) is an efficient algorithm for computing DFT coefficients.

The DFT of an N-point sequence can be obtained by multiplying the following matrix with the N-point sequence.

$$\mathrm{W} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

$$\implies \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

The complex matrix $W_{N \times N}$ is called the Fourier Matrix.

Computations required to compute N-point DFT directly
Number of complex multiplications $= (N-1)^2 \approx N^2$
Number of complex additions $= N(N-1) \approx N^2$

Both are of the order of N^2, which is why the computation will be slow and impractical even for moderately long sequences.
This is why FFT algorithm is necessary.

Another advantage in having an efficient way to calculate DFT is that the DFT and it's IDFT have a dual relationship. This means, any algorithm that is used to calculate DFT can also be used to calculate IDFT without too many modifications.

9.2.1 Decimation in Time (DIT FFT)

From $x(n)$, make two sequences $g(n)$ and $h(n)$ where, $g(n) = x(2n)$
$h(n) = x(2n+1)$

$$X(k) = \sum_{r=0}^{\frac{N}{2}-1} g(r)W_N^{(2r)k} + W_N^k \sum_{r=0}^{\frac{N}{2}-1} h(r)W_N^{(2r)k} \qquad k = 0, 1, ...N-1$$

Note that $W_N^{(2r)k} = W_{\frac{N}{2}}^{rk}$

$$\implies X(k) = G(k) + W_N^k H(k) \qquad k = 0, 1, ...N-1$$

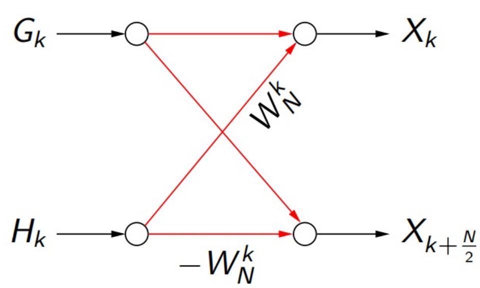
Here, G(k) and H(k) are $\frac{N}{2}$ point DFTs and are periodic with period $\frac{N}{2}$.
W_N^k is called as **Twiddle Factor**

$W_N^{k+\frac{N}{2}} = -W_N^k$

$$X_{k+\frac{N}{2}} = G(k) - W_N^k H(k) \qquad k = 0, 1, ...N-1$$

This means $W_N^k H(k)$ needs to be computed only $\frac{N}{2}$ times.

Butterfly Diagram



The following process will reduce the number of complex multiplications from N^2 to $2(\frac{N}{2})^2 + \frac{N}{2}$.

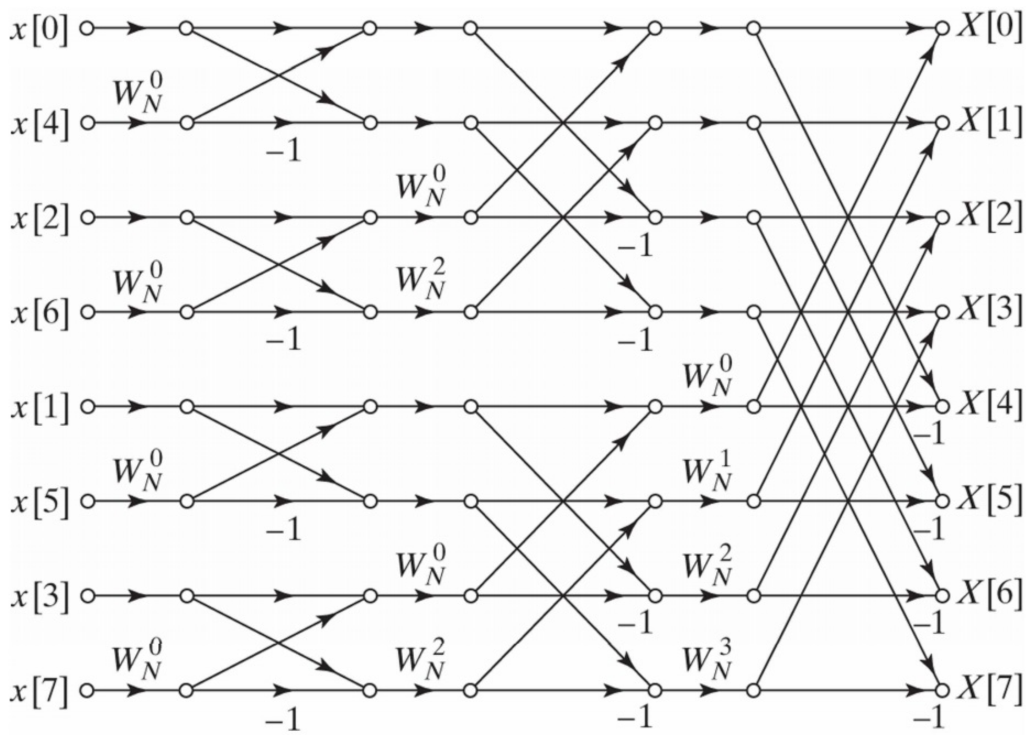
The same idea can be applied for calculating the $\frac{N}{2}$ point DFT of the sequences $g(n)$ and $h(n)$, and so on till the 2 point DFT is reached.
This idea can be applied recursively $log_2 N$ times if N is a power of 2.

Using this algorithm for $log_2 N$ stages will result in the reduction of computation burden.

Computations required to compute N-point DFT using FFT
Number of complex multiplications = $\frac{N}{2}\ log_2 N$
Number of complex additions = $N\ log_2 N$

Hence, DIT FFT is a much more convenient way of calculating DFT.

For illustration, consider 8 point sequence, where 8 point DFT has to be calculated.



Note that the original sequence $x(n)$ is not in order. This is because in each stage, the sequence is split into even and odd indexed sequences.

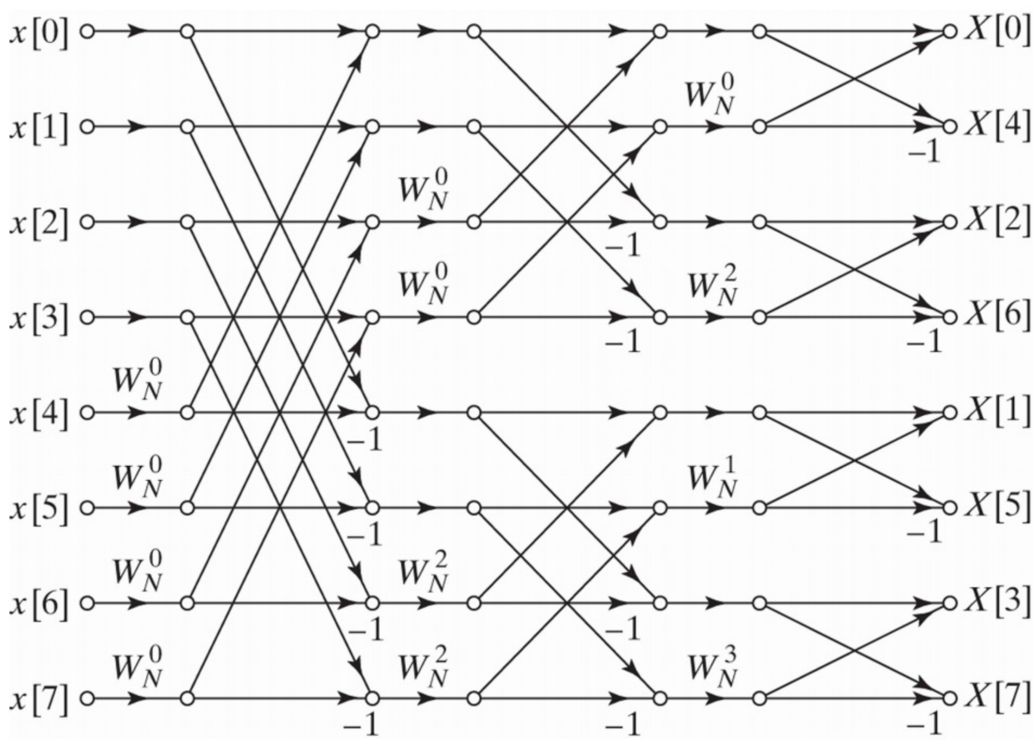
The final order is said to be in "bit reversed" form.
Meaning, convert the original indices to binary bits, reverse the bits and convert the reversed binary bits back to decimal indices to get the final order.

For example, in 8 point DFT if the original index is 3, it's binary equivalent will be 011 (3 bits because $8 = 2^3$). Reversed form is 110 which when converged gives 6. Therefore, in the final order, $x(6)$ will appear as the 3^{rd} indexed element and $x(3)$ will appear as the 6^{th} indexed element.

9.2.2 Decimation in Frequency (DIF FFT)

Another method would be to split the input sequence into two halves directly. This will result in the output DFT sequence appearing in bit reversed order.

Consider the same 8 point DFT using DIF approach.



In order to calculate IDFT of $X(k)$, the same algorithm can be used with the Twiddle factors being W_N^{-k} instead and the final result should be scaled by $\frac{1}{N}$.

DFT shortcuts:

Direct formula to find 2 point DFT

$$\begin{bmatrix} X(0) \\ X(1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \end{bmatrix}$$

Direct formula to find 4 point DFT

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & j \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$