Infi II–104281 *Due on Thursday, April 16, 2015*

Homework #2

1. Let f be integrable on $[a, b]$. Let g be a function defined on $[a, b]$ and satisfying $g(x) = f(x)$, for all $x \in [a, b]$ except for a finite number of points. Prove that g is integrable and that $\int_a^b g(x)dx = \int_a^b f(x)dx$.

2. Let f be integrable on $[a, b]$. Recall that $\Omega(P, f) = \sum_i \omega(f, I_i)\Delta_i$ is the oscillation of f with respect to P. (Here I_i is the i-th subinterval of P and $\Delta_i = |I_i|$.) Show that the upper and lower Darboux sums, $U(P, f)$ and $L(P, f)$, satisfy

$$|U(P, f) - \int_a^b f(x)dx| \leq \Omega(P, f), \quad |L(P, f) - \int_a^b f(x)dx| \leq \Omega(P, f).$$

3. Let f be differentiable on $[a, b]$ and assume that $|f'(x)| \leq M$, for all $x \in [a, b]$. Prove that for every partition P one has $\Omega(f, P) \leq M\lambda(P)(b - a)$.

4. i. Let f be a continuous function on $[a, b]$ satisfying $f \geq 0$ and $f \not\equiv 0$. Prove that $\int_a^b f(x)dx > 0$.
ii. Give an example of an integrable function f on $[a, b]$ satisfying $f \geq 0$, $f \not\equiv 0$ and $\int_a^b f(x)dx = 0$.

5 Let f be continuous on $[a, b]$. Let $M = \sup_{x \in [a,b]} |f(x)|$.
i. Show that $\int_a^b |f(x)|^n dx \leq M^n$.
ii. Show that for any $\epsilon \in (0, M)$, there exists $\delta > 0$ such that $\int_a^b |f(x)|^n dx \geq \delta(M - \epsilon)^n$. (Hint: Use the same idea as you used to prove 4-i.)
iii. Show that

$$\lim_{n \to \infty} (\int_a^b |f(x)|^n dx)^{\frac{1}{n}} = \sup_{x \in [a,b]} |f(x)|.$$

(Hint: Use (i) and (ii).)