Infi II–104281 *Due on Monday, June 1, 2015*

Homework #5

1. Prove that the following sequences of functions converge uniformly on the specified intervals.
i. $\{\frac{1}{n}\cos n^2 x\}_{n=1}^{\infty}$, $x \in R$ ii. $\{n \ln(1+\frac{1}{nx})\}_{n=1}^{\infty}$, $x \in [1,4]$.
2. Let $f_n(x) = \sqrt{n}xe^{-nx}$, for $x \geq 0$. Where does the sequence $\{f_n(x)\}_{n=1}^{\infty}$ converge pointwise and where does it converge uniformly?
3. i. Show that the sequence $\{\frac{nx}{1+n^2x^2}\}_{n=1}^{\infty}$ converges uniformly on $[a,\infty)$ for any $a > 0$, but that it only converges pointwise on $(0,\infty)$.
ii. Show that the sequence $\{\frac{1}{n}\ln(1+nx)\}_{n=1}^{\infty}$ converges uniformly on $[0,b]$ for any $b > 0$, but that it only converges pointwise on $[0,\infty)$.
4 Find the region where the following series converges uniformly and the region where it converges pointwise: $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{n(x^2-3x+2)}$.
5. Prove that the following series converges uniformly on $(0,1]$: $\sum_{n=1}^{\infty}(x\ln x)^n$.
6. Let $f_0(x)$ be integrable on $[0,a]$. For $n \geq 1$, define $f_n(x) = \int_0^x f_{n-1}(t)dt$, $x \in [0,a]$. Prove that the sequence $\{f_n(x)\}_{n=1}^{\infty}$ converges to 0 uniformly on $[0,a]$.
7. Let $f_n(x) = \frac{nx}{1+n^2x^p}$, $x \in [0,1]$, $p > 0$.
i. For which p does $\{f_n(x)\}_{n=1}^{\infty}$ converge uniformly to 0?
ii. Does $\lim_{n\to\infty}\int_0^1 f_n(x)dx = 0$ for $p = 2$? For $p = 4$?
8. Find the radii of convergence of the following power series:
i. $\sum_1^{\infty}\frac{(-1)^n}{n!}(\frac{n}{e})^n x^n$;
ii. $\sum_{n=1}^{\infty}\frac{2^n n!}{n^n}x^n$;
iii. $\sum_{n=1}^{\infty}\frac{1}{n}(1+\frac{1}{n})^{n^2}x^{3n}$.
9. Prove that for $|x| < 1$,

$$\frac{1}{(1-x)^3} = \frac{1}{2}\sum_{n=0}^{\infty}(n+1)(n+2)x^n.$$

10. Calculate the sum for the following series:
i. $\frac{x^3}{1\cdot 3} - \frac{x^5}{3\cdot 5} + \frac{x^7}{5\cdot 7} - \frac{x^9}{7\cdot 9} + \cdots$;
ii. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$.
(Hint for both parts: Find an appropriate power series that you know how to evaluate, and perform an operation on it—differentiation or integration.)