Infi II–104281 *Due on Thursday, June 25, 2015*

Homework #7

1. i. Calculate the integral $\int_0^1 \frac{x}{(1+ax)^2}dx$ by considering the function $F(y) = \int_0^1 \frac{dx}{1+xy}$ and differentiating under the integral sign on the one hand, and performing the integration directly on the other hand.
ii. Calculate the integral $\int_0^1 \frac{dx}{(x^2+a^2)^3}$ by considering the function $F(y) = \int_0^1 \frac{dx}{x^2+y^2}$.
2. i. Calculate $\frac{d}{dx}\big(\int_{-x}^{x} \frac{1-e^{-xy}}{y}dy\big)$.
ii. Calculate $\frac{d}{dx}\big(\int_0^{\frac{x}{2}} \sqrt{x^2-y^2}dy\big)$ in two different ways: (1) by differentiating under the integral sign and then integrating, and (2) by direct integration followed by differentiation.
3. Let $y(t), f(t)$ be C^2-functions satisfying

$$y(x) = 4\int_0^x (t-x)y(t)dt - \int_0^x (t-x)f(t)dt.$$

Show that y solves the differential equation $y''(x)+4y(x) = f(x)$, with the initial conditions $y(0)=0,\ y'(0)=0$.

4. Let v be continuous on $[0,1]$ and define $k(x,y) = \begin{cases} x(1-y), & x \le y; \\ y(1-x), & x > y. \end{cases}$

Prove that the function $u(x) = \int_0^1 k(x,y)v(y)dy$ satisfies the differential equation $u''(x) = -v(x),\ x \in [0,1]$.
5. Write the integral $\int_{-1}^1 \big[\int_0^{\sqrt{1-x^2}} f(x,y)dy\big]dx$ as a repeated integral with the order of the integration switched. (Don't calculate the integral!)
6. Let $D = \{(x,y) : x^2+y^2 \le 2x\}$. Write the integral $\int\int_D f(x,y)dxdy$ in polar coordinates. (Don't calculate the integral!)
7. Use polar coordinates to calculate the following integrals:
i. $\int\int_D e^{-x^2-y^2}dxdy$;
ii. $\int_0^R \Big(\int_0^{\sqrt{R^2-x^2}} \ln(1+x^2+y^2)dy\Big)dx$;
iii. $\int\int_D \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dxdy$, where $D = \{(x,y) : x^2+y^2 \le 1\}$.
8. i. Calculate $\int\int_D (x+y)^3(x-y)^3dxdy$, where D is the region bounded by the curves $x+y=1$, $x+y=3$, $x-y=1$ and $x-y=-1$.
ii. Let D be the region bounded by the curves $xy=1$, $xy=2$, $y=x$ and $y=3x$. Calculate the area of D; that is, calculate $\int\int_D dxdy$.
9. Let $f : [a,b] \to \mathbb{R}$ be continuous. Prove that the graph $\{(x,f(x)) : x \in [a,b]\}$ of f is a set of area zero. (Hint: f is uniformly continuous.)