Infi II–104281 *Due on Thursday, April 30, 2015*

Homework #3

1. Calculate:

i. $\int_0^1 \frac{1}{1+\sqrt{x}}\,dx$ ii. $\int_1^2 \ln^2 x\,dx$.

2. Calculate

i. $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right)$ ii. $\lim_{n\to\infty} n^3\sum_{j=1}^{n}\frac{1}{(n^2+j^2)^2}$.

3. Let $F(x)=\int_x^{2x} e^{t^4}dt$. Calculate $F'(x)$.

4. Calculate $\lim_{x\to 0}\frac{\int_0^{x^3}\tan t\,dt}{\int_0^{\sin^2 x} t^2 dt}$.

5. Show that

$$\frac{2}{3}-\frac{2}{5}+\frac{1}{7}-\frac{1}{27}\le\int_0^1\sqrt{x}e^{-x}dx\le\frac{2}{3}-\frac{2}{5}+\frac{1}{7}-\frac{1}{27}+\frac{1}{132}.$$

(Hint: For an appropriate value of k, expand e^{-x} in a Taylor series of order k to get the lower bound, and of order $k+1$ to get the upper bound.)

6. Calculate the area of the *astroid* A_a defined by

$$A_a=\{(x,y): x^{\frac{2}{3}}+y^{\frac{2}{3}}\le a^{\frac{2}{3}}\},$$

where $a>0$. (To make sure that things are clear: four points on the boundary of the region are $(\pm a,0)$ and $(0,\pm a)$.)

7. Calculate the length of the curve $y=\ln x$, $1\le x\le 2$. (Hint: When you arrive at an integral that has the term $(x^2+1)^{\frac{1}{2}}$ in it, it will be more efficient to let $u=(x^2+1)^{\frac{1}{2}}$ rather than to let $x=\tan\theta$.)

8. Let R denote the region bounded by the x-axis, the y-axis, the line $y=8$ and the curve $y=x^3$.

i. Calculate the volume of the three-dimensional body obtained by revolving R about the x-axis.

ii. Calculate the volume of the three-dimensional body obtained by revolving R about the y-axis.

9. Show that $\lim_{n\to\infty}\left(\frac{(2n)!}{n^n n!}\right)^{\frac{1}{n}} = \frac{4}{e}$ in two different methods. In both methods you will calculate $\lim_{n\to\infty}\ln\left(\frac{(2n)!}{n^n n!}\right)^{\frac{1}{n}}$.

First method: Use Stirling's formula: $n! \sim n^n e^{-n}\sqrt{2\pi n}$.

Second method: Interpret $\ln\left(\frac{(2n)!}{n^n n!}\right)^{\frac{1}{n}}$ as a Riemann sum.

10. i. Let f be continuous on $[0, 2\pi]$. Show that for all $\epsilon > 0$ there exists a step function f_ϵ such that $\int_0^{2\pi} |f(x) - f_\epsilon(x)|dx < \epsilon$.

ii. Let f be a step function. Show that $\lim_{n\to\infty}\int_0^{2\pi} f(x)\cos nx dx = 0$.

iii. Use (i) and (ii) to show that $\lim_{n\to\infty}\int_0^{2\pi} f(x)\cos nx dx = 0$, for all continuous functions f on $[0, 2\pi]$.