Infi II–104281 *Due on Sunday, May 17 2015*

Homework #4

ONLY problems 1-9 are to be handed in. The other problems are there to give you extra practice.

1. Calculate the value of the following improper integral: $\int_1^\infty \frac{\arctan x}{x^2} dx$.
2. Find all values of $p > 0$ and $q > 0$ for which the following integrals converge:
i. $\int_1^\infty \frac{\sin x}{x^p} dx$;
ii. $\int_0^\infty \frac{dx}{x^p + x^q}$;
iii. $\int_0^\infty \frac{dx}{x^p(1+x)^q}$.
3. Check whether the following improper integrals exist:
i. $\int_0^1 \frac{1}{\ln x} dx$;
ii. $\int_{-\infty}^\infty \frac{2x}{e^x - e^{-x}} dx$.
4. Let f be continuous on $[0, 1]$, and let $0 < a < b$. Show that the following limit exists and find the limit: $\lim_{\epsilon \to 0^+} \int_{\epsilon a}^{\epsilon b} \frac{f(x)}{x} dx$.
5. For which values of $p > 0$ does the series $\sum_{n=2}^\infty \frac{1}{n(\ln n)^p}$ converge?
6. Check whether the following series converge:
i. $\sum_{n=1}^\infty \frac{(n!)^2}{(2n)!}$;
ii. $\sum_{n=1}^\infty \frac{4n^2+5n}{n(n^2+1)^{\frac{3}{2}}}$.
7. Check whether the following series converge:
i. $\sum_{n=1}^\infty \frac{\cos n\pi}{n^{\frac{1}{4}}}$;
ii. $\sum_{n=1}^\infty (-1)^n \left[\frac{1}{\sqrt{2n+1}} - \frac{1}{\sqrt{3n+1}}\right]$.
iii. $\sum_{n=1}^\infty \left[\frac{1}{\sqrt{2n+1}} - \frac{1}{\sqrt{3n+1}}\right]$.
8. Let f be uniformly continuous on $[0, \infty)$ and assume that $\int_0^\infty f(x)dx$ exists. Prove that $\lim_{x\to\infty} f(x) = 0$.
9. Let $\{a_n\}_{n=1}^\infty$ be a positive sequence. Prove that if $\sum_{n=1}^\infty a_n < \infty$ if and only if $\sum_{n=1}^\infty \frac{a_n}{1+a_n} < \infty$.

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10. Consider the series $\sum_{n=1}^\infty \frac{1}{n(n+1)(n+2)}$. Calculate the partial sums S_n explicitly, and use this to calculate the sum of the series explicitly.
11. Check whether the following series converges: $\sum_{n=1}^\infty \frac{2\cdot 4 \cdot \ldots \cdot 2n}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n+1)}$.
12. i. Let f be a positive monotone decreasing function on $[1, \infty)$, and assume that $\int_1^\infty f(x)dx = \infty$. Let $I_n = \int_1^n f(x)dx$ and $S_n = \sum_{k=1}^n f(k)$, $n = 1, 2, \ldots$. Let $u_n = S_n - I_n$. Prove that the sequence $\{u_n\}_{n=1}^\infty$ converges.
ii. Using part (i), prove that $\lim_{n\to\infty} \left(\sum_{j=1}^n \frac{1}{j} - \ln n\right)$ exists. This limit is called *Euler's constant*, and is denoted by γ. One has $\gamma \approx .5772$.