Home Assignment II

Week IV: 15.11-23.11 [1]

Answer any 6 questions from the following 9.

Exercise 1. Consider an unfair coin for which a head occurs with probability p. The coin is tossed N times, where N is a random variable distributed Poisson with parameter λ. Let $A = \{\text{the total number of heads obtained is } n\}$ and $B = \{\text{the total number of tails obtained is } k\}$. Prove that A and B are independent for any $n, k \in \mathbb{N}$. (*The events A and B can be independent because N is a random number*)

Exercise 2. Use the Poisson approximation to calculate the probability that at most 3 of 50 people will have invalid drive license, if normally 6% of the people do.

Exercise 3. Find the most probable number of successes in a sequence of n independent Bernoulli trials with the probability p of success.

Hint: Let $X \sim B(n,p)$. Consider the fractions $\frac{P\left(X=k\right)}{P\left(X=k-1\right)}$, $k = 1, 2, \ldots n$ in order to prove that the binomial probability $P(X = k)$ first increases and then decreases, reaching its greatest value $P(X = m)$, where $m = [(n+1)p]$.

Exercise 4. Let N be a discrete integer-valued random variable with the density $f_N(n) = = c \cdot n2^{-n}$, $n = 1, 2, \ldots$ Find: (**a**) The constant c; (**b**) The expectation $E(N)$.

Exercise 5. There are two urns with w white and b black balls. One ball is drawn from the first urn and is then placed into the second urn, without looking at its color; after this a ball is drawn from the second urn. Let us denote by W_i the event "the ith drawn ball is white" and by B_i the event "the ith drawn ball is black", $i = 1, 2$. (**a**) Find the probability $P(W2)$. (**b**) Is it true that $P(B2|W1) = P(B1|W2)$?

Exercise 6. (**a**) Prove that for arbitrary events $A_1, \ldots, A_n$ the following inequality is valid:

$$P\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} P(A_i^c)$$

Let $\{B_n\}$ be an infinite sequence of events and $P(B_n) = 1$ for $n = 1, 2, \ldots$ Prove that:
(**b**) The events $\{B_n\}$ are independent; (**c**) $P\left(\bigcap_{n=1}^{\infty} B_n\right) = 1$.

Exercise 7. Consider a sequence of Bernoulli trials with probability of success p. (**a**) Find the probability that a consequent successes will occur before b failures. (**b**) Suppose that it is known that the first two trials were failures. What is the conditional probability that the first success will occur after the fifth trial but before the tenth one ?

Exercise 8. Consider an unbalanced coin such that "head" and "tail" occur with probabilities p and $q = 1 - p$, respectively. The coin is tossed n times. Let $A = \{\text{at the first toss ahead occurs}\}$, $A_k = \{\text{in } n \text{ trials exactly } k \text{ heads appear}\}$. For what relation between n, p and k are the events A and A_k are independent ?

***Exercise 9.** Let $(\Omega, \mathcal{A}, P)$ be an arbitrary probability space. Prove that there are at most countable number of elements ω such that $P(\{\omega\}) > 0$.

[1] The due date is 23.11. The solutions will be published at the net on Friday, 24.11