Home Assignment III

The due date is 17.12.

Exercise 1. Let X and Y be two random variable defined on the same probability space $(\Omega, \mathcal{A}, P)$. **(a)** We say that they are *equivalent* if $P\{\omega : X(\omega) \neq Y(\omega)\} = 0$. Give an example of two identically distributed random variables (i.e. such that $F_X(x) = F_Y(x),\ \forall x \in \mathbb{R}$) which are not equivalent. **(b)** Prove that $P(X > Y) = 1$ implies $F_X(x) \leq F_Y(x),\ x \in \mathbb{R}$. Is the converse statement true ?

Exercise 2. Let X be a random having exponential distribution with parameter λ. **(a)** Find the density of the random variable $X^\alpha,\ \alpha > 0$. **(b)** Find the probability of the event $A = \{[X] \text{ is even}\}$, where $[X]$ is the integer part of X.

Exercise 3. Let the random variable X assume only non-negative values, have continuous distribution function $F_X(x)$. Suppose that the following holds for any $x, y \geq 0:\ \ P\{X < y + x | X \geq y\} = P\{X < x\}$. Prove that X is exponentially distributed.
Hint: It is known, and you can use it, that the only continuous solution of the functional equation $G(x + y) = G(x) \cdot G(y),\ x, y \geq 0$ is $G(x) = e^{-\lambda x}$ for some $\lambda \in \mathbb{R}$.

Exercise 4. **(a)** Let X be a random variable having the normal distribution $N(\mu, \sigma^2)$. Compute $E(X^4)$. **(b)** Suppose that the points on the plane (X, Y) are distributed with the density $f_{X,Y}(x, y) = \frac{1}{12\pi} \exp\{-\frac{(x-1)^2}{8} - \frac{(y-2)^2}{18}\}$. Find the probability $P(X \geq 0, Y < 3)$.

Exercise 5. Let X be a random variable having continuous distribution function $F_X(x)$. Find the distribution function of the random variable $Y = F_X(X)$.
Hint: Consider first the case where $F_X(x)$ is a strictly increasing function. For the general case define $g(y) = \sup\limits_x \{x :\ F_X(x) \leq y\}$ for $0 < y < 1$.