**Infi II–104281** *Due on Monday, June 15, 2015*

# Homework #6

**1.** Check whether the following limits exist (possibly in the generalized sense):

i. $\lim_{x^2+y^2\to\infty}(x^2+y^2)\exp(-x-y)$; ii. $\lim_{x\to 0, y\to 2}\frac{\sqrt{x^2+(y-2)^2+1}-1}{x^2+(y-2)^2}$;

iii. $\lim_{x\to 0, y\to 0}\frac{1-\cos 2xy}{x^2 y \sin \pi y}$; iv. $\lim_{x\to 1, y\to -1}\frac{x^4-x^2-y^4+y^2}{x+y}$.

**2.** Is $f(x,y)=x^2-xy+y^2$ uniformly continuous on $R^2$?

**3** Let $f(x,y)=\begin{cases}\frac{x^3+y^3}{2x^2+y^2}, & (x,y)\neq(0,0);\\ 0, & (x,y)=(0,0).\end{cases}$

Is $f$ continuous at $(0,0)$? Do $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ exist? Is $f$ differentiable at $(0,0)$?

**4.** Let $f(x)=\begin{cases}\frac{x^2 y}{(x^2+y^2)^k}, & (x,y)\neq(0,0);\\ 0, & (x,y)=(0,0).\end{cases}$

i. For which $k>0$ is $f$ continuous at $(0,0)$?

ii. For which $k>0$ do all the directional derivatives of $f$ exist at $(0,0)$?

iii. For which $k>0$ is $f$ differentiable at $(0,0)$?

**5** Calculate $f_{xy}$ and $f_{yx}$ at $(0,0)$, where

$$f(x,y)=\begin{cases}x^2\sin\frac{1}{x}+y^2\sin\frac{1}{y}; & x,y\neq 0;\\ x^2\sin\frac{1}{x}, & x\neq 0,\ y=0;\\ y^2\sin\frac{1}{y}; & x=0,\ y\neq 0;\\ 0, & x=y=0.\end{cases}$$

**6.** Let $f(x,y)$ satisfy $|f(x,y)|\leq M(x^2+y^2)$, for some $M>0$ and for all $(x,y)$ in a neighborhood of $(0,0)$. Prove that $f$ is differentiable at $(0,0)$.

**7.** Let $g(u, v, w)$ be differentiable in $R^3$. Define $f$ by $f(x, y, z) = g(x, xy, xyz)$. Find a condition on $g$ (involving derivatives) that guarantees that $f$ does not depend on $x$.

**8.** Let $u(x, y) = \ln\sqrt{x^2 + y^2}$ and $v(x, y) = \arctan\frac{y}{x}$. Show that $u$ and $v$ solve *Laplace's equation:* $w_{xx} + w_{yy} = 0$. Such functions are called *harmonic.*

**9** Let $f$ be differentiable at $(x_0, y_0)$. Prove that there exists a direction $v$ for which the directional derivative of $f$ is equal to 0.

**10** Let $f(u, v, w)$ be differentiable on $R^3$. Assume that

$$f(x, y, 2x^2 + y^2) = 3x - 5y$$

and that $\frac{df}{dn}(1, 2, 6) = 1$, where $n = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$. Calculate $(\nabla f)(1, 2, 6)$.

**11** Find the equation of the tangent plane and find the normal to the tangent plane of the function $z = e^x \sin y$ at the point $(0, 0, 0)$.

**12** Let $m \in R$. A function $f(x, y)$ is called *homogeneous of order* $m$ if it satisfies $f(\lambda x, \lambda y) = \lambda^m f(x, y)$, for all $x, y, \lambda$.

i. Show that any function $f$ homogeneous of order $m$ can be represented in the form $f(x, y) = x^m F(\frac{y}{x})$, for all $x \neq 0$, where $F$ is some function of one variable.

ii. Show that if $f$ is homogeneous of order $m$ then it satisfies the differential equation $xf_x + yf_y = mf$.