SOLUTION TO MOED A(SKETCH)

(1) (a) $T(v) = 0 \Leftrightarrow \langle T(v), T(v)\rangle = 0 \Leftrightarrow \langle v, T^*T(v)\rangle = 0 \Leftrightarrow \langle v, TT^*(v)\rangle = 0 \Leftrightarrow \langle T^*(v), T^*(v)\rangle = 0 \Leftrightarrow T^*(v) = 0$.

(b) $(T - \lambda I)(T - \lambda I)^* = TT^* - \lambda\bar{\lambda}I - \lambda T^* - \bar{\lambda}T$ whereas $(T - \lambda I)^*(T - \lambda I) = T^*T - \lambda\bar{\lambda}I - \lambda T^* - \bar{\lambda}T$. So there is an equality since $TT^* = T^*T$.

(c) Direct consequence from (a) and (b).

(d) $\langle T(v), w\rangle = \lambda \langle v, w\rangle$ and $\langle T(v), w\rangle = \langle v, T^*w\rangle = \left\langle v, \bar{\theta}w\right\rangle = \theta \langle v, w\rangle$. Thus, $(\lambda - \theta)\langle v, w\rangle = 0$. Since $\lambda - \theta \neq 0$, we must get $\langle v, w\rangle = 0$.

(2) Define $v_1 = (1, i), v_2 = (i, -1), v_3 = (1, 0), v_4 = (0, 1)$. It is trivial that we have a basis of V. Applying Gram-Schmidt process results in

$$\frac{1}{\sqrt{2}}v_1, \frac{1}{\sqrt{2}}v_2 \frac{-1}{\sqrt{2}}(1, i), \frac{1}{\sqrt{2}}(i, -1)$$

Since we got an orthonormal basis, the last two vectors are orthogonal to the first two. Thus, the last two vectors must be an orthonormal basis of the desired subspace.

(3) (a) The associated symmetric matrix is

$$[f] = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 7 & -5 \\ 2 & -5 & 8 \end{pmatrix}.$$

We use the usual Algorithm to find a diagonal matrix congruent to this one:

$$\begin{pmatrix} 1 & -3 & 2 \\ -3 & 7 & -5 \\ 2 & -5 & 8 \end{pmatrix} \to \begin{bmatrix} R_2 + 3R_1 \to R_2 \\ C_2 + 3C_1 \to C_2 \end{bmatrix} \to \begin{pmatrix} 1 & 0 & 2 \\ 0 & -2 & 1 \\ 2 & 1 & 8 \end{pmatrix} \to \begin{bmatrix} R_3 - 2R_1 \to R_3 \\ C_3 - 2C_1 \to C_3 \end{bmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & 4 \end{pmatrix}$$

and finally

$$\to \begin{bmatrix} 0.5R_2 + R_3 \to R_3 \\ 0.5C_2 + C_3 \to C_3 \end{bmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4.5 \end{pmatrix}$$

The rank is clearly 3. The signature = number of positive diagonal values - number of negative diagonal values. So in this case it is 1.

(b) h is clearly bi-linear symmetric form so it is enough to check that it's positive define. The associated matrix (corresponding to the natural basis of $\mathbb{R}^3$) is

$$[h] = \begin{pmatrix} 2 & -3 & 2 \\ -3 & 8 & -5 \\ 2 & -5 & 9 \end{pmatrix}$$

We use the same Algorithm as before to get a diagonal matrix with positive diagonal entries.

(4) (a) It is obvious that $KerT \subseteq KerT^*T$. If $x \in KerT^*T$, then $\langle T^*T(x), x\rangle = 0 \Rightarrow \langle T(x), T(x)\rangle = 0 \Leftrightarrow T(x) = 0$. So $x \in KerT$.

(b) Let $x \in W^\perp$. Since $\langle T^*(x), w\rangle = \langle x, T(w)\rangle = 0$ for every $w \in W$ (remember that $T(w) \in W$), it follows that $T^*(x) \in W^\perp$.

(5) (a) Write $c(x) = m(x) = \prod_{i=1}^{k}(x - a_i)^{t_i}$, where $a_1, ..., a_k$ are all the different eigenvalues of A. So we get that t_i is the size of the maximal Jordan block of a_i(since $m(x) = \prod_{i=1}^{k}(x - a_i)^{t_i}$) and also that t_i is the sum of the orders of all the Jordan blocks corresponding to a_i (since $c(x) = \prod_{i=1}^{k}(x - a_i)^{t_i}$). Hence, there is only 1 Jordan block corresponding to a_i and its size is t_i. Therefore, The Jordan form of A is $diag(J_{t_1}(a_1), ..., J_{t_k}(a_k))$.

(b) Notice that $c(x) = (x - 2)^4$. So 2 is the only eigenvalue of A. Denote $T = A - 2I$. One checks that $T^2 = 0$ and that $e_1, e_4 \notin KerT$. Thus, $\{T(e_1), e_1\}$and $\{T(e_4), e_4\}$ are Jordan chain of order 2.
So

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Here the columns of P are the vectors $T(e_1), e_1, T(e_4), e_4$ respectively. It is also o.k. to consider the basis $e_1, T(e_1), e_4, T(e_4)$ and build P with regards to this ordering.

(6) See http://he.wikipedia.org/wiki/%D7%90%D7%99-%D7%A9%D7%95%D7%95%D7%99%D7%95%D7%9F_%D7%A7%D7%95%D7%A9%D7%99-%D7%A9%D7%95%D7%95%D7%A8%D7%A5.