§4.2 平稳过程的自相关函数

4.2.1平稳过程自相关函数的性质

定理4.2.1 平稳过程$\{X(t),t\in T\}$的自相关函数$R_X(\tau)$，有如下性质：

1) $R(0)\geq 0$;

2) $|R_X(\tau)|\leq R_X(0)$; $(|C_X(\tau)|\leq C_X(0);)$

3) $R_X(-\tau)=\overline{R_X(\tau)}$;

4) 非负定性 $\text{对}\forall n \geq 1, t_1, \boxed{?}, t_n \in T,$

及复数 α1,α2,…,α*n* 有

$$\sum_{k,j=1}^{n} \alpha_j \overline{\alpha_k} R_X(t_k - t_j) \geq 0.$$

证明

$$1)\quad R_X(0) = E\{X(t)\overline{X(t)}\} = E\{|X(t)|^2\} \geq 0;$$

2）由许瓦兹不等式

$$|R_X(\tau)|^2 = |R_X(t, t+\tau)|^2 = \left|E(X(t)\overline{X(t+\tau)})\right|^2$$

$$\leq E[|X(t)|^2]E[|X(t+\tau)|^2] = R_X^2(0);$$

3) $\overline{R_X(\tau)} = \overline{E[X(t)\overline{X(t+\tau)}]} = E[\overline{X(t)}X(t+\tau)]$

$$= E[X(s)\overline{X(s-\tau)}] = R_X(-\tau);$$

4) $\sum_{k,j=1}^{n} \alpha_j \overline{\alpha_k} R_X(t_k - t_j)$

$$= \sum_{k,j=1}^{n} \alpha_j \overline{\alpha_k} E[X(t_j)\overline{X(t_k)}]$$

$$= E[\sum_{k,j=1}^{n} \alpha_j \overline{\alpha_k} X(t_j)\overline{X(t_k)}] \quad = E[\left|\sum_{k=1}^{n} \alpha_k X(t_k)\right|^2] \geq 0$$

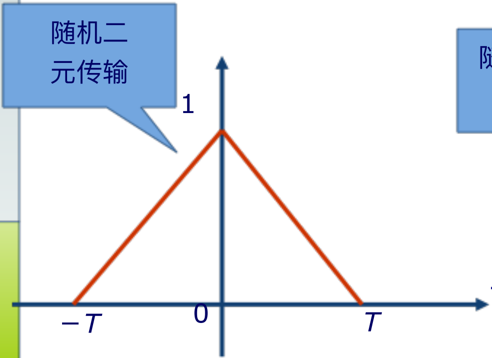
推论1 实平稳过程$\{X(t),t\in T\}$的$R_X(\tau)$有:

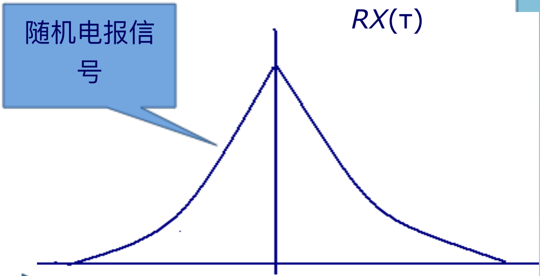
1) $R(0)\geq 0$;

2) $|R_X(\tau)|\leq R_X(0)$;

3) $R_X(-\tau)=R_X(\tau)$;

4) 具有非负定性.





定理4.2.2 如果$\{X(t),t\in T\}$是周期为L的周期平稳过程, 即有

$$P\{X(t+L)=X(t)\}=1,$$

则$R_X(\tau)$也是周期函数,有 $R(t+L)=R(t)$.

证

$$P\{X(t+L)X(t)=X^2(t)\}=1,$$

$$\Rightarrow P\{X(t+L)X(t)-X^2(t)=0\}=1.$$

$$\Rightarrow E\{X(t+L)X(t)-X^2(t)\}=0,$$

$$\Rightarrow R_X(L)=R_X(0).$$

$$\Rightarrow R_X(t+L)=R_X(t).$$

Ex.2　设平稳过程$X(t)$的相关函数为$R_X(\tau)$,且$R_X(L)= R_X(0)$, L为一个常数, $L>0$, 试证：

$X(t+L)=X(t)$ 依概率为1成立；

定理4.2.2的逆

证　因

$$R_X(0)-R_X(L)=0,$$

由切比雪夫不等式，对

$$\forall \varepsilon>0,$$

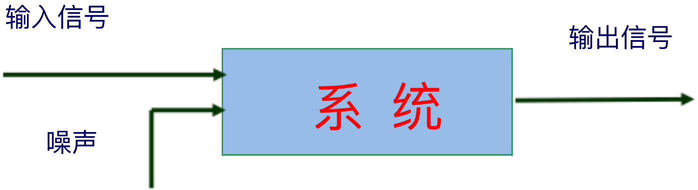
$$P\{|X(t+L)-X(t)|>\varepsilon\}\le\frac{E|X(t+L)-X(t)|^2}{\varepsilon^2}$$

$$=\frac{2}{\varepsilon^2}[R_X(0)-R_X(L)]=0,$$

$$\Rightarrow\quad P\{X(t+L)\ne X(t)\}=0,$$

4.2.2 联合平稳过程及互相关函数

实际中需同时研究多个关联随机过程的统计规律.



要从输出中检测出有用信号，需同时研究输入信号和噪声的联合统计特性.

Ex.3 设$X(t)$是雷达的发射信号, 遇到目标后的回波信号是 $aX(t-b)$, $a<<b$, b是信号返回时间,回波信号必然伴有噪音. 记噪音为$N(t)$,则接受机收到的全信号为

$$Y(t)=aX(t-b)+N(t),$$

需考虑$X(t)$与$N(t)$的联合统计特性.

又如 $\{X(t),t\in T\}$和$\{Y(t), t\in T\}$都是平稳过程,问$\{Z(t)= X(t)+ Y(t), t\in T\}$是否是平稳过程?

定义4.2.5 称平稳过程$\{X(t),t\in T\}$和平稳过程$\{Y(t),t\in T\}$为联合平稳的(平稳相关), 若对任意τ,

$$R_{XY}(s+\tau,t+\tau)=R_{XY}(s,t).$$

即 $$R_{XY}(s+\tau,t+\tau)=E[X(s+\tau)\overline{Y(t+\tau)}]$$

$$=E[X(s)\overline{Y(t)}]=R_{XY}(s,t).$$

互相关函数仅与$(t-s)$ 的大小有关.

可将联合平稳过程的互相关函数定义为

$$R_{XY}(\tau)=R_{XY}(t,t+\tau)=E\{X(t)\overline{Y(t+\tau)}\}.$$

Ex.4 设$\{X(t),t\in T\}$和$\{Y(t),t\in T\}$是平稳相关的平稳过程,讨论$\{Z(t)=X(t)+Y(t),\ t\in T\}$的平稳性.

解

$$m_Z(t)=E[X(t)+Y(t)]=m_X+m_Y;$$

$$R_Z(t,t+\tau)=E[Z(t)\overline{Z(t+\tau)}]$$

$$=E\{[X(t)+Y(t)]\overline{[X(t+\tau)+Y(t+\tau)]}\}$$

$$=R_X(\tau)+R_{XY}(\tau)+R_{YX}(\tau)+R_Y(\tau)$$

故$\{Z(t),t\in T\}$是平稳过程.

4.2.3 平稳过程的均方微积分

定理4.2.3 实平稳过程$\{X(t), t\in T\}$均方连续的充要条件是相关函数$R_X(\tau)$在$\tau=0$处连续，且此时$R_X(\tau)$处处连续.

证 充分性 设$R_X(\tau)$在$\tau=0$处连续，则

$$对\forall t_0\in T,\quad \lim_{t\to t_0} R_X(t-t_0)=R_X(0),$$

$$E[|X(t)-X(t_0)|^2]=E\{[X(t)-X(t_0)]\overline{[X(t)-X(t_0)]}\}$$

$$=E[X(t)\overline{X(t)}]+E[X(t_0)\overline{X(t_0)}]$$

$$-E[X(t)\overline{X(t_0)}]-E[X(t_0)\overline{X(t)}]$$

$$= 2[R_X(0) - R_X(t - t_0)] \to 0, \ (as \quad t \to t_0).$$

即$X(t)$在T上均方连续.

必要性 若$X(t)$在$t=t_0$处均方连续，有

$$\lim_{t \to t_0} E[|X(t) - X(t_0)|^2] = 0$$

在上式，令$\tau = t - t_0$，可得

$$\lim_{\tau \to 0}[R_X(0) - R_X(\tau)] = 0,$$

即$R_X(\tau)$在$\tau=0$处连续.

任意性 对任意τ_0,

$$|R_X(\tau)-R_X(\tau_0)|^2=\left|E\{X(t)\overline{[X(t+\tau)-X(t+\tau_0)]}\}\right|^2$$

$$\le E[|X(t)|^2]E[|X(t+\tau)-X(t+\tau_0)|^2]$$

$$=R_X^2(0)E[|X(t+\tau)-X(t+\tau_0)|^2],$$

由于$X(t)$在$t+\tau_0$处均方连续, 有

$$\lim_{\tau\to\tau_0} E[|X(t+\tau)-X(t+\tau_0)|^2]=0$$

$$\Rightarrow \lim_{\tau\to\tau_0} R_X(\tau)=R_X(\tau_0)$$

由τ_0 的任意性知 $R_X(\tau)$处处连续.

Ex.3　随机电报信号

$$X(t)=X_0(-1)^{N(t)},\qquad t\geq 0,$$

的自相关函数
$$R(\tau)=C^2e^{-2\lambda|\tau|}$$

在$\tau=0$ 处连续,从而$R(\tau)$在$(-\infty,\infty)$上连续.

故随机电报信号过程$\{X(t),t\geq 0\}$是均方连续, 均方可积的.

定理4.2.4 对于平稳过程$XT=\{X(t),\ t\in T\}$

1) XT均方可微的充要条件是$RX(\tau)$在$\tau=0$ 处二次可微;

2）XT均方可微, 其均方导数过程仍为平稳过程,有

均值函数 $$m_{X'}(t)=0,$$

相关函数 $$R_{X'}(\tau)=-R''_X(\tau).$$

证 1） 由均方可微准则, XT均方可微 $\Longleftrightarrow$

相关函数$R(s,\ t)$在$(t0\ ,t0)$处广义二阶可微,即

$$\lim_{\substack{\Delta t\to 0\\ \Delta s\to 0}}\frac{1}{\Delta t\Delta s}[R(t_0+\Delta t,t_0+\Delta s)-R(t_0+\Delta t,t_0)$$

$$-R(t_0,t_0+\Delta s)+R(t_0,t_0)]$$

$$=\lim_{\substack{\Delta t\to 0\\ \Delta s\to 0}}\frac{1}{\Delta t\Delta s}[R(\Delta s-\Delta t)-R(\Delta t)-R(\Delta s)+R(0)]$$

平稳性

存在.

2）设$X_T=\{X(t),\ t\in T\}$均方可导, 则

$$m_{X'}(t)=E[X'(t)]=\frac{d}{dt}E[X(t)]=\frac{d}{dt}(m_X)=0$$

$$R_{X'}(s,t)=E[X'(s)\overline{X'(t)}]=\frac{\partial^2}{\partial t\partial s}R_X(t-s)$$

$$=\frac{\partial}{\partial s}R'_X(t-s)=-R''_X(t-s)=-R''_X(\tau)$$

$\{X'(t),t\in T\}$是平稳过程.

续Ex.3 随机电报信号的自相关函数

$$R(\tau)=C^2e^{-2\lambda|\tau|}$$

有 $R'_X(0+)=-2\lambda C^2$, $R'_X(0-)=2\lambda C^2$

$R'_X(0)$不存在 $\Longrightarrow$ $R''_X(0)$不存在

$\Longrightarrow$ 随机电报信号$\{X(t),t\geq 0\}$均方不可导.

设{$X(t), t\in T$}是均方可微的实平稳过程,

推论1

则对$\forall t\in T, X(t)$与$X'(t)$不相关.

证 {$X(t), t\in T$}是实平稳过程

$$\Rightarrow\quad E[X(t)X'(t)] = \frac{\partial}{\partial t}R_X(t,t) = R'_X(0)$$

又因 $R_X(-\tau) = R_X(\tau) \Rightarrow \quad R'_X(-\tau) = -R'_X(\tau)$

特别 $R'_X(0) = -R'_X(0) \quad \Rightarrow R'_X(0) = 0,$

$$E[X(t)X'(t)] = 0, \text{即} X'(t) \text{与} X(t) \text{不相关.}$$

推论2

{X(t), t∈T}是均方可微的实正态平稳过程,

$$\text{则对} \forall t \in T, X(t) \text{与} X'(t) \text{相互独立.}$$

定理4.2.5 积分

设{X(t),t∈T}是均方连续的平稳过程, 则在有限区间上, 均方

$$\int_a^b X(t)dt$$

存在,且有

$$E[\int_a^b X(s)ds \overline{\int_a^b X(t)dt}] = \int_a^b \int_a^b R_X(t-s)dsdt$$

特别若{$X(t),t\in T$}是实平稳过程,则

$$1)\quad E\left[\int_a^b X(t)dt\right] = m_X(b-a);$$

$$2)\quad E\left[\int_a^b X(t)dt\right]^2 = 2\int_0^{b-a}[(b-a)-|\tau|]R(\tau)d\tau.$$

证　由均方可积准则及过程的平稳性可得

$$E\left[\int_a^b X(s)ds\overline{\int_a^b X(t)dt}\right] = \int_a^b\int_a^b R_X(t-s)dsdt$$

当{$X(t),t\in T$ }是实平稳过程

$E[X(t)] = m_X$是常数,

$R_X(s,t) = R_X(\tau)$是偶函数,

故 1) $E[\int_a^b X(t)dt] = \int_a^b m_X dt = m_X(b-a);$

2) $E[\int_a^b X(t)dt]^2 = \int_a^b\int_a^b R(t-s)dsdt$

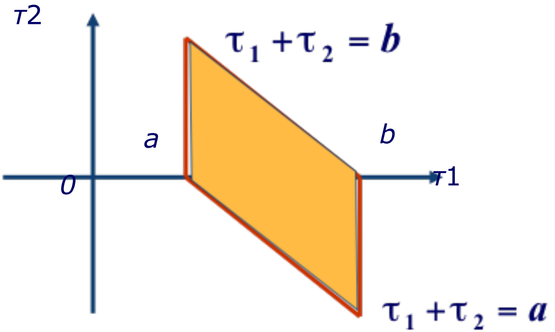
做积分变换，令

$$\begin{cases}\tau_1 = s \\ \tau_2 = t-s\end{cases} \quad 则 \quad \begin{cases}s = \tau_1 \\ t = \tau_1+\tau_2\end{cases} \quad |J| = \begin{vmatrix}1 & 0 \\ 1 & 1\end{vmatrix} = 1$$

将 $D = \{(s,t) | a \le s \le b, a \le t \le b\}$ 变换为

$$G = \{(\tau_1,\tau_2) | a \le \tau_1 \le b, a-\tau_1 \le \tau_2 \le b-\tau_1\}$$

$$G=\{(\tau_1,\tau_2)\,|\,a\le\tau_1\le b,\ a-\tau_1\le\tau_2\le b-\tau_1\}$$

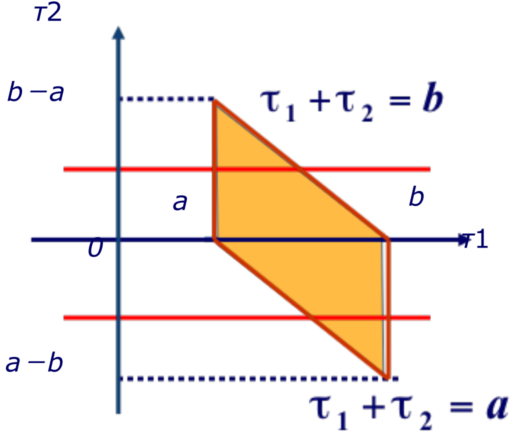


$$E[\int_a^b X(t)dt]^2=\int_a^b\int_a^b R(t-s)dsdt$$

$$=\iint_G R(\tau_2)d\tau_1 d\tau_2$$

$$= \iint_G R(\tau_2) d\tau_1 d\tau_2$$

$$= \int_{a-b}^{0} R(\tau_2) d\tau_2 \int_{a-\tau_2}^{b} d\tau_1 + \int_{0}^{b-a} R(\tau_2) d\tau_2 \int_{a}^{b-\tau_2} d\tau_1$$



$$=\int_{a-b}^{0} R(\tau_2)d\tau_2 \int_{a-\tau_2}^{b} d\tau_1 + \int_{0}^{b-a} R(\tau_2)d\tau_2 \int_{a}^{b-\tau_2} d\tau_1$$

$$=\int_{a-b}^{0} [(b-a)+\tau_2]R(\tau_2)d\tau_2$$

$$+\int_{0}^{b-a} [(b-a)-\tau_2]R(\tau_2)d\tau_2$$

$$=2\int_{0}^{b-a} [(b-a)-|\tau|]R(\tau)d\tau$$

重要公式:

$$E[\int_{a}^{b} X(t)dt]^2 = 2\int_{0}^{b-a} [(b-a)-|\tau|]R(\tau)d\tau .$$

Ex.5 设实平稳过程{$X(t), t\in T$}的相关函数为$R_X(\tau)$，均值函数为0，若

$$Y(t)=\int_0^t X(t)dt$$

求$Y(t)$的自协方差函数和方差.

解

$$m_Y(t)=\int_0^t m_X dt = 0\cdot(b-a)=0;$$

$$C_Y(s,t)=R_Y(s,t)=\int_0^s dv\int_0^t R(u-v)du;$$

$$D[Y(t)]=C_Y(t,t)=R_Y(t,t)$$

$$=\int_0^t dv\int_0^t R(u-v)du = 2\int_0^t (t-\tau)R_X(\tau)d\tau.$$

思考题:

1) 为什么需要特别研究平稳过程的自相关和互相关函数?

2) 联合平稳过程意义?