一、1. 解: 将 $f(x)=0$ 的 $f(x)$ 线性化，用线性方程的解逼近非线性方程的解

对二维方程进行泰勒展开，可将牛顿迭代法推广到二维.

$$x^{(k+1)} = x^{(k)} - F'[x^{(k)}]^{-1} \cdot F[x^{(k)}] \qquad F'(x) = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n(x)}{\partial x_1} & \cdots & \frac{\partial f_n(x)}{\partial x_n} \end{pmatrix}$$

例: $\begin{cases} x_1+2x_2-3=0 \\ 2x_1^2+x_2^2-5=0. \end{cases}$

①雅可比矩阵.

$$F'(x) = \begin{pmatrix} 1 & 2 \\ 4x_1 & 2x_2 \end{pmatrix} \Rightarrow F'(x)^{-1} = \frac{1}{2x_2-8x_1} \begin{pmatrix} 2x_2 & -2 \\ -4x_1 & 1 \end{pmatrix}$$

$$\Rightarrow x^{(k+1)} = x^{(k)} - \frac{1}{2x_2^{(k)}-8x_1^{(k)}} \begin{pmatrix} 2x_2^{(k)} & -2 \\ -4x_1^{(k)} & 1 \end{pmatrix} \begin{pmatrix} x_1^{(k)}+2x_2^{(k)}-3 \\ 2(x_1^{(k)})^2-(x_2^{(k)})^2-5 \end{pmatrix}$$

二、解: (1) 令 $\varphi_0(x)=1$ $\quad \varphi_1(x)=x$ $\quad \varphi_2(x)=x^2$

$$\begin{pmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2.9 \\ 4.2 \\ 7 \end{pmatrix} \Rightarrow \begin{cases} a = \frac{143}{350} \\ b = 0.42 \\ c = \frac{3}{35} \end{cases}$$

$$y = \frac{143}{350}x^2 + 0.42x + \frac{3}{35}$$

三、解: $x_0=0 \quad x_1=1 \quad x_2=2 \quad x_3=3$

$y_0=1 \quad y_1=3 \quad y_2=9 \quad y_3=25$

x_k	y_k	一阶差分	二阶差分	三阶差分
0	1			
1	3	2		
2	9	6	2	
3	25	16	5	1.

则 $N_3(x) = 1 + 2x + 2x(x-1) + x(x-1)(x-2)$

$= x^3 - x^2 + 2x + 1$

$\Rightarrow N_3(0.5) = \frac{15}{8}$

四、解:

$$\begin{pmatrix} 1 & -1 & 1 & -4 \\ 5 & -4 & 3 & -12 \\ 2 & 1 & 1 & 11 \end{pmatrix} \to \begin{pmatrix} 5 & -4 & 3 & -12 \\ 1 & -1 & 1 & -4 \\ 2 & 1 & 1 & 11 \end{pmatrix} \to \begin{pmatrix} 5 & -4 & 3 & -12 \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{8}{5} \\ 0 & \frac{13}{5} & -\frac{1}{5} & \frac{79}{5} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 5 & -4 & 3 & -12 \\ 0 & \frac{13}{5} & -\frac{1}{5} & \frac{79}{5} \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{8}{5} \end{pmatrix} \rightarrow \begin{pmatrix} 5 & -4 & 3 & -12 \\ 0 & \frac{13}{5} & -\frac{1}{5} & \frac{79}{5} \\ 0 & 0 & \frac{5}{13} & -\frac{5}{13} \end{pmatrix} \Rightarrow \begin{cases} x_1 = \frac{145}{13} \\ x_2 = \frac{78}{13} \\ x_3 = -1 \end{cases}$$

五解 设对 $1, x, x^2$ 精确成立

$$\Rightarrow \begin{cases} 1 = A + B \\ \frac{1}{2} = \frac{1}{2}B + C \\ \frac{1}{3} = \frac{1}{4}B \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{3} \\ B = \frac{4}{3} \\ C = -\frac{1}{6} \end{cases} \qquad \int_0^1 f(x)dx = -\frac{1}{3}f(0) + \frac{4}{3}f(\frac{1}{2}) - \frac{1}{6}f'(0)$$

对于 x^3. $\int_0^1 x^3 dx = \frac{1}{4} \neq -\frac{1}{3}\times 0 + \frac{4}{3}\times\frac{1}{8} - \frac{1}{6}\times 0 = \frac{1}{6}$ 则代数精度为2

六,解:(1).

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -2 \\ 1 & -1 & 3 \\ 2 & 2 & -1 \end{pmatrix} \Rightarrow L = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 2 & 2 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 2 & -2 \\ & -1 & 3 \\ & & -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -2 \\ & -1 & 3 \\ & & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = 1 \\ x_3 = -1 \end{cases}$$

(2)

$$\Rightarrow D = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad L = \begin{pmatrix} & & \\ -1 & & \\ -2 & -2 & \end{pmatrix} \quad U = \begin{pmatrix} & -2 & 2 \\ & & -1 \\ & & \end{pmatrix}$$

$$\text{则 } B_J = D^{-1}(L+U) = \begin{pmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix}$$

对于B. $\rho(B) = \min\{\|B\|_1, \|B\|_\infty\} = \min\{4,4\} = 4 > 1$ 发散.