第九部分：常微方程的数值解法

一阶常微分方程的欧拉方法，修正的欧拉法，局部截断误差和计算格式的精度阶概念，龙格库塔方法、常微分方程组和高阶常微分方程的数值分法。

① 欧拉方法 (一阶方法)

$$y_{n+1} = y_n + h f(x_n, y_n)$$

例1 $\begin{cases} y' = \frac{y}{x} - 2y^2 & (0 < x < 3) \\ y(0) = 0 \end{cases}$

解：$f(x_n, y_n) = \frac{y_n}{x_n} - 2y_n^2$. $\Rightarrow$ $y_{n+1} = y_n + h\left(\frac{y_n}{x_n} - 2y_n^2\right)$

取 $h=0.2$, $x_n = nh$. $f(0,0)=1$.

$\Rightarrow$	x_n	y_n
0	0	0
1	0.2	0.2
2	0.4	0.384
3	0.6	0.517
⋮		

$y_1 = y_0 + h \cdot f(0,0) = 0 + 0.2 \times 1 = 0.2$

$y_2 = y_1 + 0.2\left(\frac{y_1}{x_1} - 2y_1^2\right) = 0.2 + 0.2\left(\frac{0.2}{0.2} - 2 \times 0.2^2\right) = 0.384$

$y_3 = y_2 + 0.2\left(\frac{y_2}{x_2} - 2y_2^2\right) = 0.384 + 0.2\left(\frac{0.384}{0.4} - 2 \times 0.384^2\right) = 0.517$

② 改进欧拉

$$\begin{cases} \overline{y}_{n+1} = y_n + h f(x_n, y_n) \\ y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, \overline{y}_{n+1})\right] \end{cases}$$

例：$\begin{cases} y' = y - \frac{2x}{y} \\ y(0) = 1 \end{cases}$ $h = 0.1$.

解：$f(x_n, y_n) = y_n - \frac{2x_n}{y_n}$. $\Rightarrow \begin{cases} \overline{y}_{n+1} = y_n + h\left(y_n - \frac{2x_n}{y_n}\right) \\ y_{n+1} = y_n + \frac{h}{2}\left[\left(y_n - \frac{2x_n}{y_n}\right) + \left(\overline{y}_{n+1} - \frac{2x_{n+1}}{\overline{x}_{n+1}}\right)\right] \end{cases}$

	x_n	$\overline{y}_n$	y_n
0	0	1	1
1	0.1	1.1	1.095909
		⋮	

$\overline{y}_1 = y_0 + 0.1\left(y_0 - \frac{2x_0}{y_0}\right) = 1 + 0.1\left(1 - \frac{0}{1}\right) = 1.1$

$y_1 = y_0 + \frac{0.1}{2}\left[y_0 + \overline{y}_1 - \frac{2x_1}{\overline{y}_1}\right] = 1 + \frac{1}{20}\left(1 + 1.1 - \frac{2 \times 0.1}{1.1}\right)$

$= 1.095909$

③ 局部截断误差

① 欧拉法：$T_{n+1} = y(x_{n+1}) - y(x_n) - h f[x_n, y(x_n)]$.

$= y(x_{n+1}) - y(x_n) - h y'(x_n)$

$= \frac{h^2}{2} y''(x_n) + O(h^3)$. 一阶.

② 隐式欧拉 $T_{n+1} = y(x_{n+1}) - y(x_n) - h f[x_{n+1}, y(x_{n+1})]$

$= y(x_{n+1}) - y(x_n) - h y'(x_{n+1})$

$= -\frac{h^2}{2} y''(x_n) + O(h^3)$ 一阶

③ 梯形 $T_{n+1} = y(x_{n+1}) - y(x_n) - \frac{h}{2}\left[f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))\right]$

$= y(x_{n+1}) - y(x_n) - \frac{h}{2}\left[y'(x_n) + y'(x_{n+1})\right]$

$= -\frac{h^3}{12} y'''(x_n) + O(h^4)$ 二阶.