# General Phiscs

CXC

## 1 Classical Dynamics

**The orbit of earth is ellipse**
First deduce the motion in polar coordinates: $\vec{r} = r\vec{e}_r,\ \vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\vec{e}_r + r\frac{d\vec{e}_r}{dt} = \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta$
$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2r}{dt^2}\vec{e}_r + \frac{dr}{dt}\frac{d\vec{e}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\vec{e}_\theta + r\frac{d^2\theta}{dt^2}\vec{e}_\theta + r\frac{d\theta}{dt}\frac{d\vec{e}_\theta}{dt} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\vec{e}_r + \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\vec{e}_\theta$
Then consider $\vec{F} = \vec{F}_r + \vec{F}_\theta,\ \vec{F}_r = -G\frac{Mm}{r^2}\vec{e}_r,\ \vec{F}_\theta = 0 \Rightarrow \ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2},\ 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$, both sides of the latter equation are both multiplied by r, we have$2r\dot{r}\dot{\theta} + r^2\dot{\theta}^2 = 0 \Rightarrow \frac{d}{dt}(r^2\dot{\theta}) = 0$, which means $\dot{A} = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}h,\ \ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}$, set $u = \frac{1}{r}$, then $r^2\dot{\theta} = h \Rightarrow \dot{\theta} = hu^2$, and with$\dot{r} = \frac{d}{d\theta}\left(\frac{1}{u}\right)\dot{\theta} = -\frac{1}{u^2}\frac{du}{d\theta}\dot{\theta} = -h\frac{du}{d\theta}\ddot{r} = \frac{d}{dt}(\dot{r}) = \frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right)\dot{\theta} = -h\frac{d^2u}{d\theta^2}\dot{\theta} = -h^2u^2\frac{d^2u}{d\theta^2}$, the equation turns to $-h^2u^2\frac{d^2u}{d\theta^2} - h^2\theta^3 = -GMu^2$, namely $\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}$, thus $u = c_1 cos\theta + c_2 sin\theta + \frac{GM}{h^2}$, we get $r = \frac{p}{1+ecos(\theta-\theta_0)}$, in which $e = \frac{h^2\sqrt{c_1^2+c_2^2}}{GM},\ p = \frac{h^2}{GM}$

**Elastic collision in 2D**
Conservation of energy:$v_{1i}^2 = v_{1f}^2 + v_{2f}^2$, conservation of momentum:$\vec{v}_{1i} = \vec{v}_{1f} + \vec{v}_{2f}$
$v_{1i}^2 = (\vec{v}_{1f} + \vec{v}_{2f})(\vec{v}_{1f} + \vec{v}_{2f}) = v_{1f}^2 + v_{2f}^2 + 2\ \vec{v}_{1f}\vec{v}_{2f} \Rightarrow\ \vec{v}_{1f}\cdot\vec{v}_{2f} = 0$

**Impulse affects the tension on rope**
Rope $l_1$ hangs $m_1$ and $l_2$ connected $m_1$ with $m_2$, give $v_0$ to $m_1$, whose direction is horizontally right, save T in $m_2$:$a_1 = -\frac{v_0^2}{l_1}$, thus $m_2$ has $v_0$ horizontally left with respect to $m_1$, and $a_1$ vertically down with respect to $m_1$ $\therefore\ T = -m_2a_1 + m_2g + \frac{m_2v_0^2}{l_2} = m_2(g + \frac{v_0^2}{l_1} + \frac{v_0^2}{l_2})$

**Derivation of Coriolis force:** $\vec{F}_c = 2m\vec{v}\times\vec{\omega}$
$\frac{d\vec{i}}{dt} = \vec{\omega}\times\vec{i},\ \frac{d\vec{j}}{dt} = \vec{\omega}\times\vec{j},\ \frac{d\vec{k}}{dt} = \vec{\omega}\times\vec{k} \Rightarrow \vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega}\times\vec{r}$, the Cartesian basis vectors of the rotating frame also rotate:$\frac{d\vec{r}}{dt} = \frac{dx'}{dt}\vec{i}' + \frac{dy'}{dt}\vec{j}' + \frac{dz'}{dt}\vec{k}' + x'\frac{d\vec{i}'}{dt} + y'\frac{d\vec{j}'}{dt} + z'\frac{d\vec{k}'}{dt} \Rightarrow \vec{v}_I = \vec{v}_R + \vec{\omega}\times\vec{r},\ \frac{d\vec{r}}{dt} = \vec{v}_R + \vec{\omega}\times\vec{r} \Rightarrow \frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}_R}{dt} + \vec{\omega}\times\frac{d\vec{r}}{dt}$ where $\frac{d\vec{v}_R}{dt} = \vec{a}_R + \vec{\omega}\times\vec{v}_R \Rightarrow \vec{a}_I = \vec{a}_R + 2\vec{\omega}\times\vec{v}_R + \vec{\omega}\times(\vec{\omega}\times\vec{r}),\ \therefore\ \vec{F}_R = \vec{F}_I + \vec{F}_c \Rightarrow F_{coriolis} = 2m\ \vec{v}\times\vec{\omega}$

**Application of Coriolis force**
A boat M on equator and its anchor m, moved to the height of h, as the radius of the earth is R and its angular velocity $\omega$, With conservation of angular momentum $(M+m)\omega R^2 = M\omega' R^2 + m\omega'(R+h)^2,\ v = (\omega - \omega')R = \frac{mR(2R+h)h\omega}{MR^2+m(R+h)^2} = \frac{(2+\frac{h}{R})m\omega h}{M+(1+\frac{h}{R})^2 m}$

**Theorem .** $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c}),\ \vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

**Radius of curvature** $\rho = \frac{\left[(x')^2+(y')^2\right]^{\frac{3}{2}}}{|y''x'-x''y'|}$

**Exercise (figure 1).**
Acceleration comparison of three accompanying rope methods: $m_1g - T_1 = m_1a_{CM},\ T_1r_1 = \frac{1}{2}m_1r_1^2\beta_1,\ (T_1 - T_2)r = \frac{1}{2}m_2r_2^2\beta_2,\ T_2 - mg = ma,\ a_{CM} = a + \beta_1r_1 \Rightarrow a = \frac{\frac{1}{3}m_1 - m}{\frac{1}{3}m_1 + \frac{1}{2}m_2 + m}\cdot g$

**Theorem .** parallel axis theorem:$I = I_{CM} + mr_{CM}^2$, vertical axis theorem:$I_x + I_y + I_z = 2I_{total}$

**Exercise (figure 2).**
The inclined plane with mass M and the inclination angle is placed on a smooth horizontal surface. A uniform cylinder with mass m and radius r rolls purely from rest. Find the angular acceleration of the cylinder and the acceleration of the inclined plane. $mgrsin\theta + marcos\theta = \frac{3}{2}mr^2\beta,\ aM + am - a_{CM}mcos\theta = 0,\ a_{CM} = r\beta \therefore$ $mgrsin\theta = \frac{3a(M+m)r}{2cos\theta} - marcos\theta = \frac{ar(3(M+m)-2mcos^2\theta)}{2cos\theta} \Rightarrow \beta = \left[\frac{2(m+M)sin\theta}{3(m+M)-2mcos^2\theta}\right] \cdot \frac{g}{r},\ a = \frac{mgsin2\theta}{3(m+M)-2mcos^2\theta}$

**Exercise (figure 3).**
A disc is mounted at the midpoint of a thin rod. The friction coefficient between the thin rod and the ground is $\mu$, and the angular velocity $\omega$ is very large. Find the precession angular velocity and the nutation angular velocity.$\tau_g = \frac{l}{2} \times mg = \frac{l}{2}mgsin\theta,\ dL = Lsin\theta d\psi,\ dL = \tau_g dt \Rightarrow \Omega = \frac{d\psi}{dt} = \frac{\tau_g}{Lsin\theta} = \frac{\frac{l}{2}mgsin\theta}{Lsin\theta} = \frac{mgl}{2L} = \frac{mgl}{2I\omega} = \frac{mgl}{2\frac{1}{2}mR^2\omega} = \frac{gl}{\omega R^2}$ notice:the axis of rotation has radius r which is very small but nonnegligible, so the direction of frictional force is the opposite of the precession direction.$\tau_{f-CM} = \frac{l}{2} \times \mu mg = \frac{l}{2}\mu mg,\ dL = -Ld\theta,\ dL = \tau_{f-CM}dt \Rightarrow \Omega' = \frac{d\theta}{dt} = -\frac{\tau_{f-CM}}{L} = -\frac{\mu mgl}{2L} = -\frac{\mu mgl}{2I\omega} = -\frac{\mu mgl}{2\frac{1}{2}mR^2\omega} = -\frac{\mu gl}{\omega R^2}$

## 2 Vibration and Waves

**Theorem .**
Sinusoidal wave's various forms:$y = y_m cos(\omega t \mp kx + \psi),\ \omega \equiv \frac{2\pi}{T}, k = \frac{\omega}{v} = \frac{2\pi}{\lambda}, f = \frac{1}{T}, v = \lambda f$, in which the right direction of wave movement corresponds to the minus sign

**Theorem .** Remark:wave equation of reflection wave should plus $\pi$ because the direction of y is reversed, either mechanic waves or light waves.

**Interference of two waves**
$y_1(x,t) = Asin(k_1x - \omega_1t)\ y_2(x,t) = Asin(k_2x - \omega_2t) y_1 + y_2 = Asin(k_1x - \omega_1t) + Asin(k_2x - \omega_2t) = 2Asin\Big(\big(\frac{k_1+k_2}{2}\big)x - \big(\frac{\omega_1+\omega_2}{2}\big)t\Big)cos\Big(\big(\frac{k_1-k_2}{2}\big)x - \big(\frac{\omega_1-\omega_2}{2}\big)t\Big) = 2Asin\Big(k_{avg}x - \omega_{avg}t\Big)cos\Big(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\Big)$

**Derivation of formulas for phase velocity(figure 4)**
$\Sigma F_y = Fsin\theta_2 - Fsin\theta_1,\ \Sigma F_y \approx Ftan\theta_2 - F\tan\theta_1 = F\ \delta(tan\theta),\ F\ \delta(tan\theta) = \Sigma F_y = \delta m\ a_y = \mu\delta x\ a_y \Rightarrow \frac{\delta(tan\theta)}{\delta x} = \frac{\mu}{F}a_y \Rightarrow \frac{\delta(\frac{\partial y}{\partial x})}{\delta x} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2},\ \lim_{\delta x \to 0}\frac{\delta(\frac{\partial y}{\partial x})}{\delta x} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2},\ v = \sqrt{\frac{\partial x}{\partial t}} = \sqrt{\frac{\partial^2 y}{\partial t^2} \cdot \frac{\partial x^2}{\partial^2 y}} = \sqrt{\frac{F}{\mu}}$

**Energy transferring**
$\Delta U = \frac{1}{2}(\Delta m)(\frac{\partial y}{\partial t})^2 = \frac{1}{2}(\Delta m)\omega^2y^2 = \frac{1}{2}(\mu\Delta x)\omega^2y^2$, $\mu$ is the linear density, $dU = \frac{1}{2}\mu\omega^2[Asin(kx - \omega t)]^2dx = \frac{1}{2}\mu\omega^2A^2sin^2(kx - \omega t)dx$. The total energy E=K+U is constant, i.e., $dE = \frac{1}{2}\mu\omega^2A^2dx$ , $P = \frac{dE}{dt} = \frac{1}{2}\mu\omega^2A^2v = \frac{1}{2}A^2\omega^2\sqrt{\mu F}$

**Doppler effect**
Moving observer: $v' = v + v_0$, $\lambda$ is unchanged, $f = \frac{v'}{\lambda} = \frac{v+v_o}{\lambda} = (1 + \frac{v_o}{v})f$
Moving source: $\lambda' = \lambda - \Delta\lambda = \lambda - \frac{v_s}{f},\ f' = \frac{v}{\lambda'} = \frac{v}{\lambda - \frac{v_s}{f}} = \left(\frac{1}{1-\frac{v_s}{v}}\right)f$
Both move: $f' = \frac{v \pm v_o}{v \mp v_s}f$, moving towards is associated with frequency increase.

**Theorem .**
Streamline:line which is tangent to the instantaneous velocity field vector
Pathline:Trajectory of an individual fluid element(fluid particle)
Bernoulli's Equation: $p + \frac{1}{2}\rho v^2 + \rho gy = constant$

## 3 Einstein's Theory of Relativity

*Proof.* **Relativistic velocity addition law:** $w = \frac{u+v}{1+\frac{uv}{c^2}}$
Taking the train as the reference system:

$$\begin{cases} c(T_0+T_1) = (1+f)L \\ u(T_0+T_1) = (1-f)L \end{cases} \Rightarrow f = \frac{c-u}{c+u}$$

Taking the ground as the reference system:

$$\begin{cases} (c-v)T_0 = L \\ (c-w)T_0 = D \\ (c+v)T_1 = fL \\ (c+w)T_1 = D \end{cases} \Rightarrow \begin{cases} \frac{T_0}{T_1} = \frac{c+w}{c-w} \\ f = \frac{(c+v)T_1}{(c-v)T_0} \\ = \frac{(c+v)(c-w)}{(c-v)(c+w)} \end{cases} \Rightarrow \begin{matrix} \frac{c-w}{c+w} = \frac{(c-u)(c-v)}{(c+u)(c+v)} \\ \Rightarrow w = \frac{u+v}{1+\frac{uv}{c^2}} \end{matrix} \qquad \square$$

*Proof.* **Violation of simultaneity:** $T = \frac{Dv}{c^2}$
With the relativistic velocity addition law, $w = \frac{u+v}{1+\frac{uv}{c^2}}$

$$\begin{cases} w_L T_L + vT_L = \frac{1}{2}L_F \\ w_R T_R - vT_R = \frac{1}{2}L_F \\ D = w_R T_R + w_L T_L \\ T = T_R - T_L \end{cases} \Rightarrow \begin{cases} T = \frac{1}{2}L_F(\frac{1}{w_R-v} - \frac{1}{w_L+v}) \\ D = \frac{1}{2}L_F(\frac{w_R}{w_R-v} + \frac{w_L}{w_L+v}) \\ \frac{T}{D} = \frac{2v-(w_R-w_L)}{2w_R w_L + v(w_r - w_L)} = \frac{v}{c^2} \end{cases} \Rightarrow T = \frac{Dv}{c^2} \qquad \square$$

**The shrinking factor**

$\frac{T_M}{T_F} = \frac{L_M}{D_F} = \frac{D_M}{D_F} = s,\ D_F = L_F + vT_F = L_F + \frac{v^2 D_F}{c^2} \Rightarrow D_F = \frac{L_F}{1-\frac{v^2}{c^2}},\ sL_M + v\cdot\frac{D_F v}{c^2} = D_F \Rightarrow sD_M + D_F(\frac{v}{c})^2 = D_F,\ L_F = sL_M = sD_M = s^2 D_F \Rightarrow s = \sqrt{1-\frac{v^2}{c^2}},\ s^2 D_F = sD_M \Rightarrow D_M = sD_F$

**Lorentz transformation**

$$\begin{cases} realative\ speed\ v : \frac{c}{d} = v \\ rear\ clock\ ahead : \frac{b}{a} = \frac{v}{c^2} \\ length\ contraction : d = \frac{1}{c} \\ time\ dilation : a = \frac{1}{s} \end{cases} \Rightarrow \begin{matrix} t = \frac{t'+vx'/c^2}{s} = \frac{t'+vx'/c^2}{\sqrt{1-v^2/c^2}} \\ x = \frac{vt'+x'}{s} = \frac{x'+vt'}{\sqrt{1-v^2/c^2}} \end{matrix} \Rightarrow \begin{matrix} t' = \frac{t-vx/c^2}{\sqrt{1-v^2/c^2}} \\ x' = \frac{x-vt}{\sqrt{1-v^2/c^2}} \end{matrix}$$

y'=y, z'=z, dx'=$\gamma$(dx-vdt), dt'=$\gamma$(dt-$\frac{vdx}{c^2}$), $v'_x = \frac{dx'}{dt'} = \frac{dx-vdt}{dt-\frac{v}{c^2}dx} = \frac{\frac{dx}{dt}-v}{1-\frac{v}{c^2}\frac{dx}{dt}} = \frac{v_x - v}{1-\frac{v}{c^2}v_x},\ v'_{y,z} = \frac{v_{y,z}}{\gamma(1-\frac{v}{c^2}v_x)}$

**The invariant interval**

Galilean transformation:$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$
Lorentz transformation:$(\Delta s)^2 = c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$
For simplicity we drop $\Delta$ and ignore y and z directions:
Proof:$c^2t^2 - x^2 = \frac{c^2(t'+vx'/c^2)^2}{1-v^2/c^2} - \frac{(x'+vt')^2}{1-v^2/c^2} = \frac{t'^2(c^2-v^2)-x'^2(1-v^2/c^2)}{1-v^2/c^2} \equiv s^2$

**Extension of Newton's second law in theory of relativity**

$$\begin{aligned} \frac{d\gamma}{dt} &= \frac{d}{dt}(1-\frac{v^2}{c^2})^{-\frac{1}{2}} \\ &= -\frac{1}{2}(1-\frac{v^2}{c^2})^{-\frac{3}{2}} \cdot \frac{d}{dt}(1-\frac{v^2}{c^2}) \\ &= -\frac{1}{2}(1-\frac{v^2}{c^2})^{-\frac{3}{2}} \cdot -\frac{1}{c^2}\frac{d}{dt}v^2 \\ &= -\frac{1}{2}(1-\frac{v^2}{c^2})^{-\frac{3}{2}} \cdot -\frac{2v}{c^2}\frac{dv}{dt} \\ &= -\frac{1}{2}(1-\frac{v^2}{c^2})^{-\frac{3}{2}} \cdot -\frac{2va}{c^2} \\ &= \frac{va}{c^2}(1-\frac{v^2}{c^2})^{-\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} F &= \frac{dp}{dt} = \frac{d}{dt}(mv\gamma) = m\frac{d}{dt}(v\gamma) \\ &= m\Big(v\frac{d\gamma}{dt} + \gamma\frac{dv}{dt}\Big) \\ &= m\Big[v\cdot\frac{va}{c^2}(1-\frac{v^2}{c^2})^{-\frac{3}{2}} + \gamma a\Big] \\ &= ma\Big[\frac{v^2}{c^2}\frac{1}{(\sqrt{1-\frac{v^2}{c^2}})^3} + \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\Big] \\ &= ma\Big[\frac{v^2}{c^2(\sqrt{1-\frac{v^2}{c^2}})^3} + \frac{c^2(1-\frac{v^2}{c^2})}{c^2(\sqrt{1-\frac{v^2}{c^2}})^3}\Big] \\ &= \frac{ma}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}} \end{aligned}$$

# 4 Thermal

**Theorem .**
$\Delta L = \alpha L_0 \Delta T$, $\Delta V = \beta V_0 \Delta T \Rightarrow \beta = 3\alpha$
Ideal gas law: $PV = nRT = \frac{N}{N_A}RT = Nk_BT$, Boltzmann's constant: $k_B = \frac{R}{N_A} = 1.38 \times 10^{-23} J/K$
$PV = nRT \Rightarrow ln(V) = ln(T) + ln(n\frac{R}{P}) \Rightarrow \beta = \left(\frac{1}{V}\frac{dV}{dT}\right)_P = \left(\frac{d(lnV)}{dT}\right)_P = \frac{d(lnT)}{dT} = \frac{1}{T}$

**Van der Waals equation**
$\left(P + \frac{aN^2}{V^2}\right)(V - Nb) = Nk_BT \Rightarrow P = \frac{Nk_BT}{V-bN} - a\frac{N^2}{V^2}$, a is potential energy and b is volume of molecule
$\bar{p}A\Delta t = 2m\overline{\Sigma v_x} = 2m\overline{A \cdot v_x \Delta t \cdot \rho(= \frac{N}{V}) \cdot \frac{1}{2}} \Rightarrow \bar{p} = \rho\overline{mv_x^2} = \frac{\rho}{3}\overline{mv^2} \Rightarrow \bar{p}V = \frac{N}{3}\overline{mv^2} = \frac{nN_A}{3}\overline{mv^2}$
$\therefore \overline{\frac{1}{2}mv^2} = \frac{3\bar{p}V}{2nN_A} \xlongequal{pV=nRT} \frac{3}{2}k_BT \Rightarrow \overline{\frac{1}{2}mv_x^2} = \overline{\frac{1}{2}mv_y^2} = \overline{\frac{1}{2}mv_z^2} = \frac{1}{2}k_BT$
$\therefore$ internal energy $U = Nf(\text{degree of freedom})k_B\frac{T}{2}$, heat capacity $C_V = \left(\frac{\Delta U}{\Delta T}_{fixed\ V}\right) = \frac{f}{2}Nk_B$

**Maxwell Distribution**
$N(v) = 4\pi N\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}}v^2e^{-\frac{mv^2}{2k_BT}} \Rightarrow N = \int_0^{+\infty} N(v)dv$
$I_0 = \int_0^{+\infty} e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$ $\quad I_1 = \int_0^{+\infty} xe^{-ax^2}dx = \frac{1}{2a}$ $\quad I_{2n} = (-1)^n\frac{d^n}{da^n}I_0$ $\quad I_{2n+1} = (-1)^n\frac{d^n}{da^n}I_1$
Most probable speed: $\frac{dN(v)}{dv} = 0 \Rightarrow (2v - \frac{mv^3}{kT})e^{-\frac{mv^2}{2k_BT}} = 0 \Rightarrow v_p = \sqrt{\frac{2k_BT}{m}}$
Root mean sqaure speed: $v_{rms} = \sqrt{\bar{v^2}} = \sqrt{\frac{3k_BT}{m}} = \sqrt{\frac{3RT}{M}}$, average speed:$\bar{v} = \sqrt{\frac{8k_BT}{\pi m}} = \sqrt{\frac{8RT}{\pi M}}$

**Boltzmann Distribution**
$p = p_0e^{-\frac{\rho_0gh}{p_0}} \overset{p=n_Vk_BT}{\longrightarrow} n_V = n_0e^{-\frac{mgh}{k_BT}} \to n_V(v, h, ...) = n_0e^{-\frac{\frac{mv^2}{2}+mgh}{k_BT}} = n_0e^{-\frac{E(v,h)}{k_BT}} := n_0e^{-\beta E}$

**Derivation of speed distribution**
Define one-dimensional normalized distribution function of a single velocity $v_i$ by $f(v_i) = \frac{1}{n}\frac{dn}{dv_i}$, where n is the density of the ideal gas and $\frac{dn}{dv_i}$ represents the density of atoms with the velocity component between $v_i$ and $v_i + dv_i$
For any values of $v_x$ and $v_y$, rotate coordinate system where $v_\alpha^2 = v_x^2 = v_y^2$ and $v_\beta^2 = 0$, thus $f(v_\alpha)f(0) = f(v_x)f(v_y)$
Differentiating the expression with respect to $v_x$ gives $f'(v_x)f(v_y) = f'(v_\alpha)\frac{\partial v_\alpha}{\partial v_x}f(0)$
Because of the relationship between $v_\alpha$ and $v_x$, we get $\frac{\partial v_\alpha}{\partial v_x} = \frac{v_x}{v_\alpha}$
Similarly do with $v_y$, we give $f'(v_\alpha)f(0) = \frac{v_x}{v_\alpha}f'(v_x)f(v_y) = \frac{v_y}{v_\alpha}f(v_x)f'(v_y)$
Simplify it gives $\frac{1}{v_x}\frac{f'(v_x)}{f(v_x)} = \frac{1}{v_y}\frac{f'(v_y)}{f(v_y)}$=constant:=C
As a result we see $f(v_{x,y}) = Ae^(-\frac{C}{2}v_{x,y}^2)$
If we require $f(v_{x,y})$ to be normalized, it follows that $1 \equiv \int_{-\infty}^{+\infty} f(v_i)dv_i = A\sqrt{\frac{2}{C}}\int_{-\infty}^{+\infty} e^{-t^2}dt = A\sqrt{\frac{2\pi}{C}}$
So that $f(0) = A = \sqrt{\frac{C}{2\pi}}$
In three dimensions, the density of states with speed between v and $v + dv$ goes as $4\pi v^2dv$, namely $\frac{dn}{dv} = 4\pi nv^2f(v)f(0)$
By definition, the average of the square of the speed is $\bar{v^2} = 4\pi f^2(0)\int_0^{+\infty} v^4f(v)dv = 4\pi\left(\frac{C}{2\pi}\right)^{\frac{3}{2}}\int_0^{+\infty} v^4e^{-\frac{C}{2}v_{x,y}^2}dv$
The constant C can be determined by $\frac{m\bar{v^2}}{2} = \frac{3kT}{2}$, which gives $f(v) = 4\pi v^2\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}}e^{-\frac{mv^2}{2k_BT}}$, which is the Maxwell distribution.

**Fourier heat conduction law** $\quad \frac{Q}{\Delta t} = -\kappa_tA\frac{dT}{dx}$

**Mean free path**
Volume of the cylinder: $V = \pi d^2 vt$
Average number of collisions: $z = n_V \pi d^2 vt$
Mean free path: $l = \frac{vt}{n_V \pi d^2 vt} = \frac{1}{n_V \pi d^2}$, $pV = nK_B T \Rightarrow p = n_V k_B T$, $\therefore l = \frac{k_B T}{n_V \pi d^2 p}$
In fact, the relative motion between gas molecules leads to a correction $v \to \sqrt{2}v$
Thus mean free path: $l = \frac{vt}{n_V \pi d^2 (\sqrt{2}v)t} = \frac{1}{\sqrt{2} n_V \pi d^2} = \frac{k_B T}{\sqrt{2} n_V \pi d^2 p}$

**Isothermal expansion** $W = \int PdV = \int \frac{Nk_B T}{V} dV = Nk_B T ln \frac{V_f}{V_i}$, $\Delta U = 0 \Rightarrow Q = W = Nk_B T ln \frac{V_f}{V_i}$

**Free expansion** $\Delta U = 0$ and $Q = W = 0$

**The infinitesimal change** $dU = dQ - PdV$

**Otto cycle**
$W = Q_h - Q_c = nC_V(T_C - T_B) - nC_V(T_D - T_A) \Rightarrow e = \frac{W}{Q_h} = 1 - \frac{Q_c}{Q_h} = 1 - \frac{T_D - T_A}{T_C - T_B}$
Adiabatic expansion(dQ=0)
$C_V dT = dU = -PdV \xrightarrow{P = \frac{Nk_B T}{V}} \frac{C_V dT}{T} = -\frac{Nk_B dV}{V} \Rightarrow TV^{\gamma-1} = const$ and $PV^\gamma = const$, $\gamma = \frac{C_P}{C_V} = 1 + \frac{Nk_B}{C_V}$

**Efficiency of Otto cycle**
$A \to B$: $T_A V_1^{\gamma-1} = T_B V_2^{\gamma-1}$, $C \to D$: $T_D V_1^{\gamma-1} = T_C V_2^{\gamma-1} \Rightarrow \left(\frac{V_2}{V_1}\right)^{\gamma-1} = \frac{T_A}{T_B} = \frac{T_D}{T_C} \Rightarrow e = 1 - \frac{1}{\left(\frac{V_2}{V_1}\right)^{\gamma-1}}$

**Simplified Diesel engine**
$\frac{T_A}{T_B} = \left(\frac{V_B}{V_A}\right)^{\gamma-1} = \left(\frac{1}{r}\right)^{\gamma-1}$, $\frac{T_D}{T_C} = \left(\frac{V_C}{V_D}\right)^{\gamma-1} = \left(\frac{r_C V_B}{V_A}\right)^{\gamma-1} = \left(\frac{r_C}{r}\right)^{\gamma-1}$
$Q_{in} = nC_P(T_C - T_B)$, $Q_{out} = nC_V(T_D - T_A)$, $W_{cycle} = n[C_P(T_C - T_B) - C_V(T_D - T_A)]$

**Carnot' engine**
Isothermal expansion→Adiabatic expansion→Isothermal compression→Adiabatic compression→ ...
$e_C = 1 - \frac{Q_c}{Q_h}$, $Q_c = |W_{CD}| = nRT_c ln \frac{V_C}{V_D}$, $Q_h = W_{AB} = nRT_h ln \frac{V_B}{V_A}$, $\frac{Q_c}{Q_h} = \frac{T_c}{T_h} \frac{ln\left(\frac{V_C}{V_D}\right)}{ln\left(\frac{V_B}{V_A}\right)}$
$\begin{cases} T_h V_B \gamma - 1 = T_c V_C^{\gamma-1} \\ T_h V_A \gamma - 1 = T_c V_D^{\gamma-1} \end{cases} \Rightarrow \frac{Q_c}{Q_h} = \frac{T_c}{T_h} \Rightarrow e_C = 1 - \frac{Q_c}{Q_h} = 1 - \frac{T_c}{T_h}$
thus all Carnot engines operating between the same two temperatures have the same efficiency.

**Now putting in the proper sign**
$\frac{Q_h}{T_h} + \frac{Q_c}{T_c} = 0$, $\sum_i \frac{Q_{h_i}}{T_{h_i}} + \frac{Q_{c_i}}{T_{c_i}} = 0$, namely $\oint_{Carnot\ cycle} \frac{dQ}{T} = 0$
Thus $dS = \oint_{Carnot\ cycle} \frac{dQ}{T} = 0$ is an exact differential and T is the interating factor
$P(T,V) = \frac{RT}{V}$, $U(T) = C_V^{mol} T = \frac{fP}{2} T \Rightarrow dS = \frac{1}{T}(dU + PdV) = \frac{C_V^{mol} dT}{T} + \frac{RdV}{V}$
$\Rightarrow S(T,V) = S_0 + C_V^{mol} ln \frac{T}{T_0} + Rln \frac{V}{V_0}$