常微分方程

CXC

一阶线性 $\frac{dy}{dx}+p(x)y=f(x)\Rightarrow y=e^{-\int p(x)dx}\left[\int f(x)e^{\int p(x)dx}dx+C\right]$

伯努利

$$\frac{dy}{dx}+p(x)y=f(x)y^n\Rightarrow z=y^{1-n},\ \frac{dz}{dx}=(1-n)y^{-n}\frac{dy}{dx}$$
$$\Rightarrow\frac{dz}{dx}+(1-n)p(x)z=(1-n)f(x)$$
$$\Rightarrow z=e^{-\int(1-n)p(x)dx}\left[\int(1-n)f(x)e^{\int(1-n)p(x)dx}dx+C\right]$$

全微分

$$\mu(x,y)M(x,y)dx+\mu(x,y)N(x,y)dy\equiv du(x,y)\Rightarrow u(x,y)=C$$
$$du(x,y)=M(x,y)dx+N(x,y)dy\Rightarrow\frac{\partial u}{\partial x}=M(x,y),\frac{\partial u}{\partial y}=N(x,y)$$
$$u(x,y)=\int M(x,y)dx+\psi(y)\Rightarrow\psi'(y)=N(x,y)-\frac{\partial}{\partial y}\int M(x,y)dx$$

定理 微分方程 $M(x,y)dx+N(x,y)dy=0$ 有一个只依赖于 x 的积分因子的充要条件为 $\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=G(x)$，且此时积分因子为 $\mu(x)=e^{\int G(x)dx}$

定理 微分方程 $M(x,y)dx+Q(x,y)dy=0$ 有一个只依赖于 y 的积分因子的充要条件为 $\frac{1}{N}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=H(y)$，且此时积分因子为 $\mu(y)=e^{\int H(y)dy}$

凑微分与积分公式

$$ydx+xdy=dxy\quad\frac{ydx-xdy}{y^2}=d(\frac{x}{y})\quad\frac{-ydx+xdy}{x^2}=d(\frac{y}{x})$$
$$\frac{-ydx+xdy}{x^2+y^2}=d(arctan\frac{y}{x})\quad\frac{ydx-xdy}{x^2-y^2}=d(\frac{1}{2}ln\left|\frac{x-y}{x+y}\right|)$$
$$\int\frac{dx}{\sqrt{x^2\pm a^2}}=ln(x+\sqrt{x^2\pm a^2})+C\int\frac{dx}{\sqrt{a^2-x^2}}=arcsin(\frac{x}{a})+C$$

$$\int\sqrt{a^2-x^2}dx=\frac{1}{2}(x\sqrt{a^2-x^2}+a^2arcsin(\frac{x}{a}))+C$$
$$\int\sqrt{x^2\pm a^2}dx=\frac{1}{2}(x\sqrt{x^2\pm a^2}\pm a^2ln(x+\sqrt{x^2\pm a^2}))+C$$

常系数齐次

$$k_s\ real\ roots\ \lambda : Y = (\sum_{i=1}^{k} c_i x^{i-1}) e^{\lambda x}$$

$$k_s\ virtual\ roots\ \lambda : Y = \Big[(\sum_{i=1}^{k} a_i x^{i-1}) cos\beta + (\sum_{i=1}^{k} b_i x^{i-1}) sin\beta \Big] e^{\alpha x}$$

常系数非齐次

$$f(x) = P_m(x) e^{\alpha x} \Rightarrow y^* = x^k R_m(x) e^{\alpha x}$$

$$f(x) = P_m(x) e^{\alpha x} cos\beta x + Q_l(x) e^{\alpha x} sin\beta x \Rightarrow$$

$$y^* = x^k (R_h(x) cos\beta x + S_h(x) sin\beta x) e^{\alpha x}$$

欧拉方程

$$\sum_{i=0}^{n} a_i x^i y^{(i)} = f(x) : t = lnx,\ x = e^t$$

$$y' = \frac{1}{x}\frac{dy}{dt} \quad y'' = \frac{1}{x^2}(\frac{d^2y}{dt^2} - \frac{dy}{dt}) \quad y''' = \frac{1}{x^3}(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt})$$

刘维尔公式 $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \Rightarrow y = y_1 \Big[c_1 + c_2 \int \frac{1}{y_1^2} e^{-\int p(x)dx} dx \Big]$

常数变异 $y = y_1(c_1 - \int \frac{y_2 f}{w} dx) + y_2(c_2 + \int \frac{y_1 f}{w} dx)$

特殊情况 若 $2p' + p^2 - 4q = a$ 则令 $y = uv$ 并取 $2v' + pv = 0$ 使化为 $u'' - \frac{a}{4}u = 0$

常系数齐次

$$\vec{x}(t) = \Big(\sum_{i=0}^{k-1} \frac{t^i}{i!} \vec{v}_i \Big) e^{\lambda_0 t},\ \vec{v}_{i+1} = (A - \lambda_0 E)\vec{v}_i,\ \lambda_{1,2} = \alpha \pm \beta i, \vec{v}_{1,2} = \vec{p} \pm i\vec{q}$$

$$\vec{x}_{1,2} = e^{\alpha t}(\vec{p} cos\beta x \pm \vec{q} sin\beta x),\ \vec{x}(t) = \sum_{i=1}^{s} \sum_{j=1}^{n_i} c_{ij} \vec{p}_i^{(j)}(t) e^{\lambda_i t}$$

常系数非齐次

$$\frac{d\vec{x}}{dt} = A(t)\vec{x} \Rightarrow X(t), \frac{d\vec{x}}{dt} = A(t)\vec{x} + \vec{f}(t) \Rightarrow$$

$$\vec{x}(t) = X(t)X^{-1}(t)\vec{x}_0 + X(t)\int_{t_0}^{t} X^{-1}(\tau)\vec{f}(\tau)d\tau$$

升阶 $y = f(x', x) \Rightarrow y' = g(x'', x', x) = h(x, f(x', x))$ 从而化为一元微分方程