浙江省第二届大学生高等数学(微积分)竞赛试题

一、计算题

1、求
$$\lim_{x \to 0} rac{\int_0^x \sin(xt)^2 \,\mathrm{d}\,t}{x^5}$$
.

2、设
$$G(x) = \int_1^x t \sin t^3 dt$$
,求 $\int_1^2 G(x) dx$.

3、计算
$$\int_0^{+\infty} \frac{x^2}{1+x^4} dx$$
.

4、求
$$\lim_{n o\infty}rac{2^{-n}}{n(n+1)}{\displaystyle\sum_{k=1}^n}C_n^km{\cdot}k^2$$
 .

二、设
$$f(x)=\int_x^{x+a}\sin t^2\,\mathrm{d}\,t, a>0$$
,求证:对 $x>0$,成立 $|f(x)|<rac{1}{x}$.

三、设 $\varphi(x)$ 在 $\left[0,1\right]$ 上可导,且 $\varphi(0)=0, \varphi(1)=1$.证明:对任意正数a,b,必存在 $\left(0,1\right)$ 内的两个数 ξ 和 η ,使

$$\frac{a}{\varphi'(\xi)} + \frac{a}{\varphi'(\eta)} = a + b.$$

四、证明: 集合
$$A = \left\{ lpha \left| orall x > 0, \left(1 + rac{1}{x}
ight)^{x + lpha} > e
ight\}$$
有最小值,

并求最小值.

五、设
$$A,n>0,0\leq a < b,f(n)=\int_a^b \sin(nt-rac{A}{t^2})\,\mathrm{d}\,t$$
 ,

证明
$$|f(n)| < \frac{2}{n}$$
.

六、设
$$f(x)$$
连续, $arphi(x)=\int_0^1 f(xt)\,\mathrm{d}\,t$,且 $\lim_{x o 0}rac{f(x)}{x}=A$,

A 为有限数, 求

$$arphi'(x)$$
,并讨论 $arphi'(x)$ 在 $x=0$ 处的连续性. 《公考研竞赛数学

浙江省第二届大学生高等数学(微积分)竞赛 参考解答

一、计算题

1、【参考解析】: 先对分子定积分考虑换元,令xt=u,则

$$\int_0^{x^2} \sin(xt)^2 \,\mathrm{d}\, t = \int_0^{x^2} \sin u^2 \cdot rac{1}{x} \,\mathrm{d}\, u$$

于是由洛必达法则,得

原极限
$$=\lim_{x o 0}rac{\int_0^{x^2}\sin u^2\;\mathrm{d}\,u}{x^6}=\lim_{x o 0}rac{\sin x^4\cdot 2x}{6x^5}=rac{1}{3}$$

2、【参考解析】: 考虑分部积分法, 得

$$\int_{1}^{2} G(x) \, \mathrm{d} \, x = \left[x G(x) \right]_{1}^{2} - \int_{1}^{2} x G'(x) \, \mathrm{d} \, x$$

由已知 $G(1)=0, G(2)=\int_1^2 t \sin t^3 \,\mathrm{d}\, t$, $G'(x)=x \sin x^3$,

代入得

$$\int_{1}^{2} G(x) dx = 2G(2) - \int_{1}^{2} x^{2} \sin x^{3} dx$$

$$= 2G(2) - \frac{1}{3} \int_{1}^{2} \sin x^{3} d(x^{3})$$

$$= 2G(2) - \frac{1}{3} (-\cos x^{3})_{1}^{2} = 2G(2) + \frac{1}{3} (\cos 8 - \cos 1)$$

3、【参考解析】:由
$$\int_0^{+\infty} rac{1+x^2}{1+x^4} \mathrm{d}\, x = \int_0^{+\infty} rac{rac{1}{x^2}+1}{rac{1}{x^2}+x^2} \mathrm{d}\, x$$

$$=\int_{_0}^{_{+\infty}}\!rac{1}{(x-rac{1}{x})^2+2}\mathrm{d}(x-rac{1}{x})=\int_{_{-\infty}}^{_{+\infty}}\!rac{1}{y^2+2}\mathrm{d}\,y$$

$$=\frac{1}{\sqrt{2}}\arctan\frac{y}{\sqrt{2}}|_{-\infty}^{+\infty}=\frac{\pi}{\sqrt{2}}$$

令
$$x = \frac{1}{y}$$
及由积分的符号无关性,得

$$I = \int_0^{+\infty} rac{1}{1+x^4} \, \mathrm{d}\, x = \int_0^{+\infty} rac{y^2}{1+y^4} \, \mathrm{d}\, y = \int_0^{+\infty} rac{x^2}{1+x^4} \, \mathrm{d}\, x$$
 $2I = \int_0^{+\infty} rac{1+x^2}{1+x^4} \, \mathrm{d}\, x = rac{\pi}{\sqrt{2}} \, ,$

所以 $\int_0^{+\infty} \frac{1}{1+x^4} dx = \frac{\pi}{2\sqrt{2}}$. 由上面两个积分结果可得

$$\int_0^{+\infty} \frac{x^2}{1+x^4} \, \mathrm{d}\, x = \frac{\pi}{\sqrt{2}} - \frac{\pi}{2\sqrt{2}} = \frac{\pi}{2\sqrt{2}} \, .$$

4、【参考解析】: $(1+x)^n = 1 + \sum_{k=1}^n C_n^k k x^{k-1}$,

$$\sum_{k=1}^n C_n^k k = n \cdot 2^{n-1}$$

$$n(1+x)^{n-1} = \sum_{k=1}^n C_n^k k x^{k-1}, \sum_{k=1}^n C_n^k k = n \cdot 2^{n-1}$$

$$n(n-1)(1+x)^{n-2} = \sum_{k=1}^n C_n^k k(k-1)x^{k-2}$$

$$n(n-1)2^{n-2} = \sum_{i=1}^n C_n^k k(k-1)$$

$$egin{aligned} \sum_{k=1}^n C_n^k k^2 &= n(n-1)2^{n-2} + \sum_{k=1}^n C_n^k k \ &= n(n-1)2^{n-2} + n \cdot 2^{n-1} = n(n+1)2^{n-2} \end{aligned}$$

所以原式
$$=\lim_{n o\infty}rac{2^{-n}}{n(n+1)}\cdot n(n+1)2^{n-2}=rac{1}{4}$$
 .

二、【参考解析】: 令 $t=\sqrt{u}$,并由分部积分得

$$f(x) = \int_{x^2}^{(x+a)^2} \sin u \cdot \frac{1}{2\sqrt{u}} du$$
 $= (-\frac{\cos u}{2\sqrt{u}})\Big|_{x^2}^{(x+a)^2} - \frac{1}{4} \int_{x^2}^{(x+a)^2} u^{-\frac{3}{2}} \cos u \, du$

于是可得

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$$egin{aligned} |f(x)| &< rac{1}{2} (rac{1}{x} + rac{1}{x+a}) + rac{1}{4} \int_{x^2}^{(x+a)^2} u^{-rac{3}{2}} \, \mathrm{d} \, u \ &= rac{1}{2} (rac{1}{x} + rac{1}{x+a}) + rac{1}{2} (-u^{-rac{1}{2}}) \Big|_{x^2}^{(x+1)^2} \ &= rac{1}{2} (rac{1}{x} + rac{1}{x+a}) + rac{1}{2} (rac{1}{x} - rac{1}{x+a}) = rac{1}{x} \end{aligned}$$

三、【参考解析】:因为a,b均为正数,所以 $0<\frac{a}{a+b}<1$.又因为f(x)在 $\left[0,1\right]$ 上连续,由介值定理 $\exists \tau\in(0,1)$,使 $f(\tau)=\frac{a}{a+b}$.对f(x)在 $\left[0,\tau\right]$, $\left[\tau,1\right]$ 上分别应用拉格朗日中值定理,得

$$f(\tau) - f(0) = f'(\xi)(\tau - 0), \xi \in (0, \tau)$$

 $f(1) - f(\tau) = f'(\eta)(1 - \tau), \eta \in (\tau, 1)$

注意到f(0)=0, f(1)=1,于是两式又可表示为

$$au=rac{f(au)}{f'(\xi)}=rac{a}{a+b}$$
 , $oxed{\exists}\,f'(\xi)
eq0$,

$$1- au=rac{1-f(au)}{f'(\eta)}=rac{\dfrac{o}{a+b}}{f'(\eta)}$$
 , $oxtimes f'(\eta)
eq 0$

两式相加即得 $1=rac{a+b}{f'(\xi)}+rac{a}{f'(\eta)}$. 即

$$\frac{a}{f'(\xi)} + \frac{b}{f'(\eta)} = a + b.$$

四、【参考解析】: 不等式 $\left(1+rac{1}{x}
ight)^{x+lpha}>e$ 等价于

$$(x+lpha)\ln\!\left(\!1+rac{1}{x}\!
ight)\!>\!1$$
 ,

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$$= \lim_{x \to +\infty} \frac{x^3[x^2 - x(1+x) + (1+x)]}{1(1+x)x^2}$$

$$=rac{1}{2}\lim_{x o +\infty}rac{x}{1+x}=rac{1}{2}$$

所以 $\min A = \frac{1}{2}$.

五、【参考解析】: 考虑分部积分法, 得

$$f(n) = \int_a^b \sin(nt - \frac{A}{t^2}) dt$$

$$= \int_a^b \frac{1}{n + \frac{2A}{t^3}} \sin(nt - \frac{A}{t^2}) d(nt - \frac{A}{t^2})$$

$$egin{align} = & [-rac{1}{n+rac{2A}{t^3}}\cos(nt-rac{A}{t^2})] \left|_a^b
ight. \ & + \int_a^b (rac{1}{n+rac{2A}{t^3}})'\cos(nt-rac{A}{t^2}) \, \mathrm{d} \, t \end{array}$$

于是可得

$$egin{align} |f(n)| &< rac{1}{n + rac{2A}{a^3}} + rac{1}{n + rac{2A}{b^3}} + \int_a^b (rac{1}{n + rac{2A}{t^3}})' \, \mathrm{d} \, t \ &= 2rac{1}{n + rac{2A}{b^3}} < rac{2}{n} \end{aligned}$$

六、【参考解析】:由条件,可知f(0)=0;

当
$$x \neq 0$$
时, $\varphi(x) = \int_0^1 f(xt) dt = \frac{1}{x} \int_0^x f(u) du$,于是

$$\varphi(0) = \int_0^1 f(0) dt = f(0) = 0$$

$$arphi'(x) = rac{f(x)}{x} - rac{1}{x^2} \int_0^x f(u) \, \mathrm{d}\, u$$

即
$$lpha > rac{1}{\ln\left(1+rac{1}{x}
ight)} - x, (x>0)$$
 . 所以 $lpha \in A$ 等价于 $lpha$ 为

$$f(x)=rac{1}{\lniggl(1+rac{1}{x}iggr)}-x, (x>0)$$

的上界,按照确界的定义,即 $\min A = \sup_{x>0} f(x)$. 令

$$fig(xig) = rac{1}{\lnig(1+rac{1}{x}ig)} - x$$
 ,

则
$$f'(x) = \frac{1}{\ln^2\left(1+\frac{1}{x}\right)} \cdot \frac{1}{x(1+x)} - 1 > 0.$$

$$f(x)$$
 在 $(0,+\infty)$ 上单调递增,于是 $\sup_{x>0} f(x) = \lim_{x o +\infty} f(x)$.

由于

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \frac{1-x[\ln(1+x)-\ln x]}{\ln(1+x)-\ln x}$$

$$=\lim_{x
ightarrow+\infty}rac{-[\ln(1+x)-\ln x]-xiggl(rac{1}{1+x}-rac{1}{x}iggr)}{rac{1}{1+x}-rac{1}{x}}$$

$$\lim_{x o +\infty} rac{(1+x)-\ln x+(1+x)iggl(rac{1}{1+x}-rac{1}{x}iggr)}{-rac{1}{x}}$$

$$=\lim_{x o +\infty}rac{\ln(1+x)-\ln x-rac{1}{x}}{-rac{1}{x^2}}=\lim_{x o +\infty}rac{rac{1}{1+x}-rac{1}{x}+rac{1}{x^2}}{2rac{1}{x^3}}$$

$$arphi'(0) = \lim_{x o 0} rac{arphi(x) - arphi(0)}{x = 0} = \lim_{x o 0} rac{1}{x^2} \int_0^x (f(u) - f(0)) \, \mathrm{d} \, u$$
 $= \lim_{x o 0} rac{f(x) - f(0)}{2x} = rac{1}{2} f'(0)$ 所以 $\lim_{x o 0} arphi'(x) = \lim_{x o 0} (rac{f(x)}{x} - rac{1}{x^2} \int_0^x f(u) \, \mathrm{d} \, u)$ $= f'(0) - rac{1}{2} f'(0) = rac{1}{2} f'(0)$

即 $\varphi'(x)$ 在x=0处连续.