2005 年浙江省大学生数学竞赛试题 (数学、工科、文、专科类)

-、计算题

1、计算
$$\int_{-1}^{1} |1-2x| dx$$
.

2、设
$$f(x)=egin{cases} rac{\ln(1+x)}{x}, x>0 \ ext{可导,求常数}\,a,b$$
的值. $ax+b, x\leq 0$

3、计算
$$\lim_{n o\infty}\left(rac{\sqrt[n]{2}+\sqrt[n]{3}}{2}
ight)^n$$
.

4、计算
$$\int \frac{\sin x}{3\cos x + 4\sin x} dx$$
.

5、求函数
$$f(x) = |x| + |x-1| + |x-3|$$
的最小值.

6、计算
$$\lim_{x o 0} rac{\int_0^x \sin t \ln(1+t) \, \mathrm{d} \, t - rac{x^3}{3} + rac{x^4}{8}}{(x-\sin x) \Big(e^{x^2}-1\Big)}.$$

7、计算
$$\int_0^x \min(4, t^4) dt$$
.

8、计算
$$\int_0^{\frac{\pi}{2}} \frac{\mathrm{d} x}{1 + \tan^{2005} x}$$
.

9、设
$$z=f(x-y,x+y)+g(x+ky)$$
, f,g 具有二阶连

续偏导数,且
$$g^{\prime\prime}
eq 0$$
.如果 $rac{\partial^2 z}{\partial x^2} + 2rac{\partial^2 z}{\partial x \partial y} + rac{\partial^2 z}{\partial y^2} \equiv 4f_{22}^{\prime\prime}$,

求常数k的值.

三、证明: 当
$$0 < x < \frac{\pi}{2}$$
时,

(1)
$$\tan x > x + \frac{1}{3}x^3$$
.

(2)
$$\tan x > x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{1}{63}x^7$$
.

四、设
$$A=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1-\sin x)^2}{1+\sin^2 x} \mathrm{d}\,x, B=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{x^2+\cos^2 x} \mathrm{d}\,x,$$
 $C=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{10(1+\sin^2 x)}{4x^2+\pi^2} \mathrm{d}\,x$

试比较A, B, C的大小.

五、在某平地上向下挖一个半径为R的半球形池塘,若某点泥 土的密度为 $\rho=e^{\frac{r}{R^2}}$,其中r为此点离球心的距离,试求挖此 池塘所作的功.

六 、 设
$$x_n=rac{1}{n^2}+rac{1}{n^2+1}+rac{1}{n^2+2}+\ldots+rac{1}{n^2+2n}, \ n=1,2,3,\cdots$$

- (1) 求 $\lim_{n\to\infty}x_n$;
- (1) $\overline{x} \lim_{n o \infty} x_n$; (2) 证明数列 $\left\{x_n\right\}$ 单调减少.
- (3) 判别级数 $\sum_{n=0}^{\infty} (-1)^{\left[\sqrt{n}\right]} \cdot \frac{1}{n}$ 的敛散性.

七、对下列f(x),分别说明是否存在一个区间[a,b],(a>0), 使 $\{f(x) \mid x \in [a,b]\} = \{x \mid x \in [a,b]\}$, 并说明理由.

(1)
$$f(x) = \frac{1}{3}x^2 + \frac{2}{3}$$
 (2) $f(x) = \frac{1}{x}$

(3)
$$f(x) = 1 - \frac{1}{x}$$

八、证明对于任何连续函数f(x),有

$$\max \left\{ \int_{-1}^{1} \left| x - \sin^{2} x - f(x) \right| dx, \int_{-1}^{1} \left| \cos^{2} x - f(x) \right| dx \right\} \ge 1.$$

九、计算 $\int_L rac{y\,\mathrm{d}\,x-x\,\mathrm{d}\,y}{3x^2-2xy+3y^2}$,其中L为 $|\,x\,|+|\,y\,|=1$ 正

向一周. m+、设f(x)在m[-1,1]上二阶导数连续,证明:

$$\int_{-1}^1 x f(x) \operatorname{d} x = rac{2}{3} f'(\xi) + rac{1}{3} \xi f''(\xi).$$

2005 年浙江省大学生数学竞赛参考解答 (数学、工科、文、专科类)

一、计算题

1、【参考解析】: (文专类)去绝对值, 令1-2x=0, 得 $x=rac{1}{2}$,

于是原式
$$=\int_{rac{1}{2}}^1 (2x-1) \,\mathrm{d}\,x + \int_{-1}^{rac{1}{2}} (1-2x) \,\mathrm{d}\,x$$
 $= \left[x^2-x
ight]_{rac{1}{2}}^1 + \left[x-x^2
ight]_{-1}^{rac{1}{2}} = rac{1}{2} + 2 = rac{5}{2}$

2、【参考解析】: (文专类)函数在x=0处连续、可导,于是

$$\lim_{x o 0^+} f(x) = \lim_{x o 0^+} \ln(1+x)^{rac{1}{x}} = 1 \ \lim_{x o 0^-} f(x) = \lim_{x o 0^-} (ax+b) = b$$

因为f(x)在x=0处连续,所以b=1

$$f'_-(0) = \lim_{x o 0^-} rac{f(x) - f(0)}{x} = \lim_{x o 0^+} rac{ax + b - b}{x} = a \ f'_+(0) = \lim_{x o 0^+} rac{f(x) - f(0)}{x} = \lim_{x o 0^+} rac{\ln(1 + x) - x}{x^2} \ = \lim_{x o 0^+} rac{1}{1 + x} - 1 = \lim_{x o 0^+} \left(-rac{1}{2(1 + x)}
ight) = -rac{1}{2}$$

由 f(x) 在 x=0 处可导, $f'(0)=f'_{+}(0)$,即 $a=-rac{1}{2}$.

3、【参考解析】: (文专类) 由 $f(x) = e^{\ln f(x)}$,则

原式
$$=\lim_{n o\infty}3iggl(rac{1+\sqrt[n]{2\,/\,3}}{2}iggr)^{\!n}=3\lim_{n o\infty}iggl[1+iggl(rac{1+\sqrt[n]{2\,/\,3}}{2}-1iggr)\!iggr]^{\!n}$$

所以问题转换为计算

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$$a=\lim_{n o\infty}n\ln\left[1+\left(rac{1+\sqrt[n]{2\,/\,3}}{2}-1
ight)
ight]$$

由等价无穷小 $\ln \left(1+x\right)$ ~ x,a^x-1 ~ $x\ln a\left(x o 0
ight)$,得

$$egin{aligned} a &= \lim_{n o \infty} n \left(rac{1 + \sqrt[n]{2 \, / \, 3}}{2} - 1
ight) = \lim_{n o \infty} n \, rac{\sqrt[n]{2 \, / \, 3} - 1}{2} \ &= \lim_{n o \infty} n \, rac{rac{1}{n} \ln rac{2}{3}}{2} = rac{1}{2} \ln rac{2}{3} = \ln \sqrt{rac{2}{3}} \end{aligned}$$

所以原式 $=3\cdot e^{\ln\sqrt{\frac{2}{3}}}=3\cdot\sqrt{\frac{2}{3}}=\sqrt{6}$.

4、【参考解析】:(文专、数学、工科类)【思路一】 $\Rightarrow \tan x = u$,

则
$$x = \arctan u, \ \mathrm{d}\, x = \frac{1}{1+u^2}$$
,所以

原式=
$$\int \frac{\tan u}{4 \tan u + 3} dx = \int \frac{u du}{(3+4u)(1+u^2)}$$

对被积函数进行部分分解,得

$$rac{u}{(4u+3)\Big(u^2+1\Big)}=rac{3u+4}{25\Big(u^2+1\Big)}-rac{12}{25(4u+3)}$$

所以

原式
$$= \frac{3}{50} \ln \left(u^2 + 1 \right) - \frac{3}{25} \ln (4u + 3) + \frac{4}{25} \arctan u + C$$

$$= -\frac{3}{25} \ln |3 + 4 \tan x| + \frac{3}{50} \ln \left(1 + \tan^2 x \right) + \frac{4}{25} \arctan(\tan x) + C$$

【思路二】
$$\sin x = A(4\sin x + 3\cos x) + B(4\cos x - 3\sin x)$$
 $= (4A - 3B)\sin x + (3A + 4B)\cos x$ 考研竞赛数学

$$egin{cases} 4A-3B=0 \ 3A+4B=0 \end{cases}$$
 得 $A=rac{4}{25}, B=-rac{3}{25}$,

$$\int \frac{\sin x}{3\cos x + 4\sin x} dx = \int \left(\frac{4}{25} - \frac{3}{25} \frac{4\cos x - 3\sin x}{4\sin x + 3\cos x} \right) dx$$

$$= \frac{4}{25} x - \frac{3}{25} \ln \mid 4 \sin x + 3 \cos x \mid + C$$

5、【参考解析】: (文专类) 当x < 0时,

$$f(x) = -x - (x - 1) - (x - 3) = 4 - 3x$$
,

最小值为f(0) = 4.

当 $0 < x \le 1$ 时,f(x) = x + 1 - x + 3 - x = 4 - x,最小值为f(1) = 3.

当 $1 < x \le 3$ 时, f(x) = x + (x - 1) + 3 - x = x + 2, 最小值为f(1) = 3.

当x>3时, f(x)=x+(x-1)+(x-3)=3x-4,最小值为fig(3ig)=5.

综上可得最小值为 3.

6、【参考解析】: (数学、工科类)由等价无穷小简化分母,然 后应用洛必达法则与麦克劳林公式,得

原式
$$=\lim_{x o 0}rac{\displaystyle\int_0^x \sin t \ln(1+t) \,\mathrm{d}\, t - rac{x^3}{3} + rac{x^4}{8}}{rac{x^3}{6} \, x^2}$$

$$=\lim_{x o 0}rac{\sin x\ln(1+x)-x^2+rac{x^3}{2}}{rac{5x^4}{6}}$$

(2) 考研竞赛数学

$$=\lim_{x o 0} \left\{ \left[x - rac{x^3}{3!} + rac{x^5}{5!} + 0 \Big(x^5 \Big)
ight] \left[x - rac{x^2}{2} + rac{x^3}{3} + 0 \Big(x^3 \Big)
ight]
ight.$$
 $\left. - x^2 + rac{x^3}{2}
ight\} / rac{5x^4}{6}$

$$=\lim_{x o 0}rac{rac{x^4}{6}+0\Big(x^4\Big)}{rac{5x^4}{6}}=rac{1}{5}$$

7、【参考解析】: (工科类) 令 $4=t^4$,得 $t=\pm\sqrt{2}$,于是

$$\min\left(4,t^4
ight) = egin{cases} 4,t < -\sqrt{2},t > \sqrt{2} \ t^4,-\sqrt{2} \leq t \leq \sqrt{2} \end{cases}$$

$$\diamondsuit f(x) = \int_0^x \min(4, t^4) dt$$
. 则

当
$$x < -\sqrt{2}$$
时,

$$fig(xig) = \int_0^{-\sqrt{2}} t^4 \, \mathrm{d} \, t + \int_{-\sqrt{2}}^x \! 4 \, \mathrm{d} \, t = 4 \Big(x + \sqrt{2}\Big) - rac{4\sqrt{2}}{5}$$

当
$$-\sqrt{2} \le x \le \sqrt{2}$$
时, $f(x) = \int_0^x t^4 dt = \frac{x^5}{5}$

当
$$x > \sqrt{2}$$
时,

$$fig(xig) = \int_0^{\sqrt{2}} t^4 \, \mathrm{d} \, t + \int_{\sqrt{2}}^x 4 \, \mathrm{d} \, t = 4 \Big(x - \sqrt{2}\Big) + rac{4\sqrt{2}}{5}$$

8、【参考解析】: (数学类) 令 $x=rac{\pi}{2}-t$,则

$$\int_0^{rac{\pi}{2}} rac{\mathrm{d}\,x}{1+ an^{2005}\,x} = \int_0^{rac{\pi}{2}} rac{\mathrm{d}\,x}{1+\cot^{2005}\,x} \, .$$

所以原式
$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \left(\frac{1}{1 + \tan^{2005} x} + \frac{1}{1 + \cot^{2005} x} \right) \mathrm{d}x$$

$$=rac{1}{2}\int_{0}^{rac{\pi}{2}}\mathrm{d}x=rac{\pi}{4}$$

9、【参考解析】: (数学、工科类)令

$$x-y=u, x+y=v, x+ky=w$$
,

$$\begin{array}{ll} \boxed{ \frac{\partial z}{\partial x} = f_{1}^{'} + f_{2}^{'} + g^{'}, & \frac{\partial z}{\partial y} = -f_{1}^{'} + f_{2}^{'} + kg^{'} \\ & \frac{\partial^{2} z}{\partial x^{2}} = f_{11}^{''} + f_{12}^{''} + f_{21}^{''} + f_{22}^{''} + g^{''} \\ & \frac{\partial^{2} z}{\partial x \partial y} = -f_{11}^{''} + f_{12}^{''} - f_{21}^{''} + f_{22}^{''} + kg^{''} \\ & \frac{\partial^{2} z}{\partial y^{2}} = f_{11}^{''} - f_{12}^{''} - f_{21}^{''} + f_{22}^{''} + k^{2}g^{''} \end{array}$$

代入等式得 $(k+1)^2g^{''}=0$,所以k=-1.

二、【参考解析】: (文专、数学、工科类)

$$\lim_{x o 0}f(x)=\lim_{x o 0}iggl[rac{f(x)}{1-\cos x}\cdot(1-\cos x)iggr]=0=f(0)$$

$$\lim_{x o 0}rac{f(x)-f(0)}{x-0}=\lim_{x o 0}\left[rac{f(x)}{1-\cos x}\cdotrac{1-\cos x}{x}
ight]$$

$$= 1 \cdot \lim_{x o 0} rac{rac{1}{2} x^2}{x} = 0 = f'(0)$$

$$\lim_{x o 0}rac{f(x)}{rac{1}{2}x^2}=\lim_{x o 0}\left|rac{f(x)}{1-\cos x}\cdotrac{1-\cos x}{rac{1}{2}x^2}
ight|=1$$

$$egin{split} f(x) &= f(0) + f'(0)x + rac{1}{2}f''(0)x^2 + o(x^2) \ &= rac{1}{2}f''(0)x^2 + o(x^2) \end{split}$$

所以f''(0) = 1.

$$f'(0)=0$$
. 因为 $f'(x)=rac{1}{\cos^2 x}-(1+x^2), f(0)=0$.

$$f''(x) = \frac{2\sin x}{\cos^3 x} - 2x, f''(x) = 0$$

$$f'''(x) = 2\left[\frac{1+2\sin^2 x}{\cos^4 x} - 1\right]$$

$$=2rac{1+2\sin^2x-(1-\sin^2x)^2}{\cos^4x}$$

$$=2rac{\sin^2x(4-\sin^2x)}{\cos^4x}>0, \left[x\in\left[0,rac{\pi}{2}
ight]
ight]$$

所以f''(x) > 0,进而f'(x) > 0,f(x) > 0.即得

$$an x > x + rac{1}{3}x^3, \left[0 < x < rac{\pi}{2}
ight]$$

(2) (工科类) 由(1)
$$\tan x > x + \frac{1}{3}x^3$$

$$egin{split} \left(an x
ight)' &= 1 + an^2 \, x > 1 + \left(x + rac{1}{3} \, x^3
ight)^2 \ &= 1 + x^2 + rac{2x^4}{3} + rac{x^6}{9} \end{split}$$

所以结论成立.

四、【参考解析】: (文专类)

$$A = \int_{-rac{\pi}{2}}^{rac{\pi}{2}} rac{(1-\sin x)^2}{1+\sin^2 x} \,\mathrm{d}\,x = \int_{-rac{\pi}{2}}^{rac{\pi}{2}} rac{1-2\sin x+\sin^2 x}{1+\sin^2 x} \,\mathrm{d}\,x
onumber \ = 2 \int_{0}^{rac{\pi}{2}} \mathrm{d}\,x = \pi$$

由于
$$\frac{\sin^2 x}{r^2 + \cos^2 x}$$
<1,所以

(一) 考研竞赛数学

$$B = \int_{-rac{\pi}{2}}^{rac{\pi}{2}} rac{\sin^2 x}{x^2 + \cos^2 x} \mathrm{d}\, x \leq \int_{-rac{\pi}{2}}^{rac{\pi}{2}} \mathrm{d}\, x = \pi$$

由当
$$x \in \left(0, \frac{\pi}{2}\right)$$
时, $\frac{\pi}{2} < \frac{\sin x}{x} < 1$,所以

$$rac{10(1+\sin^2x)}{4x^4+\pi^2}\!>\!rac{10iggl[1+rac{4x^2}{\pi^2}iggr]}{4x^2+\pi^2}\!>\!rac{10}{\pi^2}\!>\!1$$
 ,

即
$$C = \int_{-rac{\pi}{2}}^{rac{\pi}{2}} rac{10(1+\sin^2 x)}{4x^2+\pi^2} \mathrm{d}\, x \geq \int_{-rac{\pi}{2}}^{rac{\pi}{2}} \mathrm{d}\, x = \pi \,.$$

综上可知B < A < C.

五、【参考解析】: (数学、工科类) 以半球球心为原点, 竖直向 上建立O - xyz 空间直角坐标系,则球面方程为

$$z=-\sqrt{R^2-x^2-y^2}, -R\leq z\leq 0$$
 ,

于是密度为 $ho=e^{rac{r^2}{R^2}}=e^{rac{x^2+y^2+z^2}{R^2}}$,则对应于ig(x,y,zig)位置

的小体积 dV 泥土提到地表面所做的功为

$$mgh =
ho \cdot \operatorname{d} V \cdot g \cdot \left(-z
ight) = -e^{rac{x^2 + y^2 + z^2}{R^2}} gz\operatorname{d} V$$

记半球体为 Ω ,则挖此池塘所作的功为

$$W=\iiint\limits_{\Omega}-e^{rac{x^2+y^2+z^2}{R^2}}gz\,\mathrm{d}\,V$$
考虑球坐标系,令

 $x = r\sin\varphi\cos\theta, y = r\sin\varphi\sin\theta, z = r\cos\varphi$

$$0 \leq heta \leq 2\pi, rac{\pi}{2} \leq arphi \leq \pi, 0 \leq r \leq R$$

则由三重积分的球坐标变换计算公式,得

(2) 考研克赛数学

$$egin{aligned} W &= -g \int_0^{2\pi} \mathrm{d}\, heta \int_{rac{\pi}{2}}^{\pi} \mathrm{d}\, arphi \int_0^R e^{rac{r^2}{R^2}} \cdot r \cos arphi \cdot r^2 \sin arphi \, \mathrm{d}\, r \ &= -2\pi g \int_{rac{\pi}{2}}^{\pi} \cos arphi \cdot \sin arphi \, \mathrm{d}\, arphi \cdot rac{1}{2} \int_0^R e^{rac{r^2}{R^2}} \cdot r^2 \, \mathrm{d} \Big(r^2 \Big) \ &= -2\pi g \cdot \left[-rac{1}{2}
ight] \cdot rac{R^4}{2} = rac{\pi g R^4}{2} \end{aligned}$$

六、【参考解析】: (1) (文专类) 由题意可知

$$x_{_{n}}=\sum_{k=0}^{2n}rac{1}{n^{^{2}}+k},n=1,2,\cdots$$
 ,

并且 $rac{2n+1}{n^2+2n} \leq x_n \leq rac{2n+1}{n^2}$,所以由夹逼定理可得 $\lim_{n \to \infty} x_n = 0$.

(2)
$$x_{n+1} = \sum_{k=0}^{2(n+1)} \frac{1}{n^2 + k}$$

$$=\!\sum_{k=0}^{2n}\!rac{1}{(n+1)^2+k}\!+\!rac{1}{(n+1)^2+2n+1}\!+\!rac{1}{(n+1)^2+2n+2}$$

$$egin{align} x_n - x_{n+1} &= \sum_{k=0}^{2n} rac{2n+1}{(n^2+k)ig((n+1)^2+kig)} \ &-igg(rac{1}{n^2+4n+2} + rac{1}{n^2+4n+3}igg) \end{aligned}$$

$$> rac{(2n+1)(2n+1)}{(n^2+2n)(n^2+4n+1)} - \left[rac{1}{(n^2+2n)}rac{1}{(n^2+4n+1)}
ight]$$

$$=rac{(2n+1)(2n+1)}{(n^2+2n)(n^2+4n+1)}\geq 0$$

所以数列 $\left\{x_{n}\right\}$ 单调减少.

(3) (数学、工科类) 由于

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$$\sum_{n=1}^{\infty} (-1)^{[\sqrt{n}]} \cdot \frac{1}{n} = -1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$$

$$+ \frac{1}{8} - \frac{1}{9} - \frac{1}{10} - \frac{1}{11} - \frac{1}{12} - \frac{1}{13}$$

$$- \frac{1}{14} - \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \cdots$$

$$= (-1) \left(1 + \frac{1}{2} + \frac{1}{3} \right) + (-1)^2 \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right)$$

$$+ (-1)^3 \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} \right) + \cdots$$

引进数列

$$u_n = \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{n^2} + \frac{1}{n^2 + 1} + \dots + \frac{1}{(n+1)^2 - 1} \right],$$

并记
$$a_n=rac{1}{n^2}+rac{1}{n^2+1}+\cdots+rac{1}{(n+1)^2-1}$$
 , 则由(1)(2)

得 $\lim_{n \to \infty} a_n = 0$ 且 $\left\{a_n\right\}$ 单调递减,于是由莱布尼兹判别法可

知 $\sum u_k$ 收敛. 设原级数的部分和为 S_n , $\sum u_k$ 的部分和为 M_N , 则有

$$\left|S_n-M_N\right| \leqslant \left|M_{N+1}-M_N\right| = \left|u_{N+1}\right| \to 0 \quad (N\to\infty)$$

因此 $S_n-M_N o 0 (n o \infty)$,即 $\lim_{n o \infty}S_n=\lim_{n o \infty}M_N$,

因此原级数收敛.

七、【参考解析】: (文专、数学、工科类) 欲使 f[a,b]=[a,b],必有 f(a)=a,f(b)=b .

$$x^2-3x+2=0$$
,即 $\left(x-rac{3}{2}
ight)^2=\left(rac{1}{2}
ight)^2$

解得 $x_1=1,x_2=2$,所以存在区间[1,2]使f[1,2]考证是要求

(2) $f(x) = \frac{1}{x}$ 在 $(0,+\infty)$ 上严格单调减少, 取 $\frac{1}{a} = b, \frac{1}{b} = a$,

所以存在区间

$$\left[a, \frac{1}{a}\right], (0 < a < 1) \oplus f\left[a, \frac{1}{a}\right] = \left[a, \frac{1}{a}\right]$$

(3) 不存在,令
$$f(x)=1-rac{1}{x}=x$$
,即 $\left(x-rac{1}{2}
ight)^2=-rac{3}{4}$,此

方程无实数解,故不存在区间[a,b],(a>0),使得 f[a,b] = [a,b].

八、【参考解析】: (数学类)

差逸
$$\geq \frac{1}{2} \left(\int_{-1}^{1} \left| x - \sin^2 x - f(x) \right| \mathrm{d}x + \int_{-1}^{1} \left| \cos^2 x - f(x) \right| \mathrm{d}x \right)$$

$$\geq \frac{1}{2} \left(\int_{-1}^{1} x - \sin^2 x - f(x) - \cos^2 x + f(x) \mid dx \right)$$

$$= \frac{1}{2} \int_{-1}^{1} \left| x - 1 \right| dx = \frac{1}{2} \int_{-1}^{1} (1 - x) dx = \frac{1}{2} \left(x - \frac{x^2}{2} \right) \Big|_{-1}^{1}$$

$$= \frac{1}{2} \left(\frac{1}{2} - \left(-\frac{3}{2} \right) \right) = 1 =$$
右边.

九、【参考解析】: (工科类)

$$\diamondsuit P = rac{y}{3x^2-2xy+3y^2}, Q = -rac{x}{3x^2-2xy+3y^2}$$
,则

可得
$$\frac{\partial P}{\partial y} = \frac{3\left(x^2 - y^2\right)}{\left(3x^2 - 2xy + 3y^2\right)^2} = \frac{\partial Q}{\partial x}$$
,即积分与路径无

关,于是取积分路径为

$$L_1: x=1, y:-1\to 1, L_2: y=1, x:1\to -1$$

$$L_3: x=-1, y:1\to -1, L_4: y=-1, x:-1\to 1$$
 于是积分为

原式 =
$$\int_{-1}^{1} \frac{-\operatorname{d} y}{3 - 2y + 3y^2} + \int_{1}^{-1} \frac{\operatorname{d} x}{3x^2 - 2x + 3}$$

$$+ \int_{1}^{-1} \frac{\operatorname{d} y}{3 + 2y + 3y^2} + \int_{-1}^{1} \frac{-\operatorname{d} x}{3x^2 + 2x + 3}$$

$$= -2 \int_{-1}^{1} \frac{\operatorname{d} y}{3 - 2y + 3y^2} - 2 \int_{-1}^{1} \frac{\operatorname{d} x}{3x^2 + 2x + 3} = -\frac{\pi}{\sqrt{2}}$$

十、【参考解析】: (数学类) 因为

$$egin{aligned} \left(xf(x)
ight)' &= f(x) + xf^{'}(x) \ \left(xf(x)
ight)'' &= 2f^{'}(x) + xf^{''}(x) \end{aligned}$$

于是

$$xf(x) = 0 + f(0)x + \left[2f^{'}\left(\xi_{x}\right) + \xi_{x}f^{''}\left(\xi_{x}\right)\right]\frac{x^{2}}{2!} \ \int_{-1}^{1} xf(x) \,\mathrm{d}\,x = \int_{-1}^{1} \left\{f(0)x + \left[2f^{'}\left(\xi_{x}\right) + \xi_{x}f^{''}\left(\xi_{x}\right)\right]\frac{x^{2}}{2!}\right\} \mathrm{d}\,x$$

由积分中值定理

$$=2f^{'}(\xi)+\xi f^{''}(\xi)$$

于是得

$$egin{align} \int_{-1}^{1}rac{x^{2}}{2}dx &= \left[2f^{'}(\xi)+\xi f^{''}(\xi)
ight]rac{x^{3}}{6}igg|_{-1}^{1} \ &=rac{2}{3}f^{'}(\xi)+rac{\xi}{3}f^{''}(\xi), \xi \in \left[-1,1
ight]. \ &\lesssim lpha$$