浙江省第二届大学生高等数学（微积分）竞赛试题

一、计算题

1、求$\lim\limits_{x\to 0}\dfrac{\int_0^x \sin(xt)^2\,\mathrm{d}t}{x^5}$.

2、设$G(x)=\int_1^x t\sin t^3\,\mathrm{d}t$，求$\int_1^2 G(x)\,\mathrm{d}x$.

3、计算$\int_0^{+\infty}\dfrac{x^2}{1+x^4}\,\mathrm{d}x$.

4、求$\lim\limits_{n\to\infty}\dfrac{2^{-n}}{n(n+1)}\sum\limits_{k=1}^{n}C_n^k\cdot k^2$.

二、设$f(x)=\int_x^{x+a}\sin t^2\,\mathrm{d}t, a>0$，求证：对$x>0$，成立$|f(x)|<\dfrac{1}{x}$.

三、设$\varphi(x)$在$\left[0,1\right]$上可导，且$\varphi(0)=0,\varphi(1)=1$.证明：对任意正数$a,b$，必存在$(0,1)$内的两个数$\xi$和$\eta$，使

$$\frac{a}{\varphi'(\xi)}+\frac{a}{\varphi'(\eta)}=a+b.$$

四、证明:集合$A=\left\{\alpha\left|\forall x>0,\left(1+\dfrac{1}{x}\right)^{x+\alpha}>e\right.\right\}$有最小值,并求最小值.

五、设$A,n>0,0\le a<b,f(n)=\int_a^b\sin(nt-\dfrac{A}{t^2})\,\mathrm{d}t$，证明$|f(n)|<\dfrac{2}{n}$.

六、设$f(x)$连续，$\varphi(x)=\int_0^1 f(xt)\,\mathrm{d}t$，且$\lim\limits_{x\to 0}\dfrac{f(x)}{x}=A$，$A$为有限数，求$\varphi'(x)$，并讨论$\varphi'(x)$在$x=0$处的连续性.

浙江省第二届大学生高等数学（微积分）竞赛

参考解答

一、计算题

1、【参考解析】：先对分子定积分考虑换元，令 $xt=u$，则

$$\int_0^{x^2}\sin(xt)^2\,\mathrm{d}t=\int_0^{x^2}\sin u^2\cdot\frac{1}{x}\,\mathrm{d}u$$

于是由洛必达法则，得

$$原极限=\lim_{x\to0}\frac{\int_0^{x^2}\sin u^2\,\mathrm{d}u}{x^6}=\lim_{x\to0}\frac{\sin x^4\cdot2x}{6x^5}=\frac{1}{3}$$

2、【参考解析】：考虑分部积分法，得

$$\int_1^2G(x)\,\mathrm{d}x=\left[xG\left(x\right)\right]_1^2-\int_1^2xG'(x)\,\mathrm{d}x$$

由已知 $G(1)=0,G(2)=\int_1^2t\sin t^3\,\mathrm{d}t$，$G'(x)=x\sin x^3$，

代入得

$$\begin{aligned}&\int_1^2G(x)\,\mathrm{d}x=2G(2)-\int_1^2x^2\sin x^3\,\mathrm{d}x\\&=2G(2)-\frac{1}{3}\int_1^2\sin x^3\,\mathrm{d}\left(x^3\right)\\&=2G(2)-\frac{1}{3}\left(-\cos x^3\right)_1^2=2G(2)+\frac{1}{3}(\cos8-\cos1)\end{aligned}$$

3、【参考解析】：由 $\int_0^{+\infty}\frac{1+x^2}{1+x^4}\,\mathrm{d}x=\int_0^{+\infty}\frac{\frac{1}{x^2}+1}{\frac{1}{x^2}+x^2}\,\mathrm{d}x$

$$=\int_0^{+\infty}\frac{1}{(x-\frac{1}{x})^2+2}\,\mathrm{d}(x-\frac{1}{x})=\int_{-\infty}^{+\infty}\frac{1}{y^2+2}\,\mathrm{d}y$$

$$=\frac{1}{\sqrt2}\arctan\frac{y}{\sqrt2}\Big|_{-\infty}^{+\infty}=\frac{\pi}{\sqrt2}$$

令 $x=\frac{1}{y}$ 及由积分的符号无关性，得

$$I=\int_0^{+\infty}\frac{1}{1+x^4}\mathrm{d}x=\int_0^{+\infty}\frac{y^2}{1+y^4}\mathrm{d}y=\int_0^{+\infty}\frac{x^2}{1+x^4}\mathrm{d}x$$

$$2I=\int_0^{+\infty}\frac{1+x^2}{1+x^4}\mathrm{d}x=\frac{\pi}{\sqrt{2}},$$

所以$\int_0^{+\infty}\frac{1}{1+x^4}\mathrm{d}x=\frac{\pi}{2\sqrt{2}}$. 由上面两个积分结果可得

$$\int_0^{+\infty}\frac{x^2}{1+x^4}\mathrm{d}x=\frac{\pi}{\sqrt{2}}-\frac{\pi}{2\sqrt{2}}=\frac{\pi}{2\sqrt{2}}.$$

4、【参考解析】: $(1+x)^n=1+\sum_{k=1}^{n}C_n^k k x^{k-1}$,

$$\sum_{k=1}^{n}C_n^k k=n\cdot 2^{n-1}$$

$$n(1+x)^{n-1}=\sum_{k=1}^{n}C_n^k k x^{k-1},\sum_{k=1}^{n}C_n^k k=n\cdot 2^{n-1}$$

$$n(n-1)(1+x)^{n-2}=\sum_{k=1}^{n}C_n^k k(k-1)x^{k-2}$$

$$n(n-1)2^{n-2}=\sum_{k=1}^{n}C_n^k k(k-1)$$

$$\begin{aligned}\sum_{k=1}^{n}C_n^k k^2&=n(n-1)2^{n-2}+\sum_{k=1}^{n}C_n^k k\\&=n(n-1)2^{n-2}+n\cdot 2^{n-1}=n(n+1)2^{n-2}\end{aligned}$$

所以原式$=\lim_{n\to\infty}\frac{2^{-n}}{n(n+1)}\cdot n(n+1)2^{n-2}=\frac{1}{4}$.

二、【参考解析】: 令$t=\sqrt{u}$，并由分部积分得

$$\begin{aligned}f(x)&=\int_{x^2}^{(x+a)^2}\sin u\cdot\frac{1}{2\sqrt{u}}\mathrm{d}u\\&=(-\frac{\cos u}{2\sqrt{u}})\Big|_{x^2}^{(x+a)^2}-\frac{1}{4}\int_{x^2}^{(x+a)^2}u^{-\frac{3}{2}}\cos u\,\mathrm{d}u\end{aligned}$$

于是可得

$$
\begin{aligned}
|f(x)| &< \frac{1}{2}(\frac{1}{x}+\frac{1}{x+a})+\frac{1}{4}\int_{x^2}^{(x+a)^2} u^{-\frac{3}{2}}\,\mathrm{d}u \\
&= \frac{1}{2}(\frac{1}{x}+\frac{1}{x+a})+\frac{1}{2}(-u^{-\frac{1}{2}})\Big|_{x^2}^{(x+1)^2} \\
&= \frac{1}{2}(\frac{1}{x}+\frac{1}{x+a})+\frac{1}{2}(\frac{1}{x}-\frac{1}{x+a})=\frac{1}{x}
\end{aligned}
$$

三、【参考解析】: 因为a,b均为正数, 所以$0<\dfrac{a}{a+b}<1$. 又因为$f(x)$在$\left[0,1\right]$上连续，由介值定理$\exists\tau\in(0,1)$，使$f(\tau)=\dfrac{a}{a+b}$. 对$f(x)$在$\left[0,\tau\right],\left[\tau,1\right]$上分别应用拉格朗日中值定理，得

$$
\begin{aligned}
f(\tau)-f(0)=f'(\xi)(\tau-0),\xi\in(0,\tau) \\
f(1)-f(\tau)=f'(\eta)(1-\tau),\eta\in(\tau,1)
\end{aligned}
$$

注意到$f(0)=0,f(1)=1$，于是两式又可表示为

$$
\tau=\frac{f(\tau)}{f'(\xi)}=\frac{\frac{a}{a+b}}{f'(\xi)}\text{，且}f'(\xi)\neq 0\text{，}
$$

$$
1-\tau=\frac{1-f(\tau)}{f'(\eta)}=\frac{\frac{b}{a+b}}{f'(\eta)}\text{，且}f'(\eta)\neq 0
$$

两式相加即得$1=\dfrac{\frac{a}{a+b}}{f'(\xi)}+\dfrac{\frac{a}{a+b}}{f'(\eta)}$. 即

$$
\frac{a}{f'(\xi)}+\frac{b}{f'(\eta)}=a+b.
$$

四、【参考解析】: 不等式$\left(1+\dfrac{1}{x}\right)^{x+\alpha}>e$等价于

$$
(x+\alpha)\ln\left(1+\frac{1}{x}\right)>1,
$$

$$=\lim_{x\to+\infty}\frac{x^3[x^2-x(1+x)+(1+x)]}{1(1+x)x^2}$$

$$=\frac{1}{2}\lim_{x\to+\infty}\frac{x}{1+x}=\frac{1}{2}$$

所以$\min A=\dfrac{1}{2}$.

五、【参考解析】：考虑分部积分法，得

$$f(n)=\int_a^b\sin(nt-\frac{A}{t^2})\,\mathrm{d}\,t$$

$$=\int_a^b\frac{1}{n+\dfrac{2A}{t^3}}\sin(nt-\frac{A}{t^2})\,\mathrm{d}(nt-\frac{A}{t^2})$$

$$=[-\frac{1}{n+\dfrac{2A}{t^3}}\cos(nt-\frac{A}{t^2})]\Big|_a^b$$

$$+\int_a^b(\frac{1}{n+\dfrac{2A}{t^3}})'\cos(nt-\frac{A}{t^2})\,\mathrm{d}\,t$$

于是可得

$$|f(n)|<\frac{1}{n+\dfrac{2A}{a^3}}+\frac{1}{n+\dfrac{2A}{b^3}}+\int_a^b(\frac{1}{n+\dfrac{2A}{t^3}})'\,\mathrm{d}\,t$$

$$=2\frac{1}{n+\dfrac{2A}{b^3}}<\frac{2}{n}$$

六、【参考解析】：由条件，可知$f(0)=0$；

当$x\neq0$时，$\varphi(x)=\int_0^1 f(xt)\,\mathrm{d}\,t=\dfrac{1}{x}\int_0^x f(u)\,\mathrm{d}\,u$，于是

$$\varphi(0)=\int_0^1 f(0)\,\mathrm{d}\,t=f(0)=0$$

$$\varphi'(x)=\frac{f(x)}{x}-\frac{1}{x^2}\int_0^x f(u)\,\mathrm{d}\,u$$

即 $\alpha > \dfrac{1}{\ln\left(1+\dfrac{1}{x}\right)} - x, (x>0)$. 所以 $\alpha \in A$ 等价于 α 为

$$f(x) = \frac{1}{\ln\left(1+\dfrac{1}{x}\right)} - x, (x>0)$$

的上界，按照确界的定义，即 $\min A = \sup\limits_{x>0} f(x)$. 令

$$f(x) = \frac{1}{\ln\left(1+\dfrac{1}{x}\right)} - x,$$

则
$$f'(x) = \frac{1}{\ln^2\left(1+\dfrac{1}{x}\right)} \cdot \frac{1}{x(1+x)} - 1 > 0.$$

$f(x)$ 在 $(0,+\infty)$ 上单调递增，于是 $\sup\limits_{x>0} f(x) = \lim\limits_{x\to+\infty} f(x)$.

由于

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{1 - x[\ln(1+x) - \ln x]}{\ln(1+x) - \ln x}$$

$$= \lim_{x\to+\infty} \frac{-[\ln(1+x) - \ln x] - x\left(\dfrac{1}{1+x} - \dfrac{1}{x}\right)}{\dfrac{1}{1+x} - \dfrac{1}{x}}$$

$$\lim_{x\to+\infty} \frac{(1+x) - \ln x + (1+x)\left(\dfrac{1}{1+x} - \dfrac{1}{x}\right)}{-\dfrac{1}{x^2}}$$

$$= \lim_{x\to+\infty} \frac{\ln(1+x) - \ln x - \dfrac{1}{x}}{-\dfrac{1}{x^2}} = \lim_{x\to+\infty} \frac{\dfrac{1}{1+x} - \dfrac{1}{x} + \dfrac{1}{x^2}}{2\dfrac{1}{x^3}}$$

$$\varphi'(0)=\lim_{x\to 0}\frac{\varphi(x)-\varphi(0)}{x=0}=\lim_{x\to 0}\frac{1}{x^2}\int_0^x (f(u)-f(0))\,\mathrm{d}\,u$$

$$=\lim_{x\to 0}\frac{f(x)-f(0)}{2x}=\frac{1}{2}f'(0)$$

所以$\lim\limits_{x\to 0}\varphi'(x)=\lim\limits_{x\to 0}(\dfrac{f(x)}{x}-\dfrac{1}{x^2}\displaystyle\int_0^x f(u)\,\mathrm{d}\,u)$

$$=f'(0)-\frac{1}{2}f'(0)=\frac{1}{2}f'(0)$$

即$\varphi'(x)$在$x=0$处连续.