浙江省大学生高等数学(微积分)竞赛试题 (2004年工科、数学专业合编)

一、计算题

1. 计算:
$$\lim_{x o 0} rac{\int_0^x e^t \cos t \, \mathrm{d} \, t - x - rac{x^2}{2}}{\left(x - \tan x
ight)\!\left(\sqrt{x+1} - 1
ight)}.$$

2. 计算
$$\lim_{n \to \infty} \sqrt[n]{2^n + a^{2n}}$$
 , 其中 a 为常数.

3. 计算
$$\int_0^\pi \frac{\pi + \cos x}{x^2 - \pi x + 2004} dx$$
.

4.求函数
$$fig(x,yig)=x^2+4y^2+15y$$
在 $\Omega=\left\{ig(x,yig)ig|4x^2+y^2\leq 1
ight\}$

上的最大、小值.

5. 计算:
$$\iint_D \max\left(xy,x^3\right) \mathrm{d}\,\sigma$$
,其中 $D = \left\{\left(x,y\right)\middle| -1 \le x \le 1, 0 \le y \le 1\right\}.$

二.设
$$fig(xig)=rctanrac{1-x}{1+x}$$
,求 $f^{ig(nig)}ig(0ig)$.

$$oxed{oxed}$$
 三.设椭圆 $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$ 在 $Aigg(1,\dfrac{3\sqrt{3}}{2}igg)$ 点的切线交 y 轴于 B

点,设l为从A到B的直线段,试计算

$$\int_{I} \left(\frac{\sin y}{x+1} - \sqrt{3}y \right) \mathrm{d}\,x + \left[\cos y \ln \left(x+1 \right) + 2\sqrt{3}x - \sqrt{3} \right] \mathrm{d}\,y.$$

四. 设函数fig(xig)连续,a < b,且 $\int_a^b \left| fig(xig) dx = 0$,试证

明:
$$fig(xig)\equiv 0, x\in ig[a,big].$$

五.判别级数 $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{(n\,!)^{\alpha}}}$ 的敛散性,其中 $\alpha>0$ 为常数.

六. 设函数f(x)在[0,1]上连续,证明:

$$\left(\int_0^1 rac{f\left(x
ight)}{t^2+x^2} \,\mathrm{d}\,x
ight)^2 \leq rac{\pi}{2t} \int_0^1 rac{f^2\left(x
ight)}{t^2+x^2} \,\mathrm{d}\,x\,ig(t>0ig).$$

七.已知函数fig(xig)在ig(0,1ig)上三阶可导,且fig(0ig)=-1, fig(1ig)=0, f'ig(0ig)=0,

试证至少存在一点 $\xi\in ig(0,1ig)$,使

$$fig(xig) = -1 + x^2 + rac{x^2ig(x-1ig)}{3!}f'''ig(\xiig), x \in ig(0,1ig)$$

浙江省大学生高等数学 (微积分) 竞赛试题 (2004 年工科、数学专业合编) 参考解答

一、计算题

1.【参考解析】:【思路一】由等价无穷小

$$\left(1+x
ight)^{lpha}-1$$
 - $lpha x\left(x
ightarrow0
ight)$,

并基于洛必达法则,可得(记原式极限为A)

$$A = \lim_{x o 0} rac{\int_0^x e^t \cos t \, \mathrm{d} \, t - x - rac{x^2}{2}}{\left(x - an x
ight) \cdot rac{1}{2} x}$$

$$=\lim_{x o 0}rac{2e^{x}\cos x-2-2x}{x-x an^{2}x- an x}=\lim_{x o 0}rac{2e^{x}\cos x-2-2x}{x-x an^{2}x- an x} =\lim_{x o 0}rac{2e^{x}\cos x-2-2x}{x-x an^{2}x- an x} =\lim_{x o 0}rac{2e^{x}\cos x-2-2x}{x^{3}}$$

其中

$$\lim_{x \to 0} \left(\frac{x - \tan x}{x^3} - \frac{x \tan^2 x}{x^3} \right) = \lim_{x \to 0} \frac{x - \tan x}{x^3} - \lim_{x \to 0} \frac{x \tan^2 x}{x^3}$$

$$= \lim_{x \to 0} \frac{1 - \sec^2 x}{3x^2} - \lim_{x \to 0} \frac{x \tan^2 x}{x^3}$$

$$= \lim_{x \to 0} \frac{-\tan^2 x}{3x^2} - \lim_{x \to 0} \frac{x \tan^2 x}{x^3} = -\frac{4}{3}$$

将结果代入以上极限式,并应用两次洛必达法则,得

$$egin{aligned} A &= -rac{3}{4} \lim_{x o 0} rac{2e^x \cos x - 2 - 2x}{x^3} \ &= rac{1}{2} \lim_{x o 0} rac{e^x \cos x - e^x \sin x - e^x \sin x - e^x \cos x}{2x} \ &= -rac{1}{4} \lim_{x o 0} rac{-2e^x \sin x}{x} = rac{1}{2}. \end{aligned}$$

【思路二】考虑带皮亚诺余项的麦克劳林公式,可以简化计算, 比如可以得到

$$x - \tan x \sim -\frac{x^3}{3} (x \rightarrow 0)$$

这样可以直接得到

$$A = \lim_{x \to 0} \frac{\int_0^x e^t \cos t \, \mathrm{d} \, t - x - \frac{x^2}{2}}{\left(-\frac{x^3}{3}\right) \cdot \frac{1}{2} x} = \lim_{x \to 0} \frac{e^x \cos x - x - 1}{-\frac{2x^3}{3}}$$

进一步依据 e^x , $\cos x$ 的三阶带皮亚诺余项的麦克劳林公式

$$e^{x} = 1 + x + rac{x^{2}}{2} + rac{x^{3}}{6} + o\left(x^{3}
ight) \ \cos x = 1 - rac{x^{2}}{2} + o\left(x^{3}
ight)$$

将它们代入极限式,可得

$$A = \lim_{x o 0} rac{1 + x - rac{x^3}{3} + o\left(x^3
ight) - x - 1}{-rac{2x^3}{3}} = \lim_{x o 0} rac{-rac{x^3}{3} + o\left(x^3
ight)}{-rac{2x^3}{3}} = rac{1}{3} \cdot rac{3}{2} = rac{1}{2}.$$

【参考解析】: 令 $b=a^2$,则 $b\geq 0$,则

若
$$0 \leq b \leq 2$$
,则 $2 \leq \sqrt[n]{2^n + b^n} \leq \sqrt[n]{2 \cdot 2^n} o 2$,即

$$\lim_{n o\infty}\sqrt[n]{2^n+a^{2n}}=2.$$

若
$$b>2$$
,则 $b\leq \sqrt[n]{2^n+b^n}\leq \sqrt[n]{2b^n} o b$,即

$$\lim_{n o\infty}\sqrt[n]{2^n+b^n}=b=a^2$$
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因此,原式
$$=\lim_{n \to \infty} \sqrt[n]{2^n + b^n} = \max\left\{2, a^2\right\}$$
.

3.【参考解析】: 原式
$$=\int_0^\pi \frac{\pi + \cos x}{\left(x - \frac{\pi}{2}\right)^2 - \frac{\pi^2}{4} + 2004}$$

$$(\diamondsuit x - \frac{\pi}{2} = t)$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi - \sin t}{t^2 - \frac{\pi^2}{4} + 2004} dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi}{t^2 - \frac{\pi^2}{4} + 2004} dx - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin t}{t^2 - \frac{\pi^2}{4} + 2004} dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi}{t^2 + \left[2004 - \frac{\pi^2}{4}\right]} dx$$

$$= \frac{1}{\sqrt{2004 - \pi^2 / 4}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi}{\left(\frac{t}{\sqrt{2004 - \pi^2 / 4}}\right)^2 + 1} d\frac{t}{\sqrt{2004 - \pi^2 / 4}}$$

$$=\frac{\pi}{\sqrt{2004-\pi^2/4}}\arctan\frac{t}{\sqrt{2004-\pi^2/4}}\begin{vmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \end{vmatrix}$$

$$=\frac{2\pi}{2\pi}\arctan\frac{\pi}{2}$$

$$= \frac{2\pi}{\sqrt{2004 - \frac{\pi^2}{4}}} \arctan \frac{\pi}{2\sqrt{2004 - \pi^2/4}}$$

$$= \frac{4\pi \arctan \frac{\pi}{\sqrt{8016 - \pi^2}}}{\sqrt{8016 - \pi^2}}.$$

4.【参考解析】: (1)在区域内考虑无条件极值: 令

$$egin{cases} f_x'ig(x,yig)=2x=0,\ f_y'ig(x,yig)=8y+15=0. \end{cases}$$

解得唯一驻点 $\left(0,-rac{15}{8}
ight)$,因为 $4\cdot 0^2+\left(-rac{15}{8}
ight)^2>1$,驻点不

在区域内, 所以在区域内没有需要考虑的最值点.

(2) 考虑区域边界上的有条件极值,即

$$f\!\left(x,y\right) = x^2 + 4y^2 + 15y$$

在约束条件 $4x^2 + y^2 = 1$ 下可能极值点. 令拉格朗日函数为

$$Lig(x,y,\lambdaig)=x^2+4y^2+15y+\lambda(4x^2+y^2-1)$$

并求解方程组 $egin{cases} F_x'ig(x,yig) = 2x + 8\lambda x = 0, \ F_y'ig(x,yig) = 8y + 15 + 2\lambda y = 0, \end{aligned}$ 得驻点为 $F_\lambda'ig(x,yig) = 4x^2 + y^2 - 1 = 0$

(0,1),(0,-1) . 并且 f(0,1)=19,f(0,-1)=-11 . 所以最大值为 f(0,1)=19 , 最小值为 f(0,-1)=-11 .

5.【参考解析】:用 $xy=x^3$ 分割积分区域分割成从左到右四个部分,分别记作 D_1,D_2,D_3,D_4 四个部分,于是

$$egin{aligned} &\iint_D \max\left(xy,x^3
ight) \mathrm{d}\,\sigma \ &= \iint_{D_1} xy\,\mathrm{d}\,\sigma + \iint_{D_2} x^3\,\mathrm{d}\,\sigma + \iint_{D_3} xy\,\mathrm{d}\,\sigma + \iint_{D_4} x^3\,\mathrm{d}\,\sigma \ &= \int_{-1}^0 \mathrm{d}\,x \int_0^{x^2} xy\,\mathrm{d}\,y + \int_{-1}^0 \mathrm{d}\,x \int_{x^2}^1 x^3\,\mathrm{d}\,y \ &+ \int_0^1 \mathrm{d}\,x \int_{x^2}^1 xy\,\mathrm{d}\,y + \int_0^1 \mathrm{d}\,x \int_0^{x^2} x^3\,\mathrm{d}\,y \ &= -rac{1}{12} - rac{1}{12} + rac{1}{6} + rac{1}{6} = rac{1}{6} \end{aligned}$$

二.【参考解析】:【思路一】由
$$f'(x) = -\frac{1}{1+x^2}$$
,由

$$rac{1}{1-x}=\sum_{n=0}^{+\infty}x^n$$
 ,

得
$$f'(x) = -rac{1}{1+x^2} = \sum_{n=0}^{+\infty} \left(-1
ight)^{n+1} x^{2n}$$
. 两端在 $0, x$ 积分,

得
$$f(x) - f(0) = \sum_{n=0}^{+\infty} \frac{\left(-1\right)^{n+1}}{2n+1} x^{2n+1}$$

由于
$$f(0) = \arctan 1 = \frac{\pi}{4}$$
,所以

$$f\left(x
ight) = rac{\pi}{4} + \sum_{n=0}^{+\infty} rac{\left(-1
ight)^{n+1}}{2n+1} x^{2n+1}$$

比较系数得

$$egin{align} f^{(2n+1)}(0) &= rac{\left(-1
ight)^{n+1}}{2n+1} \cdot \left(2n+1
ight)! \ &= \left(-1
ight)^{n+1} \cdot \left(2n
ight)!, n = 0, 1, \cdots \ f^{(2n)}(0) &= 0, n = 1, 2, \cdots . \ f\left(0
ight) &= rac{\pi}{4} \, . \end{aligned}$$

【思路二】由
$$f'(x) = -\frac{1}{1+x^2}$$
,则

$$(1+x^2)f'(x) = -1$$
,

对上式两边对x 求(n-1) 阶导数,则由莱布尼茨公式得

$$(1+x^2)f^{(n)}(x) + 2(n-1)xf^{(n-1)}(x) + n(n-1)f^{(n-2)}(x) = 0$$

令 x = 0,得:

$$f^{(n)}(0) = -n(n-1)f^{(n-2)}(0)$$
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而
$$f'(0) = -1, f''(0) = 0$$
,则

$$f^{(n)}(0) = egin{cases} \pi \ / \ 4, & n = 0 \ 0, & n = 2k \ (-1)^{rac{n+1}{2}} ig(n-1)!, n = 2k-1 \end{cases}$$

三.【参考解析】: 方程 $\frac{x^2}{4} + \frac{y^2}{9} = 1$ 两边对x求导得

$$rac{x}{2}+rac{2y}{9}y'=0$$
 ,

代入 $x=1,y=rac{3\sqrt{3}}{2}$,解得 $y'ig|_{x=1}=-rac{\sqrt{3}}{2}$.所以切线方程

为
$$y-rac{3\sqrt{3}}{2}=-rac{\sqrt{3}}{2}ig(x-1ig)$$
 ,所以 B 点的坐标为 $\Big(0,2\sqrt{3}\Big)$,

设 $C\left[0, rac{3\sqrt{3}}{2}
ight]$,则CABC构成逆时钟方向,围成封闭区域为

D,则由格林公式,得

$$\int\limits_{l}=\int\limits_{D}-\int\limits_{C o A}-\int\limits_{B o C}$$
令 $Q(x,y)=\cos y\lnig(x+1ig)+2\sqrt{3}x-\sqrt{3}$,则 $rac{\partial Q}{\partial x}-rac{\partial P}{\partial y}=3\sqrt{3}$, $C o A:y=rac{3\sqrt{3}}{2},x:0 o 1$

$$B o C: x=0, y: 2\sqrt{3} o rac{3\sqrt{3}}{2}$$

将上面等式与曲线方程直接代入积分,得

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$$\int_{l} = 3\sqrt{3} \iint_{D} d\sigma + \sqrt{3} \int_{2\sqrt{3}}^{\frac{3}{2}\sqrt{3}} dy$$

$$- \int_{0}^{1} \left[\frac{\sin \frac{3}{2}\sqrt{3}}{x+1} - \sqrt{3} \cdot \frac{3}{2}\sqrt{3} \right] dx$$

$$= \frac{9}{4} - \frac{3}{2} - \ln 2 \cdot \sin \frac{3\sqrt{3}}{2} + \frac{9}{2} = \frac{21}{4} - \ln 2 \cdot \sin \frac{3\sqrt{3}}{2}$$

四.【参考解析】: 由定积分的定义, 有

$$\int_a^b \Bigl| f \Bigl(x \Bigr) \! \! \left| \operatorname{d} x = \lim_{\lambda o 0} \sum_{i=1}^n \Bigl| f (\xi_i) \Bigr| \Delta x_i = 0$$

以上等式无论 $\left[a,b\right]$ 如何分割, $\xi_i\in\left[x_{i-1},x_i\right]$ 如何取值都成立.

假设存在 $x_0\in (a,b)$,使得 $f(x_0)\neq 0$,不妨设 $f\left(x_0\right)>0$,则由于函数 $f\left(x\right)$ 连续,则 $\exists \delta>0$, $\forall x\in [x_0-\delta,x_0+\delta]$,都有 $f\left(x\right)>0$,则 $f\left(x\right)\cdot 2\delta>0$,与 $\int_a^b \left|f\left(x\right)\right|\mathrm{d}\,x=0$ 矛盾,所以 $f\left(x\right)\equiv 0, x\in (a,b)$.

同理可证端点取值也等于 0,即 $f(x) \equiv 0, x \in [a,b]$.

【或】由于函数 $f\left(x\right)$ 连续,故在 $[x_0-\delta,x_0+\delta]$ 内存在最大、最小值分别为 M_0,m_0 ,显然 $M_0>0,m_0>0$,并且有

$$\int_a^b \left| f\!\left(x
ight) \! \! \! \right| \mathrm{d}\,x \geq \int_{x_\circ - \delta}^{x_0 + \delta} \left| f\!\left(x
ight) \! \! \! \! \! \! \right| \mathrm{d}\,x \geq 2\delta m_0^{} > 0$$

与 $\int_a^b |f(x)| dx = 0$ 矛盾,故假设错误.

五.【参考解析】:【思路一】由正项级数的对数判别法:记

$$a_n = \left(rac{1}{n!}
ight)^{rac{lpha}{n}}$$
,则

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$$\begin{split} \lim_{n \to \infty} \frac{\ln \frac{1}{a_n}}{\ln n} &= \alpha \lim_{n \to \infty} \frac{\ln 1 + \ln 2 + \dots + \ln n}{n \ln n} \\ &= \alpha \lim_{n \to \infty} \frac{\ln (n+1)}{(n+1) \ln (n+1) - n \ln n} \\ &= \alpha \lim_{n \to \infty} \frac{\ln (n+1)}{n \ln (n+1)} = \alpha \end{split}$$

若 $\alpha>1$,则由极限保号性可知, $\exists k$,使得当n>k时,有

$$\frac{\ln\frac{1}{a_n}}{\ln n} > \frac{\alpha+1}{2} = 1 + \frac{\alpha-1}{2}$$

则由对数判别法知,当lpha>1时,级数收敛.

【注】正项级数的对数判别法:

- (1) 若 $\exists k$ 和常数 $\alpha>0$ 使得当n>k时, $\dfrac{\ln\dfrac{1}{a_n}}{\ln n}\geq 1+\alpha$,则级数 $\sum_{n=1}^\infty a_n$ 收敛;
- (2) 若 $\exists k$ 使得当n > k时, $\frac{\ln \frac{1}{a_n}}{\ln n} \le 1$,则级数 $\sum_{n=1}^{\infty} a_n$ 发散.

【定理证明】:(1)
$$\dfrac{\ln\dfrac{1}{a_n}}{\ln n}\geq 1+lpha$$
,则 $\ln\dfrac{1}{a_n}\geq (1+lpha)\ln n=(\ln n)^{1+lpha}$,《多研克要数学

即 $\dfrac{1}{a_n} \geq n^{1+lpha}, \dfrac{1}{n^{1+lpha}} \geq a_n$,由比较判别法可知级数收敛.

(2) 由
$$\dfrac{\ln\dfrac{1}{a_n}}{\ln n} \leq 1$$
 , 则 $\ln\dfrac{1}{a_n} \leq \ln n$, 即 $a_n \geq \dfrac{1}{n}$, 由比较

判别法可知级数发散

【思路二】由斯特林公式:

$$n! = \sqrt{2n\pi} \cdot \left(rac{n}{e}
ight)^n \cdot e^{rac{ heta}{12n}}, 0 < heta < 1$$

因为
$$(n!)^{rac{lpha}{n}} = \left(\sqrt{2\pi n} \cdot n^n \cdot e^{-n + rac{ heta}{12n}}
ight)^{rac{lpha}{n}}$$
 $= n^{lpha}(2\pi)^{rac{lpha}{2n}} n^{rac{lpha}{2n}} e^{-lpha + rac{ heta}{12n^2}lpha}$

而
$$(2\pi)^{rac{lpha}{2n}}n^{rac{lpha}{2n}}e^{-lpha+rac{ heta}{12n^2}lpha}
ightarrowrac{1}{e^lpha}$$
,所以级数 $\sum_{n=1}^\inftyrac{1}{\sqrt[n]{\left(n\,!
ight)^lpha}}$ 与

$$\sum_{n=1}^{\infty} rac{1}{n^{lpha}}$$
有相同敛散性. 因此当 $lpha > 1$ 时,级数 $\sum_{n=1}^{\infty} rac{1}{\sqrt{\left(n\,!
ight)^{lpha}}}$

收敛; 当 $0 < \alpha \le 1$ 时, 级数发散.

【注】由此可得
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{(n!)^2}}$$
收敛, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n!}}$ 发散.

六.【参考解析】: 由柯西-施瓦兹不等式,得

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$$\left(\int_{0}^{1} \frac{f(x)}{t^{2} + x^{2}} dx\right)^{2} = \left(\int_{0}^{1} \frac{1}{\sqrt{t^{2} + x^{2}}} \frac{f(x)}{\sqrt{t^{2} + x^{2}}} dx\right)^{2}$$

$$\leq \int_{0}^{1} \frac{1}{t^{2} + x^{2}} dx \cdot \int_{0}^{1} \frac{f^{2}(x)}{t^{2} + x^{2}} dx$$

$$= \left[\frac{1}{t} \arctan \frac{1}{t}\right]_{0}^{1} \cdot \int_{0}^{1} \frac{f^{2}(x)}{t^{2} + x^{2}} dx \leq \frac{\pi}{2t} \cdot \int_{0}^{1} \frac{f^{2}(x)}{t^{2} + x^{2}} dx$$

七.【参考解析】: $\forall x \in (0,1)$, 令

$$g(t) = f(t) + 1 - t^2 - \frac{f(x) + 1 - x^2}{x^2(x - 1)}t^2(t - 1), \ t \in (0, 1)$$

由已知可得 $g\left(x\right)=g\left(1\right)=g\left(0\right)=0$,从而

$$\exists oldsymbol{\xi}_1 \in (0,x), oldsymbol{\xi}_2 \in (x,1)$$
 ,

使得
$$g'ig(\xi_1ig) = g'ig(\xi_2ig) = 0, 0 < \xi_1 < x < \xi_2 < 1$$
. 对 $gig(tig)$

求导,得
$$g'(t) = f'(t) - 2t - \frac{f(x) + 1 - x^2}{x^2(x-1)} (3t^2 - 2t)$$

由
$$f'(0)=0$$
,所以 $g'ig(0ig)=0=g'ig(\xi_1ig)$.于是 $\exists \eta_1\inig(0,\xi_1ig)$

使得
$$g''ig(\eta_1ig)=0$$
. 同理,由 $g'ig(\xi_1ig)=g'ig(\xi_2ig)=0$ 和 $\xi_1<\xi_2$

知,
$$\exists \eta_2 \in (\xi_1, \xi_2)$$
 使得 $g^{\prime\prime}(\eta_2) = 0$.

于是可知 $\exists \xi \in \left(\eta_1, \eta_2\right)$,使得 $g^{\prime\prime\prime}(\xi) = 0$.由于

$$g''(t) = f''(t) - 2 - \frac{f(x) + 1 - x^2}{x^2(x-1)}(6t-2)$$

$$g^{\prime\prime\prime}(t) = f^{\prime\prime\prime}(t) - 6\frac{f(x) + 1 - x^2}{x^2(x-1)}$$

由
$$g^{\prime\prime\prime}(\xi)=0$$
代入可得 $f^{\prime\prime\prime}(\xi)=6rac{f(x)+1-x^2}{x^2(x-1)}$,即

$$f(x)=-1+x^2+rac{x^2(x-1)}{3!}f^{\prime\prime\prime}(\xi)$$