INTELLIGENT ROBOTICS

CHAPTER 1: PROBABILISTIC ROBOTICS

Learning Objectives

1、What are the probabilistic models of perception and action?

3、What is conditional independence?

4、What are the Bayes rule and Bayesian estimation?

5、What is the Markov model?

6、What is the Bayes Filter？

7、What is the Markov Decision Process?

Outline

- Probability Fundamentals
- Modeling Perception
- Modeling Actions
- Bayes Filters
- Markov Decision Process

Robotics Paradigms

Classical Robotics (mid-70’ s)

- exact models
- no sensing necessary

Reactive Paradigm (mid-80’ s)

- no models
- relies heavily on good sensing

Hybrids (since 90’ s)

- model-based at higher levels
- reactive at lower levels

Probabilistic Robotics (since mid-90’ s)

- integration of models and sensing
- inaccurate models, inaccurate sensors

Intelligent Robotics (since 00’ s)

- deep learning and deep reinforcement learning
- vision based sensing

Probabilistic Robotics

Key idea:
Explicit representation of uncertainty
(using the calculus of probability theory)

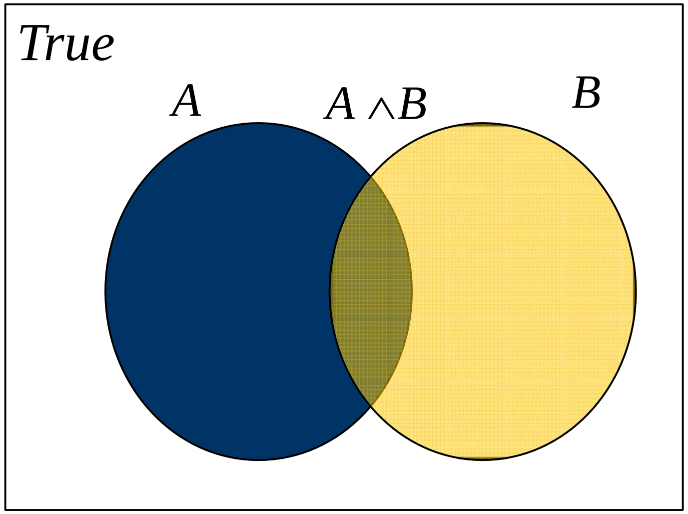
- Perception = state estimation (higher posterior)
- Action = utility optimization (higher utility)

Reward – Cost

Axioms of Probability

Pr*(A)* denotes probability that proposition *A* is true.

- $0 \leq \Pr(A) \leq 1$
- $\Pr(True) = 1 \quad \Pr(False) = 0$



- $\Pr(A \vee B) = \Pr(A) + \Pr(B) - \Pr(A \wedge B)$

Using the Axioms

$$
\begin{aligned}
\Pr(A \vee \neg A) &= \Pr(A) + \Pr(\neg A) - \Pr(A \wedge \neg A) \\
\Pr(\mathit{True}) &= \Pr(A) + \Pr(\neg A) - \Pr(\mathit{False}) \\
1 &= \Pr(A) + \Pr(\neg A) - 0 \\
\Pr(\neg A) &= 1 - \Pr(A)
\end{aligned}
$$

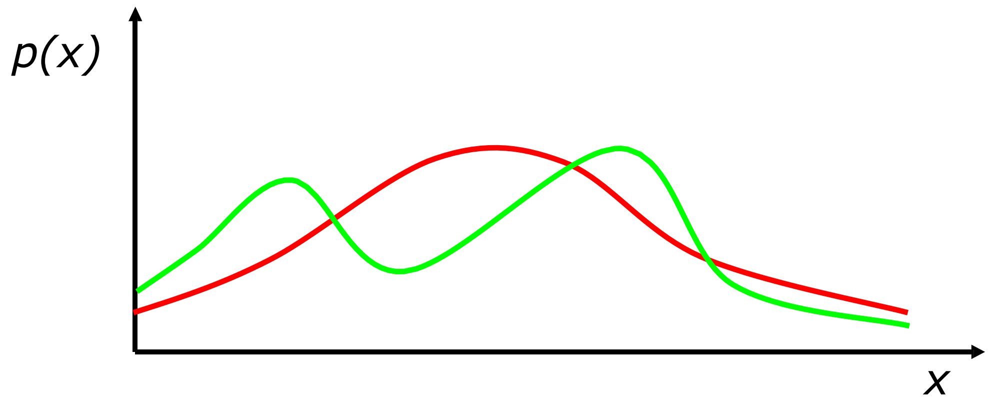
Discrete Random Variables

- *X* denotes a random variable
- *X* can take on a countable number of values in $\{x_1, x_2, \ldots, x_n\}$
- $P(X=x_i)$ or $P(x_i)$ is the probability that the random variable *X* takes on value x_i
- $P(\cdot)$ is called probability mass function
- E.g. $P(Room) = \{0.7, 0.2, 0.08, 0.02\}$

Continuous Random Variables

- *X* takes on values in the continuum.
- *p(X=x)* or *p(x)* is a probability density function

$$\Pr(x \in (a,b)) = \int_a^b p(x)\,dx$$



Continuous Random Variables

Discrete case

$$\sum_{x} P(x) = 1$$

Continuous case

$$\int p(x)\, dx = 1$$

Joint and Conditional Probability

- $P(X{=}x \text{ and } Y{=}y) = P(x,y)$
- If X and Y are independent then
  $P(x,y) = P(x)\,P(y)$
- $P(x \mid y)$ is the probability of x given y

  $P(x \mid y) = P(x,y) \,/\, P(y) \quad P(x,y) = P(x \mid y)\,P(y)$
- If X and Y are independent then
  $P(x \mid y) = P(x)$

Law of Total Probability

Discrete case

$$P(x) = \sum_{y} P(x \mid y) P(y)$$

Continuous case

$$p(x) = \int p(x \mid y)\, p(y)\, dy$$

Marginalization

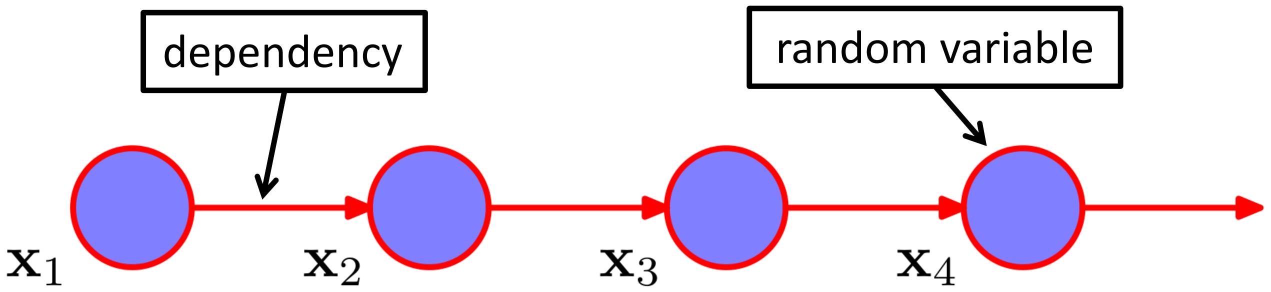
Discrete case

$$P(x) = \sum_{y} P(x, y)$$

Continuous case

$$p(x) = \int p(x, y)\, dy$$

Markov Model

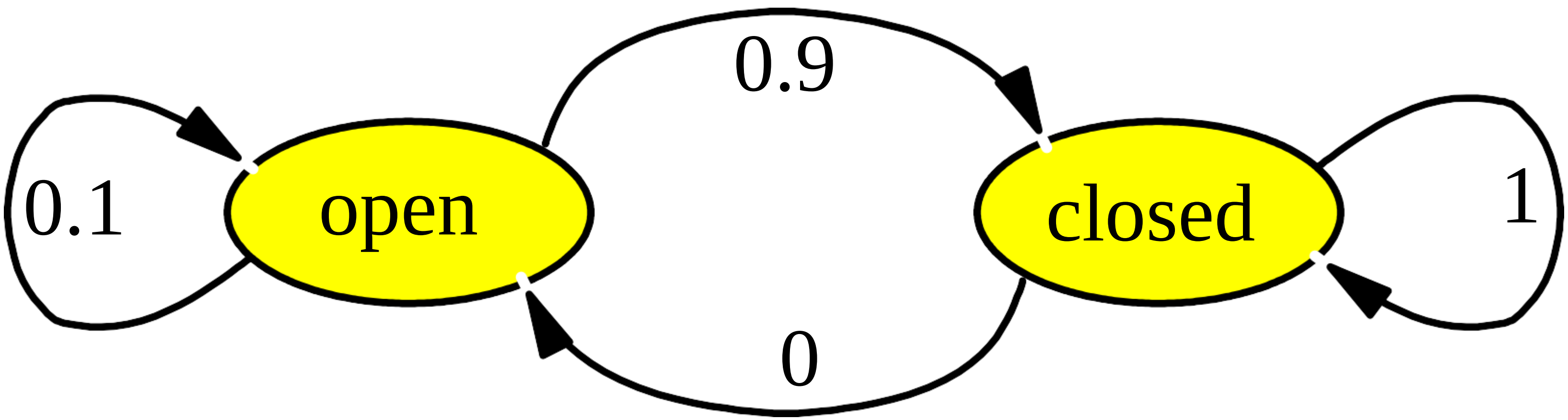


$\{x_1, .. x_N\}$ have only two states: "open" and "closed" $p(x) = \begin{bmatrix} p_o \\ p_c \end{bmatrix}$

Markovian assumption: $$p(\mathbf{x}_1, \ldots, \mathbf{x}_N) = p(\mathbf{x}_1) \prod_{n=2}^{N} p(\mathbf{x}_n | \mathbf{x}_{n-1})$$

$$p(\mathbf{x}_n | \mathbf{x}_1, \ldots, \mathbf{x}_{n-1}) = p(\mathbf{x}_n | \mathbf{x}_{n-1})$$

Markov Model



Transition probabilities:

$$\begin{bmatrix} 0.1 & 0 \\ 0.9 & 1 \end{bmatrix}$$

State probability:

$$\begin{bmatrix} p_o \\ p_c \end{bmatrix}$$

Updating:

$$\begin{bmatrix} p_o \\ P_c \end{bmatrix}' = \begin{bmatrix} 0.1 & 0 \\ 0.9 & 1 \end{bmatrix} \begin{bmatrix} p_o \\ p_c \end{bmatrix}$$

Steady State:

$$\begin{bmatrix} p_o \\ P_c \end{bmatrix} = \begin{bmatrix} 0.1 & 0 \\ 0.9 & 1 \end{bmatrix} \begin{bmatrix} p_o \\ p_c \end{bmatrix}$$

Bayes Formula

$$P(x, y) = P(x \mid y)P(y) = P(y \mid x)P(x)$$

$$\Rightarrow$$

$$\boxed{P(x|y) = \frac{P(y \mid x)\, P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}}$$

Normalization

$$P(x \mid y) = \frac{P(y \mid x)\, P(x)}{P(y)} = \eta\, P(y \mid x)\, P(x)$$

$$\eta = P(y)^{-1} = \frac{1}{\sum_x P(y \mid x) P(x)}$$

Bayes Rule with Background Knowledge

$$P(x \mid y, z) = \frac{P(y \mid x, z)\ P(x \mid z)}{P(y \mid z)}$$

Conditional Independence

$$P(x, y \mid z) = P(x \mid z) P(y \mid z)$$

- Equivalent to $P(x \mid z) = P(x \mid z, y)$

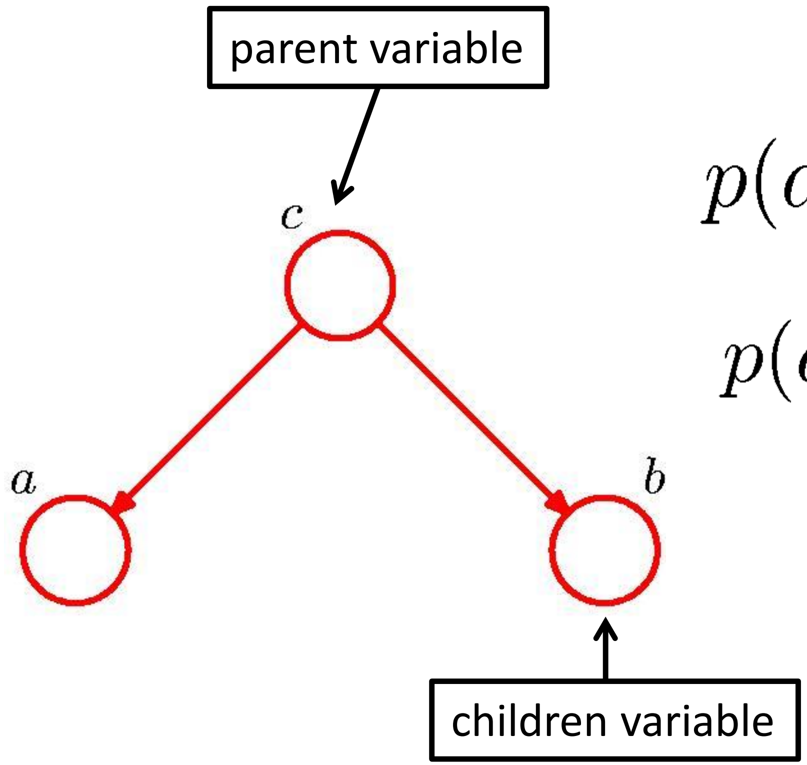
  and $P(y \mid z) = P(y \mid z, x)$

- But this does not necessarily mean

$$P(x, y) = P(x) P(y)$$

(real independence)

Conditional Independence: Example 1

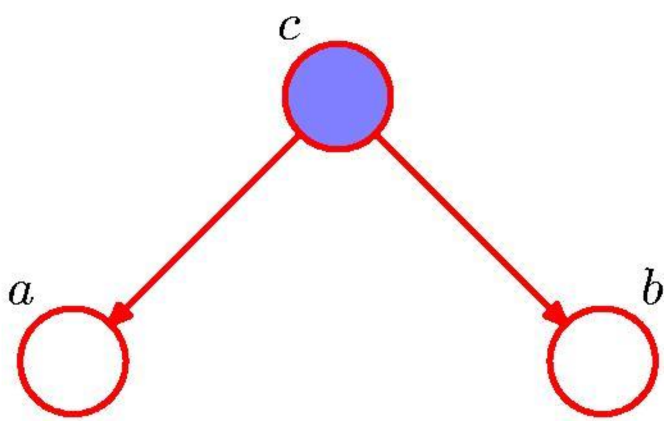


$$p(a, b, c) = p(a|c)p(b|c)p(c)$$

$$p(a, b) = \sum_c p(a|c)p(b|c)p(c)$$

$$a \not\perp\!\!\!\perp b \mid \emptyset$$

Conditional Independence: Example 1

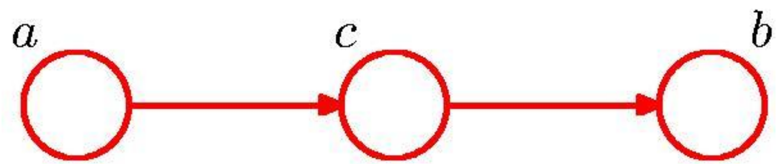


$$p(a, b|c) = \frac{p(a, b, c)}{p(c)}$$

$$= p(a|c)p(b|c)$$

$$a \perp\!\!\!\perp b \mid c$$

Conditional Independence: Example 2

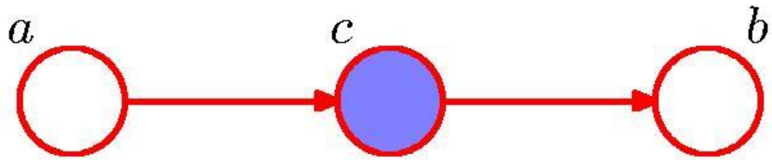


$$p(a, b, c) = p(a)p(c|a)p(b|c)$$

$$p(a, b) = p(a) \sum_c p(c|a)p(b|c) = p(a)p(b|a)$$

$$a \not\perp\!\!\!\perp b \mid \emptyset$$

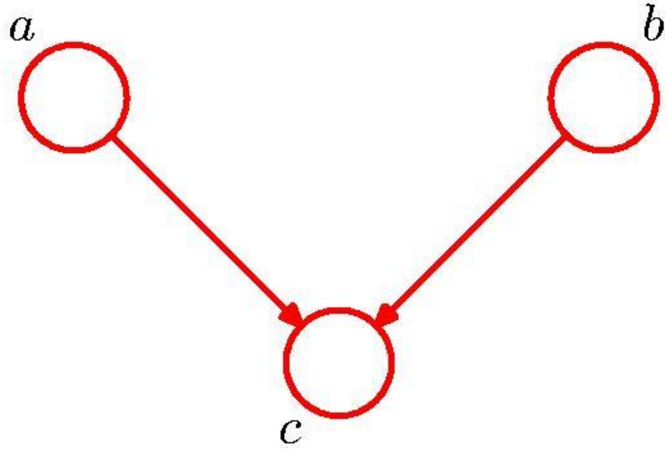
Conditional Independence: Example 2



$$
\begin{aligned}
p(a, b|c) &= \frac{p(a, b, c)}{p(c)} \\
&= \frac{p(a)p(c|a)p(b|c)}{p(c)} \\
&= p(a|c)p(b|c)
\end{aligned}
$$

$$a \perp\!\!\!\perp b \mid c$$

Conditional Independence: Example 3



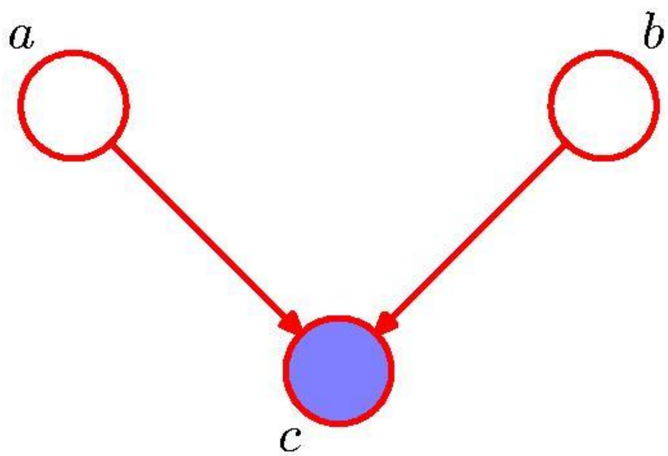
$$p(a, b, c) = p(a)p(b)p(c|a, b)$$

$$p(a, b) = p(a)p(b)$$

$$a \perp\!\!\!\perp b \mid \emptyset$$

Note: this is the opposite of Example 1, with c unobserved.

Conditional Independence: Example 3



$$
\begin{aligned}
p(a, b|c) &= \frac{p(a, b, c)}{p(c)} \\
&= \frac{p(a)p(b)p(c|a, b)}{p(c)}
\end{aligned}
$$

$$a \not\perp\!\!\!\perp b \mid c$$

Note: this is the opposite of Example 1, with c observed.

Outline

- Probability Fundamentals
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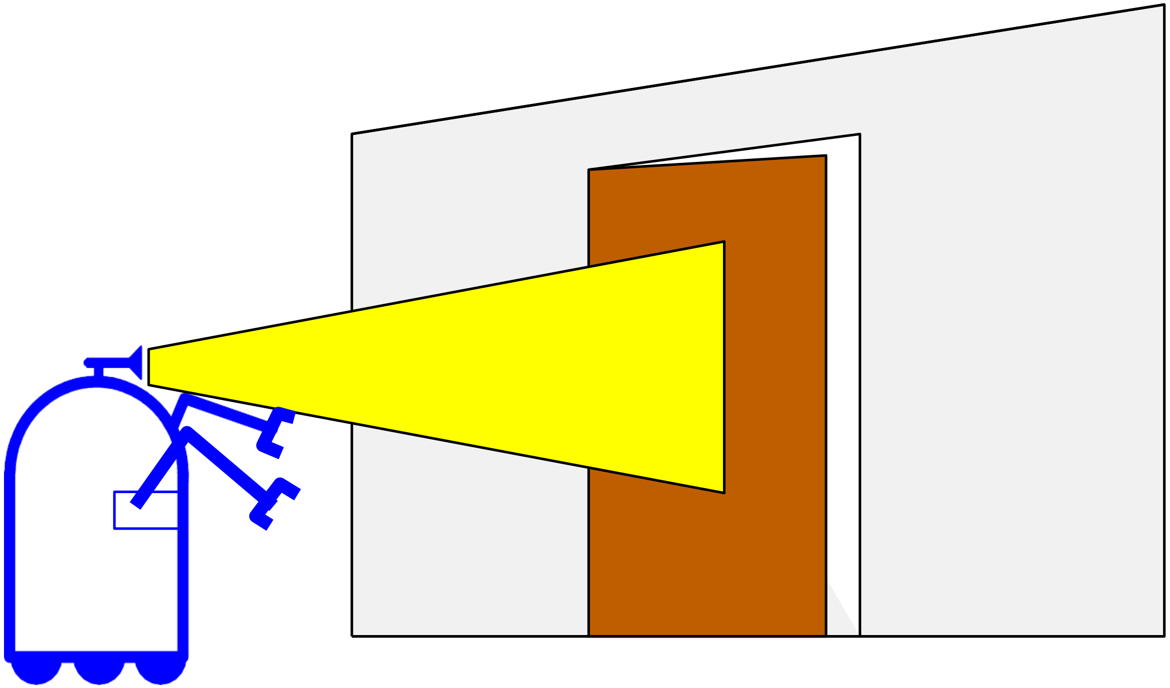
- Modeling Actions
- Bayes Filters
- Markov Decision Process

State Estimation: Modeling Perception

- Suppose a robot obtains measurement z
- What is $P(open|z)$?



Casual v.s. Diagnostic Reasoning

- $P(open|z)$ is diagnostic
- $P(z|open)$ is causal
- Often causal knowledge is easier to obtain

count frequencies!

- Bayes rule allows us to use causal knowledge:

$$P(open \mid z) = \frac{P(z \mid open)P(open)}{P(z)}$$

Example

- $P(z|open) = 0.6$ $\qquad$ $P(z|\neg open) = 0.3$
- $P(open) = P(\neg open) = 0.5$

$$P(open \mid z) = \frac{P(z \mid open)P(open)}{P(z \mid open)\, p(open) + P(z \mid \neg open)\, p(\neg open)}$$

$$P(open \mid z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{0.3}{0.3 + 0.15} = 0.67$$

- z raises the probability that the door is open

Combining Evidence

- Suppose our robot obtains another observation z_2

- How can we integrate this new information?

- More generally, how can we estimate $P(x| z_1 \ldots z_n)$?

Combining Evidence

$$P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x, z_1, \ldots, z_{n-1})\, P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})}$$

Markov assumption:
z_n is independent of $z_1, \ldots, z_{n-1}$ if we know x

$$P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x)\, P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})}$$

$$= \eta\, P(z_n \mid x)\, P(x \mid z_1, \ldots, z_{n-1})$$

Conditional independence

$$= \eta_{1 \ldots n} \prod_{i=1 \ldots n} P(z_i \mid x)\, P(x)$$

Example

- $P(z_2|open) = 0.5$ $P(z_2|\neg open) = 0.6$
- $P(open|z_1)=2/3$

$$P(open \mid z_2, z_1) = \frac{P(z_2 \mid open)\, P(open \mid z_1)}{P(z_2 \mid open)\, P(open \mid z_1) + P(z_2 \mid \neg open)\, P(\neg open \mid z_1)}$$

$$= \frac{\frac{1}{2} \cdot \frac{2}{3}}{\frac{1}{2} \cdot \frac{2}{3} + \frac{3}{5} \cdot \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{5}} = \frac{\frac{1}{3}}{\frac{8}{15}} = \frac{5}{8} = 0.625$$

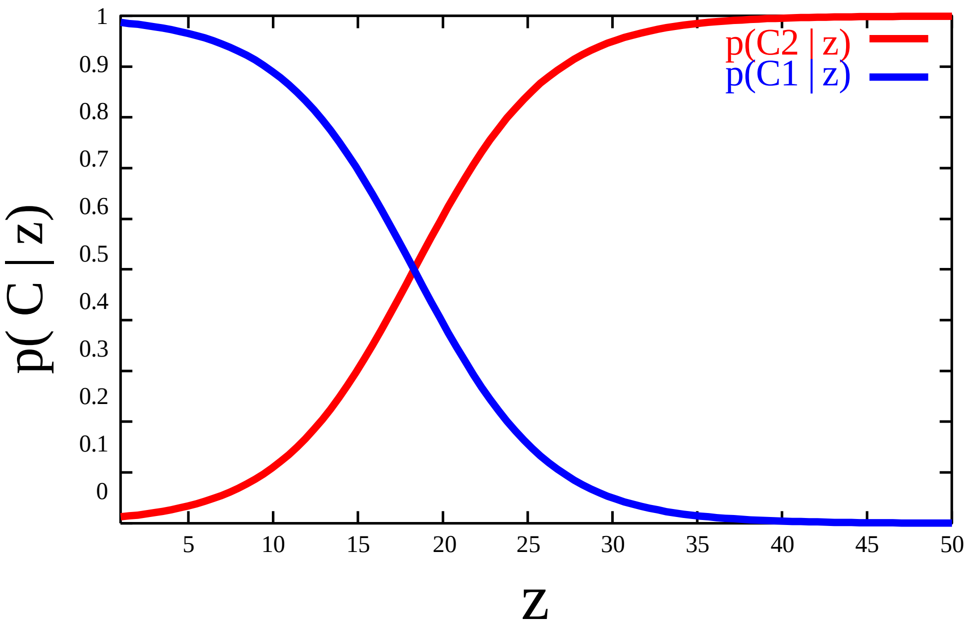
- z_2 *lowers the probability that the door is open*

A Typical Dual-Mode Posterior

- Two possible locations C_1 and C_2, z is a sensor measurement
- $P(C_1)=0.99$ $\quad$ $P(C_2)=0.01$
- $P(z|C_1)=0.01$ $\quad$ $P(z|C_2)=0.999$

$\Rightarrow P(C_1|z)=0.502$ $\quad$ $P(C_2|z)=0.498$

$$\boxed{C^* = \max_{C_i} P(C_i|z)}$$



$$p(C_1|z) = \frac{p(z|C_1)p(C_1)}{p(z|C_1)p(C_1) + p(z|C_2)p(C_2)}$$

$$p(C_2|z) = \frac{p(z|C_2)p(C_2)}{p(z|C_1)p(C_1) + p(z|C_2)p(C_2)}$$

Probabilistic Generative Models

- Use a separate generative model of the input vectors for each class, and see which model makes a test input vector most probable.
- The posterior probability of class 1 is given by:

$$p(C_1 \mid \mathbf{x}) = \frac{p(C_1)\, p(\mathbf{x} \mid C_1)}{p(C_1)\, p(\mathbf{x} \mid C_1) + p(C_0)\, p(\mathbf{x} \mid C_0)} = \frac{1}{1+e^{-z}} = \sigma(z)$$

Logistic function

$$\textit{where} \quad z = \ln \frac{p(C_1)\, p(\mathbf{x} \mid C_1)}{p(C_0)\, p(\mathbf{x} \mid C_0)} = \boxed{\ln \frac{p(C_1 \mid \mathbf{x})}{1 - p(C_1 \mid \mathbf{x})}}$$

z is called the logit and is given by the log odds

A Simple Example

- Assume that the input vectors for each class are from a Gaussian distribution, and all classes have the same covariance matrix.

$$p(\mathbf{x} \mid C_k) = \overset{\text{normalizing constant}}{\downarrow \atop a} \ \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_k)^T \overset{\text{inverse covariance matrix}}{\downarrow \atop \Sigma^{-1}}(\mathbf{x}-\boldsymbol{\mu}_k)\right\}$$

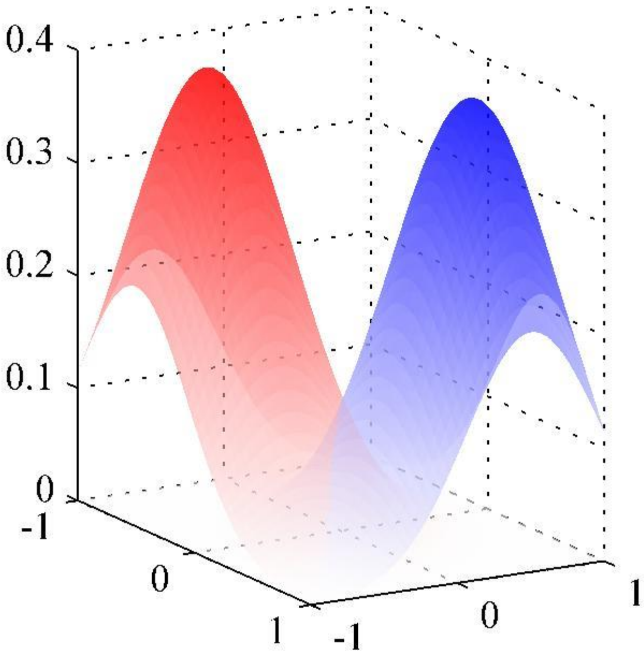
- For two classes, C_1 and C_0, the posterior is a logistic:

$$p(C_1 \mid \mathbf{x}) = \sigma(\mathbf{w}^T\mathbf{x} + w_0)$$

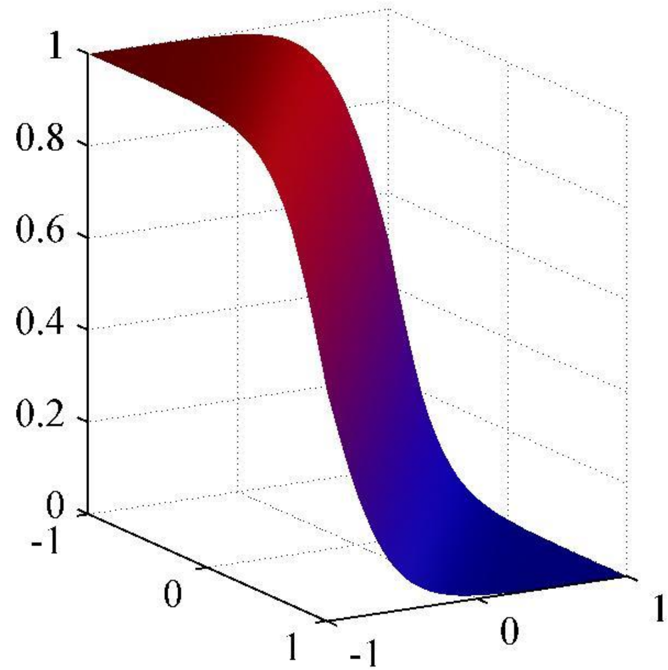
$$\mathbf{w} = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)$$

$$w_0 = -\frac{1}{2}\boldsymbol{\mu}_1^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_1 + \frac{1}{2}\boldsymbol{\mu}_0^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_0 + \ln\frac{p(C_1)}{p(C_0)}$$

Likelihood and Posterior



$p(x|C_0)$ $p(x|C_1)$



$p(C_0|x)$ $1 - p(C_1|x)$

Outline

- Probability Fundamentals
- Modeling Perception
- Modeling Actions
- Bayes Filters
- Markov Decision Process

Actions

- Often the world is **dynamic** since
  - **actions carried out by the robot**,
  - **actions carried out by other agents**,
  - or just the **time** passing by change of the world

- How can we **incorporate** such **actions**?

Typical Actions

- The robot **turns its wheels** to move
- The robot **uses its manipulator** to grasp an object
- Plants grow over **time**…

- Actions are never carried out with **absolute certainty**
- In contrast to measurements, actions generally **increase the uncertainty**

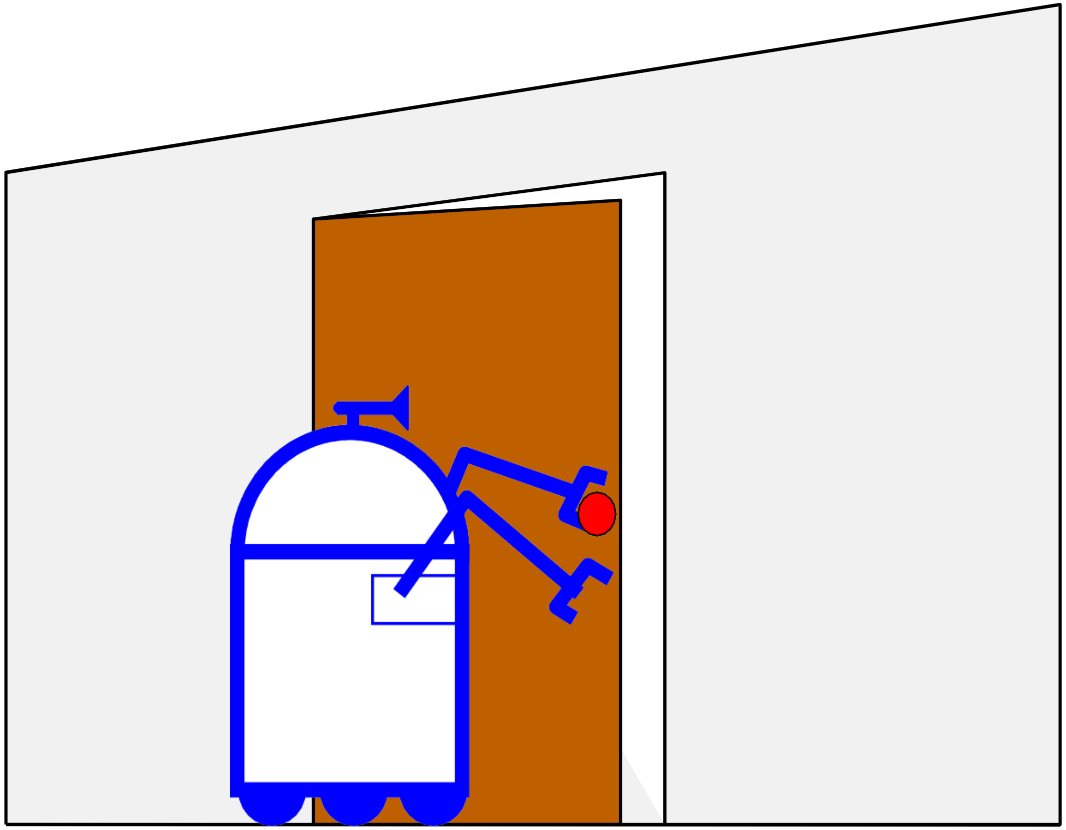
Modeling Actions

- To incorporate the outcome of an action u into the current "belief", we use the conditional pdf

$$P(x'|u,x)$$

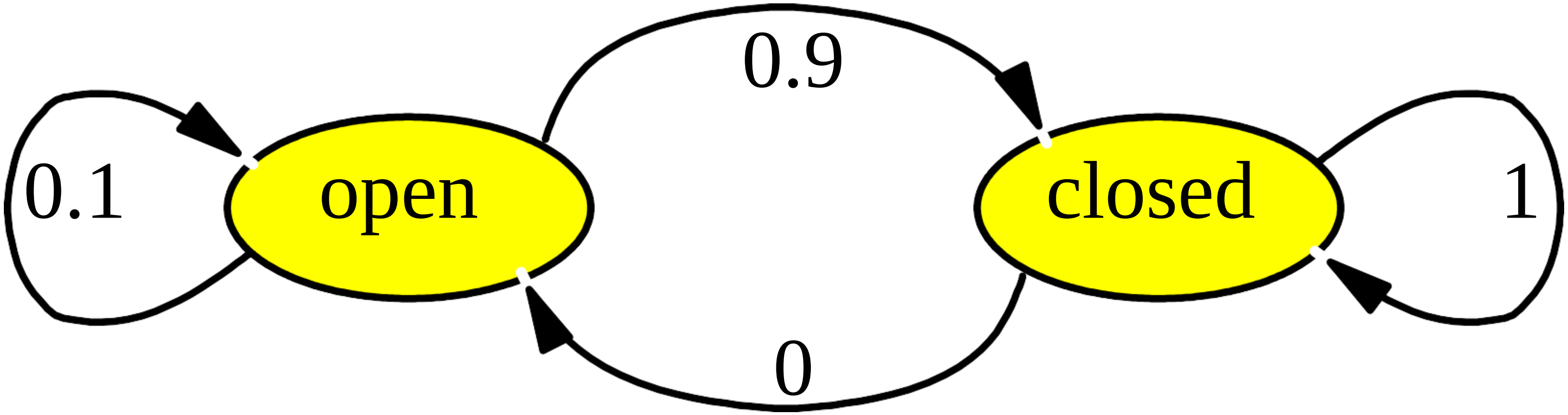
- This term specifies the pdf that executing u changes the state from x *to* x'.

Example: Closing the Door



State Transitions

P(x'|u,x) for *u* = “close door”:



If the door is open, the action “close door” succeeds in 90% of all cases

Integrating the Outcome of Actions

Continuous case:

$$P(x' \mid u) = \int P(x' \mid u, x) P(x) dx$$

Discrete case:

$$P(x' \mid u) = \sum P(x' \mid u, x) P(x)$$

Example: The Resulting Belief

$$P(closed \mid u) = \sum P(closed \mid u, x) P(x)$$

$$= P(closed \mid u, open) P(open) + P(closed \mid u, closed) P(closed)$$

$$= 9/10 * 5/8 + 1/1 * 3/8 = 15/16$$

$$P(open \mid u) = \sum P(open \mid u, x) P(x)$$

$$= P(open \mid u, open) P(open) + P(open \mid u, closed) P(closed)$$

$$= 1/10 * 5/8 + 0/1 * 3/8 = 1/16$$

for u = "close door"

State Reward and Utility Optimization

If the reward for the state of the door being *closed is Rc* and the reward for being open is *Ro*, and the cost of any action is *C*,

then the utility of the action of "close the door" is

$$U(u_1=\text{“close the door”}) = Rc^*\, P(closed \mid u_1) + Ro^*\, P(open \mid u_1) - C$$

the utility of the action of "open the door" is

$$U(u_2=\text{“open the door”}) = Rc^*\, P(closed \mid u_2) + Ro^*\, P(open \mid u_2) - C$$

$$\boxed{u^* = \max_{u_i} U(u_i)}$$

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- Probability Fundamentals
- Bayes Rules and State Estimation
- Modeling Actions
- Bayes Filters
- Markov Decision Process

Bayes Filters: Framework

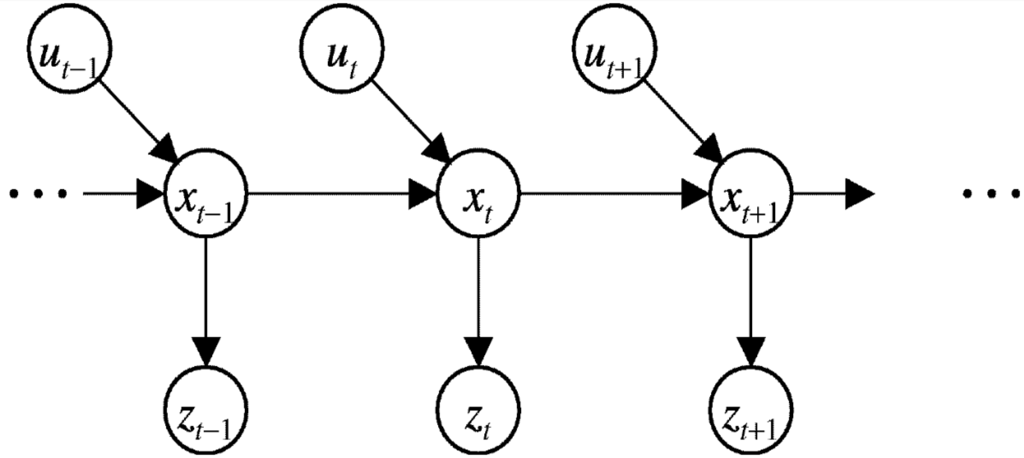
- **Given:**
  - Stream of observations *z* and action data *u:*

$$d_t = \{u_1, z_1, \ldots, u_t, z_t\}$$

  - Sensor model $P(z|x)$
  - Action model $P(x|u,x')$
  - Prior probability of the system state $P(x)$
- **Wanted:**
  - Estimate of the state *X* of a dynamical system
  - The posterior of the state is also called **Belief**:

$$Bel(x_t) = P(x_t \mid u_1, z_1, \ldots, u_t, z_t)$$

Markov Assumption



$$p(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t}) = p(z_t \mid x_t)$$

$$p(x_t \mid x_{0:t-1}, z_{1:t}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$$

Underlying Assumptions

- Static world
- Independent noise
- Perfect model, no approximation errors

Bayes Filter

z = observation
u = action
x = state

$$\boxed{Bel(x_t)} = P(x_t \mid u_1, z_1, \ldots, u_t, z_t)$$

Bayes $$= \eta\ P(z_t \mid x_t, u_1, z_1, \ldots, z_{t\text{-}1}, u_t)\ P(x_t \mid u_1, z_1, \ldots, z_{t\text{-}1}, u_t)$$

Markov $$= \eta\ P(z_t \mid x_t)\ P(x_t \mid u_1, z_1, \ldots, z_{t\text{-}1}, u_t)$$

Total prob. $$= \eta\ P(z_t \mid x_t) \int P(x_t \mid u_1, z_1, \ldots, z_{t\text{-}1}, u_t, x_{t-1})\ P(x_{t-1} \mid u_1, z_1, \ldots, z_{t\text{-}1}, u_t)\ dx_{t-1}$$

Markov $$= \eta\ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1})\ P(x_{t-1} \mid u_1, z_1, \ldots, z_{t-1}, u_t)\ dx_{t-1}$$

Markov $$= \eta\, P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1})\, P(x_{t-1} \mid u_1, z_1, \ldots, z_{t-1})\ dx_{t-1}$$

$$\boxed{= \eta\ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1})\ Bel(x_{t-1})\ dx_{t-1}}$$

Correction

Prediction

Algorithm

1. Algorithm **Bayes_filter**(*Bel(x),d*):
2. η=0
3. If *d* is a perceptual data item *z* then
4. For all *x* do
5. $Bel'(x) = P(z \mid x) Bel(x)$
6. $\eta = \eta + Bel'(x)$
7. For all *x* do
8. $Bel'(x) = \eta^{-1} Bel'(x)$
9. Else if *d* is an action data item *u* then
10. For all *x* do
11. $Bel'(x) = \int P(x \mid u, x')\, Bel(x')\, dx'$
12. Return *Bel′(x)*

$$Bel(x_t) = \eta\ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1})\, Bel(x_{t-1})\, dx_{t-1}$$

Bayes Filters

$$Bel(x_t) = \eta \; P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \; Bel(x_{t-1}) \; dx_{t-1}$$

- Kalman filters
- Particle filters
- Hidden Markov models
- Dynamic Bayesian networks
- Partially Observable Markov Decision Processes (POMDPs)

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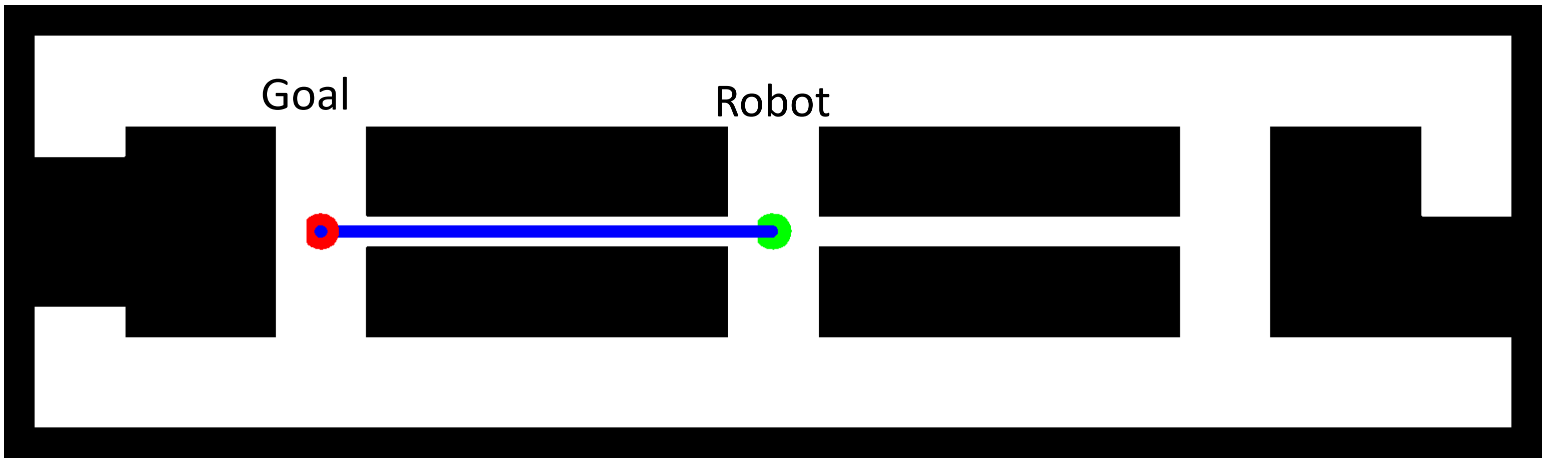
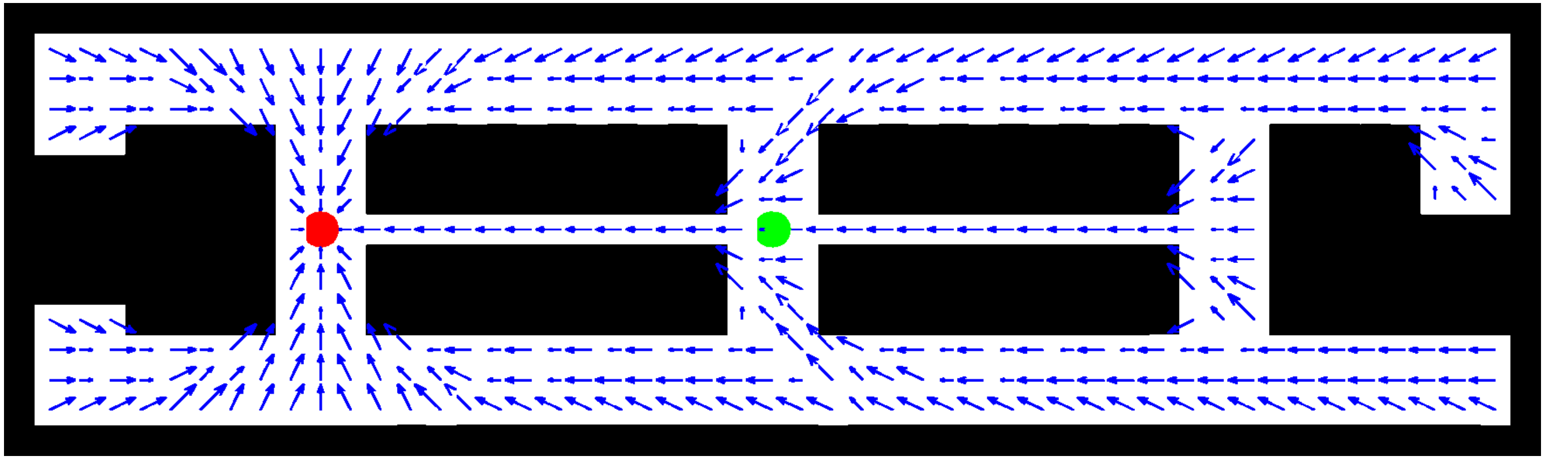
Robot Navigation Problem

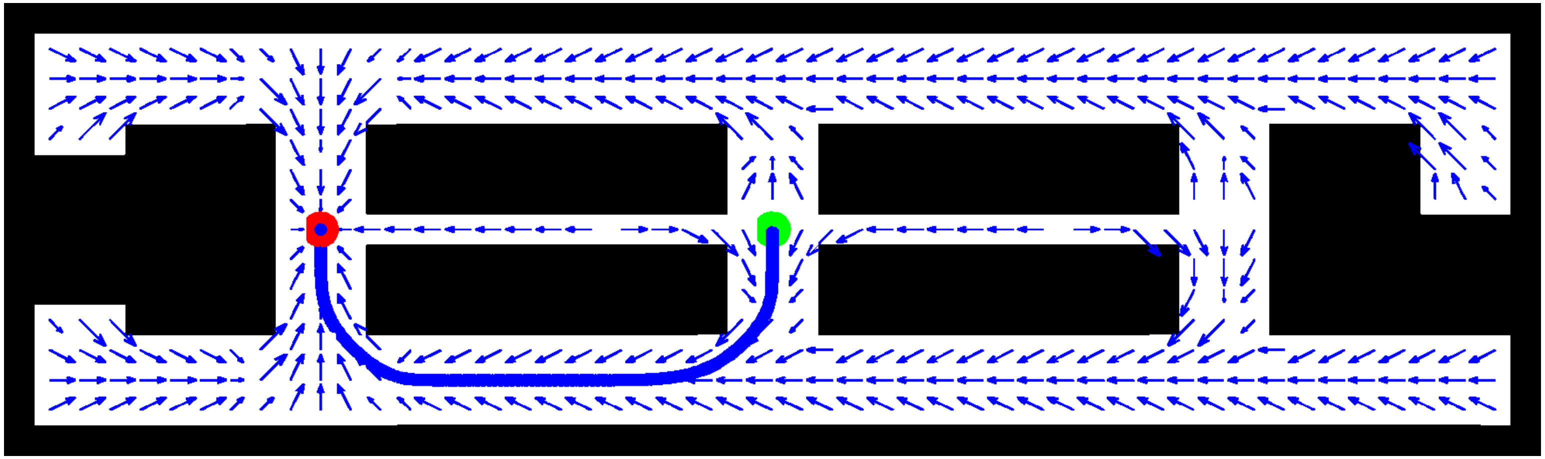


Uncertainty in Motion

without any uncertainty

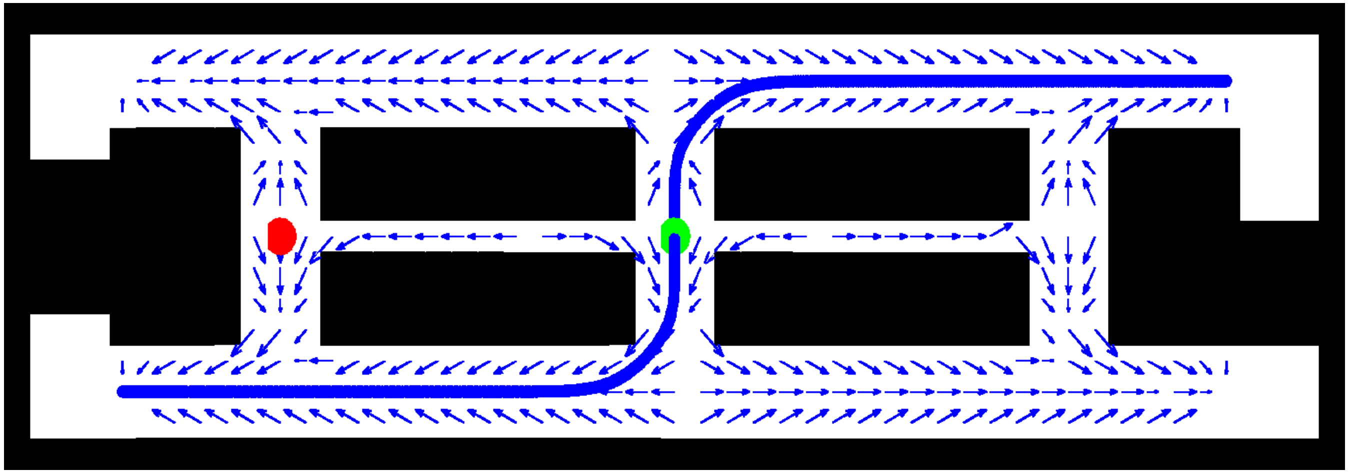


Given motion uncertainty, chose a path with larger space, avoiding the narrow corridor

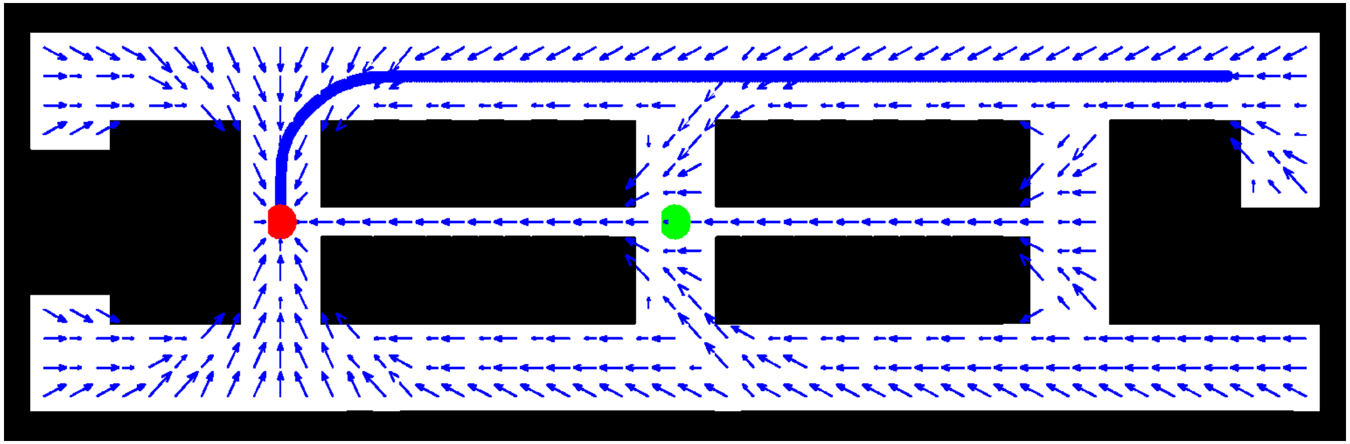


Uncertainty in Motion and Observation

1st Step:
Go to a state with less uncertainty in observation



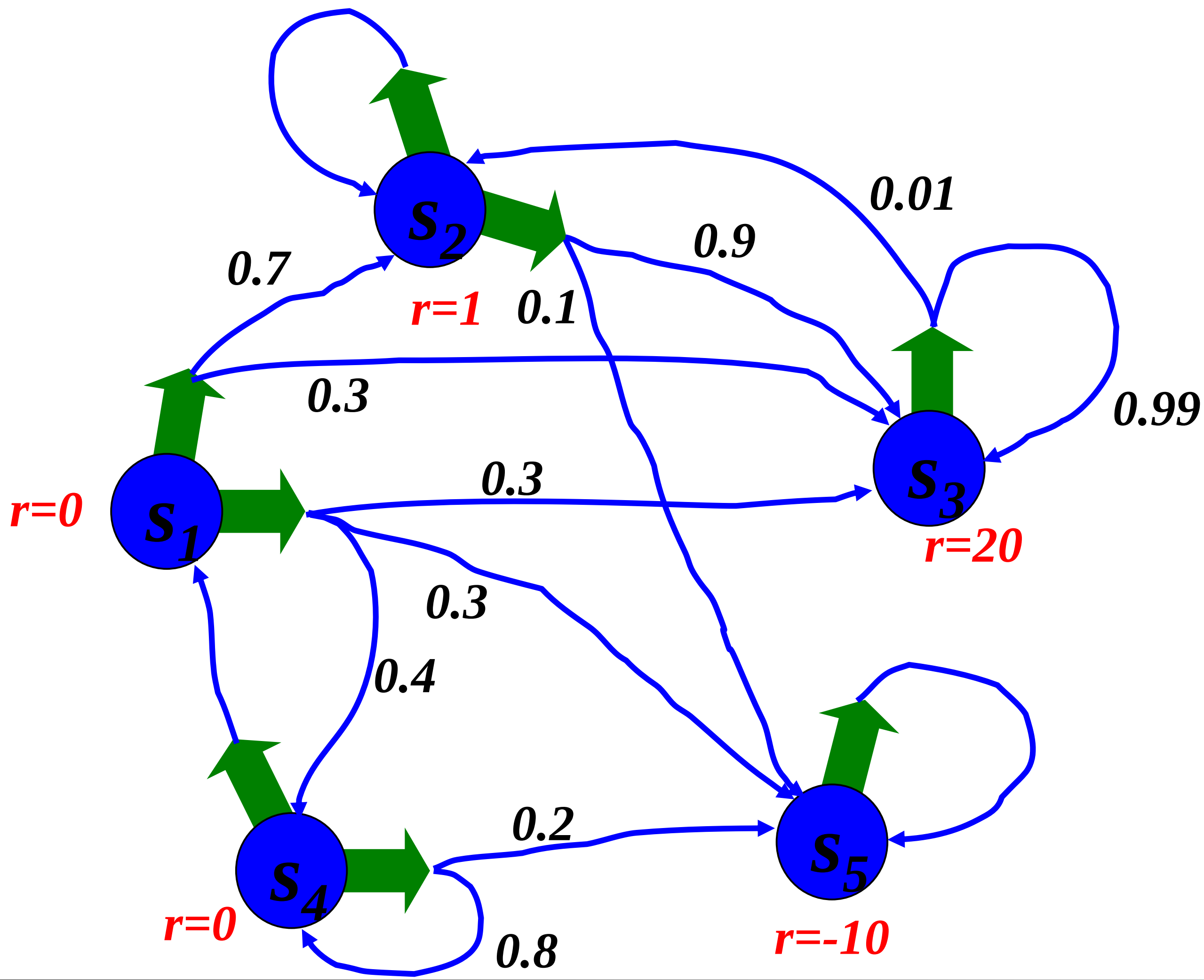
2nd Step:
Go to the goal along a path accommodating more uncertainty



Markov Decision Process

			RIGHT GOAL
	OBSTACLE		WRONG GOAL
START POSITION			

Markov Decision Process



Markov Decision Process Setup

- Given：

  States x, Actions u

  Transition probabilities $p(x'|u, x)$

  Reward function $r(x, u)$

- Wanted： $\pi: \; x_t \rightarrow u_t$

  Policy $\pi(x)$, mapping from states to actions, that maximizes the future expected reward

Summary I

- Bayes rule allows us to compute probabilities that are hard to assess otherwise.
- Under the Markov assumption, recursive Bayesian updating can be used to efficiently combine evidence.
- Bayes filters are a probabilistic tool for estimating the state of dynamic systems.

Summary II

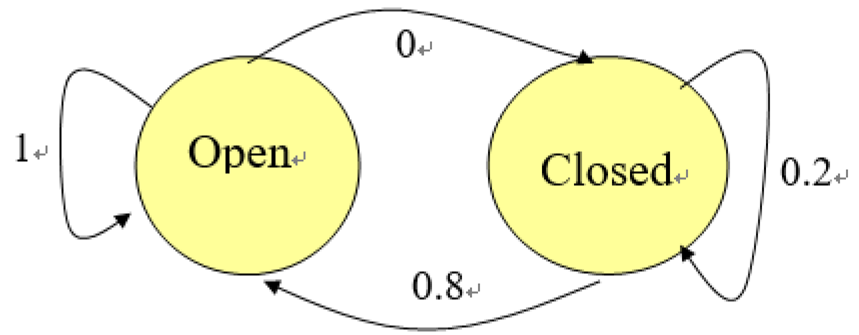
- Markov decision allows us to change the state with the probabilistic actions.

- Under the full model assumption, recursive Markovian decisions can be used to efficiently reach maximum rewards.

- Markov decision is a probabilistic tool for changing the state of dynamic systems.

Homework 1

Problem1： The state transition for the action “open the door” is as shown in Fig. 1. If the door is closed, the action “open the door” succeeds in 80% of all cases. Assume the probabilities of the closed door and open door are 50% respectively.



Calculate the probability of

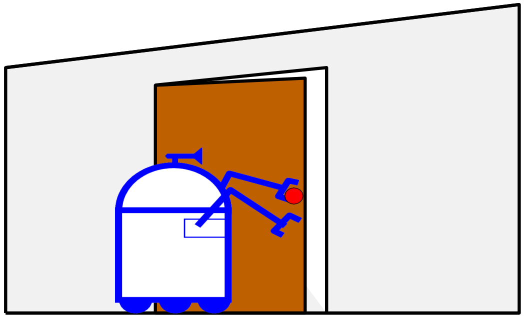
P(open|u) for *u* = “open door”:

Homework 1

Problem 2: A robot is going through a door, where the state of the door is x={open, closed}, the measurement of the door by the robot is z={open, closed}, and the action of the robot is u={push, do_nothing}. We assume that

(1) the robot doesn't know the state of the door initially;
(2) the measurement noise: p(z=open|x=open) = 0.8; p(z=closed|x=open) = 0.2;
(3) the measurement noise: p(z=open|x=closed) = 0.3; p(z=closed|x=closed) = 0.7;
(4) when the robot pushes the closed door, the chance to make it open is 0.9;
(5) when the robot pushes the open door, nothing will be changed;
(6) when the robot does nothing, the state of the door will not be changed;

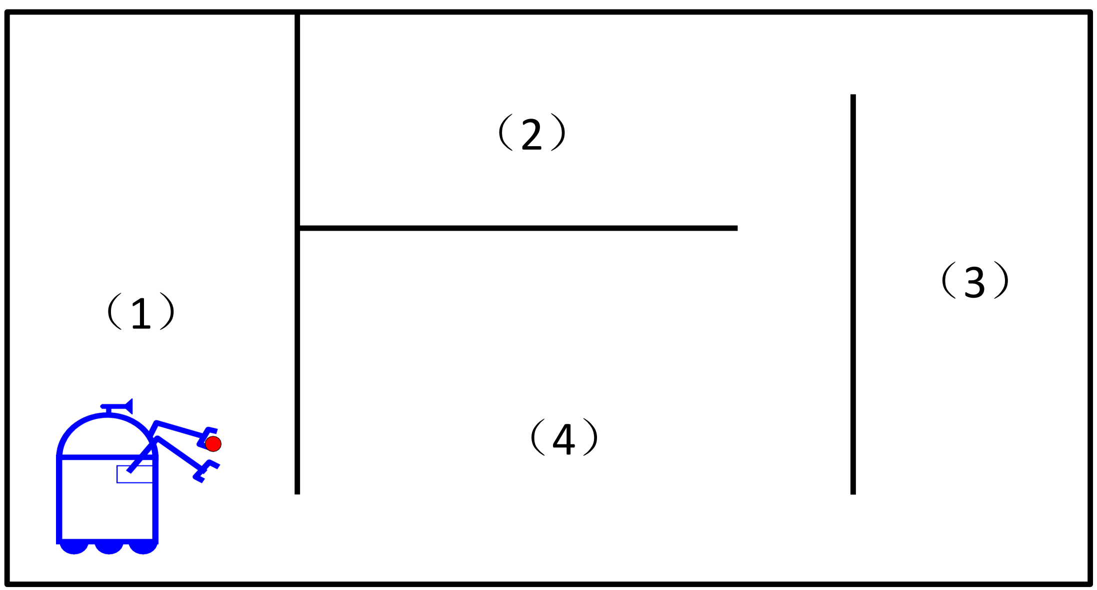
Then, what is the state distribution of the door, after the robot's measurements are {open, open}, and its actions are {do_nothing, push}?



Homework 1

Problem 3: A robot cleaner is roaming within an apartment with four rooms. The map of the apartment is given as follows. The probability of the robot going through each door is 0.1. Please answer the following questions:

(1) what is the Markov model for the robot roaming?
(2) what is the probability of the robot staying at each room?
(3) what is the probability of the robot going through the door between (1) and (4) when the robot is going through a door?



Homework 1

Problem 4

Given the observation model $p(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t}) = p(z_t \mid x_t)$ and the dynamics model $p(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$

Please derive how to estimate the belief over x_t, that is, $bel(x_t) = p(x_t \mid u_1, z_1, \ldots, u_t, z_t)$, with the terms of $bel(x_{t-1})$, observation and dynamics models.

Homework 1

Problem 5

Given the observation model $p(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t}) = p(z_t \mid x_t)$ and the dynamics model $p(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$

Please derive how to estimate the belief over $x_{0:t}$, that is, $bel(x_{0:t}) = p(x_{0:t} \mid u_1, z_1, \ldots, u_t, z_t)$, with the terms of $bel(x_{0:t-1})$, observation and dynamics models.