Stochastic Newton with Arbitrary Sampling

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Abstract

We analyse stochastic Newton-type methods for solving Empirical Risk Minimization problem. We prove fast local convergence rates independent of the condition number. Unlike most other stochastic variants of second order methods, which require the evaluation of a large number of gradients and/or Hessians in each iteration to guarantee convergence, the method do not have this shortcoming. We investigate the performance of the method by applying existing sampling strategies.

1 Introduction

$$\min_{x\in\mathbb{R}^d}\left[f\left(x\right):=\frac{1}{n}\sum_{i=1}^{n}f_i\left(x\right)\right]. \tag{1}$$

Here n is the number of data points that is typically extremely large in real problems; d is the number of model parameters. Ususally, f_i denotes the value of a loss function on i-th data point (a_i, b_i). One of the examples of the problem that has the form of (1) is Logistic Regression problem where

$$f_i(x) = \log\left(1+\exp(-b_i a_i^\top x)\right), \tag{2}$$

where $a_i \in \mathbb{R}^d$ and $b_i \in \{-1, 1\}$.

As n is large the problem (1) is typically solved by First-order methods that uses only one data point per iteration. These methods are extensively

studied [3] and there are a wide variety of variations of such techniques. In particular, the Stochastic Gradient Descent(SGD) is often used, the distinguishing feature of which is cheap iterations independent of n. Nevertheless, SGD with constant-stepsize has a number of disadvantages, the main of which is that it converges only up to the neighbourhood of the solution, not the exact solution. This problem arises since stochastic gradient estimator has non-zero variance. Radius of this convergence area is proportional to the variance of the stochastic gradient. The so-called variance-reduced methods [2, 8] are used to solve this problem. They have the same iteration cost as SGD, but now the algorithm converges to the exact solution. However, all first-order methods known to us are characterized by the dependence of the required number of iterations on the condition number[1]. This makes impossible using SGD and its variants for ill-conditioned problems.

In classic optimization one of the solutions is to use second-order information about the objective. Classic Newton's method adapts to the curvature of the problem and thereby decrease the dependence on the condition number. The step of Newton's method has the following form

$$x^{k+1} = x^k - \left(\nabla^2 f\left(x^k\right)\right)^{-1} \nabla f\left(x^k\right) \tag{3}$$

In the case of ERM we need to compute n Hessians per iteration which is extremely costly in practice. Our desire is to use only a few Hessians in each iteration. One of the most popular directions is so called Subsampled Stochastic Newton's methods [1]. Despite first-order methods, these methods are poorly understood, and the theory usually requires a large batch sizes. To the best of our knowledge there are just a few works that provable work with arbitrary batch sizes [5, 6, 7].

In this work we focus on the Algorith 1 of [4]. They presentes the following algorithm:

Algorithm 1 Stochastic Newton (SN)

Initialize: Choose starting iterates $w_1^0, w_2 a^0, \ldots, w_n^0 \in \mathbb{R}^d$ and minibatch size $\tau \in \{1, 2, \ldots, n\}$

for $k = 1, \ldots$ **do**

$$x^{k+1} = \left(\sum_{i=1}^{n} \nabla^2 f_i\left(w_i^k\right)\right)^{-1} \sum_{i=1}^{n} \left(\nabla^2 f_i\left(x^k\right) w_i^k - \nabla f_i\left(w_i^k\right)\right)$$

Choose a subset $k \subseteq \{1, \ldots, n\}$ of size τ uniformly at random

$$w_i^{t+1} = \begin{cases} w_i^t & i \notin S^t \\ x^{t+1} & i \in S^t \end{cases}$$

end for

[1]For a continuously Differentiable function f condition number is defined as $\lim\limits_{\varepsilon \to 0} \sup\limits_{\|\partial x\| \leq \varepsilon} \frac{\|\partial f(x)\|}{\|\partial x\|}$,

for L-smooth and μ-convex function the condition number is $\frac{L}{\mu}$

We investigate how the sampling strategies affect the performance of Algorithm (1). In practice, the uniform sampling is not the best choice, and we need to use another strategies how to choose a set S^t.

2 Problem Statement

Assume we have n training points (a_i, b_i) for $i \in \overline{1,n}$. We also assume n to be large. Let $f_i(x)$ be a loss function on i-th training point . We analyze second order methods solving Empirical Risk Minimization problem of the form.

One of the examples of the problem that has the form of (1) is Logistic Regression problem where

$$f_i(x) = \log\left(1 + \exp(-b_i a_i^\top x)\right), \tag{4}$$

where $a_i \in \mathbb{R}^d$ and $b_i \in \{-1, 1\}$.

$$\min_{x \in R} \left[f(x) := \frac{1}{n} \sum_{i=1}^{n} f_i(x) \right]. \tag{5}$$

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