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Сэмплирование по важности в оптимизации с вероятностными ограничениями

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Выпускная квалификационная работа Бакалавра

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Аннотация

Данная бакалаврская диссертация основана на статье «Importance Sampling Approach to Chance-Constrained DC Optimal Power Flow» [17] за авторством Вячеслава Горчакова, Александра Лукашевича и Юрия Максимова.

Моя бакалаврская работа посвящена анализу сэмплирования по важности в применении к решению задач оптимизации с вероятностыми ограничениями. Областью применения данного исследования являются уравнения энергосистем и, в частности, задача оптимального потока мощности переменного тока. Несмотря на значительные экономические и экологические эффекты, более высокий уровень производства возобновляемой энергии приводит к повышенной неопределенности и изменчивости при подаче мощности, тем самым ставя под угрозу надежность сети. Для улучшения бузопасности энергосистемы, мы исследуем "chance-constrained"(CC) аппроксимацию постоянного тока (DC) в контексте задачи оптимальной мощности (OPF) переменного тока (AC). Задача состоит в том, чтобы найти экономически оптимальное производство электроэнергии, гарантируя, что значения мощности генераций, линейных потоков и напряжений остаются в пределах своих допустимых границ одновременно с заранее заданной вероятностью. К сожалению, задача CC-DC-OPF является вычислительно сложной, даже если распределение возобновляемых источников энергии известно и зафиксированно для всей энергитической системы. Более того, существующие аппроксимирующие решения к проблеме нахождения общей мощности (OPF) в постановке "chance-constrained"оптимизационной задачи являются чрезмерно консервативными и требовательны к вычислительным ресурсам и, следовательно, имеют меньшую ценность для применения на практике. Эта работа посвящена применению подхода сэмплирования по важности для построения эффективной и надежной аппроксимации по сценариям для CC-DC-OPF. Получена теоретическая оценка на минимальное количество необходимых сэмплов, что дает лучшую оценку сложности сэмплов и их точность, чем современные "state-of-the-art"методы. Наши теоретические оценки также подтверждаются проведенные экспериментами на тестовых наборах данных. Алгоритм эффективно снижает количество необходимых сценариев за счет создания и использования только нескольких наиболее важных, что позволяет получать решениев режиме реального времени для тестовых энергосетей размеров до нескольких сотен точек генераций.

Abstract

This Bachelor thesis is focused on importance sampling analysis in the context of chance-constrained optimization problems. The application area of this study is power systems equations and the AC optimal power flow problem in particular. Despite significant economic and ecological effects, a higher level of renewable energy generation leads to increased uncertainty and variability in power injections, thus compromising grid reliability. To improve power grid security, we investigate a joint chance-constrained (CC) direct current (DC) approximation of the AC optimal power flow (OPF) problem. The problem aims to find economically optimal power generation while guaranteeing that all power generation, line flows, and voltages remain within their bounds at the same time with a pre-defined probability. Unfortunately, the CC-DC-OPF problem is computationally intractable even if the distribution of renewables' fluctua- tions is specified. Moreover, existing approximate solutions to the joint CC OPF problem are overly conservative and computationally demanding, and, therefore, have less value for the operational practice. This paper proposes an importance sampling approach for constructing an efficient and reliable scenario approximation for CC-DC-OPF with theoretical guarantees on the number of samples required, which yields better sample complexity and accuracy than current state-of-the-art methods. The algorithm efficiently reduces the number of scenarios by generating and using only a few most important, thus enabling real-time solu- tions for test cases with up to several hundred buses.

Contents

1 Introduction

1.1 Related work

In 2020 electricity produced approximately 25% of greenhouse gas emissions in the USA, and integration of a higher volume of renewable energy generation is seen as the primary tool to reduce the emission level [10]. In turn, a higher amount of renewable generation increases the power grid uncertainty, compromises its security, and challenges classical power grid operation and planning policies [13].

The optimal power flow (OPF) problem, which determines the economically optimal operating level of power generation under given power balance equations and security constraints, is one of the most fundamental problems in grid operation and planning [26]. Several extensions are proposed for the optimal power flow problem for addressing the uncertainty of power generation, and consumption [8]. Robust and chance-constrained power flow formulations are among the most popular ones. The robust OPF problem assumes bounded uncertainty and requires a solution to be feasible against any possible uncertainty realization within a given uncertainty set [1, 6].

A more flexible and general chance-constrained approach requires security constraints to be satisfied a high probability while assuming the distribution of renewables is known in whole or in part [15, 24]. This paper considers a *joint chance-constrained* optimal power flow problem, where the joint probability of at least one failure of the security constraints (line load limits, voltage stability bounds) is bounded from above by a confidence threshold (JCC-OPF).

In contrast to a *single chance-constrained* formulation (SCC-OPF), which imposes individual failure probability thresholds for each of the constrains, the joint chance constraint is computationally hard even for (linear) direct current (DC) power balance equations, linear security limits and Gaussian uncertainty [12]. Several tractable convex approximations have been proposed [21, 22] for joint chance-constrained optimization to overcome the computational hardness of the problem; however, they often lead to conservative solutions inapplicable for operational practice. Because the equations representing alternating current (AC) are substantially more difficult to deal with, and there is little to gain from the complexity of an

AC model, it is typical to utilize a simpler linear direct current (DC) model of the electrical grid. It's also usual to characterize the grid's randomness as Gaussian, particularly for short time horizons.

Other approaches such as scenario and sample average approximations [3, 7, 21] consist of substituting the stochastic part with a set of deterministic inequalities based on the uncertainty realization. This approach can be distributionally robust and allow exploiting uncertainties beyond the Gaussian ones. A combination of an analytical approximation and sampling [11] can further improve the accuracy of the solution. However it may require a large number of samples. Scenario curation/modification heuristics have been designed to improve the sample complexity of JCC-OPF [19], although without formal analysis of the methodology. In work outside power grids, statistical learning has been used to approximate uncertain convex programs [4, 29].

Nevertheless, the scenario approximation approach remains the most accurate algorithm for solving the joint chance-constrained DC optimal power flow. At the same time, its complexity is often unacceptable for large-scale power grids [25]. To this end, the paper suggests using importance sampling to reduce the complexity and improve the accuracy of the scenario approximation to chance-constrained optimal power flow.

The importance sampling approach generates more informative samples and results in an optimization problem with fewer constraints. Importance sampling is a valuable alternative to MonteCarlo sampling, which allows adjusting distribution for generating more samples in the areas of interest, e.g., close to the reliability boundary. Pmvnorm is one of the most efficient importance sampling algorithms in general, but its performance is often limited for rare events probability estimation, which is of the utmost importance to power systems study.

ALOE is another efficient method designed especially for computing a rare event probability. However, it does not fully respect the geometry of reliability constraints. It thus requires a large number of samples to estimate the risk of constraint overflow, especially for large power grids and multi-line overloads. One may fail to obtain any points where the rare event occurs, which is a common way for rare event sampling to be erroneous. As a result, the likelihood of a rare event is grossly underestimated. The failure to sample any sites where two or more of the rare constituent events occur is the comparable difficulty in ALOE. In that situation, instead of zero, ALOE will report the union bound as the projected rare event probability. In this case, the union bound is also likely to be a decent approximation. As a result, ALOE is resistant to extreme underestimation of the chance of a rare event. The second typical issue with rare event sampling is that the likelihood ratio weighting applied to the observations has an extreme value. The biggest conceivable weight in ALOE is only J times that of the smallest.

Finally, a convex optimization-based algorithm for adaptive importance sampling from exponential families is also a competitive approach. At each step, the algorithm adjusts the distribution parameters so that the sampler's variance is minimized. Unfortunately, the distribution of output power that leads to an overload is far from the exponential family, limiting the algorithm's efficiency in power systems.

1.2 Contributions summary

The contribution of the paper is as follows.

1. we propose a novel computationally efficient approach to the joint chance-constrained DC OPF problem. The algorithm exploits physics-informed importance sampling to refine the classical scenario approximation [3];

2. we prove the algorithm to converge to a solution of JCC-DC-OPF with a guaranteed accuracy under mild technical assumptions;

3. we demonstrate the proposed algorithm's superior computational efficiency and accuracy relative to standard methods over multiple real and synthetic test cases.

2 Background and Problem Setup

2.1 Notation

The direct current (DC) power flow approximation remains extremely popular yet simple for analysis because of a linear relation between powers and phase angles in typical high-voltage power grids.

In the following, we consider a power grid given by a graph $= (V, E)$ with a set of nodes/buses V and edges/lines E. Assume n is a number of buses, and m is a number of lines. Let p be a vector of power injections $p = (p_F, p_R, p_S)^\top$, where p_F corresponds to buses with deterministic/fixed (F) power injections, p_R to buses with random (R) injections, and p_S is the injection at the slack bus (S). The power system is balanced, i.e. the sum of all power injections equals to zero, $p_S + \sum_{r\in R} p_r + \sum_{f\in F} p_f = 0$. Let θ be a vector of phase angles. Without a loss of generalization, We assume that the phase angle on the reference slack bus $\theta_S = 0$. Let $B \in \mathbb{R}^{N\times N}$ be an admittance matrix of the system, $p = B\theta$. The components B_{ij} are such that $B_{ij} \neq 0$ if there is a line between nodes i and j, the diagonal elements $B_{ii} = -\sum_{j\neq i} B_{ij}$, i.e. B is a Laplacian matrix. Let $B^\dagger$ be the pseudo-inverse of B, i.e. $\theta = B^\dagger p$.

The DC power flow equations, generation and stability constraints are

$$p = B\theta \tag{1}$$

$$p_i^{\min} \leq p_i \leq p_i^{\max}, i \in V \tag{2}$$

$$|\theta_i - \theta_j| \leq \bar{\theta}_{ij},\ (i,j) \in E \tag{3}$$

Let $A \in \{-1, 0, 1\}^{m\times n}$ be the incidence matrix of grid G, i.e. if nodes i and j are connected by edge k then $A_{ki} = +1$, $A_{jk} = -1$ and all other elements are equal to zero. Then the phase angle constraints in (3) can be represented as $AB^\dagger p \leq \bar{\theta}$, $-AB^\dagger p \leq \bar{\theta}$.

The following system of inequalities defines the DC OPF operational constraints

$$\underbrace{(AB^\dagger, -AB^\dagger, I, -I)^\top}_{W} p \leq \underbrace{(\bar{\theta}, \bar{\theta}, p_{\max}, -p_{\min})^\top}_{b},$$

where I is $n \times n$ identity matrix. Let J be a number of constraints,

E	set of lines	$\nu(p)$	nominal distribution PDF
V	set of buses	$\mathcal{V}(p)$	nominal distribution CDF
B	admittance matrix	$\nu(p, x)$	parametric distribution CDF
m	number of lines	$\mathcal{V}(p, x)$	parametric distribution CDF
n	number of buses	$p^{\max}$	generation upper limits
p	power injections	$p^{\min}$	generation lower limits
θ	voltage phases	$\mathbb{P}(\cdot)$, $\mathbb{E}(\cdot)$	probability, expectation
I_n	$n \times n$ identity matrix	ξ	vector of stochastic
J	number of constraints		power fluctuations
$\bar{\theta}_{ij}$	voltage angle limits	$\mathcal{N}(\mu, \Sigma)$	Gaussian distribution
ξ	power injection unce-	$\mathcal{P}$	feasibility polytope
	rtainty, $p \sim (0, \Sigma)$		$\mathcal{P} = \{p : \bigcap_{i \leq J} \omega_i^\top p \leq b_i\}$
p	power injections	x	deterministic part of
	$p = x + \xi$		power injections, $x = \mathbb{E}_\xi p$

Table 1: Paper notation.

$J = 2m + 2n$, then $W \in \mathbb{R}^{J \times n}$ and $b \in \mathbb{R}^{n \times 1}$. We refer a feasibility polytope as a set $\mathcal{P}$, $\mathcal{P} = \big\{p : Wp \leq b\big\} = \big\{p : \bigcap_{i=1}^{J} \omega_i^\top p \leq b_i\big\}$.

Finally, we assume that fluctuations of power injections p are Gaussian, $p = x + \xi$, where ξ is a Gaussian uncertainty, $\xi \sim (0, \Sigma)$ and x is the deterministic part of power injections.

The paper notation is summarized in Table 1. We use lower indices for coordinates of vectors and matrices, lower-case letters for probability density functions (PDFs), and upper-case letters for cumulative distribution functions (CDFs). When it does not lead to confusion, we use $\mathbb{P}$ and $\mathbb{E}$ to denote probability and expectation without explicitly mentioning a distribution.

2.2 Problem Setup

The joint chance-constrained optimal power flow problem is:

$$\min_x \mathbb{E}_{\xi \sim (0,\Sigma)} \operatorname{cost}(x, \xi) \tag{4}$$
$$\mathbb{P}_{\xi \sim (0,\Sigma)}(x + \xi \in \mathcal{P}) \geq 1 - \eta, 0 < \eta \leq 1/2,$$

where η is a preset confidence parameter, and cost is a cost function convex in x for any realization of ξ. In other words, we assume that power flow balance equations (Eqs. (1)) are satisfied almost surely, and the probability that at least one of the security constraints (Eqs. (2) and (3)) fails is at most η.

Notice that despite the convexity of the cost function in problem (4) is non-convex as its feasibility set is non-convex for a sufficiently high level of uncertainty.

2.3 Scenario Approach

Over the last two decades, the scenario approach [3, 21] remains the state-of-the-art method for solving joint chance-constrained optimization. The scenario approach consists in substituting the probabilistic constraints with a larger number of deterministic ones with each constraint standing for some uncertainty realization:

$$\begin{aligned} \min_x \ & \frac{1}{N}\sum_{t=1}^{N} \mathrm{cost}(x, \xi_t) && (5)\\ & p_t^{\min} \le x + \xi^t \le p_t^{\max},\ 1 \le t \le N \\ & |\theta_i(\xi^t) - \theta_j(\xi^t)| \le \bar{\theta}_{ij}, (i,j) \in,\ 1 \le t \le N \\ & x + \xi^t = B\theta(\xi^t),\ 1 \le t \le N \end{aligned}$$

where N is a number of scenarios, and $\{\xi^t\}_{i=1}^N$ is a series of uncertainty realization. We assume below that the generation cost, $\mathrm{cost}(x, \xi)$, does not depend on the randomness in power fluctuation ($\mathrm{cost}(x, \xi) = \mathrm{cost}(x)$ for any uncertainty realization ξ), but may depend on the uncertainty distribution (i.e., on its mean or variance).

The key disadvantage of the scenario approximation (5) is the number of constraints induced by adding scenarios ξ^t. A classical theory, developed by Calafiore and Campi [3], suggests to include a large number of scenarios N. This will allow the solution of (5) to be feasible for original problem (4) with high probability $1 - \beta$. However, the number of required scenarios N grows as $O\left(\frac{1}{\eta}\left(\log\frac{1}{\beta} + \log\frac{1}{\eta}\right)\right)$. For example, with $\eta = 10^{-2}, \beta = 10^{-2}$, the number of samples required would be $N \approx 10^3$. In particular, for

IEEE-54 case one would end up with $3.34 \cdot 10^3$ constraints which is time- and memory-consuming to solve.

The major contribution of this paper is a significant reduction of the requirement on the number of samples by sampling the most informative scenarios. The latter reduces the computational complexity of the scenario approximation and comes up with tight accuracy guarantees for the joint chance-constrained DC optimal power flow problem.

3 Algorithm

3.1 Idea and Sketch

Our algorithm consists of several steps: (1) constructing an inner approximation to the feasibility set, (2) generating samples outside of the approximation, (3) and, finally, solving the scenario approximation Problem (5) with a new collection of samples.

First, using the fact that *the probability of a union of events is bounded from below by the maximum of individual event probabilities*, we construct a lower bound on the probability of constraint feasibility $\mathbb{P}(x + \xi \in \mathcal{P})$. The latter allows to add a set of constraints, $x \in \mathcal{P}_m$, so that if $x \notin \mathcal{P}_m$ then $\mathbb{P}(x + \xi \notin \mathcal{P}) > \eta$. In other words, if the solution $\bar{x}$ of the scenario approximation (5) with samples from the nominal distribution $(0, \Sigma)$ satisfies $\mathbb{P}(x + \xi \in \mathcal{P}) \geq 1 - \eta$, then adding additional inequalities $x \in \mathcal{P}_m$ does not change $\bar{x}$. Second, using the aforementioned bound, we design a polytope $\mathcal{P}_{in} = \{p : W_{in}\xi \leq b_{in}, p = x + \xi\}$ around x with derived W_{in} and b_{in} independent of x. Figure 1 illustrates the idea.

Then, we show that for any sample ξ^t and feasible x, if $p = x + \xi^t \in \mathcal{P}_{in}$, then p also necessarily belongs to the constraint feasibility set $\mathcal{P}$. To this end scenarios $\xi^t : x + \xi^t \in \mathcal{P}_{in}$ can be eliminated from the optimization problem (see Eq. (5)) without impacting the approximation accuracy.

Finally, we sample scenarios outside of the polytope $\mathcal{P}_{in}$ using the state-of-the-art importance sampling methods [16, 23] and solve the Scenario Approximation Problem (5) with the collection of samples generated from importance distribution. Later in this section, we provide rigorous proof of

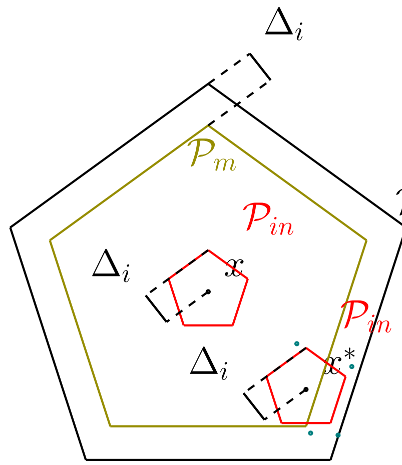

Figure 1: We consider a deterministic feasibility set $\mathcal{P}$ and derive a polytope $\mathcal{P}_m$ inside of it so that no optimal solution x^* of Prob. (4) belongs to $\mathcal{P} \setminus \mathcal{P}_m$. The polytope $\mathcal{P}_{in}$, defined in terms of noise ξ only, depicts the distances between the planes of $\mathcal{P}$ and $\mathcal{P}_m$. Samples of uncertainty outside $\mathcal{P}_{in}$ are used to determine optimal solution using importance sampling.

the algorithm's efficiency and justify its empirical performance in Section 4.

3.2 Inner Approximation

Consider a probability for the power generation p of being inside the feasibility polytope, $p \in \mathcal{P}$:

$$\mathbb{P}(p \in \mathcal{P}) = \mathbb{P}(p : Wp \leq b) =$$
$$\mathbb{P}\left(p : \bigcap_{i=1}^{J} \omega_i^\top p \leq b_i\right) = 1 - \mathbb{P}\left(p : \bigcup_{i=1}^{J} \omega_i^\top p > b_i\right) \leq$$
$$1 - \max_{1 \leq i \leq J} \mathbb{P}\left(p : \omega_i^\top p > b_i\right).$$

So, if there exists x such that for some i, $\mathbb{P}(\omega_i^\top p > b_i) > \eta$, then $\mathbb{P}(p \in \mathcal{P}) < 1 - \eta$. Thus, to satisfy the joint chance constraint for $p = x + \xi$,

$\xi \sim (0, \Sigma)$, we need

$$\begin{aligned}\eta \ &\geq \mathbb{P}(\omega_i^\top p > b_i) = \mathbb{P}(\omega_i^\top \xi + \omega_i^\top x > b_i) = \\ &\mathbb{P}\left(\frac{\omega_i^\top \xi}{\|\Sigma^{1/2}\omega\|_2} > \frac{b_i - \omega_i^\top x}{\|\Sigma^{1/2}\omega\|_2}\right) = \\ &\mathbb{P}\left(\zeta > \frac{b_i - \omega_i^\top x}{\|\Sigma^{1/2}\omega\|_2}\right) = \Phi\left(\frac{\omega_i^\top x - b_i}{\|\Sigma^{1/2}\omega\|_2}\right),\end{aligned} \tag{6}$$

where $\zeta \sim (0, 1)$, Φ is a CDF of the standard normal distribution. Notice, that the function $\Phi(\cdot)$ is convex as soon as its argument is negative, i.e., Ineq. (6) is convex as soon as $x \in \mathcal{P}$.

A set of inequalities $\eta \geq \Phi\left((\omega_i^\top x - b_i)/(\|\Sigma^{1/2}\omega_i\|_2)\right)$, $1 \leq i \leq J$, defines a polytope $\mathcal{P}_m$ as follows:

$$\mathcal{P}_m = \left\{x : \omega_i^\top x \leq b_i - \Delta_i\right\}, \tag{7}$$

where $\Delta_i = \|\Sigma^{1/2}\omega_i\|_2 \Phi^{-1}(1-\eta)$, $\eta \leq 1/2$. Figure 1 illustrates mutual arrangement of the polytopes $\mathcal{P}_m \subset \mathcal{P}$. Note that $\mathcal{P}_m$ defines an outer approximation of the non-convex chance-constrained feasibility set, which is itself an inner approximation of the constraint feasiblity set without any uncertainty. Eq. (7) implies Theorem 3.1.

Theorem 3.1. *The joint chance-constrained optimal power flow problem* (4) *has the same set of optimal solutions X as*

$$\begin{aligned}&\min_x \mathbb{E}_{\xi\sim(0,\Sigma)} \operatorname{cost}(x, \xi) \\ &\quad \mathbb{P}_{\xi\sim(0,\Sigma)}(x + \xi \in \mathcal{P}) \geq 1 - \eta, 0 < \eta \leq 1/2, \\ &\quad x \in \mathcal{P}_m\end{aligned} \tag{8}$$

Proof. A set of additional equations $x \in \mathcal{P}_m$ does not affect the solution of Problem (4) as the feasiblity set of the chance-constrained optimization problem is inside $\mathcal{P}_m$ as the probability for a scenario to be outside of the polytope $\mathcal{P}$ is lower bounded by the probability of being outside of its single face. The latter determines polytope $\mathcal{P}_m$, see. Eq. 7 for details. □

3.3 Redundant Scenarios

Another practical consequence of the fact that the optimal solution of the chance-constrained optimal power flow problem is well separated from the boundary of polytope $\mathcal{P}$ is that some scenarios may be removed as they do not improve the accuracy of scenario approximation (5).

Let the optimal solution $\bar{x}$ of the problem (5) be feasible for the chance-constrained OPF problem. Then by Theorem 3.1, it necessarily also belongs to $\mathcal{P}_m$. Theorem 3.2 provides a mathematical justification of the latter.

Theorem 3.2. *Any solution of the Scenario approximation problem* (5) *that is feasible for the Chance-constrained OPF problem* (4) *is also a solution of*

$$\min_x \ \text{cost}(x)$$

$$p_t^{\min} \leq x + \xi^t \leq p_t^{\max},\ 1 \leq t \leq N \tag{9a}$$

$$|\theta_i(\xi^t) - \theta_j(\xi^t)| \leq \bar{\theta}_{ij}, (i,j) \in,\ 1 \leq t \leq N \tag{9b}$$

$$x + \xi^t = B\theta(\xi^t),\ 1 \leq t \leq N \tag{9c}$$

$$x \in \mathcal{P}_m \tag{9d}$$

Proof. Let $\mathcal{P}_{in}(x) = \{p = x + \xi : \omega_i^\top \xi \leq \Delta_i, \forall i \leq J\}$ with $\Delta_i = \|\Sigma^{1/2}\omega_i\|\Phi^{-1}(1-\eta)$. Notice, that

$$\mathcal{P}_{in}(x) = \mathcal{P}_{in}(x_0) + (x - x_0).$$

Thus for any $x \in \mathcal{P}_m$, if $\xi^t \in \mathcal{P}_{in}(0)$ one immediately gets $p = x + \xi^t \in \mathcal{P}$. In other words, one can exclude scenario $\xi \in \mathcal{P}_{in}(0)$ from Problem (5) as soon as $p = x + \xi \in \mathcal{P}$. □

Figure 1 illustrates the idea and the geometry of $\mathcal{P}, \mathcal{P}_m$, and $\mathcal{P}_{in}$.

The following Theorem 3.4 follows from the result of Calafiore and Campi [3] and establishes approximation properties of a solution of the Problem (9). Assumption 3.3 is the main technical assumption used in the proof of Theorem 3.4.

Assumption 3.3. *Assume that for all possible uncertainty realizations* $\xi^1, \ldots, \xi^N$*, the optimization problem* (9) *is either infeasible or, if feasible, it attains a unique optimal solution.*

Theorem 3.4. *Let $\bar{x}_N$ be a unique solution of the Scenario optimization Problem* (9) *with N i.i.d. samples, so that none of the samples belong to $\mathcal{P}_{in}$. Moreover, assume that for any N the assumption 3.3 is fulfilled. Then for any $\delta \in (0,1)$ and any $\eta \in (0, 1/2]$, $\bar{x}_N$ is also a solution for the Chance-constrained optimal power flow Problem* (4) *with probability at least $1-\delta$ if*

$$N \geq \left\lceil 2\frac{(1-\pi)\ln\frac{1}{\delta}}{\eta} + 2d + 2d(1-\pi)\frac{\ln\frac{2(1-\pi)}{\eta}}{\eta} \right\rceil,$$

where d is a dimension of the space of controllable generators, and π is a probability of a random scenario ξ to belong to $\mathcal{P}_{in}$, and $\pi < 1$.

Proof. First, notice that discarding scenarios $\xi^t \sim (0, \Sigma)$ is equivalent in solving the scenario approximation problem with $\xi \sim D \Leftrightarrow \xi \sim (0,\Sigma)\xi \notin \mathcal{P}_{in}$.

According to the result of Calafiori and Campi [3], for any probability $\delta \in (0,1)$ and any confidence threshold probability ε, and dimension of the space of parameters d one has, for N_1

$$N_1 \geq \left\lceil \frac{2}{\varepsilon}\ln\frac{1}{\delta} + 2d + \frac{2d}{\varepsilon}\ln\frac{2}{\varepsilon} \right\rceil \tag{10}$$

scenarios from D and the optimal solution $\bar{x}$ of the Problem (9), the probability of failure is bounded as

$$\mathbb{P}_D(p \notin \mathcal{P}) \leq \varepsilon$$

with probability at least $1-\delta$.

Notice, that the bounds on the number of samples (see Eq. (10)) is strictly decreasing in ε for $\varepsilon \in (0,1)$. As scenarios in $\mathcal{P}_{in}$ do not cause failure, to get a probability of failure η according to measure $(0,\Sigma)$, we need the failure probability according to D to be at least $\varepsilon = \eta/(1-\pi)$. Thus, taking $\varepsilon = \eta/(1-\pi)$ and using monotonicity of Ineq. (10) one gets the statement of the theorem. $\square$

Theorem 3.4 establishes the number of scenarios sufficient for the scenario approximation solution being feasible for the chance-constrained optimal power flow problem. This number significantly decreases if one can come up with a sufficiently restrictive inner approximation of the feasibility set, that is close to the true chance-constrained set. Notice, that without an inner approximation, i.e. for $\pi = 0$, one gets the result of Calafiore and Campi [3, Theorem 1].

3.4 Importance Sampling

Although scenario optimization with scenarios that do not belong to the polytope $\mathcal{P}_{in}$ obey a nice complexity bound, it requires on average $1/(1-\pi)$ or more samples to generate at least one point outside of $\mathcal{P}_{in}$ and decreases the overall efficiency of the approach. The problem is especially challenging when dealing with rare events, i.e., the confidence level $\eta \to 0$.

Importance sampling is a general technique that helps to improve the efficiency of scenario generation [28]. It consists of changing the probability distribution to sample rare events with a higher probability. Figure 2 illustrates the concept.

$$x\phi(x)\psi(x)f(x)$$

Figure 2: Our goal is to sample points in the area $\{x : f(x) = 1\}$. Sampling from the nominal distribution $\phi(x)$ is inefficient as one needs to sample from the distribution tail. The probability distribution $\psi(x)$ is a better choice as the probability of getting $f(x) = 1$ is substantially higher when sampling from it.

Unfortunately, there is no exact and time-efficient algorithm for sampling outside of a convex polytope from Gaussian distribution [12]; however, the ALOE algorithm [16, 23] proposes an elegant way for approximating the distribution of interest by sampling from a mixture of distributions.

We consider Gaussian fluctuations of power injections, $\xi \sim (0, \Sigma)$, with known covariance $\Sigma \in \mathbb{R}^{n\times n}$ and aim to sample scenarios outside of $\mathcal{P}_{in}$ so that the probability distribution to sample from is as close as possible to the conditional Gaussian distribution $\xi \sim (0, \Sigma)\xi \notin \mathcal{P}_{in}$.

The method essentially samples from a weighted mixture of conditional Gaussian distributions D_i

$$\xi \sim D_i \Longleftrightarrow \xi \sim (0, \Sigma)\omega_i^\top \xi > \Delta_i.$$

Consider the set of inequalities $\{\omega_i^\top \xi > \Delta_i\}_{i \le J}$ in more detail. First, let $\zeta \sim (0, I_n)$ then the system is equivalent to $\{(\Sigma^{1/2}\omega_i)^\top \zeta > \Delta_i\}_{i \le J}$.

Distribution D_i can be simulated exactly using the inverse transform method [14, 20] that admits conditional sampling $\xi \sim (0, \Sigma)$ s.t. $\omega_i^\top \xi \ge \Delta_i$.

1. Sample $z \sim (0, I)$ and sample $u \sim U(0, 1)$
2. Compute $y = \Phi^{-1}(\Phi(-\Delta_i) + u(1 - \Phi(-\Delta_i)))$
3. Set $\phi = \phi y + (I - \phi\phi^\top) z$, $\phi = \Sigma^{1/2}\omega_i / \|\Sigma^{1/2}\omega_i\|_2$
4. Set $\xi = \Sigma^{1/2}\phi$.

In [16, 23], the authors proposed a slightly refined version of the algorithm above that exhibits better numerical stability. We refer to the same papers for the corresponding proofs and analysis. Figure 3 illustrates the idea.

Finally, ALOE proposes to sample scenarios from a weighted mixture

$$D = \sum_{i=1}^{J} \alpha_i D_i,\ \alpha_i \ge 0, \sum_{i=1}^{J} \alpha_i = 1,$$
$$\alpha_i \propto \Phi(-\Delta_i / \|\Sigma^{1/2}\omega_i\|_2), \tag{11}$$

where Φ is a CDF of the standard normal distribution. Let $q_D(\xi)$ be a PDF of distribution D, then Theorem 3.5 established a maximal ratio of the conditional Gaussian density $\xi \sim (0, \Sigma) \xi \notin \mathcal{P}_{in}$ and $q_D(\xi)$.

Theorem 3.5. *Let $p(\xi)$, $q_D(\xi)$ be PDFs of $\xi \sim (0, \Sigma)\xi \notin \mathcal{P}_{in}$, and a mixture density (see Eq. (11)) resp. Then for any $\xi \notin \mathcal{P}_{in}$, we have*

$$p(\xi) \le M q_D(\xi), M = \frac{\sum_{i \le J} \Phi(-\Delta_i / \|\Sigma^{1/2}\omega_i\|_2)}{\max_{i \le J} \Phi(-\Delta_i / \|\Sigma^{1/2}\omega_i\|_2)}, \tag{12}$$

where D and α_i are given in Eq. (11).

Proof. Let $\phi(\xi)$ be PDF of $\xi \sim (0, \Sigma)$. Notice, that the conditional densities D_i have probability density functions

$$q_{D_i}(\xi) = \begin{cases} \phi(\xi)/\Phi(-\Delta_i/\|\Sigma^{1/2}\omega_i\|_2), & \omega_i^\top \xi > \Delta_i \\ 0, & \text{otherwise} \end{cases}$$

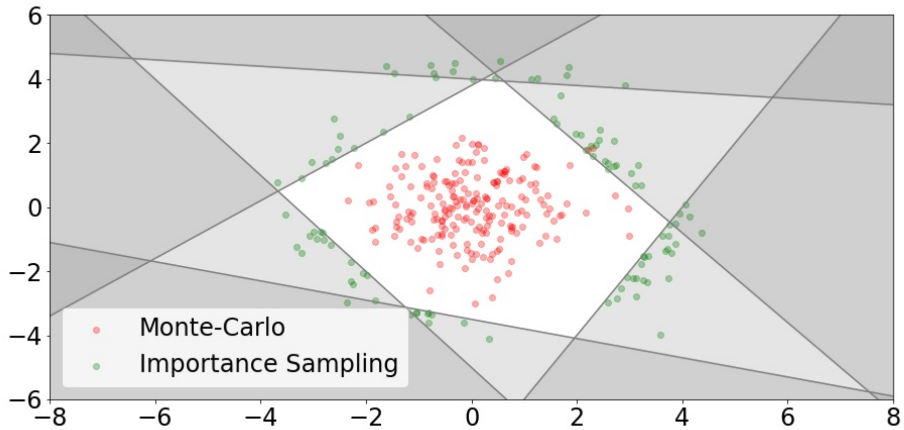

Figure 3: The white area stands for generations that do not exceed operating limits. Power generations that lead to at least one constraint violation are in grey. Two or more reliability constraints are not satisfied in the dark grey area. We mark samples from the nominal distribution and the constructed mixture in red and green, respectively.

Thus the density of distribution D is $\sum_{i \le J} \alpha_i q_{D_i}(\xi)$. Similarly, density $p(\xi)$ outside of the polytope $\mathcal{P}_{in}$ is $\phi(\xi)/\mathbb{P}_{\xi \sim (0,\Sigma)}(\xi \notin \mathcal{P}_{in})$ which is less or equal then $\phi(\xi)/\max_{i \le J} \Phi(-\Delta_i/\|\Sigma^{1/2}\omega_i\|_2)$ for any $\xi \notin \mathcal{P}_{in}$.

Finally, taking the ratio of $q_D(\xi)$ and $p(\xi)$ and using the value of α_is, we get the lemma statement. □

Theorem 3.5 implies that if a probability of a set w.r.t. measure q_D is less or equal then ε, then the probability of the same set w.r.t. measure p does not exceed $M\varepsilon$.

3.5 Scenario Approximation with Importance Sampling

In this section, we present a scenario approximation for the chance-constrained optimal power flow with a set of scenarios generated by the ALOE algorithm [23]. A particular advantage of this approach is that every scenario is generated outside of $\mathcal{P}_{in}$. The latter substantially improves the accuracy and efficiency of the scenario approximation. In particular, we solve the following optimization problem instead of Problem (5):

$$\min_x \ \mathrm{cost}(x)$$

$$p_t^{\min} \le x + \xi^t \le p_t^{\max},\ 1 \le t \le N \tag{13a}$$

$$|\theta_i(\xi^t) - \theta_j(\xi^t)| \le \bar{\theta}_{ij}, (i,j) \in,\ 1 \le t \le N \tag{13b}$$

$$x + \xi^t = B\theta(\xi^t),\ 1 \le t \le N \tag{13c}$$

$$x \in \mathcal{P}_m \tag{13d}$$

$$\xi^1, \xi^2, \ldots, \xi^N \sim D, \tag{13e}$$

where D is the probability distribution defined by Eq. (11).

Notice, that sampling from distribution D allows to efficiently generate scenarios outside of the polytope $\mathcal{P}_{in}$; however, they follow distribution D instead of $\xi \sim (0, \Sigma)\xi \notin \mathcal{P}_{in}$. As these distributions are close to each other, Theorem 3.6 establishes efficient complexity bounds for the scenario approximation with importance sampling.

Theorem 3.6. *Let $\bar{x}_N$ be a unique solution of the Scenario optimization Problem* (13) *with N i.i.d. samples follow distribution D. Moreover, assume that for any N the assumption 3.3 is fulfilled. Then for any $\delta \in (0,1)$ and any $\eta \in (0, 1/2]$, $\bar{x}_N$ is also a solution for the Chance-constrained optimal power flow Problem* (4) *with probability at least* $1 - \delta$ *if*

$$N \ge \left\lceil 2M \frac{(1-\pi)\ln\frac{1}{\delta}}{\eta} + 2d + 2dM(1-\pi)\frac{\ln\frac{2M(1-\pi)}{\eta}}{\eta} \right\rceil,$$

where d is a dimension of the problem and π is a probability of a random scenario ξ to belong to $\mathcal{P}_{in}$, $\pi < 1$, and constant M is defined by Theorem 3.5.

Proof. The proof is similar to the one of Theorem 3.4. Application of theorem 3.5 allows to upper-bound the probability of an event in measure D with respect to its probability in measure $(0, \Sigma)\xi \notin \mathcal{P}_{in}$. □

4 Empirical Study

We compare our importance sampling-based approach (referred to as SA-IS) with the classical scenario approximation (SA) [3] over real and simulated test cases. Empirical results justify our theoretical results: *importance sampling based CC-OPF required much fewer samples to achieve a highly reliable solution than classical scenario approximation.*.

We limit the empirical setting to considering Gaussian distributions and linear feasibility constraints only. We omit detailed comparison with other importance sampling strategies [2, 9, 16] when generating scenarios because of the paper space limitation and for the sake of empirical study clarity. A profound discussion on the efficiency of importance samplers is given in [16].

4.1 Implementation details

We use Python 3.9. and pandapower 2.8.0 [27] on MacBook Pro (M1 Max, 64 GB RAM). In the experiments, the computational time for each case does not exceed five minutes, which makes it applicable for the operational practice. Our code is available online on Github [1]. When solving the optimization problem, we use CVX [5] and GLPK [18] optimization solvers.

4.2 Test Cases and Numerical Results

Synthetic Example For elucidation of the theory, we first study the efficiency of importance sampling based scenarios over a one dimensional test case:

$$\max x$$
$$\mathbb{P}_\xi(x + \xi \leq a) \geq 1 - \eta, \xi \sim (0, 1)$$

for $0 < \eta < 1/2$ and a positive constant a. In this case, the chance-constrained optimization problem admits an exact solution, $x^* = a - \Phi^{-1}(1 - \eta)$.

[1] https://github.com/vjugor1/IS-SA

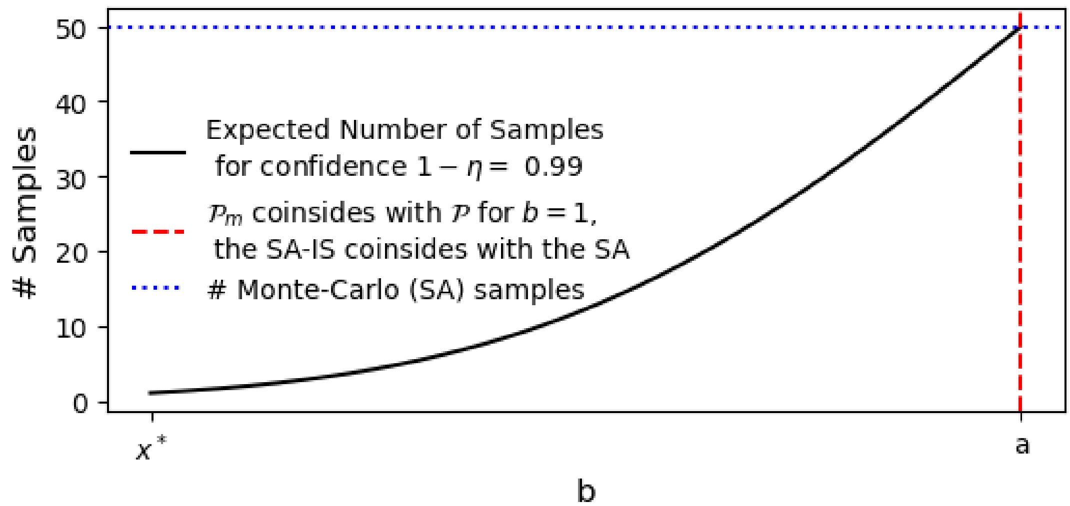

Figure 4: Feasibility of the scenario approximation with importance sampling depending on the size of $\mathcal{P}_m = \{x : x \leq b\}$, where $a \geq b \geq x^* = a - \Phi^{-1}(1-\eta)$. The more accurate approximation $\mathcal{P}_m$ is, the smaller is the number of samples required by the SA-IS algorithm.

The polytopes $\mathcal{P}_m$ and $\mathcal{P}_{in}$ are $\{x : x \leq x^*\}$ and $\{\xi : \xi \leq \Phi^{-1}(1-\eta)\}$ respectively. To illustrate the role of an approximation $\mathcal{P}_m$, $x^* \in \mathcal{P}_m \in \mathcal{P}$, we consider different polytopes $\mathcal{P}_m^b = \{x : x \leq b\}$ and corresponding polytope $\mathcal{P}_{in}^b = \{\xi : \xi \leq a - b\}$, $a - b \leq \Phi^{-1}(1-\eta)$.

Figure 4 illustrates that the efficiency of sampling improves (i.e. less number of samples needed) as the polytope $\mathcal{P}_m$ better approximates $\mathcal{P}_{in}(0)$. Indeed, the probability of a sample from the nominal distribution outside of $\mathcal{P}_{in}^b$ being also outside $\mathcal{P}_{in}$ is proportional to $\Phi(a-b)$, which becomes negligible as $a - b$ decays and leads to a high number of samples. Note, that we reach the standard scenario approximation when $b = a$ and get the best possible approximation for $a - b = \Phi^{-1}(1-\eta)$.

Our approach crucially relies on a non-conservative/tight approximation of the joint chance-constrained feasible set is crucial for the success of the importance sampling approach, as discussed next for power grid test cases.

Power grid test cases We address the scenario-based chance-constrained optimal power flow problem (CC-OPF) under Gaussian fluctuations in four different test cases (IEEE 30, IEEE 57, IEEE 118 and IEEE 300 bus systems with 30, 57, 118 and 300 buses/nodes, respectively). For all considered

Case	η	SA No	SA Cost	IS-SA No	IS-SA Cost	DC-OPF Cost
grid30	0.05	160	5.89e+03	**60**	5.87e+03	5.67e+03
grid57	0.05	210	2.52e+04	**160**	2.52e+04	2.50e+04
grid118	0.05	1300	8.72e+04	**1050**	8.72e+04	8.48e+04
grid300	0.05	1550	4.72e+05	**1250**	4.72e+05	4.71e+05
grid30	0.01	800	5.94e+03	**300**	5.96e+03	5.67e+03
grid57	0.01	1300	2.52e+04	**300**	2.53e+04	2.50e+04
grid118	0.01	6000	8.74e+04	**3600**	8.74e+04	8.48e+04
grid300	0.01	9000	4.72e+05	**4500**	4.72e+05	4.71e+05

Table 2: Number of SA-IS and SA samples required in CC-OPF with confidence threshold $1-\eta$ to ensure desired empirical reliability level $1-\hat{\delta} = .99$. The confidence threshold $1-\eta$ is estimated using Monte Carlo samples (out of sample) and empirical reliability is computed by averaging over $L = 100$ independent CC-OPF problem instances as stated in Algorithm 1. The costs of solutions obtained are depicted alongside with the corresponding costs of deterministic DC-OPF solution. It is clear that SA-IS requires much less samples compared to SA, while maintaning the same cost of solution.

cases, we assume the power generation and consumption level fluctuate with the standard deviation of 0.07 of its nominal value.

To demonstrate the practical benefits of importance sampling (SA-IS) versus standard scenario approximation (SA), we compare their respective number of samples N needed to solve CC-OPF for different test cases, given prescribed η, δ. As stated in theorems in Section 3, $1-\eta$ is the required *confidence threshold* for constraint feasibility (of Joint Chance Constraint) by a solution, while $1-\delta$ is the required reliability of the Scenario Approximation's solution obtained.

We consider the setting of fixed confidence threshold $1-\eta$ and N, the number of samples used in CC-OPF (with SA-IS or SA), and determine their effect on empirical reliability $1-\hat{\delta}$ using Algorithm 1. In Algorithm 1, we independently form $L = 100$ different scenario approximation problems, each with N scenarios constructed using SA or SA-IS. We thus obtain L different outcomes $(x_N^*)_l,\ l = 1, \ldots, L$. Using 10^4 Monte Carlo samples of uncertainty for each $(x_N^*)_l$, we then estimate each solution's probability of constraint satisfaction $(\hat{\mathbb{P}}_N)_l$. The estimated reliability $1 - \hat{\delta}$ is then given by the fraction of L $(x_N^*)_l$ solutions with $(\hat{\mathbb{P}}_N)_l \geq 1 - \eta$.

Algorithm 1 $\hat{\delta}$ – an empirical estimate

L – number of trials, DC-OPF problem parameters, η – confidence level, N_0 – initial size of scenario approximation, $N_{\max}$ – maximal size of scenario approximation $N \leftarrow N_0$ $\hat{\boldsymbol{\delta}}$ – storage for $\hat{\delta}_N$ $N \leq N_{\max}$ $C_N \leftarrow 0$ – feasibility counter $l \leftarrow 1$ $l \leq L$ Obtain $(x_N^*)_l$ – scenario approximation with N samples (using SA-IS or SA) Compute constraint satisfaction probability $(\hat{\mathbb{P}}_N)_l$ using Monte Carlo samples. $(\hat{\mathbb{P}}_N)_l \geq 1-\eta$ $C_N \leftarrow C_N+1$ $1 - \hat{\delta}_N \leftarrow C_N/L$ – fraction of trials turned out to be feasible Append $\hat{\delta}_N$ to $\hat{\boldsymbol{\delta}}$ $n \leftarrow n + N_{\max}/10$ $\hat{\boldsymbol{\delta}}$

Table 2 summarizes the number of samples needed using SA and SA-IS to ensure an empirical reliability of 0.99, for two confidence thresholds $1-\eta$ (.95 and .99). Our experiments show that SA-IS requires much fewer samples to provide a reliable CC-OPF feasible solution while maintaining the same cost for each test case compared to CC-OPF with SA. The improvement in the number of scenarios is bigger for the higher confidence threshold value $(1 - \eta = .99)$.

We illustrate the dependence between empirical reliability $1 - \hat{\delta}$ and N over a range of values for the IEEE 118 bus system in Fig. 7. Here, we keep a confidence threshold of joint chance constraint feasibility $1 - \eta = 0.99$. In addition to SA-IS and SA, we also consider an intermediate setting -Scenario Approximation with Polygon-Set (SA-O), as described in Eq. (9).

Compared to SA, SA-O includes the inner approximation constraints $x \in \mathcal{P}_m$. However, unlike SA-IS, SA-O does not involve importance sampling-based samples. It is clear from Fig. 7 that at all values of N, SA-IS's reliability $1 - \hat{\delta}$ is much higher than that of SA or SA-O. In fact, for $40 \leq N \leq 80$, the SA-IS has more than twice the reliability of SA. As a result, the solution of SA-IS also becomes conservative faster (highlighted by the decrease in slope at higher reliability). On the other hand, SA and SA-O have almost similar reliability, which follows from Theorem 3.2.

Finally, for the same L optimization instances used for Fig. 7, we present box-plots for the spread of $(\hat{\mathbb{P}}_N)_l$ for different N, in Fig. 8. Here, $(\hat{\mathbb{P}}_N)_l$ is

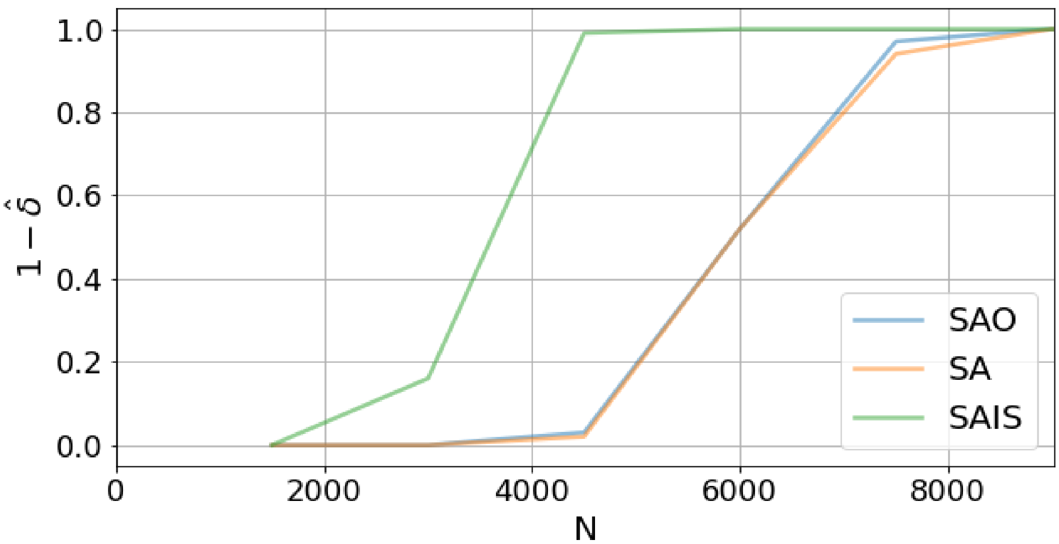

Figure 5: Empirical reliability $(1 - \hat{\delta})$ versus number of samples in CC-OPF (N) for IEEE 300 bus system.

the probability of constraint satisfaction, empirically computed using 10^4 Monte Carlo samples of Algorithm 1, see Eq. 1 . Note that SA-IS reaches a higher reliability level $(1 - \hat{\delta})$ at a fewer number of samples N, observed when almost all of the box is above the $1 - \eta$ threshold. The boxplots also indicate that the variance in the obtained solution's chance-constraint feasibility reduces faster for SA-IS, noted by thinner boxplots and lack of outliers, compared to SA and SA-O.

5 Conclusion

In this paper, we investigated the scenario approximation for the chance-constrained optimal power flow. We showed that the importance sampling technique used for scenario generation leads to better accuracy and time complexity in theory and practice, namely, for stochastic OPF. It was addressed both theoretically and practically.

The theoretical study indicates benefit from using different samples are presented and proven alongside with numerical experiments that indicate significant reduction of sample size in scenario approximation required to reach high reliability level. Finally, the approach can be extended to

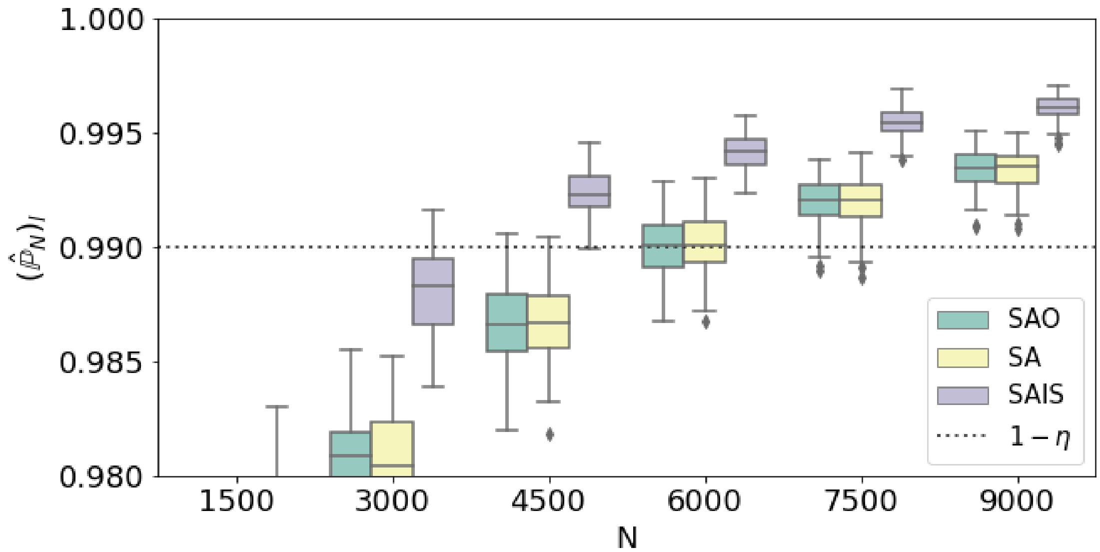

Figure 6: Spread of probability of constraint feasibility ($(\hat{\mathbb{P}}_N)_l$) versus number of samples in CC-OPF (N) for IEEE 300 bus system.

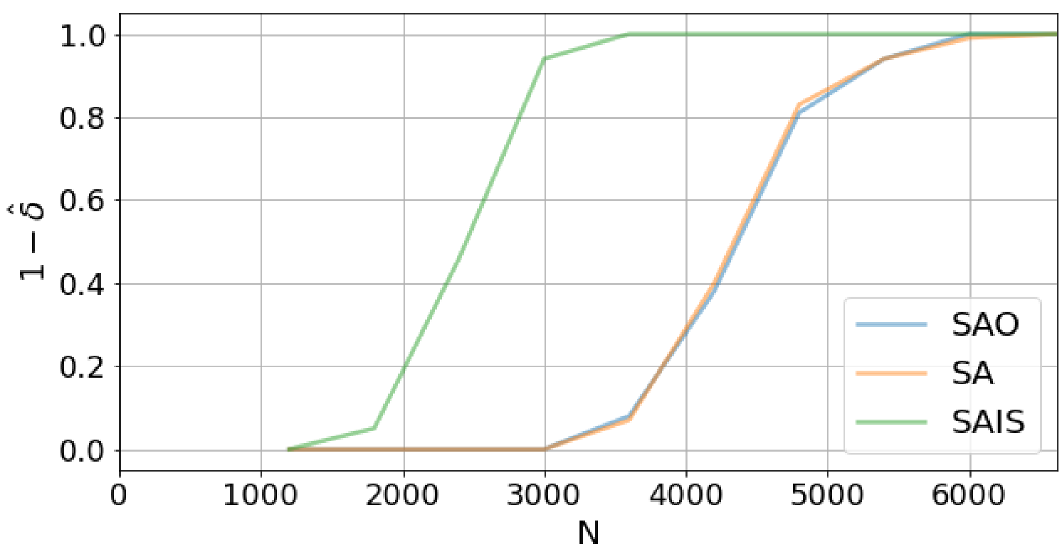

Figure 7: Empirical reliability ($1 - \hat{\delta}$) versus number of samples in CC-OPF (N) for IEEE 118 bus system.

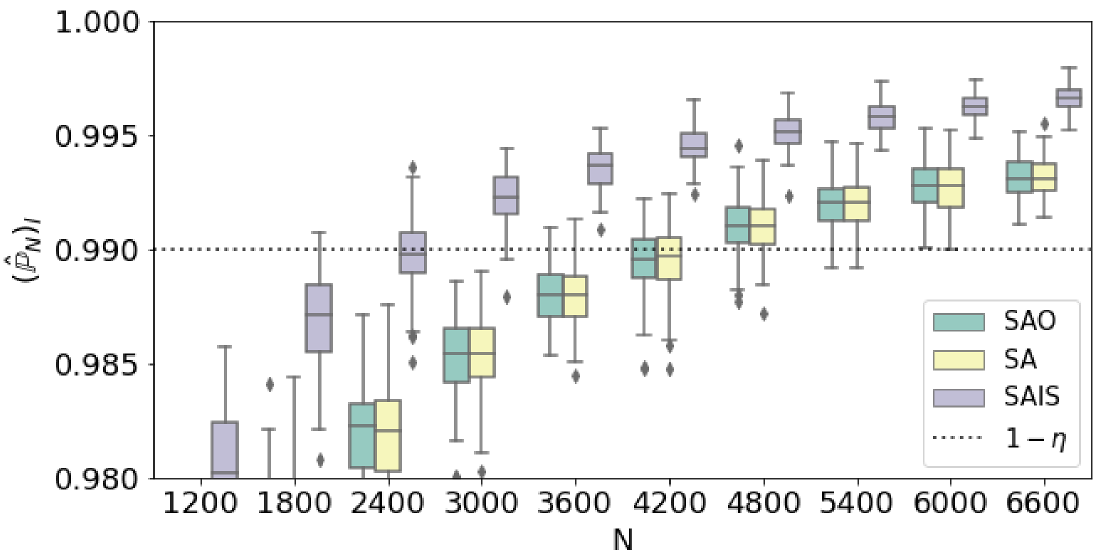

Figure 8: Spread of probability of constraint feasibility ($(\hat{\mathbb{P}}_N)_l$) versus number of samples in CC-OPF (N) for IEEE 118 bus system.

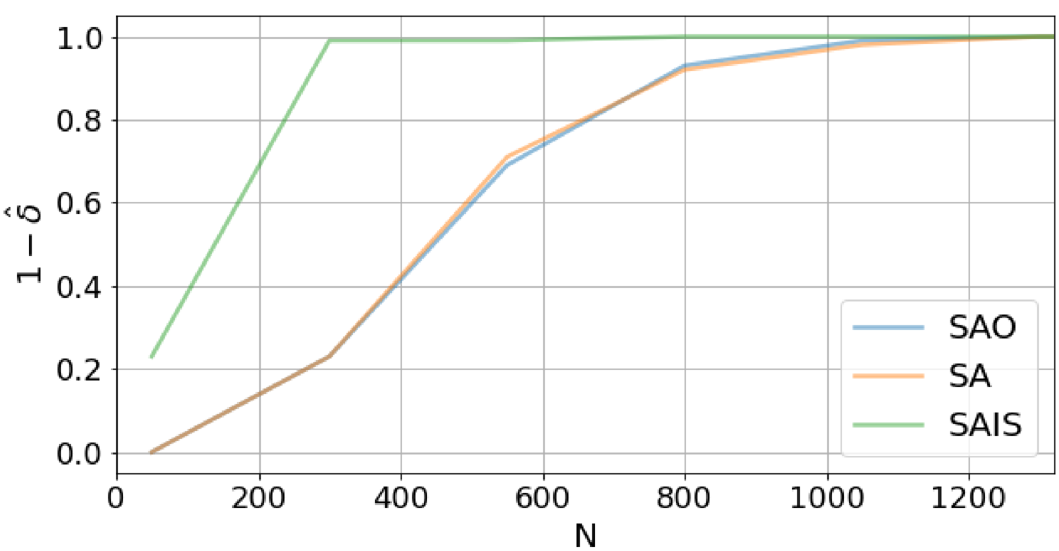

Figure 9: Empirical reliability $(1 - \hat{\delta})$ versus number of samples in CC-OPF (N) for IEEE 57 bus system.

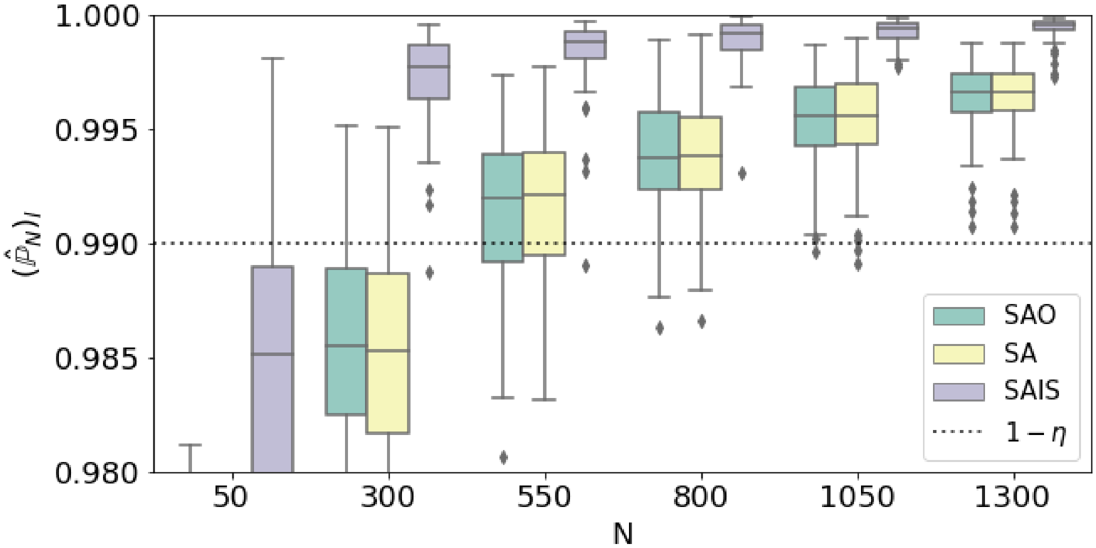

Figure 10: Spread of probability of constraint feasibility $((\hat{\mathbb{P}}_N)_l)$ versus number of samples in CC-OPF (N) for IEEE 57 bus system.

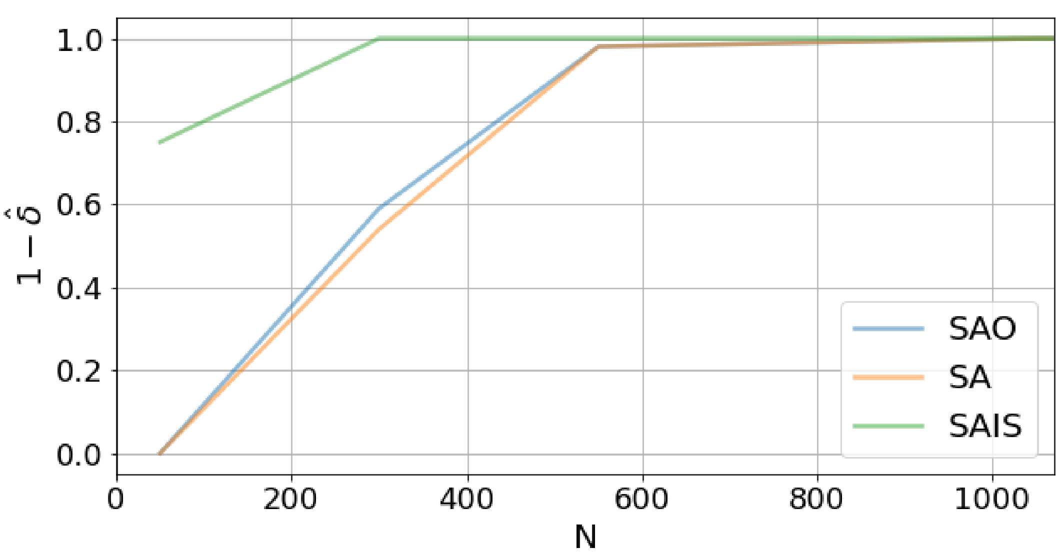

Figure 11: Empirical reliability $(1 - \hat{\delta})$ versus number of samples in CC-OPF (N) for IEEE 30 bus system.

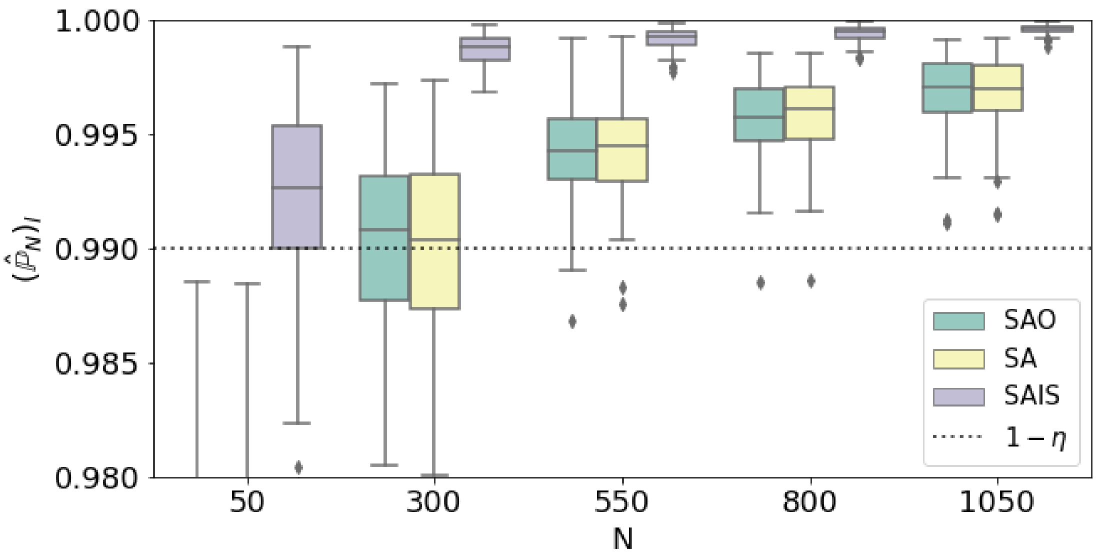

Figure 12: Spread of probability of constraint feasibility ($(\hat{\mathbb{P}}_N)_l$) versus number of samples in CC-OPF (N) for IEEE 30 bus system.

automated real-time control of bulk power systems.

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