Basis Matters: Better Communication-Efficient Second Order Methods for Federated Learning*

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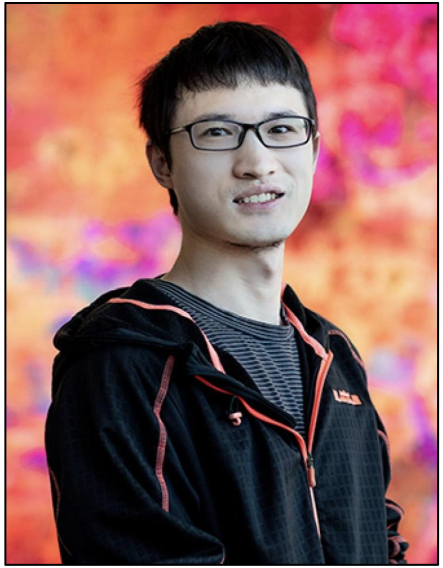
Co-Authors



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The Problem

$$\min_{x\in\mathbb{R}^d}\left\{f(x) := \frac{1}{n}\sum_{i=1}^{n} f_i(x)\right\}$$

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#model parameters

The Problem

machines/devices

$$\min_{x\in\mathbb{R}^d} \left\{ f(x) := \frac{1}{n}\sum_{i=1}^{n} f_i(x) \right\}$$

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The Problem

μ-strongly convex

\# machines/devices

$$\min_{x\in\mathbb{R}^d} \left\{ f(x) := \frac{1}{n}\sum_{i=1}^{n} f_i(x) \right\}$$

\#model parameters

empirical loss/risk

The Problem

μ-strongly convex

machines/devices

has Lipschitz Hessian

$$\min_{x \in \mathbb{R}^d} \left\{ f(x) := \frac{1}{n} \sum_{i=1}^{n} f_i(x) \right\}$$

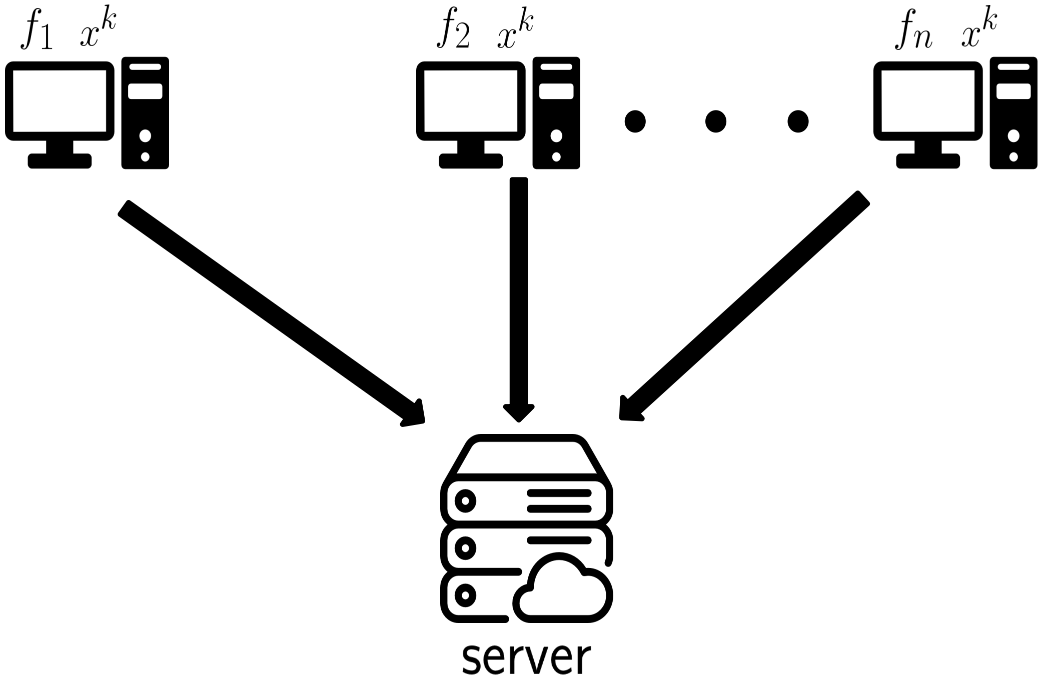
#model parameters

empirical loss/risk

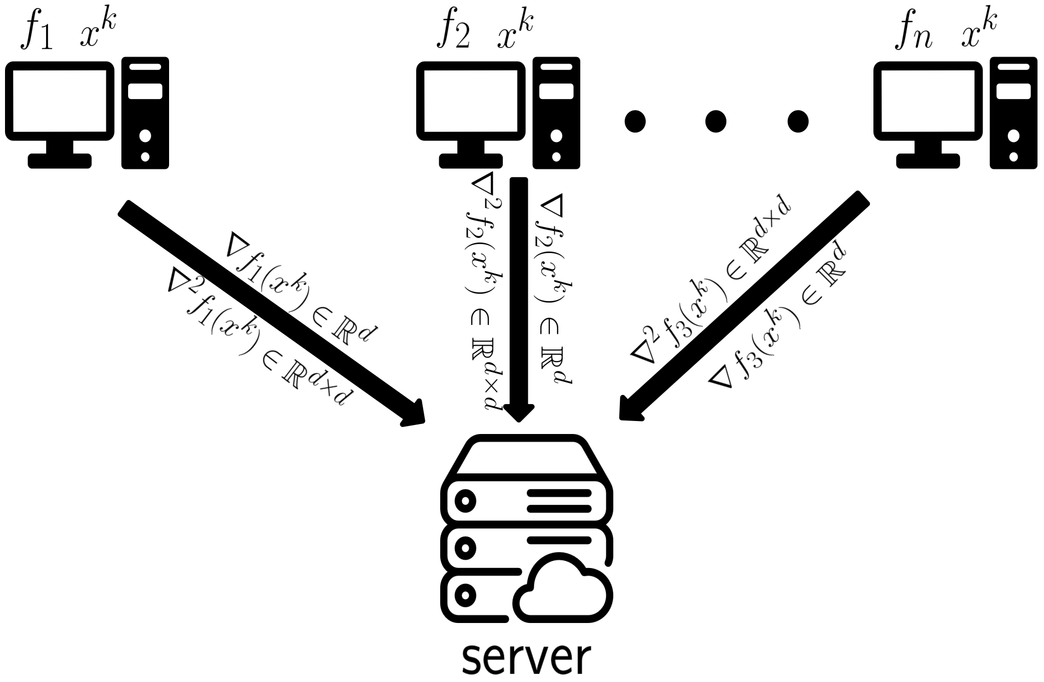
local training data

local loss function $f_i(x) = \mathbb{E}_{\xi \sim \mathcal{D}_i}[f_\xi(x)]$

Centralized Setting



Centralized Setting



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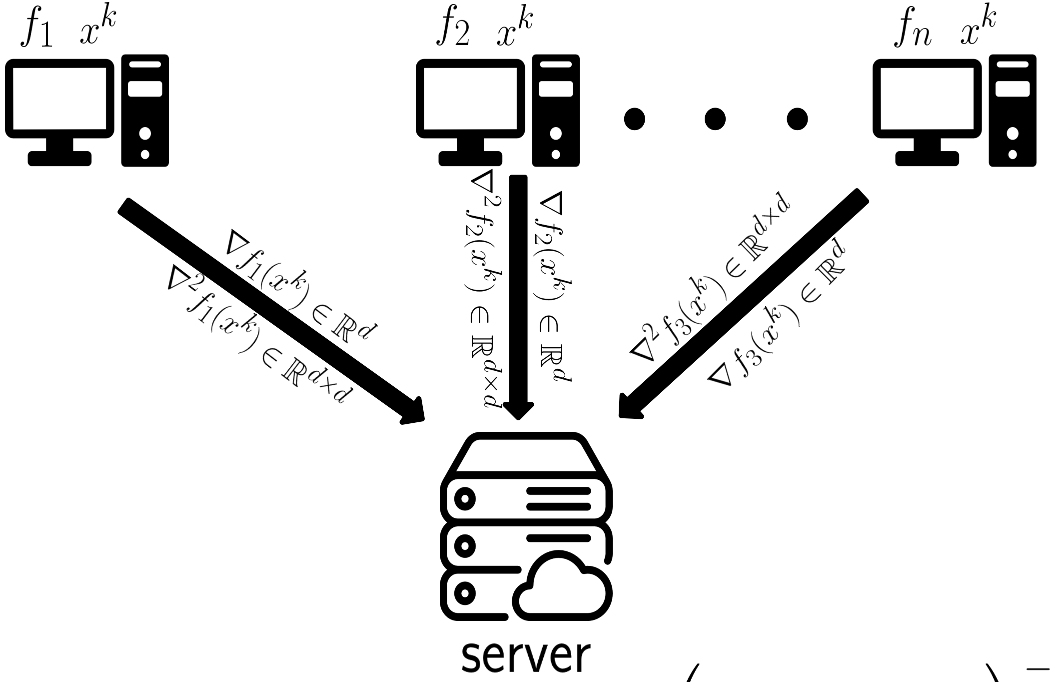
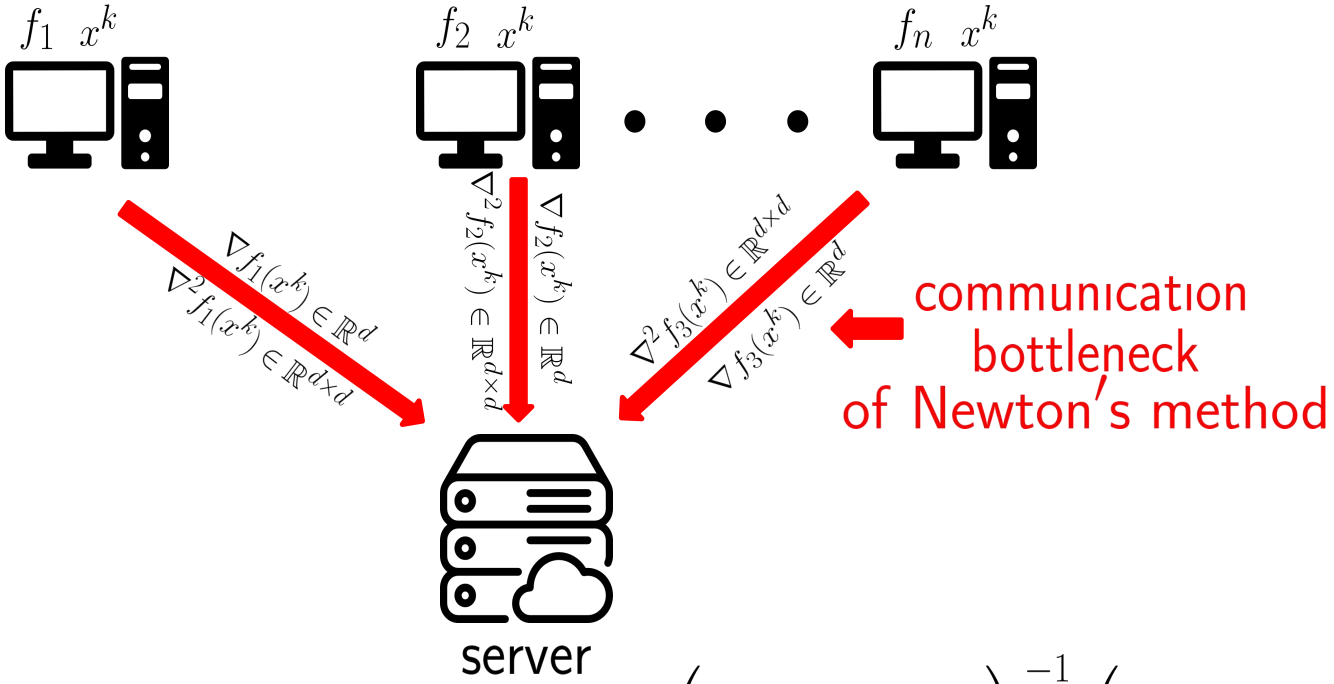


$$x^k \Rightarrow x^{k+1} = x^k - \left(\frac{1}{n}\sum_{i=1}^{n} \nabla^2 f_i(x^k)\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} \nabla f_i(x^k)\right)$$

Centralized Setting



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Existing approaches and their disadvantages

First order methods

- ❌ Rates depend on the condition number
- ❌ Hard to find optimal stepsizes

Second order methods

- ❌ Rates depend on the condition number
- ❌ Communication cost is high

Existing approaches and their disadvantages

First order methods

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Second order methods

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GOAL

Develop a communication-efficient distributed Newton-type method whose (local) convergence rate is independent of the condition number

- ✓ Can provably benefit from communication compression
- ✓ Rate independent of the condition number
- ✓ Supports bidirectional compression
- ✗ Good rate for local convergence only

Related Works

Table 1: Theoretical comparison of 7 second order methods (including ours). Advantages are written in green, while limitations are colored in red.

Method	Problem	Assumptions	CC[1]	Rate	Comments
GIANT [Wang et al., 2018]	GLM[3]	LipC[2] Hessian, convex $+ l_2$ reg., $\approx$ i.i.d. data	$\mathcal{O}(d)$	Local κ-dependent linear. Global $\mathcal{O}(\log \kappa/\epsilon)$, quadratics	Big data regime (#data $\gg d$)
DINGO [Crane and Roosta, 2019]	GFS[4]	Moral Smoothness[5], $\approx$ strong convexity[6]	$\mathcal{O}(d)$	Global linear rate. No fast local rate.	Operates full gradients, Hessian-vector products, Hessian pseudo-inverse and vector products.
DAN [Zhang et al., 2020]	GFS	LipC Hessian, strong convexity	$\mathcal{O}(nd^2)$	Global quadratic rate after $\mathcal{O}(L/\mu^2)$ iterations.	Operates full gradients and Hessian matrices.
DAN-LA [Zhang et al., 2020]	GFS	LipC Hessian, LipC gradient, strong convexity	$\mathcal{O}(nd)$	Asymptotic and implicit global superlinear rate.	$\lim_{k\to\infty} \frac{\|x_{k+1}-x^*\|}{\|x_k-x^*\|} = 0$ Independent of κ ? Better non-asymptotic complexity over linear rate ?
NL [Islamov et al., 2021]	GLM	LipC Hessian, convex $+ l_2$ reg.	$\mathcal{O}(d)$	Local superlinear rate independent of κ, but dependent on #data. Global linear rate.	reveals local data to server
Quantized Newton [Alimisis et al., 2021]	GFS	LipC Hessian, LipC gradient, strong convexity[6]	$\tilde{\mathcal{O}}(d^2)$	Local (fixed) linear rate. No global rate.	Operates full gradients and Hessian matrices.
FedNL [Safaryan et al., 2021]	GFS	LipC Hessian, strong convexity	$\mathcal{O}(d)$	Local (fixed) linear rate. Local superlinear rate independent of κ, independent of #data. Global linear rate.	Operates full gradients and Hessian matrices. Supports contractive Hessian compression. Extensions[†]
Basis Learn (this work)	GFS	LipC Hessian, strong convexity	$\mathcal{O}(d)$	Local (fixed) linear rate. Local superlinear rate independent of κ, independent of #data.	Operates full gradients and Hessian matrices. Supports contractive Hessian compression. Extensions[†]

[1] **CC** = **C**ommunication **C**ost per iteration. [2] **LipC** = **Lip**schitz **C**ontinuous.
[3] GLM = Generalized Linear Model, e.g. $loss_j(x; a_j) = \phi_j(a_j^\top x) + \lambda\|x\|^2$. [4] GFS = General Finite Sum.
[5] Moral Smoothness: $\|\nabla^2 f(x)\nabla f(x) - \nabla^2 f(y)\nabla f(y)\| \le L\|x-y\|$. [6] Applies to local loss functions for all clients.
[†] Partial Participation and Bidirectional Compression.

Motivation: utilizing the data structure

Naïve Implementation: $\nabla f_i(x^k) \in \mathbb{R}^d, \nabla^2 f_i(x^k) \in \mathbb{R}^{d\times d}$

Expensive communication $\mathcal{O}(d^2)$

Local Quadratic Rate

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→ Local Quadratic Rate

$$\min_x \left\{ f(x) := \frac{1}{n}\sum_{i=1}^{n} f_i(x) \right\}, \quad f_i(x) = \frac{1}{m}\sum_{j=1}^{m} \varphi_{ij}(a_{ij}^\top x)$$

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Assumption: $\{a_{ij} \in \mathbb{R}^d : j \in [m]\} \subseteq G_i, \quad \mathrm{dim} G_i = r$ ➡ Fix basis $\{v_{it}\}_{t=1}^r$

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New Hessian representation:

$$\nabla^2 f_i(x) = \frac{1}{m}\sum_{j=1}^m \varphi''_{ij}(a_{ij}^\top x) a_{ij} a_{ij}^\top = \frac{1}{m}\sum_{j=1}^m \varphi''_{ij}(a_{ij}^\top x) \sum_{t,l=1}^r \alpha_{ijt}\alpha_{ijl} v_{it} v_{il}^\top$$

$$= \sum_{t,l=1}^r \left[\frac{1}{m}\sum_{j=1}^m \varphi''_{ij}(a_{ij}^\top x)\alpha_{ijt}\alpha_{ijl} \right] v_{it} v_{il}^\top$$

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New basis in the space of matrices

Motivation: utilizing the data structure

Naïve Implementation: $\nabla f_i(x^k) \in \mathbb{R}^d, \nabla^2 f_i(x^k) \in \mathbb{R}^{d\times d}$ → Expensive communication $\mathcal{O}(d^2)$ → Local Quadratic Rate

$$\min_x \left\{ f(x) := \frac{1}{n}\sum_{i=1}^{n} f_i(x) \right\}, \quad f_i(x) = \frac{1}{m}\sum_{j=1}^{m} \varphi_{ij}(a_{ij}^\top x)$$

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New representation can be derived for gradients too

Motivation: utilizing the data structure

Naïve Implementation: $\nabla f_i(x^k) \in \mathbb{R}^d, \nabla^2 f_i(x^k) \in \mathbb{R}^{d\times d}$

Expensive communication $\mathcal{O}(d^2)$

Local Quadratic Rate

New Hessian and gradient representations requires $\mathcal{O}(r^2)$ floats. In the extreme cases of $r = \mathcal{O}(1)$ we run Newton's method with $\mathcal{O}(1)$ cost per iteration!

Assu

New data representation: $a_{ij} = \sum_{t=1}^{r} \alpha_{ijt} v_{it}, j \in [m]$

New Hessian representation:

$$\nabla^2 f_i(x) = \frac{1}{m}\sum_{j=1}^{m} \varphi''_{ij}(a_{ij}^\top x) a_{ij} a_{ij}^\top = \frac{1}{m}\sum_{j=1}^{m} \varphi''_{ij}(a_{ij}^\top x) \sum_{t,l=1}^{r} \alpha_{ijt}\alpha_{ijl} v_{it} v_{il}^\top$$

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Newton Star

$$x^{k+1} = x^k - \left(\frac{1}{n}\sum_{i=1}^{n} \nabla^2 f_i(x^*)\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} \nabla f_i(x^k)\right)$$

PDF

Rustem Islamov, Xun Qian and Peter Richtárik
Distributed Second Order Methods with Fast Rates and Compressed Communication,
ICML 2021.

Newton Star

$$x^* := \arg\min_x f(x)$$

$$x^{k+1} = x^k - \Big(\underbrace{\frac{1}{n}\sum_{i=1}^{n} \nabla^2 f_i(x^*)}_{\nabla^2 f(x^*)}\Big)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} \nabla f_i(x^k)\right)$$

We assume this is known

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We assume this is known

Can be computed by machine *i*

Newton Star

$$x^{k+1} = x^k - \left(\frac{1}{n}\sum_{i=1}^{n}\nabla^2 f_i(x^*)\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\nabla f_i(x^k)\right)$$

- ✅ Local quadratic convergence rate independent of the condition number
- ✅ Cheap $O(d)$ communication cost
- ❌ The Hessian at the optimum in unknown

Hessian Lipschitz constant

$$\|x^{k+1} - x^*\| \leq \frac{H}{2\mu}\|x^k - x^*\|^2$$

Strong convexity constant

Basis Learn: Idea

Fix basis: $\{\mathbf{B}_i^{jl} : j, l \in [d]\} \longrightarrow \forall \mathbf{A} \in \mathbb{R}^{d\times d} \hookrightarrow \mathbf{A} = \sum_{j,l} h_{j,l}^i(\mathbf{A})\mathbf{B}_i^{jl}$

Basis Learn: Idea

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Define: $h^i(\mathbf{A}) \in \mathbb{R}^{d \times d} : h^i(\mathbf{A})_{j,l} = h_{j,l}^i(\mathbf{A})$

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Wish list:

- $\mathbf{H}_i^k \to \nabla^2 f_i(x^*)$ as $k \to \infty$
- **Local rate independent of the condition number**

Basis Learn: Idea

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Wish list:

- $\mathbf{H}_i^k \to \nabla^2 f_i(x^*)$ as $k \to \infty \longleftrightarrow \mathbf{L}_i^k \to h^i(\nabla^2 f_i(x^*))$
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Wish list:

- $\mathbf{H}_i^k \to \nabla^2 f_i(x^*) \text{ as } k \to \infty$ $\longleftrightarrow$ $\boxed{\mathbf{L}_i^k \to h^i(\nabla^2 f_i(x^*))}$ $\longleftarrow$ **Desire: Communication-efficient learning mechanism**
- **Local rate independent of the condition number**

Rustem Islamov, Xun Qian and Peter Richtárik
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BL1: Basis Learn with Bidirectional Compression

Parameters: Hessian learning rate $\alpha \geq 0$, model learning rate $\eta \geq 0$, gradient compression probability $p \in (0, 1]$

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Initialization: $x^0 = w^0 = z^0$

Model parameters on server side

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Model parameters on client side

BL1: Basis Learn with Bidirectional Compression

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Model parameters when gradient was computed last time

BL1: Basis Learn with Bidirectional Compression

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Initialization: $x^0 = w^0 = z^0$, $\mathbf{L}_i^0 \in \mathbb{R}^{d \times d}$, $\mathbf{H}_i^0 = \sum_{j,l} (\mathbf{L})_{j,l} \mathbf{B}_i^{j,l}$

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$$\mathbf{H}^0 = \frac{1}{n} \sum_{i=1}^{n} \mathbf{H}_i^0, \xi^0 = 1$$

If one, we compute gradients
If zero, we use gradients from previous iteration

Learning mechanism

$$\mathbf{L}_i^{k+1} = \mathbf{L}_i^k + \alpha \mathcal{C}_i^k (h^i(\nabla^2 f_i(z^k)) - \mathbf{L}_i^k)$$

Compressing the update inspired (by first-order method DIANA)

Konstantin Mishchenko, Eduard Gorbunov, Martin Takáč and Peter Richtárik, **Distributed learning with compressed gradient differences,** arXiv:1901.09269, 2019.

Learning mechanism

$$\mathbf{L}_i^{k+1} = \mathbf{L}_i^k + \alpha \boxed{\mathcal{C}_i^k} \left(h^i \left(\nabla^2 f_i(z^k) \right) - \mathbf{L}_i^k \right)$$

Compression operator

Learning mechanism

$$\mathbf{L}_i^{k+1} = \mathbf{L}_i^k + \alpha \mathcal{C}_i^k \left(h^i(\nabla^2 f_i(z^k)) - \mathbf{L}_i^k \right)$$

Compression operator

Contractive compressor

$$\|\mathcal{C}(\mathbf{M})\|_{\mathrm{F}} \leq \|\mathbf{M}\|_{\mathrm{F}}$$

$$\|\mathcal{C}(\mathbf{M}) - \mathbf{M}\|_{\mathrm{F}}^2 \leq (1-\delta)\|\mathbf{M}\|_{\mathrm{F}}^2 \quad \forall\, \mathbf{M} \in \mathbb{R}^{d\times d}$$

Unbiased compressor

$$\mathbb{E}[\mathcal{C}(\mathbf{M})] = \mathbf{M}$$

$$\mathbb{E}\left[\|\mathcal{C}(\mathbf{M}) - \mathbf{M}\|_{\mathrm{F}}^2\right] \leq \omega \|\mathbf{M}\|_{\mathrm{F}}^2 \quad \forall\, \mathbf{M} \in \mathbb{R}^{d\times d}$$

Learning mechanism

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Don't need Error Feedback Mechanism

Learning mechanism

$$\mathbf{L}_i^{k+1} = \mathbf{L}_i^k + \alpha \mathcal{C}_i^k\left(h^i(\nabla^2 f_i(z^k)) - \mathbf{L}_i^k\right)$$

Stepsize depends only on the compression

BL1: Algorithm. Client Side

for each device $i = 1, \ldots, n$ in parallel **do**

  if $\xi^k = 1$

    $w^{k+1} = z^k$, compute local gradient $\nabla f_i(z^k)$ and send to the server

  if $\xi^k = 0$

    $w^{k+1} = w^k$

  Compute local Hessian $\nabla^2 f_i(z^k)$ and send $\mathbf{S}_i^k := \mathcal{C}_i^k(h^i(\nabla^2 f_i(z^k)) - \mathbf{L}_i^k)$ to the server

  Update local Hessian shifts $\mathbf{L}_i^{k+1} = \mathbf{L}_i^k + \alpha \mathbf{S}_i^k$, $\mathbf{H}_i^{k+1} = \mathbf{H}_i^k + \alpha \sum_{jl} (\mathbf{S}_i^k)_{jl} \mathbf{B}_i^{jl}$

end for

BL1: Algorithm. Server Side

on server

if $\xi^k = 1$

$w^{k+1} = z^k, \; g^k = \nabla f(z^k)$

if $\xi^k = 0$

$w^{k+1} = w^k, \; g^k = [\mathbf{H}^k]_\mu (z^k - w^k) + \nabla f(w^k)$

$x^{k+1} = z^k - [\mathbf{H}^k]_\mu^{-1} g^k$

$\mathbf{H}^{k+1} = \mathbf{H}^k + \frac{\alpha}{n} \sum_{i=1}^n \sum_{jl} (\mathbf{S}_i^k)_{jl} \mathbf{B}_i^{jl}$

Send $v^k := \mathcal{Q}^k(x^{k+1} - z^k)$ to all devices $i \in [n]$

Update the model $z^{k+1} = z^k + \eta v^k$

Send $\xi^{k+1} \sim \text{Bernoulli}(p)$ to all devices $i \in [n]$

for each device $i = 1, \ldots, n$ in parallel **do**

Update the model $z^{k+1} = z^k + \eta v^k$

Basis Learn: Assumptions

Assumption 4.3. *(i)* $\mathcal{Q}^k$ $(\mathcal{Q}_i^k)$ *is an unbiased compressor with parameter* $\omega_{\rm M}$ *and* $0 < \eta \leq 1/(\omega_{\rm M}+1)$. *(ii) For all* $j \in [d]$, $(z^k)_j$ *in Algorithm 1 is a convex combination of* $\{(x^t)_j\}_{t=0}^k$ *for* $k \geq 0$.

Assumption 4.4. *(i)* $\mathcal{Q}^k$ $(\mathcal{Q}_i^k)$ *is a contraction compressor with parameter* $\delta_{\rm M}$ *and* $\eta = 1$. *(ii)* $\mathcal{Q}^k$ $(\mathcal{Q}_i^k)$ *is deterministic, i.e.,* $\mathbb{E}[\mathcal{Q}^k(x)] = \mathcal{Q}^k(x)$ *for any* $x \in \mathbb{R}^d$.

Assumption 4.5. *(i)* $\mathcal{C}_i^k$ *is an unbiased compressor with parameter* ω *and* $0 < \alpha \leq 1/(\omega+1)$. *(ii) For all* $i \in [n]$ *and* $j, l \in [d]$, $(\mathbf{L}_i^k)_{jl}$ *is a convex combination of* $\{h^i(\nabla^2 f_i(z^t))_{jl}\}_{t=0}^k$ *in Algorithm 1*

Assumption 4.6. *(i)* $\mathcal{C}_i^k$ *is a contraction compressor with parameter* δ *and* $\alpha = 1$. *(ii)* $\mathcal{C}_i^k$ *is deterministic, i.e.,* $\mathbb{E}[\mathcal{C}_i^k(\mathbf{A})] = \mathcal{C}_i^k(\mathbf{A})$ *for any* $\mathbf{A} \in \mathbb{R}^{d\times d}$.

Assumption 4.7. *We have* $\|\nabla^2 f_i(x) - \nabla^2 f_i(y)\| \leq H\|x-y\|$, $\|\nabla^2 f_i(x) - \nabla^2 f_i(y)\|_{\rm F} \leq H_1\|x-y\|$, $\|h^i(\nabla^2 f_i(x)) - h^i(\nabla^2 f_i(y))\|_{\rm F} \leq M_1\|x-y\|$, $\max_{jl}\{|h^i(\nabla^2 f_i(x))_{jl} - h^i(\nabla^2 f_i(y))_{jl}|\} \leq M_2\|x-y\|$, $\max_{jl}\{\|\mathbf{B}_i^{jl}\|_{\rm F}\} \leq R$ *for any* $x, y \in \mathbb{R}^d$ *and* $i \in [n]$.

Local Convergence Theory

$$\mathbb{E}[\Phi_2^k] \leq \left(1 - \frac{\min\{4A, A_M\}}{4}\right)^k \Phi_2^0$$

$$\mathbf{E}\left[\frac{\|x^{k+1} - x^*\|^2}{\|x^k - x^*\|^2}\right] \leq \left(1 - \frac{\min\{4A, A_M\}}{4}\right)^k \left(\frac{A_M H^2}{8BM_1^2\mu^2} + \frac{2N_B R^2}{\mu^2}\right) \Phi_2^0$$

Local Convergence Theory

$$\mathbb{E}[\Phi_2^k] \boxed{\leq} \left(1 - \frac{\min\{4A, A_M\}}{4}\right)^k \Phi_2^0$$

Local linear rate

$$\mathbf{E}\left[\frac{\|x^{k+1} - x^*\|^2}{\|x^k - x^*\|^2}\right] \boxed{\leq} \left(1 - \frac{\min\{4A, A_M\}}{4}\right)^k \left(\frac{A_M H^2}{8BM_1^2\mu^2} + \frac{2N_B R^2}{\mu^2}\right)\Phi_2^0$$

Local superlinear rate

Local Convergence Theory

Lyapunov function

$$\Phi_2^k := \frac{1}{n}\sum_{i=1}^{n} \|\mathbf{L}_i^k - \mathbf{L}_i^*\|_{\mathrm{F}}^2 + \frac{4BM_1^2}{A_M}\|x^k - x^*\|^2$$

$$\mathbb{E}[\boxed{\Phi_2^k}] \leq \left(1 - \frac{\min\{4A, A_M\}}{4}\right)^k \boxed{\Phi_2^0}$$

Local linear rate

$$\mathbf{E}\left[\frac{\|x^{k+1} - x^*\|^2}{\|x^k - x^*\|^2}\right] \leq \left(1 - \frac{\min\{4A, A_M\}}{4}\right)^k \left(\frac{A_M H^2}{8BM_1^2\mu^2} + \frac{2N_B R^2}{\mu^2}\right)\boxed{\Phi_2^0}$$

Local superlinear rate

Local Convergence Theory

Lyapunov function

$$\Phi_2^k := \frac{1}{n}\sum_{i=1}^{n}\|\mathbf{L}_i^k - \mathbf{L}_i^*\|_{\mathrm{F}}^2 + \frac{4BM_1^2}{A_M}\|x^k - x^*\|^2$$

$$\mathbb{E}[\boxed{\Phi_2^k}] \leq \left(1 - \frac{\min\{4A, A_M\}}{4}\right)^k \boxed{\Phi_2^0}$$

Local linear rate

Constants depending on the choice of the compressor and stepsize

$$A_M = \begin{cases} \eta & \text{if Asm. } 4.3(i) \text{ holds} \\ \frac{\delta_M}{4} & \text{if Asm. } 4.4(i) \text{ holds} \end{cases}$$

$$A = \begin{cases} \alpha & \text{if Asm. } 4.5(i) \text{ holds} \\ \frac{\delta}{4} & \text{if Asm. } 4.6(i) \text{ holds} \end{cases}$$

$$\mathbf{E}\left[\frac{\|x^{k+1} - x^*\|^2}{\|x^k - x^*\|^2}\right] \leq \left(1 - \frac{\min\{4A, A_M\}}{4}\right)^k \left(\frac{A_M H^2}{8BM_1^2\mu^2} + \frac{2N_B R^2}{\mu^2}\right)\boxed{\Phi_2^0}$$

Local superlinear rate

Local Convergence Theory

Lyapunov function

$$\Phi_2^k := \frac{1}{n}\sum_{i=1}^{n}\|\mathbf{L}_i^k - \mathbf{L}_i^*\|_{\mathrm{F}}^2 + \frac{4BM_1^2}{A_M}\|x^k - x^*\|^2$$

Provably learn the Hessian at the optimum

$$\mathbb{E}[\boxed{\Phi_2^k}] \leq \left(1 - \frac{\min\{4A, A_M\}}{4}\right)^k \boxed{\Phi_2^0}$$

Local linear rate

Constants depending on the choice of the compressor and stepsize

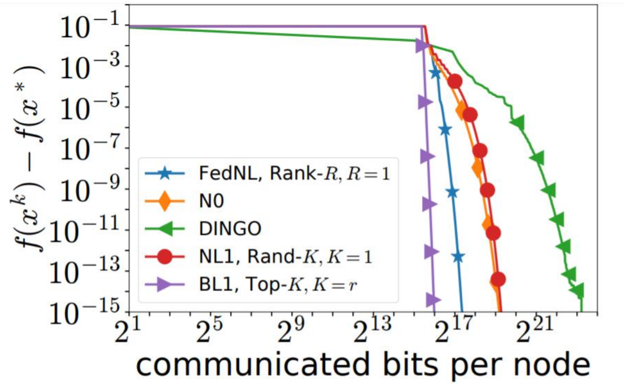
$$A_M = \begin{cases} \eta & \text{if Asm. 4.3}(i) \text{ holds} \\ \frac{\delta_M}{4} & \text{if Asm. 4.4}(i) \text{ holds} \end{cases}$$

$$A = \begin{cases} \alpha & \text{if Asm. 4.5}(i) \text{ holds} \\ \frac{\delta}{4} & \text{if Asm. 4.6}(i) \text{ holds} \end{cases}$$

$$\mathbf{E}\left[\frac{\|x^{k+1} - x^*\|^2}{\|x^k - x^*\|^2}\right] \leq \left(1 - \frac{\min\{4A, A_M\}}{4}\right)^k \left(\frac{A_M H^2}{8BM_1^2\mu^2} + \frac{2N_B R^2}{\mu^2}\right)\boxed{\Phi_2^0}$$

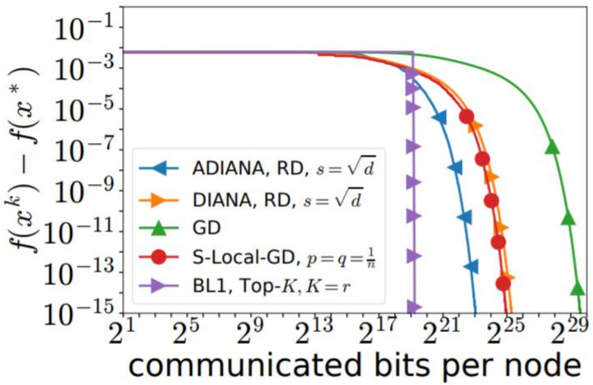
Local superlinear rate

Experiments: Logistic Regression



(a) covtype, $\lambda = 10^{-3}$

$r \sim 0.44d$



(b) w2a, $\lambda = 10^{-4}$

$r \sim 0.2d$

The End

For more details:

Xun Qian, Rustem Islamov, Mher Safaryan, and Peter Richtárik
Basis Matters: Better Communication-Efficient Second Order Methods for Federated Learning
arXiv preprint arXiv: 2106.02969, 2021