Jack's Car Rental

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Machine and Reinforcement Learning in Control Applications

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Problem

Manage cars between two locations.

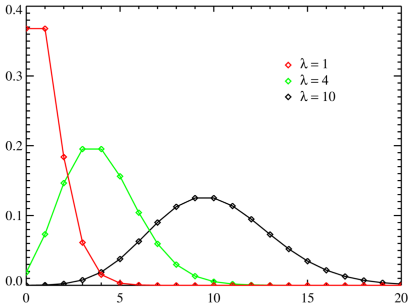
Problem formulation

- Jack manages two locations for a car rental company.
- Each day, some customers arrive at each location to rent cars.
- If Jack has a car available, he rents it out and is credited \$10.
- Cars are available for renting the day after they are returned.
- Jack can move cars between the two locations overnight.
- The cost of moving a car is \$2.
- Each location is capable of accommodating 20 cars.

Problem data

- Cars requested and returned at each location are Poisson random variables

$$\mathbb{P}[\text{cars} = n] = \frac{\lambda^n}{n!} \exp(-\lambda).$$



Problem data

- Cars requested and returned at each location are Poisson random variables

$$\mathbb{P}[\text{cars} = n] = \frac{\lambda^n}{n!} \exp(-\lambda).$$

- λ is 4 and 3 for rental requests.
- λ is 2 and 3 for returns.
- Jack's foresight modeled with discount $\gamma = 0.9$.
- Jack can move up to 5 cars between the two locations.

Model

- We can model the process as an MDP.
- The state is the number of car at each location
  - is it a Markov state?
  - we have $\#1 \cdot \#2$ states.
- The action is the number of cars moved
  - we have $2\#c + 1$ actions.

Transition and reward

- Let S and A be the number of states and actions.
- Transition probabilities can be stored in a $S \times A \times S$ matrix P

$$P_{s,a,s'} = \mathbb{P}[S_{t+1} = s' | S_t = s, A_t = a] = p(s'|s,a).$$

- Rewards can be stored in a $S \times A$ matrix R

$$R_{s,a} = \mathbb{E}[R_{t+1} | S_t = s, A_t = a] = r(s,a).$$

State update

1. Jack reintroduces car returned at previous day.
2. Jack rents available cars.
3. Jack moves cars between the two locations.

Transition probabilities

1. Probability of returns do not depend on actions
   - define the return probability matrix P_{ret}

$$[P_{\text{ret}}]_{s,s'} = \mathbb{P}[S_{\text{after return}} = s' | S_t = s].$$

2. Probability of rentals do not depend on actions
   - define the rental probability matrix P_{ren}

$$[P_{\text{ren}}]_{s,s'} = \mathbb{P}[S_{\text{after rental}} = s' | S_{\text{after return}} = s].$$

3. Probability of movement depend on actions
   - define the movement probability matrix P_{mov}

$$[P_{\text{mov}}]_{s,a,s'} = \mathbb{P}[S_{\text{after movement}} = s' | S_{\text{after rental}} = s, A_t = a].$$

4. By the law of total probability

$$P = P_{\text{ret}} \cdot P_{\text{ren}} \cdot P_{\text{mov}}.$$

Expected rewards

- Expected earning do not depend on action

$$\mathbb{E}\left[\text{earning}_{t+1} | S_{\text{after return}} = s\right] = \sum_r r \mathbb{P}\left[r | S_{\text{after return}} = s\right].$$

- By the law of total probability

$$\begin{aligned}
&\mathbb{E}\left[\text{earning}_{t+1} | S_t = s\right] \\
&= \sum_{s'} \mathbb{P}[S_{\text{after return}} = s' | S_t = s] \sum_r r \mathbb{P}\left[r | S_{\text{after return}} = s'\right].
\end{aligned}$$

- The expected reward is given by

$$R_{s,a} = [P_{\text{ret}} \cdot \text{earning}]_s - \text{cost}_a.$$

Matrix formulation

- Given a deterministic policy π, define
  - $P^\pi = \begin{bmatrix} P_{1,1,\pi(1)} & P_{1,2,\pi(1)} & \cdots & P_{1,S,\pi(1)} \\ P_{2,1,\pi(2)} & P_{2,2,\pi(2)} & \cdots & P_{2,S,\pi(2)} \\ \vdots & \vdots & \ddots & \vdots \\ P_{S,1,\pi(S)} & P_{S,2,\pi(S)} & \cdots & P_{S,S,\pi(S)} \end{bmatrix};$
  - $R^\pi = \begin{bmatrix} R_{1,\pi(1)} \\ R_{2,\pi(2)} \\ \vdots \\ R_{S,\pi(S)} \end{bmatrix}.$

PI and VI revisited

- Recall classical Bellman expectation update

$$v^\pi(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)\left(r + \gamma v(s')\right)$$

- Given a deterministic policy π

$$\begin{aligned} v^\pi(s) &= \sum_{s',r} p(s',r|s,\pi(s))\left(r + \gamma v(s')\right) \\ &= \sum_{s',r} p(s',r|s,\pi(s))r + \gamma \sum_{s',r} p(s',r|s,\pi(s))v(s') \\ &= \sum_r r \sum_{s'} (s',r|s,\pi(s)) + \gamma \sum_{s'} v(s') \sum_r p(s',r|s,\pi(s)) \\ &= r(s,\pi(s)) + \gamma \sum_{s'} p(s'|s,\pi(s))v(s'). \end{aligned}$$

- Bellman expectation update can be rewritten as

$$v^\pi \leftarrow R^\pi + \gamma P^\pi v^\pi.$$

PI and VI revisited

- Recall classical Bellman optimality update

$$v^\star(s) \leftarrow \max_a \left\{ r(s,a) + \gamma \sum_{s'} p(s'|s,a) v(s') \right\}.$$

- Bellman optimality update can be rewritten as

$$v^\star \leftarrow \max_\pi \{R^\pi + \gamma P^\pi v^\pi\}$$

Assignment #2

- Write a code for PI (in class).
- Write a code for VI (in class).
- Model gambler's problem
  - a gambler has the opportunity to make bets on the outcomes of a sequence of coin flips;
  - if the coin comes up heads, he wins as many dollars as he has staked on that flip; if it is tails, he loses his stake;
  - the game ends when the gambler wins by reaching his goal of $100, or loses by running out of money;
  - reward $+1$ for winning.