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Direct data-driven model-reference control with Lyapunov stability guarantees

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***Abstract*— We introduce a novel data-driven model-reference control design approach for unknown linear systems with fully measurable state. The proposed control action is composed by a static feedback term and a reference tracking block shaped from data to reproduce the desired behavior in closed-loop. By focusing on the case where the reference model and the plant share the same order, we propose an optimal design procedure with Lyapunov stability guarantees, tailored to handle state measurements with additive noise. Two simulation examples are illustrated to show the potential of the proposed strategy.**

I. INTRODUCTION

In many control applications, the desired performance is expressed in terms of an *a-priori specified* closed-loop model, to be matched using a controller with a given structure [16]. When a model of the plant to be controlled is available, the model-matching issue can be formulated as an *interpolation* problem [23]. More often, the mathematical description of the system is not given and thus a model needs to be identified from a set of experimental data [17]. Such a procedure, namely the sequence of system identification and model-based control design, has been proven to potentially lead to sub-optimal solutions, as the model that best fits the data is not necessarily the best for controller tuning [14].

Alternative approaches have thus been proposed to map the data *directly* onto the controller parameters without undertaking a full modeling study, see, *e.g.,* [2]. Such techniques are usually referred to as "direct data-driven" approaches. Albeit appealing and effective (see, *e.g.,* their application in [12]), these approaches suffer from few drawbacks, which prevent them from being real competitors of model-based strategies. For instance, data-driven methods are mostly conceived to handle SISO (Single Input Single Output) systems, or at most MIMO (Multiple Inputs Multiple Outputs) ones with few input/output channels [13]. Moreover, stability is guaranteed only asymptotically, *i.e.,* as the number of data goes to infinity, see, *e.g.,* [2], [12], [18].

In this work, we propose a new direct data-driven design strategy for model-reference control, which is endowed with stability guarantees. The key technical step is the data-based representation of linear systems originally proposed in [21]

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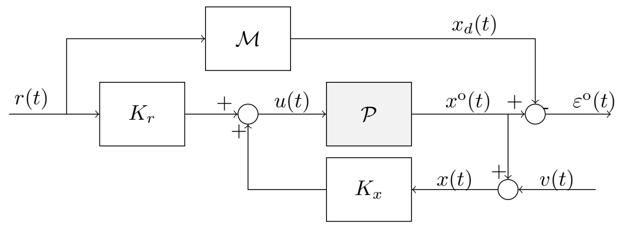


Fig. 1. Matching scheme. The unknown plant $\mathcal{P}$ is highlighted in gray. The mismatch error is denoted with $\varepsilon^{o}(t)$.

within the behavioral framework, and recently reconsidered in [8]. This data-based representation has emerged as a powerful tool for tackling many important control problems, such as predictive [3], [7] and optimal and robust control [1], [4], [8], [20], [22]. Inspired by [8], we tackle the model-reference control problem in a state-space setting. For noise-free data, we show that this problem can be cast as a semi-definite program. We also account for possible model mismatches by considering a regularization-based strategy that moves the matching constraints to the objective function (see [11] for a discussion on the use of regularization techniques in the context of data-driven control). In the spirit of [9], we finally address the case of noisy measurements by considering a strategy based on averaging multiple experiments and derive sufficient conditions for closed-loop stability. The strategy is tested on two numerical examples.

The remainder of the paper is as follows. In Section II, the model-reference control design problem is formally stated. Section III describes the key technical steps allowing us to formulate the problem as a data-based optimization task. The main results are presented in Section IV and Section V, and some examples are discussed in Section VI. Section VII ends the paper with concluding remarks.

II. SETTING AND PROBLEM FORMULATION

Let $\mathcal{P}$ be a discrete-time *linear time invariant* (LTI) system, described by:

$$\mathcal{P}: \quad \begin{cases} x^{o}(t+1) = Ax^{o}(t) + Bu(t), \\ x(t) = x^{o}(t) + v(t), \end{cases} \tag{1a}$$

where $x^{o}(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is an exogenous input signal fed into the system at time $t \in \mathbb{N}$, and we measure the noisy state $x(t) \in \mathbb{R}^n$, where $v(t) \in \mathbb{R}^n$ is measurement noise.

We aim at designing a *stabilizing* controller that matches a user-defined reference model $\mathcal{M}$, dictating the desired closed-loop response to a customizable set point $r(t) \in \mathbb{R}^n$. The ~~reference model is characterized by the equation~~

$$\boxed{\mathcal{M}: \quad x_d(t+1) = A_M x_d(t) + B_M r(t)} \tag{2}$$

with $x_d(t) \in \mathbb{R}^n$ indicating the desired state at time t. Here, the reference model $\mathcal{M}$ is assumed to be stable and provided at design time, thus being fixed and known. The matching problem can be formally stated as follows.

Problem 1 (Matching problem): Let $\mathcal{P}$ be an LTI system as in (1) and $\mathcal{M}$ be a stable reference model as in (2). The matching problem amounts to finding two control matrices $K_x, K_r \in \mathbb{R}^{m\times n}$ such that

$$A + BK_x = A_M \quad (3a)$$

$$BK_r = B_M, \quad (3b)$$

If they exist, we say that the matching problem is *feasible*. This definition comes from the fact that, if the equations in (3) have a solution, then the control law

$$u(t) = K_x x(t) + K_r r(t), \quad (4)$$

ensures that the noiseless behavior of $\mathcal{P}$ matches that of $\mathcal{M}$. Note that K_x is responsible for closed-loop stability while K_r is a feed-forward term allowing one to attain the desired response to $r(t)$. The considered matching scheme is depicted in the block diagram of Figure 1.

In principle, solving Problem 1 requires the knowledge of the plant matrices (A, B) characterizing (1). In this paper, we are interested in solving the matching problem when A and B are *unknown*, and we have access to a finite set of input-state pairs only. The available data are obtained by applying an input signal $\mathcal{U}_{T-1} = \{u(t)\}_{t=0}^{T-1}$ to $\mathcal{P}$ and measuring the response of the system $\mathcal{X}_T = \{x(t)\}_{t=0}^{T}$, where T denotes the length of the experiment[1]. This new matching problem is formalized as follows.

Problem 2 (Data-driven matching problem): Let $\mathcal{P}$ be an LTI system as in (1), and let $\mathcal{M}$ be a stable reference model as in (2). The data-driven matching problem amounts to finding two control matrices $K_x, K_r \in \mathbb{R}^{m\times n}$ satisfying the conditions in (3) by using a set of data $(\mathcal{U}_{T-1}, \mathcal{X}_T)$ only.

III. Data-based Description of the Closed-loop

Given an experiment of length T carried out on $\mathcal{P}$, define the following data matrices:

$$X_{0,T-1} = \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix}, \quad (5a)$$

$$U_{0,T-1} = \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix}, \quad (5b)$$

$$V_{0,T-1} = \begin{bmatrix} v(0) & v(1) & \cdots & v(T-1) \end{bmatrix}, \quad (5c)$$

$$X_{1,T} = \begin{bmatrix} x(1) & x(2) & \cdots & x(T) \end{bmatrix}, \quad (5d)$$

$$V_{1,T} = \begin{bmatrix} v(1) & v(2) & \cdots & v(T) \end{bmatrix}, \quad (5e)$$

and let $X_{0,T-1}^{\mathrm{o}}$ and $X_{1,T}^{\mathrm{o}}$ be the noiseless counterparts of the matrices in (5a) and (5d), *i.e.,*

$$X_{0,T-1}^{\mathrm{o}} = \begin{bmatrix} x^{\mathrm{o}}(0) & x^{\mathrm{o}}(1) & \cdots & x^{\mathrm{o}}(T-1) \end{bmatrix}, \quad (6a)$$

$$X_{1,T}^{\mathrm{o}} = \begin{bmatrix} x^{\mathrm{o}}(1) & x^{\mathrm{o}}(2) & \cdots & x^{\mathrm{o}}(T) \end{bmatrix}. \quad (6b)$$

Consider now the following assumption, which is related to the richness of the data.

[1]A stabilizing controller is assumed to be available to perform closed-loop experiments when the plant is unstable.

Assumption 1: The following condition holds:

$$\operatorname{rank}\left(\begin{bmatrix} U_{0,T-1} \\ X_{0,T-1} \end{bmatrix}\right) = n + m. \quad (7)$$

As shown next, condition (7) makes it possible to express the behavior of $\mathcal{P}$ in feedback with (4) purely in terms of the data matrices in (5) for *any* control gains K_x and K_r. For controllable systems and noiseless data this condition can be enforced at the design stage by choosing $\mathcal{U}_{T-1}$ as a *persistently exciting* signal [21]. It is also simple to see that Assumption 1 can be verified from data and that it holds even with noisy ones, whenever the noise is sufficiently small in magnitude.

Proposition 1 (Data-driven closed-loop representation): Let Assumption 1 be satisfied. For any matrices K_x, K_r, the closed-loop dynamics resulting from (4) can be equivalently expressed in terms of data as

$$x^{\mathrm{o}}(t+1) = A_{cl} x^{\mathrm{o}}(t) + B_{cl} r(t) + D_{cl} v(t) \quad (8)$$

with $A_{cl} = (X_{1,T} + W_{0,T})G^x$, $B_{cl} = (X_{1,T} + W_{0,T})G^r$ and $D_{cl} = (X_{1,T} + W_{0,T})G^v$, where G^x, G^v and G^r satisfy

$$\begin{bmatrix} K_x \\ I_n \end{bmatrix} = \begin{bmatrix} U_{0,T-1} \\ X_{0,T-1} \end{bmatrix} G^x, \quad (9)$$

$$\begin{bmatrix} K_r \\ \mathbf{0}_n \end{bmatrix} = \begin{bmatrix} U_{0,T-1} \\ X_{0,T-1} \end{bmatrix} G^r, \quad (10)$$

$$\begin{bmatrix} K_x \\ \mathbf{0}_n \end{bmatrix} = \begin{bmatrix} U_{0,T-1} \\ X_{0,T-1} \end{bmatrix} G^v, \quad (11)$$

and where $W_{0,T} = AV_{0,T-1} - V_{1,T}$. □

Proof: By combining (1) and (4), the closed-loop dynamics becomes

$$x^{\mathrm{o}}(t+1) = (A + BK_x)x^{\mathrm{o}}(t) + BK_r r(t) + BK_x v(t), \quad (12)$$

which can be equivalently written as:

$$x^{\mathrm{o}}(t+1) = \begin{bmatrix} B & A \end{bmatrix} \left\{ \begin{bmatrix} K_x \\ I_n \end{bmatrix} x^{\mathrm{o}}(t) + \begin{bmatrix} K_r \\ \mathbf{0}_n \end{bmatrix} r(t) + \begin{bmatrix} K_x \\ \mathbf{0}_n \end{bmatrix} v(t) \right\}. \quad (13)$$

Since the rank condition in (7) holds, by the Rouché-Capelli theorem there exist $G^x, G^r, G^v \in \mathbb{R}^{T\times n}$ such that (9)-(11) are satisfied. Therefore, the following holds:

$$\begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K_x \\ I_n \end{bmatrix} = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_{0,T-1} \\ X_{0,T-1}^{\mathrm{o}} \end{bmatrix} G^x + AV_{0,T-1}G^x, \quad (14a)$$

$$\begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K_r \\ \mathbf{0}_n \end{bmatrix} = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_{0,T-1} \\ X_{0,T-1}^{\mathrm{o}} \end{bmatrix} G^r + AV_{0,T-1}G^r, \quad (14b)$$

$$\begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K_x \\ \mathbf{0}_n \end{bmatrix} = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_{0,T-1} \\ X_{0,T-1}^{\mathrm{o}} \end{bmatrix} G^v + AV_{0,T-1}G^v. \quad (14c)$$

Because of the dynamics in (1), it follows that

$$\begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_{0,T-1} \\ X_{0,T-1}^{\mathrm{o}} \end{bmatrix} = X_{1,T}^{\mathrm{o}},$$

which, in turn, implies that (14) is equivalent to:

$$\begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K_x \\ I_n \end{bmatrix} = X_{1,T}^{\mathrm{o}} G^x + AV_{0,T-1}G^x, \quad (15a)$$

$$\begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K_r \\ \mathbf{0}_n \end{bmatrix} = X_{1,T}^{\mathrm{o}} G^r + AV_{0,T-1}G^r, \quad (15b)$$

$$\begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K_x \\ \mathbf{0}_n \end{bmatrix} = X_{1,T}^{\mathrm{o}} G^v + AV_{0,T-1}G^v. \quad (15c)$$

Finally, since $X_{1,T} = X^{\circ}_{1,T} + V_{1,T}$ we obtain

$$\begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K_x \\ I_n \end{bmatrix} = (X_{1,T} + W_{0,T})\, G^x, \tag{16a}$$

$$\begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K_r \\ \mathbf{0}_n \end{bmatrix} = (X_{1,T} + W_{0,T})\, G^r, \tag{16b}$$

$$\begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K_x \\ \mathbf{0}_n \end{bmatrix} = (X_{1,T} + W_{0,T})\, G^v, \tag{16c}$$

which easily leads to the data-based representation in (8). ■

Remark 1 (Relaxing Assumption 1): Having $X_{0,T-1}$ full row rank is necessary to have (9)-(11) fulfilled. In contrast, for some K_x and K_r, (9)-(11) might have a solution even when $U_{0,T-1}$ is not full row rank, in line with what has been shown in [19]. Nonetheless, Assumption 1 ensures that the data-based representation of Proposition 1 is valid for *any* control matrices K_x, K_r. ■

By Proposition 1, the design problem can thus be cast in a data-driven fashion as

$$(X_{1,T} + W_{0,T})G^x = A_M, \tag{17a}$$

$$(X_{1,T} + W_{0,T})G^r = B_M, \tag{17b}$$

which have to be paired with the two consistency conditions

$$X_{0,T-1}G^x = I_n, \quad X_{0,T-1}G^r = \mathbf{0}_n, \tag{17c}$$

needed for the constraints (17a) and (17b) to be equivalent to the model-based ones in (3). If the matching problem is feasible, K_x and K_r can be retrieved using the relations $K_x = U_{0,T-1}G^x$ and $K_r = U_{0,T-1}G^r$, while G^v is not explicitly needed for computing K_x and K_r . We summarize this fact in the following result.

Theorem 1: Consider system (1) along with a reference model as (2). If Assumption 1 holds, the matching problem is feasible, namely there exist two control matrices K_x and K_r satisfying (3), if and only if there exist matrices G^x and G^r such that (17) holds. In this case, K_x and K_r are given by $K_x = U_{0,T-1}G^x$ and $K_r = U_{0,T-1}G^r$, respectively. ■

IV. Data-driven model matching

With noiseless data, the conditions in (17) reduce to

$$X_{1,T}G^x = A_M, \quad X_{1,T}G^r = B_M, \tag{18a}$$

$$X_{0,T-1}G^x = I_n, \quad X_{0,T-1}G^r = \mathbf{0}_n \tag{18b}$$

Therefore, in this case, Theorem 1 gives a complete and easy-to-implement data-based solution for the matching problem. We summarize this result next.

Theorem 2: Consider system (1) and a reference model as (2). Let Assumption 1 hold for a noise-free dataset. Then, the matching problem is feasible if and only if (18) is satisfied. In this case, any solution (G^x, G^r) is such that $K_x = U_{0,T-1}G^x$ and $K_r = U_{0,T-1}G^r$ solve the matching problem. ■

Note that, in the noiseless case, any solution of (18) ensures the stability of the closed-loop system. Indeed, matrix A_M is stable by hypothesis and $X_{1,T}G^x = A + BK_x$.

A. Handling model mismatch

Theorem 2 rests on the assumption that the matching problem is feasible. In case one has selected a reference model $\mathcal{M}$ for which perfect matching is not possible, (18) will have no solutions. One way to remedy this situation is to modify (18) so as to search for a stabilizing controller that best matches $\mathcal{M}$ in some suitable sense, as specified below.

An intuitive way for relaxing the matching constraints is to lift them to a cost function, recasting the problem as

$$\begin{aligned} &\underset{G^x, G^r}{\text{minimize}} \quad \|X_{1,T}G^x - A_M\| + \lambda \|X_{1,T}G^r - B_M\| \\ &\text{subject to} \quad (18b) \end{aligned} \tag{19}$$

where $\lambda > 0$ weights the relative importance between the two matching objectives and $\|\cdot\|$ is any norm. In this new formulation, we have no longer guarantees of perfect matching. Hence, we have to incorporate a stability constraint. To this end, note that (18) can be equivalently written as

$$X_{1,T}Q^x = A_M P, \quad X_{1,T}Q^r = B_M P, \tag{20a}$$

$$X_{0,T-1}Q^x = P, \quad X_{0,T-1}Q^r = \mathbf{0}_n, \tag{20b}$$

having defined $Q^x = G^x P$ and $Q^r = G^r P$, where $P \succ 0$ but otherwise arbitrary. Moreover, recall that in the noiseless case $X_1 G^x = A + BK_x$, which can be rewritten as $X_1 Q^x P^{-1} = A + BK_x$. Hence, the closed-loop system with feedback controller K_x is stable if there exists $P \succ 0$ that satisfies the Lyapunov inequality $X_{1,T}Q^x P^{-1}(X_{1,T}Q^x)' - P \prec 0$ which, in turn, can be rewritten as

$$\begin{bmatrix} P & X_{1,T}Q^x \\ (X_{1,T}Q^x)' & P \end{bmatrix} \succ \mathbf{0}_{2n}, \tag{21}$$

by using Schur complement. This immediately leads to the following formulation:

$$\begin{aligned} &\underset{Q^x, Q^r, P}{\text{minimize}} \quad \|X_{1,T}Q^x - A_M P\| + \lambda \|X_{1,T}Q^r - B_M P\| \\ &\text{subject to} \quad (20b), (21), \end{aligned} \tag{22}$$

where (21) ensures $P \succ 0$. Compared with (19), this new formulation is such that any solution returns a stabilizing controller and the matching problem in (22) corresponds to a *semi-definite program*, that can be efficiently handled by many existing solvers. The specific properties of this formulation are summarized in the next theorem.

Theorem 3: Consider system (1) along with a reference model in (2). Let the data be gathered over a noise-free experiment and let Assumption 1 hold. Then:

(i) If there exists a stabilizing controller for (1), the program (22) is feasible and any solution is (Q^x, Q^r, P) such that $K_x = U_{0,T-1}Q^x P^{-1}$ ensures closed-loop stability.

(ii) If the matching problem is feasible, the program (22) is also feasible and any solution (Q^x, Q^r, P) is such that $K_x = U_{0,T-1}Q^x P^{-1}$ and $K_r = U_{0,T-1}Q^r P^{-1}$ solve the matching problem.

Proof: (i) Let $\underline{K}_x$ be any stabilizing controller and let $\underline{K}_r$ be an arbitrary matrix of dimension $m \times n$. By Assumption 1, there exists a matrix G^x such that $X_{0,T-1}G^x = I_n$ and $U_{0,T-1}G^x = \underline{K}_x$. This implies $A + B\underline{K}_x = X_{1,T}G^x$.

Since $\underline{K}_x$ is stabilizing, there exists a matrix $P \succ 0$ such that $X_{1,T}G^x P(X_{1,T}G^x)' - P \prec 0$. Further, there exists G^r such that $X_{0,T-1}G^r = \mathbf{0}_n$ and $U_{0,T-1}G^r = \underline{K}_r$. Hence, $Q^x = G^x P$ and $Q^r = G^r P$ ensure the fulfillment of (20b) and (21), meaning that (22) is feasible. Let now $Q^x, Q^r, P \succ 0$ be any solution to (22). In view of the first constraint in (20b) and because $K_x = U_{0,T-1}Q^x P^{-1}$, we have $X_{1,T}Q^x P^{-1} = (AX_{0,T-1} + BU_{0,T-1})Q^x P^{-1} = A + BK_x$ and, thus, (21) ensures that K_x is stabilizing. (ii) The proof of the second part of the theorem follows directly from the previous point and the fact that, in this case, the minimum of the objective function is zero. ∎

V. Noise-aware data-based matching strategy

With noisy data, the solution to (22) might not exist or it may lead to a controller K_x that is not stabilizing. Nonetheless, suppose that the noise samples have zero-mean and are *independent* and *identically distributed* (i.i.d.). By the *Strong Law of Large Numbers*,

$$\lim_{N \longrightarrow \infty} \frac{v(0) + v(1) + \ldots + v(N)}{N} = 0. \tag{23}$$

with probability 1. This simple, yet fundamental, property suggests that if we perform multiple experiments on the system and we then average the collected data, the effect of noise will become decreasingly marked as the number of experiment increases. In this light, suppose that we make N experiments on system (1), each of length T. Let $(U_{0,T-1}^{(i)}, X_{0,T-1}^{(i)}, X_{1,T}^{(i)})$, $i = 1, \ldots, N$, be the data matrices resulting from the i-th experiment and $(V_{0,T-1}^{(i)}, V_{1,T}^{(i)})$ be the corresponding noise matrices. Denote with $\bar{X}_{0,T-1}$, $\bar{X}_{1,T}$, $\bar{U}_{0,T-1}$, $\bar{V}_{0,T-1}$ and $\bar{V}_{1,T}$ the average of N matrices $X_{0,T-1}^{(i)}$, $X_{1,T}^{(i)}$, $U_{0,T-1}^{(i)}$, $V_{0,T-1}^{(i)}$ and $V_{1,T}^{(i)}$, respectively. In light of the considerations made for the noiseless case, we can cast the following optimization problem:

$$\begin{aligned}
&\underset{Q^x, Q^r, P}{\text{minimize}} && \|\bar{X}_{1,T}Q^x - A_M P\| + \lambda \|\bar{X}_{1,T}Q^r - B_M P\| && \text{(24a)}\\
&\text{subject to} && \bar{X}_{0,T-1}Q^x = P, && \text{(24b)}\\
& && \bar{X}_{0,T-1}Q^r = \mathbf{0}_n, && \text{(24c)}\\
& && \begin{bmatrix} P & \bar{X}_{1,T}Q^x \\ (\bar{X}_{1,T}Q^x)' & P \end{bmatrix} \succ \mathbf{0}_{2\cdot n}, && \text{(24d)}
\end{aligned}$$

where $\lambda > 0$. If a solution to (24) is found, the control matrices are given by $K_x = \bar{U}_{0,T-1}Q^x P^{-1}$ and $K_r = \bar{U}_{0,T-1}Q^r P^{-1}$. We can term (24) a *certainty-equivalence* solution since the design is carried out as if the noise were zero, because of the use of averages. Note that, from (24) we exactly recover Theorem 3 as $N \to \infty$, since $\bar{W}_{0,T} = 0$.

Hereafter, we provide a stability result and some consideration on the properties of (24). To this end, we introduce the following assumption.

Assumption 2: The data satisfy the following

$$\begin{bmatrix} \mathbf{0}_{m,T} \\ \bar{V}_{0,T-1} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{m,T} \\ \bar{V}_{0,T-1} \end{bmatrix}' \preceq \gamma_1 \begin{bmatrix} \bar{U}_{0,T-1} \\ \bar{X}_{0,T-1} \end{bmatrix} \begin{bmatrix} \bar{U}_{0,T-1} \\ \bar{X}_{0,T-1} \end{bmatrix}' \tag{25a}$$

$$\bar{V}_{1,T}\bar{V}_{1,T}' \preceq \gamma_2 \bar{X}_{1,T}\bar{X}_{1,T}' \tag{25b}$$

for some $\gamma_1 \in (0, 0.5)$ and $\gamma_2 > 0$.

Under this hypothesis, we can provide sufficient conditions for closed-loop stability under noisy data as follows.

Theorem 4: Consider system (1) and a reference model as in (2). Suppose that Assumption 2 holds and that (24) is feasible. Let (Q^x, Q^r, P) be any solution and let α and β be positive constants such that

$$\Xi + \alpha \bar{X}_{1,T}(\bar{X}_{1,T})' \preceq 0, \quad M \preceq \beta I, \tag{26}$$

where $\Xi = \bar{X}_{1,T}M\bar{X}_{1,T}' - P$, $\quad M = Q^x P^{-1}(Q^x)'$. If

$$\frac{6\gamma_1 + 3\gamma_2}{1 - 2\gamma_1} < \frac{\alpha^2}{2\beta(2\beta + \alpha)}, \tag{27}$$

then $K_x = \bar{U}_{0,T-1}Q^x P^{-1}$ ensures closed-loop stability.

The proof follows the same steps as the one of [8, Theorem 5] and [8, Corollary 1], so we will omit it. Note that α and β can be found in practice by bisection.

Besides recovering Theorem 3 as $N \to \infty$, the averaging strategy plays a key role also for finite N, as remarked next.

1) Hypothesis fulfillment: Consider $v \sim \mathcal{N}(0, \sigma^2 I_n)$. As shown in [9, Lemma 9], for any $\mu > 0$ it holds that[2]

$$\|\bar{V}_{0,T-1}\| \leq \sigma \sqrt{\frac{T}{N}} \left(1 + \mu + \sqrt{\frac{n}{T}}\right) \tag{28}$$

with probability at least $1 - e^{-T\mu^2/2}$. This inequality suggests that Assumption 2 is eventually satisfied if the experiments ensure that $\bar{U}_{0,T-1}$, $\bar{X}_{0,T-1}$ and $\bar{X}_{1,T}$ do not vanish as N increases. This is the case of *repeated* experiments, when it is simple to see that for any probability level there exists a finite N such that Assumption 2 is satisfied as long as $\mathcal{U}_{T-1}$ is persistently exciting, *c.f.* [9, Section 5.1].

2) Hypothesis verification: Since the right-hand sides of (25) are known from data, Assumption 2 can be checked whenever an upperbound on the noise is known. For Gaussian noise, (28) makes it possible to have high-confidence bounds on the noise, hence on the probability that Assumption 2 holds. Note that, to have high confidence bounds we need $T\mu^2$ large. Averaging allows to take μ^2 large and compensate its effect with N.

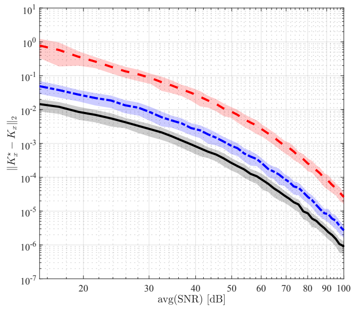
3) Problem complexity: Averaging allows us to benefit from large datasets. Instead of running longer experiments, our strategy works by using several tests. This enables us to keep the number of constraints equal to the one in (22), not increasing the complexity of the data-driven problem.

VI. Numerical examples

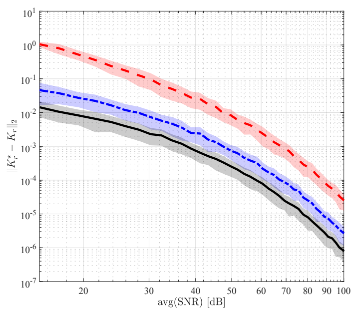
We consider two third-order systems with three inputs, one open-loop stable and the other unstable. Our goal is to decouple the dynamics of each state and enforce closed-loop stability. The examples have been designed so that the matching conditions in (3) are fulfilled by a unique pair of optimal gains $\boxed{K_x^\star}$ and $\boxed{K_r^\star}$. Since perfect matching can in principle be achieved, in (24) we minimize L_1-norms, with $\lambda = 1$. Within each scenario, the performance is assessed by carrying out 100 Monte Carlo simulations for increasing

[2] An analogous bound holds for $\bar{V}_{1,T}$.

$\Sigma\, S(\gamma, j)$



(a) $\|K_x - K_x^\star\|_2$ *vs* average SNR over states



(b) $\|K_r - K_r^\star\|_2$ *vs* average SNR over states

Fig. 2. Stable system: Average (solid lines) and standard deviation (colored areas) of the norms of the gain estimation errors over Monte Carlo runs resulting in a stable closed-loop for N=1 (red), N=100 (blue) and N=1000 (black).

levels of noise, which is quantified via the *Signal-to-Noise Ratio* (SNR) over the three output channels, *i.e.,*

$$\mathrm{SNR}_j = 10\log\left(\frac{\sum_{t=0}^{T}(x_j^{\mathrm{o}}(t))^2}{\sum_{t=0}^{T}(v_j^{(i)}(t))^2}\right), \ [dB] \tag{29}$$

with $j=1,2,3$ and $i=1,\dots,N$. Independently of the framework, the sequence $\{v(t)\}_{t=0}^{T}$ is zero-mean and Gaussian distributed, *i.e.,* $v \sim \mathcal{N}(0,\sigma^2 I)$, with increasing variance so as to span the interval SNR $\in [3,100]$ dB. By considering initial datasets of length $T=30$ samples, we also evaluate the matching performance for a growing number N of experiments, from $N=1$ up to $N=1000$. These conditions correspond to datasets comprising a minimum of 30 state samples, up to $30\cdot 10^3$ state points. The design problem was solved with the CVX package [15], by imposing

$$\begin{bmatrix} P & \bar{X}_{1,T}Q^x \\ (\bar{X}_{1,T}Q^x)' & P \end{bmatrix} \succeq 10^{-10}\cdot I_{2\cdot n}.$$

A. Model-reference control of a stable system

We initially consider a randomly generated open-loop stable system of the form in (1), characterized by the matrices

$$\begin{aligned} A &= \begin{bmatrix} 0.1344 & 0.2155 & -0.1084 \\ 0.4585 & 0.0797 & 0.0857 \\ -0.5647 & -0.3269 & 0.8946 \end{bmatrix}, \\ B &= \begin{bmatrix} 0.9298 & 0.9143 & -0.7162 \\ -0.6848 & -0.0292 & -0.1565 \\ 0.9412 & 0.6006 & 0.8315 \end{bmatrix}, \end{aligned} \tag{30}$$

with A having three distinct real eigenvalues $\{0.9536, -0.2118, 0.3670\}$. Our aim is to design a static law (4) such that the behavior of the reference model in (2) with $A_M = 0.2\cdot I_3$ and $B_M = 0.8\cdot I_3$ is matched. This choice allows us to speed up the dynamics of the open-loop system and to attain zero steady-state error when tracking step like references. Perfect matching is achieved with $K_x^\star$ and $K_r^\star$ reported in [5]. To retrieve them from data, the system is fed with a random input sequence uniformly distributed within $[-2,2]$, which guarantees that the plant is persistently excited.

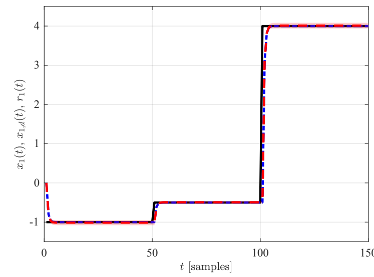


Fig. 3. Stable system: reference (solid black), desired response (dotted dashed blue) *vs* average (dashed red) closed-loop output for $N=100$ over the first channel. The red area indicates the standard deviation with respect to the average response. The other channels share similar behaviors.

As shown in Figure 2, the lower the noise corrupting the state measurements is, the better the quality of the retrieved gains results. Moreover, an increasing number N of experiments leads to a considerable reduction in the error between the optimal and data-driven gains, as we get closer to the limit condition. Indeed, the plant is never destabilized when $N \geq 100$, even for an average SNR ≈ 4 dB, while the attained closed-loop is always stable for $N \geq 2$ whenever the average SNR over the states is above 11 dB.

We then assess the actual matching performance, by comparing the desired and achieved closed-loop output for a piecewise constant reference. This comparison is shown in Figure 3, where we focus on the controllers retrieved for an average SNR $\in [11.2, 14.3]$ dB. The desired and closed-loop behavior closely match on average and the variance over the Monte Carlo runs is almost negligible on transients, with a slight mismatch occurring at steady-state.

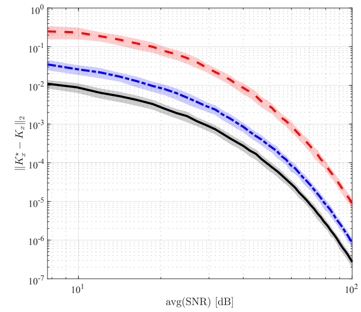
B. Model-reference control of an unstable system

We now consider the system introduced in [10], with

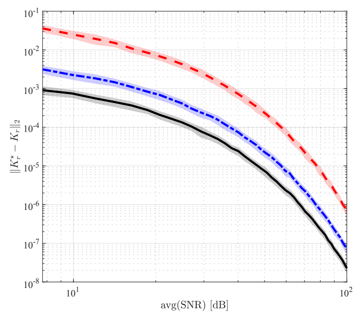
$$A = \begin{bmatrix} 1.01 & 0.01 & 0 \\ 0.01 & 1.01 & 0.01 \\ 0 & 0.01 & 1.01 \end{bmatrix}, \quad B = I_3. \tag{31}$$

In this case, the matrices of the reference model in (2) are chosen as $A_M = 0.9\cdot I_3$ and $B_M = 0.1\cdot I_3$, so as to dictate a stable behavior and guarantee that step references are perfectly tracked. This reference model is matched with $K_x^\star$ and $K_r^\star$ shown in [5]. Due to the open-loop instability of the plant, experiments are carried out in closed-loop by stabilizing the system with a static controller of the same form as (4), with $K_x = -I_n$ and $K_r = I_n$. The reference to be tracked is selected as a sequence of uniformly distributed samples within $[-5,10]$ generated at random.

Figure 4 shows that, for an increasing level of noise and different N, larger sets of data lead to a better reconstruction of the matching gains, as the average formulation in



(a) $\|K_x - K_x^\star\|_2$ *vs* average SNR over states



(b) $\|K_r - K_r^\star\|_2$ *vs* average SNR over states

Fig. 4. Unstable system: Average (solid lines) and standard deviation (colored areas) of the norm of the gain estimation errors over the Monte Carlo runs resulting in a stable closed-loop for $N = 1$ (red), $N = 100$ (blue) and $N = 1000$ (black).

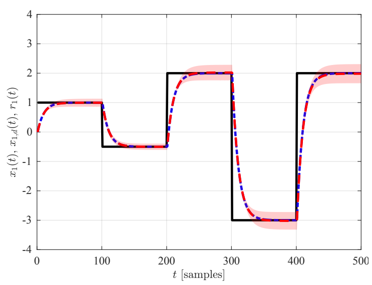


Fig. 5. Unstable system: reference (solid black), desired response (dotted dashed blue) *vs* average (dashed red) closed-loop output for $N = 100$ over the first channel. The red area indicates the standard deviation with respect to the average response. The other components share similar features.

(24) increasingly resemble the noiseless ones. This result is aligned with the reduction in the number of unstable instances obtained when more experiments are performed. Note that the controllers are always stabilizing for datasets yielding average SNRs above 20 dB, independently of N.

When the average SNR $\in [10.33, 13.53]$ dB, as shown in Figure 5, the desired behavior and the attained one match exactly on average, when a piecewise constant reference is considered. While the transient behavior is generally not affected by differences in the realization of the dataset, there is a variation in the steady state values of the closed-loop output over the 100 gains retrieved. This problem can be handled with an integrator, which we aim at introducing in future works.

VII. Conclusions

We have introduced a data-driven design strategy for model-reference control guaranteeing Lyapunov stability. Unlike existing data-based approaches, the one proposed here allows us to obtain non-asymptotic stability guarantees also in case of unfeasible perfect matching and noisy measurements. Numerical studies have confirmed the effectiveness of the proposed strategy.

Future work will be devoted to the analysis of scenarios where the order of the reference model does not coincide with that of the plant and to introduce automatic reference model tuning procedures, inspired by [6].

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