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Calculation of some Étale cohomology groups.

Theorem 1:

$$H^p_{et}(X, \mathbb{G}_m) = \begin{cases} \Gamma(X, \mathcal{O}_X^*), & p=0 \\ \mathrm{Pic}(X), & p=1 \\ 0, & n \geq 2 \end{cases}$$

for X a ~~scheme~~ curve: separated, integral scheme of finite type over a field k of dimension 1.

idea: step 1: There is a short exact sequence

$$0 \longrightarrow \mathbb{G}_{m,X} \longrightarrow j_* \mathbb{G}_{m,\eta} \longrightarrow \bigoplus_{x \in |X|} i_{x*}\mathbb{Z} \longrightarrow 0 \quad (*)$$

Step 2: Lemma 1: $H^p_{et}(X, j_* \mathbb{G}_{m,\eta}) = 0$ for $p \geq 1$,

Lemma 2: $H^p_{et}(X, \bigoplus_{x \in |X|} i_{x*}\mathbb{Z}) = 0$ for $p \geq 1$,

where $j: \eta \longrightarrow X$, η is the generic point of X, $i: \{x\} \longrightarrow X$.

Then the long exact sequence associated by $(*)$ is

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X^*) \longrightarrow k(X)^* \longrightarrow \Gamma(X, \bigoplus_{x \in |X|} i_{x*}\mathbb{Z}) \longrightarrow$$
$$H^1_{et}(X, \mathbb{G}_{m,X}) \longrightarrow H^1_{et}(X, j_* \mathbb{G}_{m,\eta}) \longrightarrow H^1_{et}(X, \bigoplus i_{x*}\mathbb{Z}) \longrightarrow \cdots$$

It follows that $H^p_{et}(X, \mathbb{G}_m) = 0$ for $p \geq 2$. The long exact sequence then reduces to

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X^*) \longrightarrow k(X)^* \longrightarrow \mathrm{Div}(X) \longrightarrow 0.$$

Therefore, $H^0_{et}(X, \mathbb{G}_m) = \Gamma(X, \mathcal{O}_X^*)$, $H^1_{et}(X, \mathbb{G}_m) = \mathrm{Pic}(X)$.

Lemma 2: i_{x*} is exact $\Rightarrow$ $\cancel{H^p_{et}(X, \mathrm{Div})} =$

$$H^p_{et}(X, \bigoplus i_{x*}\mathbb{Z}) = H^p_{et}(x, \mathbb{Z}) = 0.$$

$\to$ Galois cohomology $\quad K_y$???

Lemma 1: $(R^p j_* \mathbb{G}_{m,\eta})_y = H^p(\operatorname{Spec} K_y, \mathbb{G}_m)$
$= 0, \quad p \geq 1$

$\Rightarrow R^p j_* \mathbb{G}_{m,\eta} = 0, \ p \geq 1$

Therefore, $H^p(*, \mathbb{G}_m) = H^p_{et}(X, j_* \mathbb{G}_{m,\eta}) = 0$

Theorem 2:

$$H^p_{et}(X, \mu_n) = \begin{cases} \mu_n, & p = 0 \\ \operatorname{Pic}^0(X)[n], & p = 1 \\ \mathbb{Z}/n\mathbb{Z}, & p = 2 \\ 0, & p \geq 3 \end{cases} \qquad k \text{ alg. closed}$$

for X a curve.

idea: Step 1: Kummer sequence $\quad 0 \to \mu_n \to \mathbb{G}_m \xrightarrow{(\cdot)^n} \mathbb{G}_m \to 0 \quad (*)$
where n is invertible (e.g. over k $n \nmid \operatorname{char} k$)
Step 2: l.e.s associated to $(*) \Rightarrow H^p_{et}(X, \mu_n) = 0$ for $p \geq 3$, as $H^p_{et}(X, \mathbb{G}_m) = 0$ for $p \geq 2$. So l.e.s reduces to

$$0 \to \mu_n \to \underbrace{\Gamma(X, \mathcal{O}_X^*)}_{k^*} \xrightarrow{(\cdot)^n} \underbrace{\Gamma(X, \mathcal{O}_X^*)}_{k^*} \to H^1_{et}(X, \mu_n) \to \operatorname{Pic}(X)$$
$$\xrightarrow{(\cdot)^n} \operatorname{Pic}(X) \to H^2_{et}(X, \mu_n) \to 0.$$

Consider

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Pic}^0(X) \arrow[r] \arrow[d, "(\cdot)^n = \alpha"] & \operatorname{Pic}(X) \arrow[r, "\deg"] \arrow[d, "(\cdot)^n = \beta"] & \mathbb{Z} \arrow[r] \arrow[d, "\cdot n = \theta"] & 0 \\
0 \arrow[r] & \operatorname{Pic}^0(X) \arrow[r] & \operatorname{Pic}(X) \arrow[r, "\deg"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

abelian variety (Mumford.)

BATMAN

Snake lemma $\Rightarrow$

$$0 \to \ker\alpha \to \ker\beta \to \ker\theta \to \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\theta \to 0$$

$$\ker\theta = 0, \quad \operatorname{coker}\alpha = 0$$

$$\Rightarrow \ker\alpha \xrightarrow{\sim} \ker\beta, \quad \operatorname{coker}\beta \xrightarrow{\sim} \operatorname{coker}\theta$$

$$\ker\alpha = \operatorname{Pic}^0(X)[n], \quad \ker\beta = H^1_{et}(X, \mu_n), \quad \operatorname{coker}\beta = H^2_{et}(X, \mu_n), \quad \operatorname{coker}\theta = \mathbb{Z}/n\mathbb{Z}$$

$\Rightarrow$ Theorem 2.

Theorem 3:

$$H^p_{et}(X, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z}, & p=0 \\ (\mathbb{Z}/n\mathbb{Z})^{2g}, & p=1 \\ \mathbb{Z}/n\mathbb{Z}, & p=2 \\ 0, & p\geq 3 \end{cases}$$

Proof: $\operatorname{Pic}^0(X)[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$ (abelian variety)

$$H^p_{et}(X, R^q j_* \mathbb{Q}_\eta) \Rightarrow H^{p+q}_{et}(\eta, \mathbb{Q})$$

~~Galois cohomology~~

$$R^q j_* \mathbb{Q}_\eta = 0 \;?? \Rightarrow H^p(\eta, \mathbb{Q}) = H^p(X, j_*\mathbb{Q}) = 0$$

$$(R^q j_* \mathbb{Q}_\eta)_y = H^q(\operatorname{Spec} K_y, \mathbb{Q})$$ = Galois cohomology

$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$

$\Rightarrow$ long exact sequence:

$$0 \to H^0_{et}(X,\mathbb{Z}) \to H^0_{et}(X,\mathbb{Q}) \to H^0_{et}(X,\mathbb{Q}/\mathbb{Z}) \to$$
$$H^1_{et}(X,\mathbb{Z}) \to H^1_{et}(X,\mathbb{Q}) \to H^1_{et}(X,\mathbb{Q}/\mathbb{Z}) \to \cdots$$

Since $H^p_{et}(X,\mathbb{Q}) = 0$ for $p > 0$, then

$$H^p_{et}(X,\mathbb{Q}/\mathbb{Z}) \cong H^{p+1}_{et}(X,\mathbb{Z}) \text{ for } p \geq 1.$$

$H^p_{et}(X,\mathbb{Q}) = 0$ is done by the same method as Thm.

$H^p_{et}(X,\mathbb{Q}/\mathbb{Z})$ can be calculated via n-torsion.