Convex Optimization Homework

Intereswing

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1 Homework 1

1

$$\nabla^2 f(x) = H \in \mathbf{R}^{n\times n}, \text{ where } H_{ij} = \frac{1}{n^2 x_i x_j}(x_1 \cdot x_2 \cdot \cdots \cdot x_n)^{\frac{1}{n}} \text{ for } i \neq j$$
$$\text{and } H_{ii} = \frac{1-n}{n^2 x_i^2}(x_1 \cdot x_2 \cdot \cdots \cdot x_n)^{\frac{1}{n}}$$

So $H = \frac{1}{n^2}(x_1 \cdot x_2 \cdot \cdots \cdot x_n)^{\frac{1}{n}}([\frac{1}{x_1}\ \frac{1}{x_2}\ \ldots\ \frac{1}{x_n}]^\top [\frac{1}{x_1}\ \frac{1}{x_2}\ \ldots\ \frac{1}{x_n}] - n\mathbf{diag}(\frac{1}{x_1^2}, \frac{1}{x_2^2}, \ldots, \frac{1}{x_n^2}))$.
For any $\omega \in \mathbf{R}^n$,

$$w^\top H w = \frac{1}{n^2}(x_1 \cdot x_2 \cdot \cdots \cdot x_n)^{\frac{1}{n}}((\sum_{i=1}^n \frac{w_i}{x_i})^2 - \sum_{i=1}^n 1^2 \sum_{i=1}^n \frac{w_i^2}{x_i^2}) \leq 0$$

, which follows from the Cauchy-Schwarz inequality.

So $H \preceq 0$. Consequently, $f(x)$ is concave on $\mathbf{dom} f = \mathbf{R}^n_{++}$.

2 Let $A = \{x \mid \|x\|_2 \leq 2\}$. For any $x_1, x_2 \in A$ and any θ with $0 \leq \theta \leq 1$, we have

$$\|\theta x_1 + (1-\theta)x_2\|_2 \leq \|\theta x_1\|_2 + \|(1-\theta)x_2\|_2 = \theta\|x_1\|_2 + (1-\theta)\|x_2\|_2 \leq 2\theta + 2(1-\theta) = 2$$

So $\theta x_1 + (1-\theta)x_2 \in A$, which means A is convex. Thus $\mathbf{dom} f = \mathbf{R}^n_{++} \cap A$ is also convex.

Similarly to problem 1, we can conclude that $f(x)$ is concave on $\mathbf{dom} f = \mathbf{R}^n_{++} \cap \{x \mid \|x\|_2 \leq 2\}$.

3 For any $x_1, x_2 \in \mathbf{R}$ and any θ with $0 \leq \theta \leq 1$, we have

$$|\theta x_1 + (1-\theta)x_2|^p \leq ||\theta x_1| + |(1-\theta)x_2||^p = (\theta|x_1| + (1-\theta)|x_2|)^p$$

If $x_1 x_2 = 0$, supposing $x_1 = 0$,

$$|\theta x_1 + (1-\theta)x_2|^p = (1-\theta)^p|x_2|^p \leq \theta|x_1|^p + (1-\theta)|x_2|^p$$

On the other hand, if $x_1 x_2 \neq 0$, we consider $g(x) = x^p$ on $\mathbf{dom} g = \mathbf{R}_{++}$. $g''(x) = p(p-1)x^{p-2} \geq 0$, so $g(x)$ is convex. Thus for any $v_1, v_2 > 0$, and any θ with $0 \leq \theta \leq 1$, we have

$$(\theta v_1 + (1-\theta)v_2)^p \leq \theta v_1^p + (1-\theta)v_2^p$$

Thus

$$|\theta x_1 + (1-\theta)x_2|^p \leq (\theta|x_1| + (1-\theta)|x_2|)^p \leq \theta|x_1|^p + (1-\theta)|x_2|^p$$

In summary, $|\theta x_1 + (1-\theta)x_2|^p \leq \theta|x_1|^p + (1-\theta)|x_2|^p$. Thus $f(x) = |x|^p$ is convex.