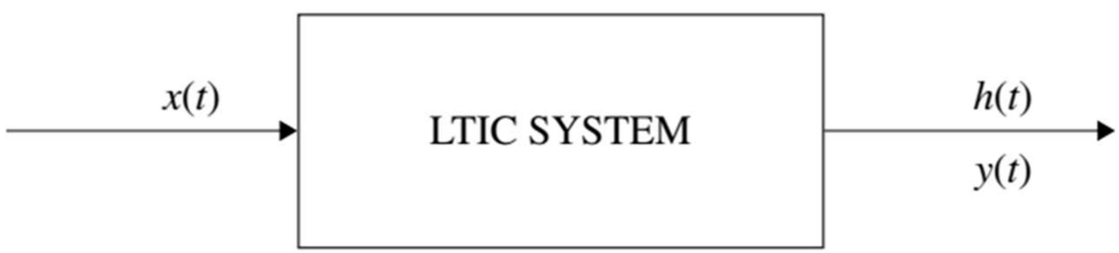
Convolution : convolution is a mathematical operation that defines the relationship between an input signal, the output signal, and the impulse response of a Linear Time-Invariant (LTI) system. It

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$



Step by Step Procedure to Solve Convolution

Step 1. Let $x(t)$ be the input signal and $h(t)$ the impulse response. Substitute $t = \tau$ where τ is an independent dummy variable and represent $x(\tau)$ and $h(\tau)$.

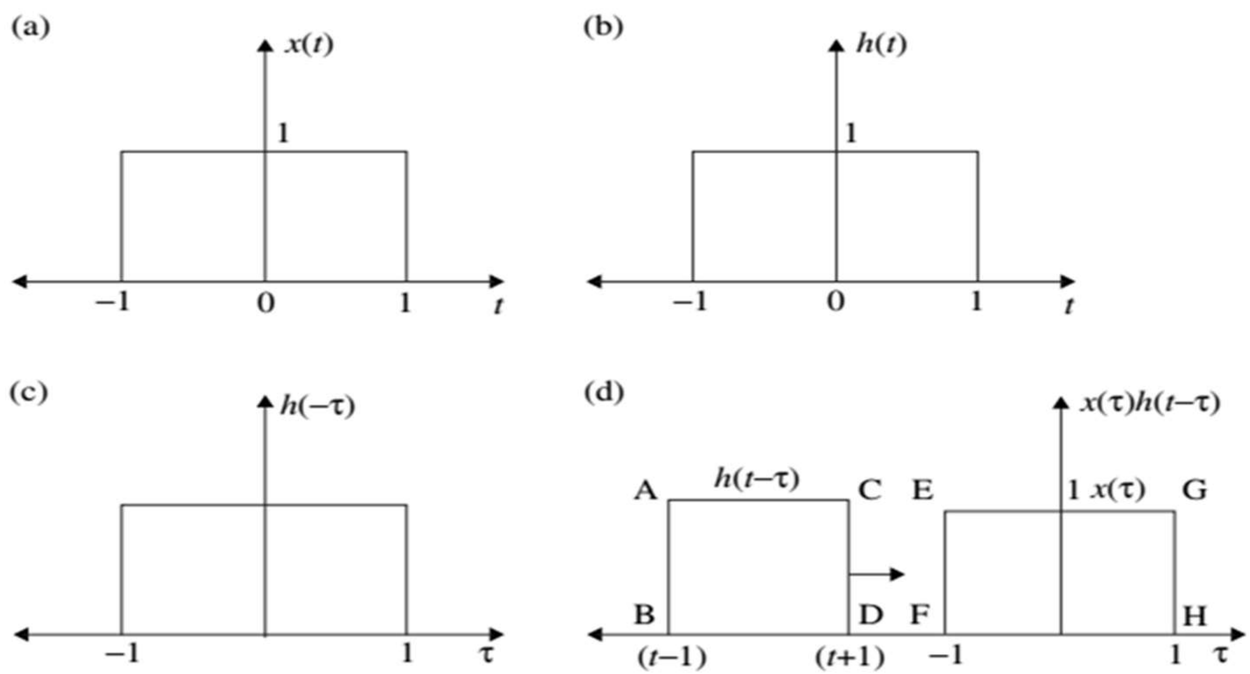
Step 2. Represent $x(\tau)$ in figure, invert $h(\tau)$ as $h(-\tau)$ and represent it in figure. This is called folding of $h(\tau)$. Shift the inverted $h(-\tau)$ along the τ axis and obtain $h(t-\tau)$ by giving very long negative shift.

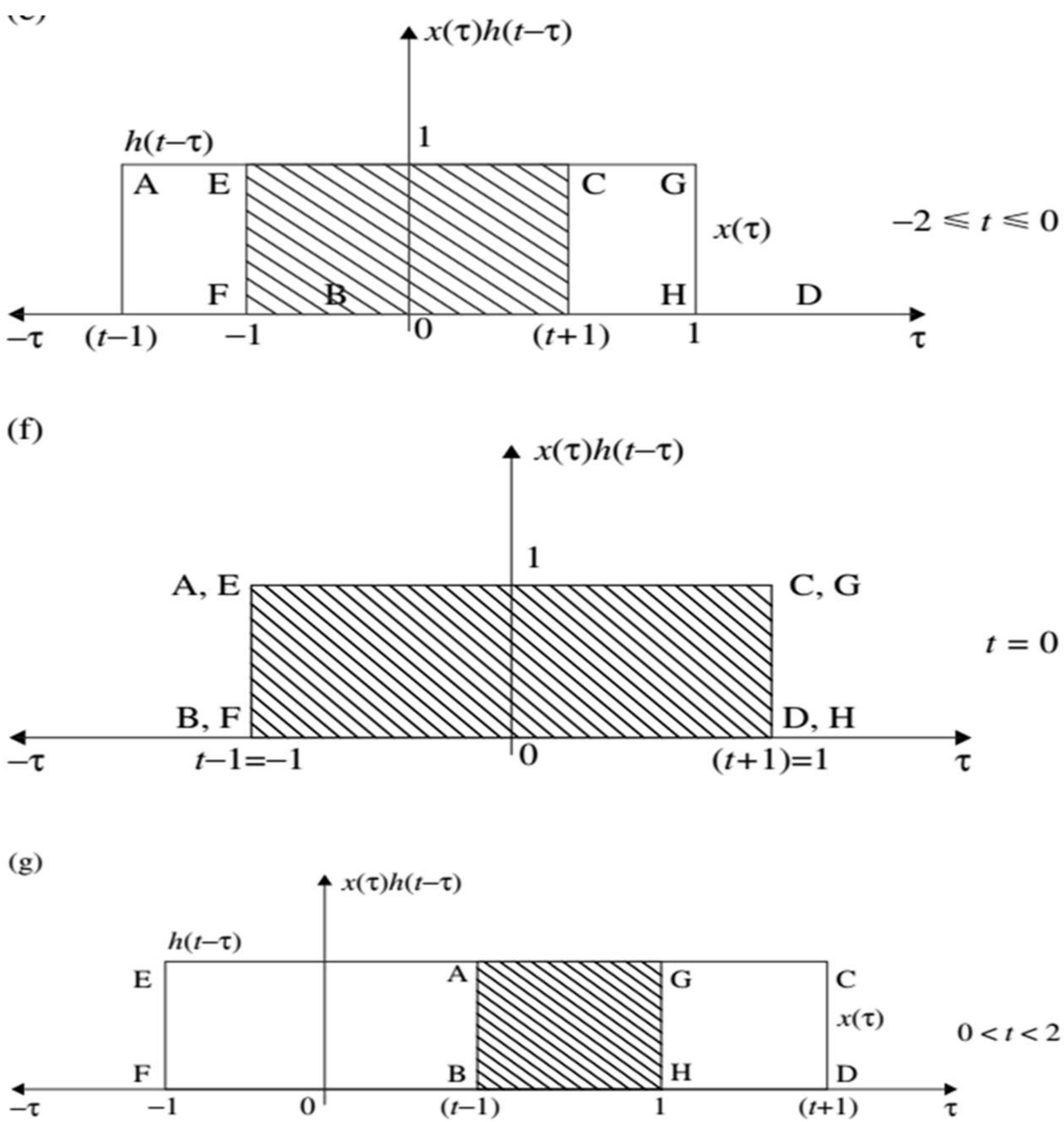
Step 3. Multiply the two signals $x(\tau)$ and $h(t-\tau)$ and integrate over the overlapping interval. For this $x(\tau)$ is fixed and $h(t-\tau)$ is moved toward the right so that $x(\tau)$ and $h(t-\tau)$ overlap.

Step 4. Whenever either $x(\tau)$ or $h(t-\tau)$ changes, the new time shift occurs. Identify the end of the current interval and the beginning of the new interval. The output response $y(t)$ is calculated using Step 3.

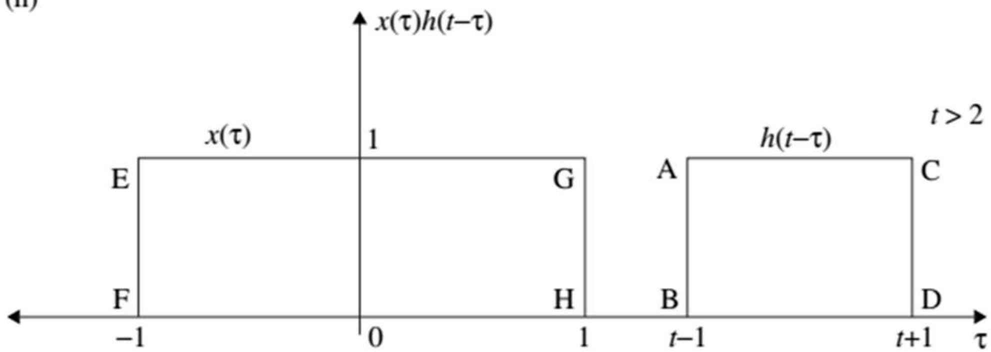
Step 5. Steps 3 and 4 are repeated for all intervals.

Find the convolution of the signal shown in Fig.   and b by graphical method.

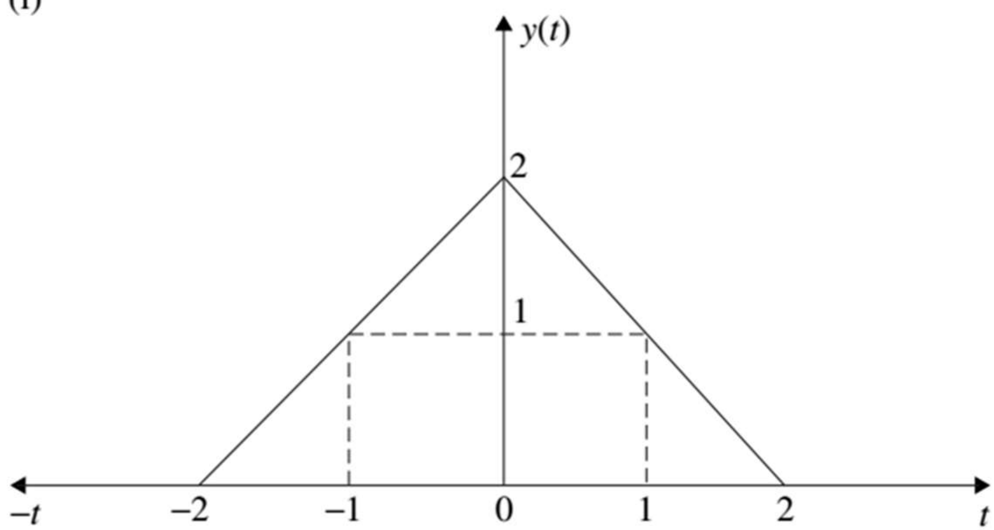



(e)
x(τ)h(t−τ)
h(t−τ)
1
A
E
C
G
x(τ)
−2 ⩽ t ⩽ 0
F
B
H
D
−τ
(t−1)
−1
0
(t+1)
1
τ
(f)
x(τ)h(t−τ)
1
A, E
C, G
t = 0
B, F
D, H
−τ
t−1=−1
0
(t+1)=1
τ
(g)
x(τ)h(t−τ)
h(t−τ)
E
A
G
C
x(τ)
0 < t < 2
F
B
H
D
−τ
−1
0
(t−1)
1
(t+1)
τ

(h)

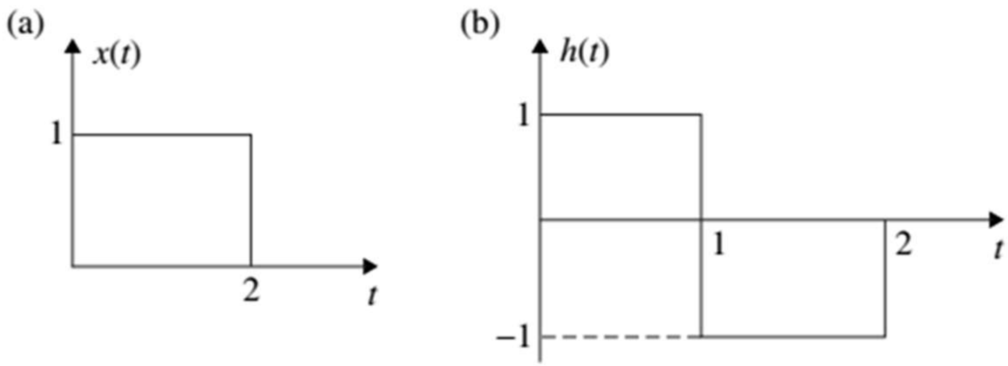


(i)



HW

Find the convolution of the two signals given below



$$x(t) = u(t) - u(t-4)$$
$$h(t) = u(t) - u(t-1)$$

$$x(t) = e^{-2|t|}$$
$$h(t) = u(t)$$

$$\begin{aligned} x(t) &= 2t && 0 < t < 1 \\ &= (3-t) && 1 < t < 3 \\ &= 0 && \text{elsewhere.} \end{aligned}$$
$$h(t) = u(t) - u(t-2)$$

Properties of Convolution

- The commutative property;
- The distributive property;
- The associative property;
- The shift property;
- The width property and
- The convolution with unit impulse.

The Commutative Property

if $y(t) = x(t) * h(t)$ then $y(t) = h(t) * x(t)$.

Proof

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Put $t - \tau = p$ and $d\tau = -dp$.

For $\tau = \infty$; $p = -\infty$; and $\tau = -\infty$; $p = \infty$

$$\begin{aligned} y(t) &= -\int_{+\infty}^{-\infty} x(t-p)h(p)dp \\ &= \int_{-\infty}^{\infty} x(t-p)h(p)dp \\ &= \int_{-\infty}^{\infty} h(p)x(t-p)dp \\ y(t) &= h(t) * x(t) \end{aligned}$$

The Distributive Property

$$x(t) * h_1(t) + x(t) * h_2(t) = x(t) * (h_1(t) + h_2(t))$$

Proof

We want to show for signals x, h, and g:

$$x * (h + g) = x * h \;+\; x * g.$$

By definition,

$$(x * (h + g))(t) \;=\; \int_{-\infty}^{\infty} x(\tau)\,[h + g](t - \tau)\,d\tau.$$

Use the fact that addition distributes inside the integrand:

$$[x(\tau)\,(h + g)(t - \tau)] = x(\tau)\,h(t - \tau) + x(\tau)\,g(t - \tau).$$

So

$$(x * (h + g))(t) = \int_{-\infty}^{\infty} \big(x(\tau)h(t - \tau) + x(\tau)g(t - \tau)\big)\,d\tau.$$

$$(x * (h + g))(t) = \int_{-\infty}^{\infty} \big(x(\tau)h(t - \tau) + x(\tau)g(t - \tau)\big)\, d\tau.$$

By linearity of the integral (you may split the integral into the sum of two integrals),

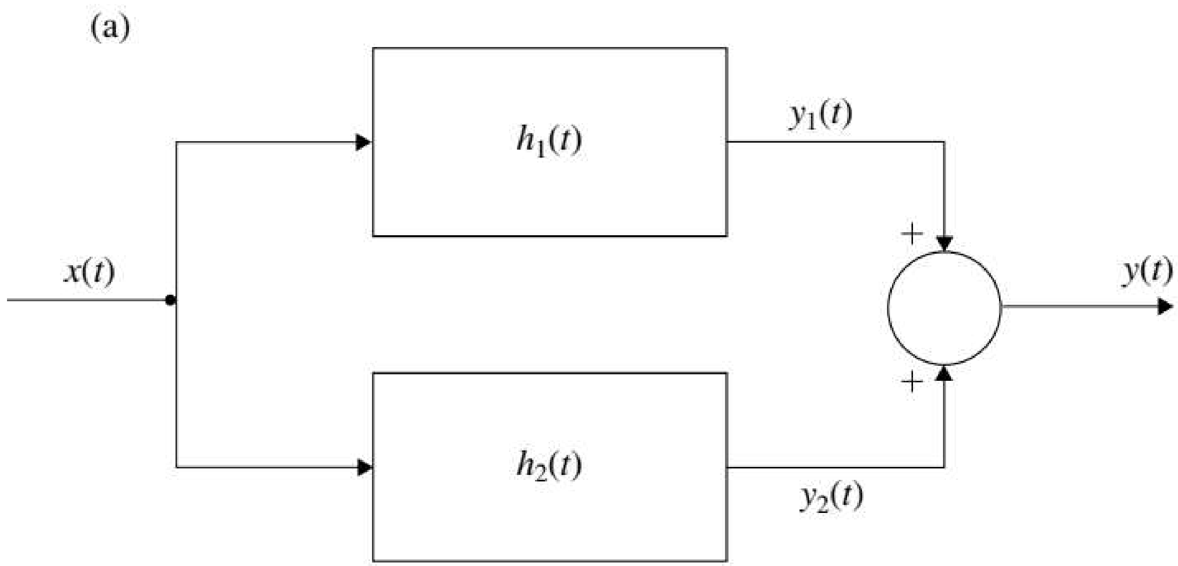
$$(x * (h + g))(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)\, d\tau \; + \; \int_{-\infty}^{\infty} x(\tau)g(t - \tau)\, d\tau.$$

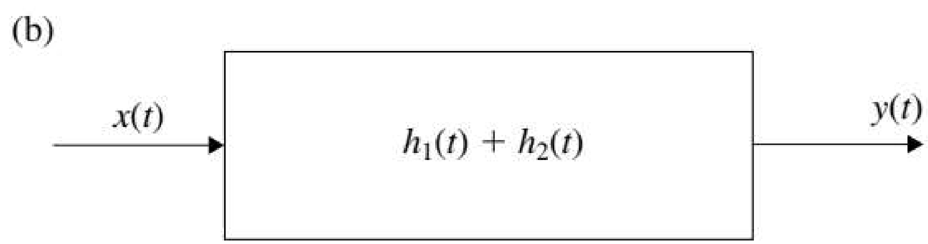
Recognize each integral as a convolution:

$$(x * (h + g))(t) = (x * h)(t) + (x * g)(t).$$

The Distributive Property

$$x(t) * h_1(t) + x(t) * h_2(t) = x(t) * (h_1(t) + h_2(t))$$



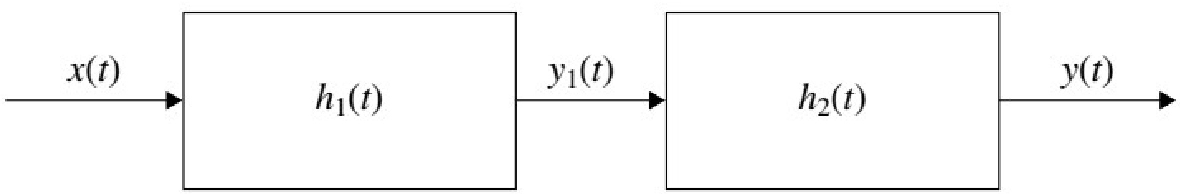


The Associative Property

$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$$

Proof

$$y_1(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h_1(t-\tau)d\tau$$



The output of the second system is

$$y(t) = y_1(t) * h_2(t) = \int_{m=-\infty}^{\infty} y_1(m)h_2(t-m)dm$$

$$y(t) = y_1(t) * h_2(t) = \int_{m=-\infty}^{\infty} y_1(m) h_2(t-m) dm$$

$$= \int_{m=-\infty}^{\infty} \left[\int_{\tau=-\infty}^{\infty} x(\tau) h_1(m-\tau) d\tau \right] h_2(t-m) dm$$

Put $m - \tau = q$ and $dm = dq$

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau) \left[\int_{q=-\infty}^{\infty} h_1(q) h_2[(t-(q+\tau)] dq \right] d\tau$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau) \left[\int_{q=-\infty}^{\infty} h_1(q) h_2[(t-\tau)-q] dq \right] d\tau$$

But

$$\int_{q=-\infty}^{\infty} h_1(q) h_2[(t-\tau)-q] dq = h_1(t) * h_2(t) = h(t-\tau)$$

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau) \left[\int_{q=-\infty}^{\infty} h_1(q) h_2[(t-\tau) - q] dq \right] d\tau$$

$$\begin{aligned} y(t) &= \int_{\tau=-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \\ &= x(t) * h(t) \\ y(t) &= x(t) * [h_1(t) * h_2(t)] \end{aligned}$$

$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$$

The Shift Property

$$x(t) * h(t - T) = y(t - T)$$

Proof

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

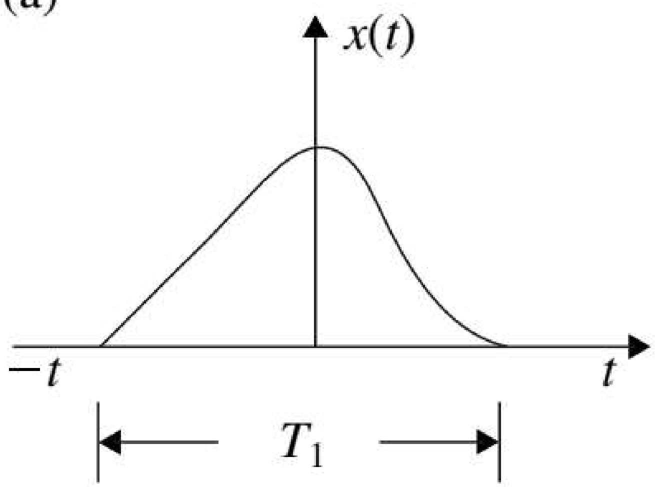
then

$$\begin{aligned} x(t) * h(t - T) &= \int_{-\infty}^{\infty} x(\tau)h(t - \tau - T)d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)h[(t - T) - \tau]d\tau \\ &= y(t - T) \end{aligned}$$

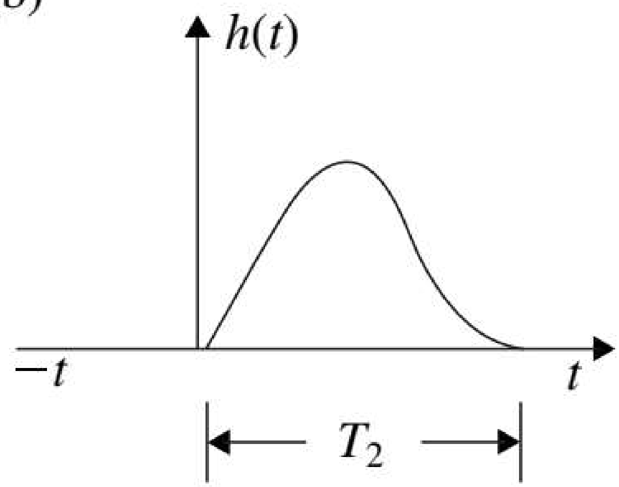
The Width Property

Let T_1 be the width of $x(t)$ and T_2, the width of $h(t)$. The width of $y(t) = x(t) * h(t)$ is $T_1 + T_2$.

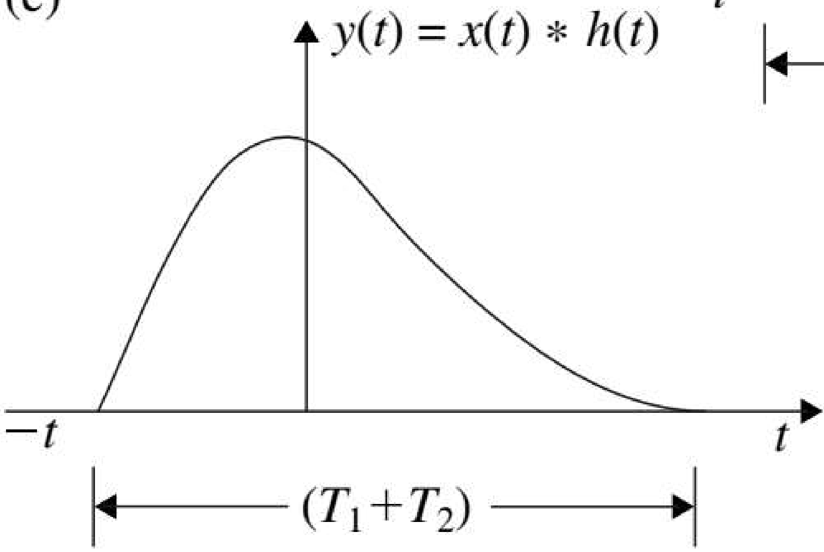
(a)



(b)



(c)



Find the output of a LTIC system with the impulse responses $h(t) = \delta(t-3)$ and $x(t) = (\cos 4t + \cos 7t)$.

Solution:

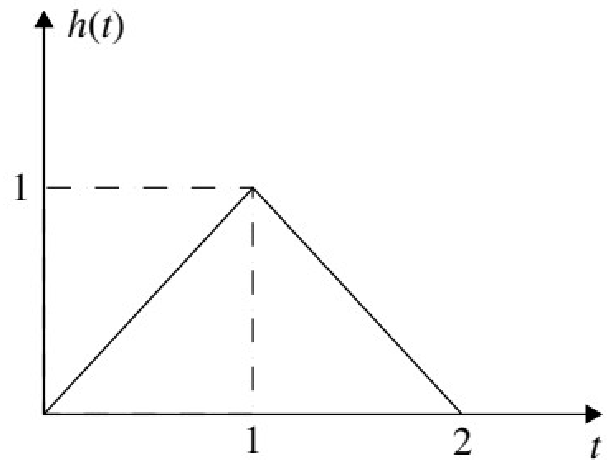
According to shifting property of convolution

$$\begin{aligned} y(t) &= x(t) * \delta(t - t_0) \\ &= x(t - t_0) \end{aligned}$$

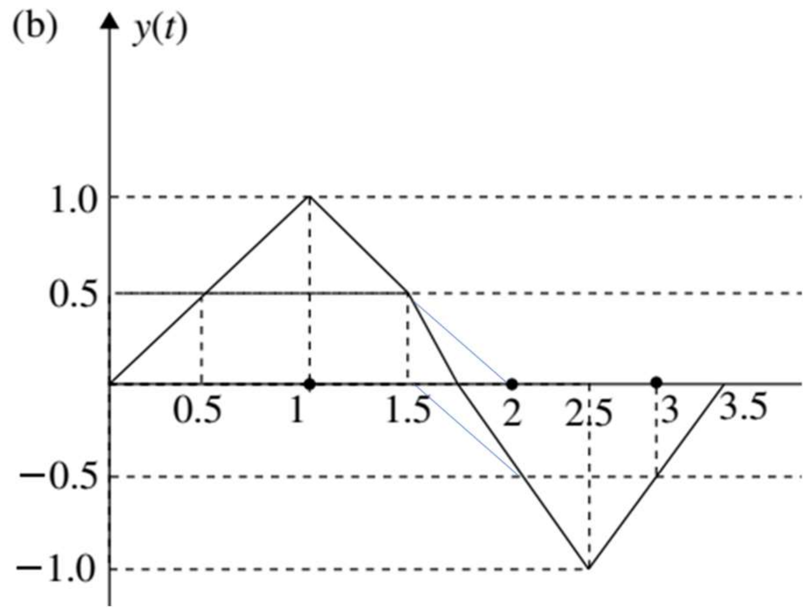
$$y(t) = \cos 4(t-3) + \cos 7(t-3)$$

The impulse response of an LTIC system is shown in Fig. a. The input $x(t) = \delta(t) - \delta(t - 1.5)$. Find the response $y(t)$ of the system.

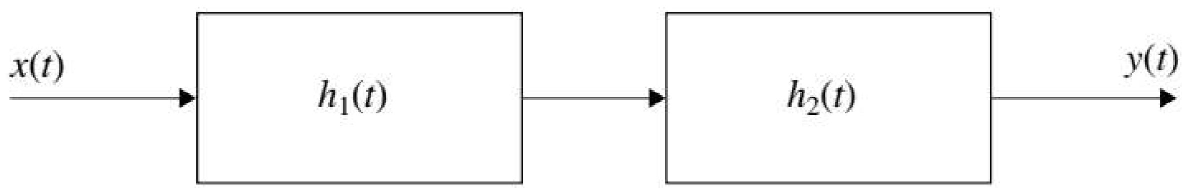
(a)



(b)



The system shown in Fig.   is formed in connecting two systems in cascade. The impulse response of the systems are given by $h_1(t) = e^{-2t}u(t)$ and $h_2(t) = 2e^{-t}u(t)$. Find the overall impulse response of the system.



Solution:

$$h(t) = h_1(t) * h_2(t)$$

$$h(t) = \int_0^t 2e^{-2\tau} e^{-(t-\tau)} d\tau$$

$$= 2e^{-t} \int_0^t [e^{-\tau}] d\tau$$

$$= -2e^{-t} \left[e^{-\tau} \right]_0^t$$

$$h(t) = -2e^{-t}[e^{-t} - 1]$$

The impulse response of a certain system is given by

$$h(t) = e^{-3t}u(t-2)$$

Determine the stability of the system.

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} |h(t)| dt \\ &= \int_{2}^{\infty} e^{-3t} dt \\ &= -\frac{1}{3}\left[e^{-3t}\right]_2^{\infty} \\ &= \frac{1}{3}e^{-6} \end{aligned}$$

$$\boxed{y(t) = 8.262 \times 10^{-4} < \infty}$$

system is BIBO stable.

HW

The impulse response of an LTIC system is

$$h(t) = e^{-2t} \sin 3t\, u(t)$$

Determine whether the given system is BIBO stable.

BIBO stable.

A certain LTIC system has the following impulse response. Determine whether the system is BIBO stable.

$$h(t) = e^{-3t} \cos 2t\, u(t)$$

BIBO stable.

The impulse response of a certain LTIC system is given by

$$h(t) = te^{-2t}u(t)$$

Determine the BIBO stability of the system.

BIBO stable.

The Convolution Sum

Convolution sum is a way to compute the output of a discrete-time LTI system by “sliding” one sequence over another, multiplying where they overlap, and adding those products to get each output sample.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$y(n) = x(n) * h(n)$$

Properties of Convolution Sum

$x[n]*(h1[n]+h2[n]) = (x[n]*h1[n]) + (x[n]*h2[n])$

Let RHS

$$y(n) = y_1(n) + y_2(n)$$
$$= x(n) * h_1(n) + x(n) * h_2(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h_1(n-k) + \sum_{k=-\infty}^{\infty} x(k)h_2(n-k)$$

$$= \sum_{k=-\infty}^{\infty} x(k)[h_1(n-k) + h_2(n-k)]$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

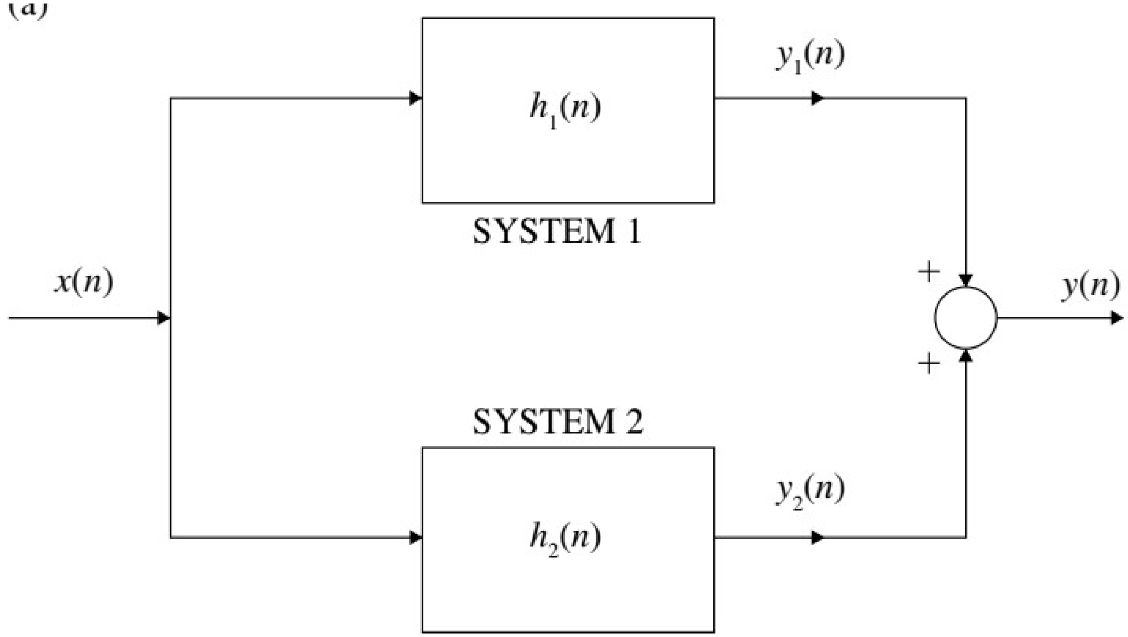
$$= x(n) * h(n)$$

$$y(n) = x(n) * [h_1(n) + h_2(n)]$$ LHS

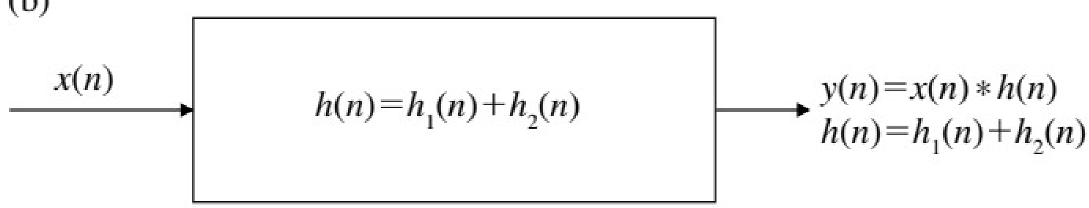
Associative Property of Convolution

$$y(n) = x(n) * [h_1(n) * h_2(n)] = [x(n) * h_1(n)] * h_2(n)$$

(a)



(b)



$$y_1(n) = x(n) * h_1(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k)h_1(n-k)$$

$$y_(n) = y_1(n) * h_2(n)$$

$$= \sum_{k=-\infty}^{\infty} y_1(k)h_2(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} x(p)h_1(k-p)h_2(n-k)$$

Putting $(k-p) = q$

$$y(n) = \sum_{p=-\infty}^{\infty} x(p) \sum_{q=-\infty}^{\infty} h_1(q)h_2(n-p-q)$$

$$= \sum_{p=-\infty}^{\infty} x(p) \qquad h(n-p)$$

$$= \sum_{p=-\infty}^{\infty} x(p)h(n-p)$$

$$\boxed{y(n) = x(n) * h(n)}$$

Commutative Property of Convolution

$$h_1(n) * h_2(n) = h_2(n) * h_1(n)$$

Proof

$$h_1(n) * h_2(n) = \sum_{k=-\infty}^{\infty} h_1(k) h_2(n-k)$$

Put $n - k = p$

$$h_1(n) * h_2(n) = \sum_{p=-\infty}^{\infty} h_1(n-p) h_2(p)$$

$$= \sum_{p=-\infty}^{\infty} h_2(p) h_1(n-p)$$

$$\boxed{h_1(n) * h_2(n) = h_2(n) * h_1(n)}$$

Shifting Property of Convolution

If $y[n] = (x * h)[n]$ and $x_1[n] = x[n - n_0]$ $(x_1 * h)[n] = y[n - n_0]$

Let $y[n] = (x * h)[n]$

Define $h_1[n] = h[n - n_0]$.

$$
\begin{aligned}
(x * h_1)[n] &= \sum_{k=-\infty}^{\infty} x[k]\, h_1[n-k] \\
&= \sum_{k=-\infty}^{\infty} x[k]\, h\big(n - k - n_0\big) \\
&= \sum_{k=-\infty}^{\infty} x[k]\, h\big((n - n_0) - k\big) \\
&= (x * h)[\, n - n_0\,] = y[n - n_0].
\end{aligned}
$$

The Width Property of Convolution

Let the width of x(n) be T1 and that of h(n) be T2.
Then, the width
of y(n) = x(n) ∗ h(n) is T = T1 + T2

Convolution with an Impulse

$$x(n) * \delta(n) = x(n)$$

$$x(n) * \delta(n - n_0) = x(n - n_0)$$

Response Using Convolution Sum

- — Using mathematical expression for the convolution sum
- — Multiplication method
- — Tabulation method.

- Graphical method

Using mathematical expression for the convolution sum

The impulse response $h(n)$ of a certain LTID system is given by $h(n) = a^n u(n)$ where $0 < a < 1$. The system is excited by $x(n) = u(n)$, a step sequence. Find $y(n)$ using convolution sum.

$$x(n) = u(n)$$

$$h(n) = a^n u(n)$$

$$y(n) = \sum_{k=-0}^{n} x(k)h(n-k)$$

$$= \sum_{k=-0}^{n} (a^{n-k})$$

$$= a^n \sum_{k=-0}^{n} \left(\frac{1}{a}\right)^k$$

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots,\ ar^{n-1}$$

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}}.$$

$$y(n) = \frac{a^n\left(1 - \frac{1}{a^{n+1}}\right)}{\left(1 - \frac{1}{a}\right)}$$

$$y(n) = \frac{(1-a^{n+1})}{(1-a)} u(n)$$

A linear time invariant discrete time system has the following impulse response.

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

The system is excited by the signal $x(n)$=$u(n)$ determine the output of the system

$$y(n) = x(n) * h(n)$$

$$= \sum_{k=0}^{n} x(n)h(k)$$

$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{n-k}$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} (2)^k$$

$$= \left(\frac{1}{2}\right)^n \frac{1-2^{n+1}}{(1-2)} \qquad = \left(\frac{1}{2}\right)^n (2^{n+1}-1)u(n)$$

$$x(n) = u(n)$$

$$h(n) = u(n-4)$$

$$y(n) = \sum_{k=0}^{n-4} x(k)h(n-k) \qquad n \geq 4$$

$$= \sum_{k=0}^{n-4} 1 = (n-3)$$

$$\boxed{y(n) = (n-3)} \qquad n \geq 4$$

HW

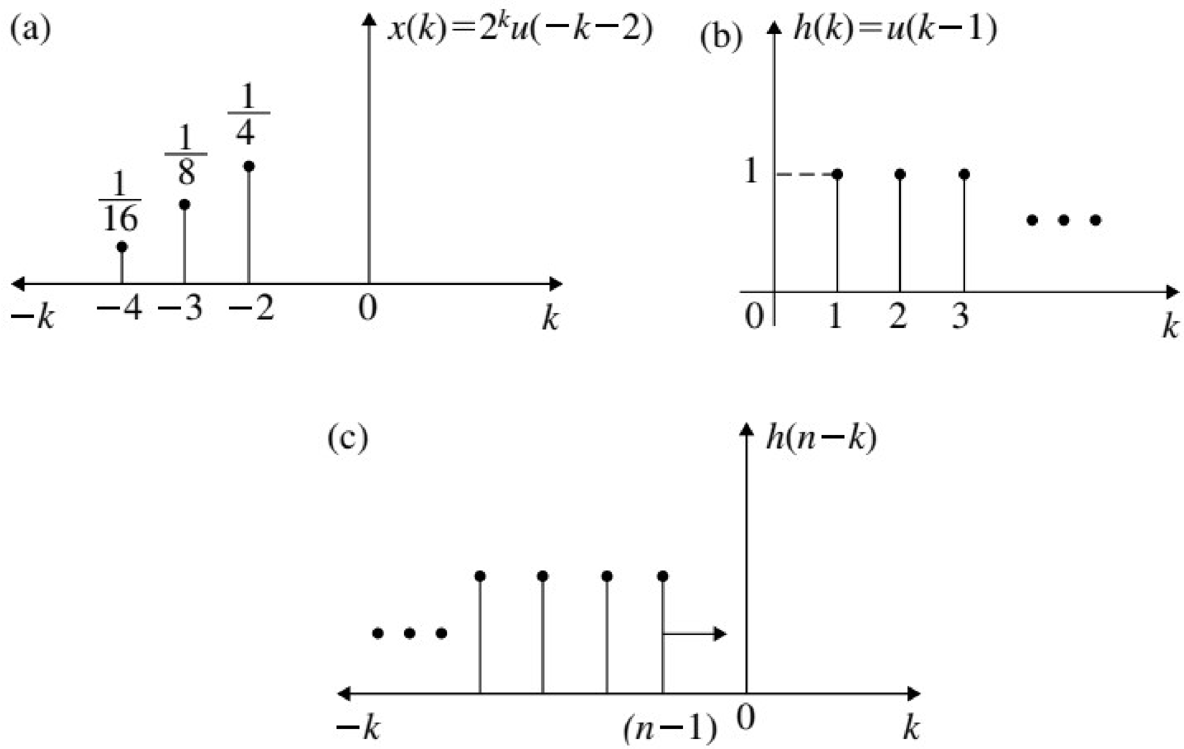
$$x(n) = u(n)$$
$$h(n) = (0.6)^n u(n)$$

$$x(n) = a^n u(n)$$
$$h(n) = b^n u(n)$$

$$x(n) = \left(\frac{1}{5}\right)^n u(n)$$
$$h(n) = 3^n u(n)$$

$$x(n) = 2^n u(-n-2)$$ Non causal

$$h(n) = u(n-1)$$



Time interval $-\infty < n < -1$.

$$y(n) = \sum_{k=-\infty}^{n-1} 2^k \qquad - \infty < n < -1$$
$$= 2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots$$

$$y(n) = 2^{n-1}\left[1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right]$$
$$= 2^{n-1}\left[\frac{1}{1-\frac{1}{2}}\right]$$

$$y(n) = 2^n \qquad - \infty < n < -1$$

Time interval $n > -1$.

$$y(n) = \sum_{k=-\infty}^{-2} 2^k \qquad n > -1$$

$$= \sum_{2}^{\infty} \left(\frac{1}{2}\right)^k$$

We know $\displaystyle \sum_{k=n}^{\infty} (a)^k = \frac{a^n}{(1-a)}$

$$y(n) = \left(\frac{1}{2}\right)^2 \frac{1}{1-\left(\frac{1}{2}\right)}$$

$$= 0.5 \qquad n > -1$$

$$\boxed{y(n) = 2^n} \qquad n < -1$$

$$\boxed{y(n) = 0.5} \qquad n > -1$$

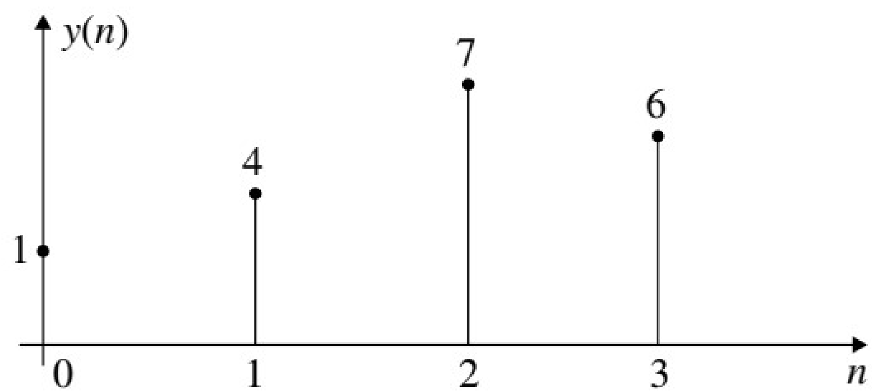
What is the response of an LTID system with impulse response $h(n) = \delta(n) + 2\delta(n-1)$ for the input $x(n) = \{1, 2, 3\}$?

$$\begin{aligned} h(n) &= \delta(n) + 2\delta(n-1) \\ x(n) &= \{1, 2, 3\} \\ &= \delta(n) + 2\delta(n-1) + 3\delta(n-2) \end{aligned}$$

$$\begin{aligned} y(n) &= h(n) * x(n) \\ &= h(n) * \delta(n) + h(n) * 2\delta(n-1) + h(n) * 3\delta(n-2) \\ &= y_1(n) + y_2(n) + y_3(n) \end{aligned}$$

$$y_1(n) = h(n) * \delta(n) = h(n)$$
$$y_2(n) = h(n) * 2\delta(n-1) = 2h(n-1)$$
$$y_3(n) = h(n) * 3\delta(n-2) = 3h(n-2)$$
$$y_1(n) = h(n) = \delta(n) + 2\delta(n-1)$$
$$y_2(n) = 2h(n-1) = 2\delta(n-1) + 4\delta(n-2)$$
$$y_3(n) = 3h(n-2) = 3\delta(n-2) + 6\delta(n-3)$$

$$y(n) = \delta(n) + 4\delta(n-1) + 7\delta(n-2) + 6\delta(n-3)$$



HW

Find the overall impulse response of the system

$$h_1(n) = \left(\frac{1}{3}\right)^n u(n)$$

$$h_2(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$h_3(n) = \left(\frac{1}{5}\right)^n u(n)$$

Convolution Sum of Two Sequences by Multiplication Method

$$x(n) = \{-1/2,\ \underset{\uparrow}{2},\ 1/3,\ 3/2\}$$

$$h(n) = \{1,\ -1/2,\ \underset{\uparrow}{2/3}\}$$

Here $N_1 = 1$; $N_2 = 2$ and $N_1 + N_2 = 3$;

$T_1 = 4,\ T_2 = 3$ and $T = 4 + 3 - 1 = 6$.

$$\begin{array}{cccccc}
 & & -\frac{1}{2} & 2 & \frac{1}{3} & \frac{3}{2} \\
 & & & 1 & -\frac{1}{2} & \frac{2}{3} \\
\hline
 & & -\frac{1}{3} & \frac{4}{3} & \frac{2}{9} & 1 \\
 & \frac{1}{4} & -1 & -\frac{1}{6} & -\frac{3}{4} & \\
-\frac{1}{2} & 2 & \frac{1}{3} & \frac{3}{2} & & \\
\hline
-\frac{1}{2} & \frac{9}{4} & -1 & \frac{8}{3} & -\frac{19}{36} & 1 \\
\hline
\end{array}$$

$$y(n) = \left\{-\frac{1}{2},\ \ \frac{9}{4},\ \ -1,\ \ \underset{\uparrow}{\frac{8}{3}},\ \ -\frac{19}{36},\ \ 1\right\}$$

HW

Find the convolution of the following

$$x(n) = u(n) - 3u(n-2) + 2u(n-4)$$
$$h(n) = u(n+1) - u(n-8)$$

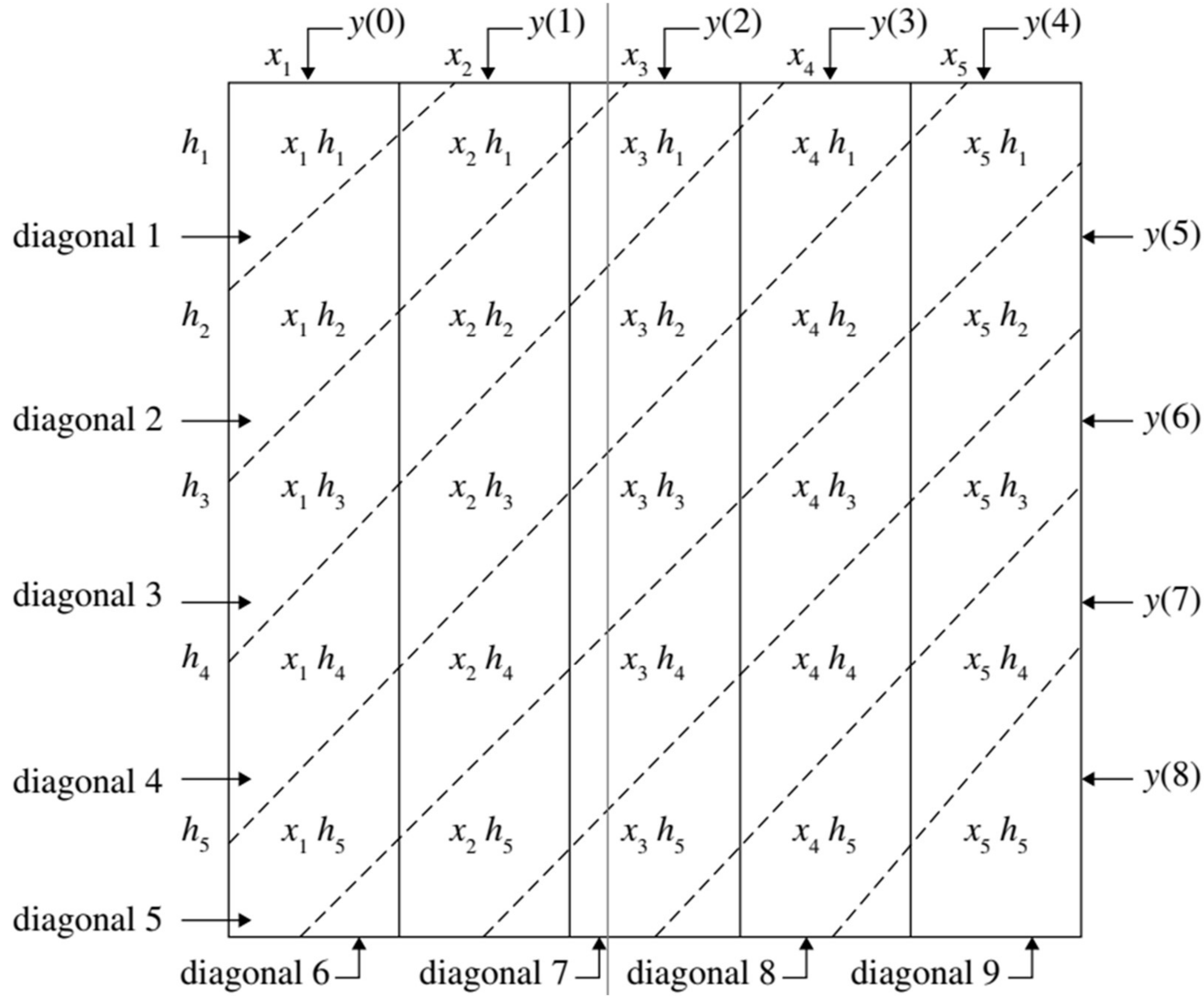
$$x(n) = \{\underset{\uparrow}{1},\ 2,\ 3,\ 4,\ 5,\ 6\}$$

$$h(n) = \{\underset{\uparrow}{2},\ -4,\ 6,\ -8\}$$

Find the convolution of $x(n) = \{1, 2, 3, 4, 5\}$ with $h(n) = \{1, 2, 3, 3, 2, 1\}$.

Convolution Sum

by Tabulation Method

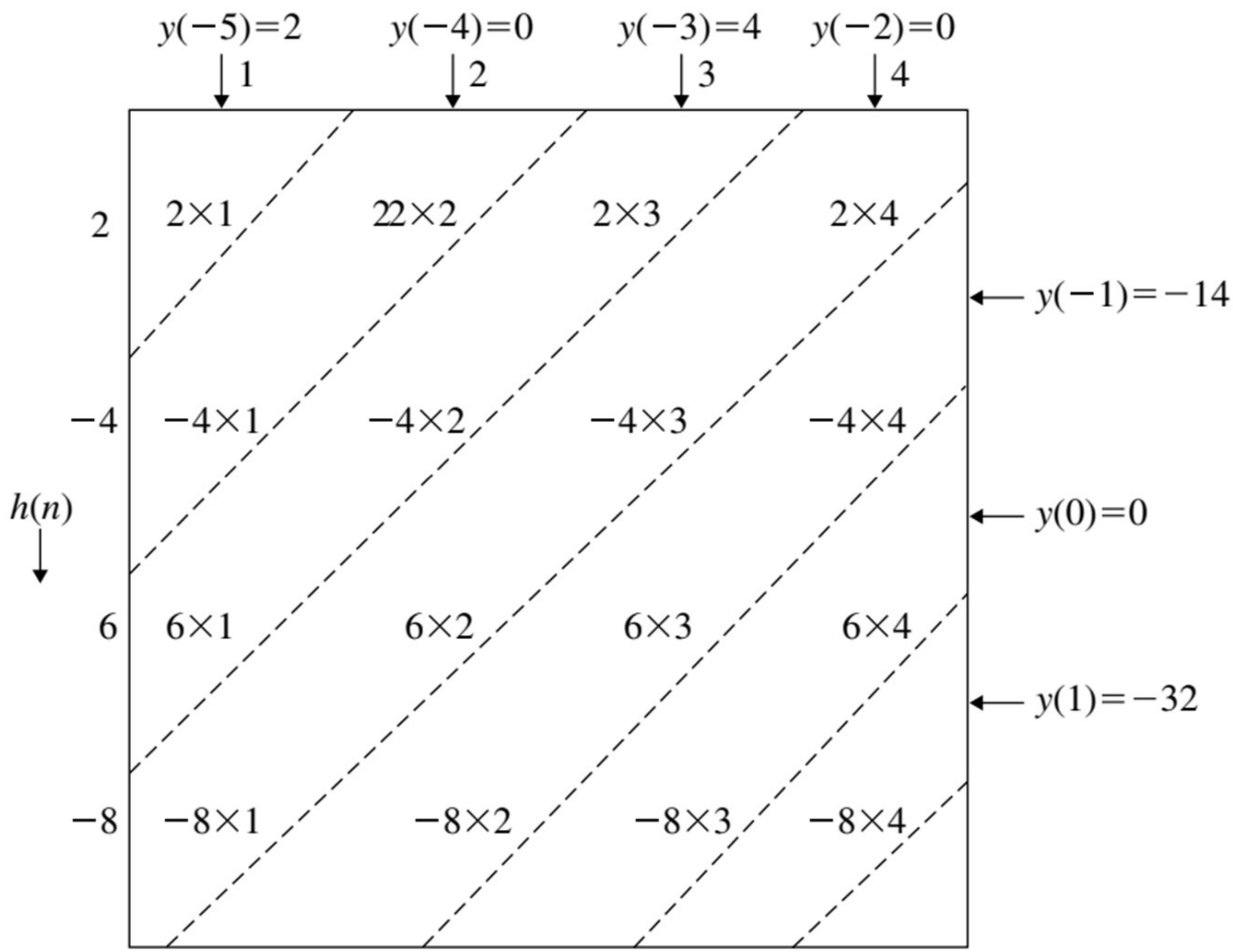


$$x(n) = \{1,\ 2,\ \underset{\uparrow}{3},\ 4\}$$

$$h(n) = \{2,\ -4,\ 6,\ \underset{\uparrow}{-8}\}$$

$$N = N_1 + N_2 = 2 + 3 = 5.$$

$$y(n) = \{2, 0, 4, 0, -14, \underset{\uparrow}{0}, -32\}$$



HW

$$x(n) = \{1, 2, 3, 4, 5\}$$
$$h(n) = \{1, 2, 3, 3, 2, 1\}$$

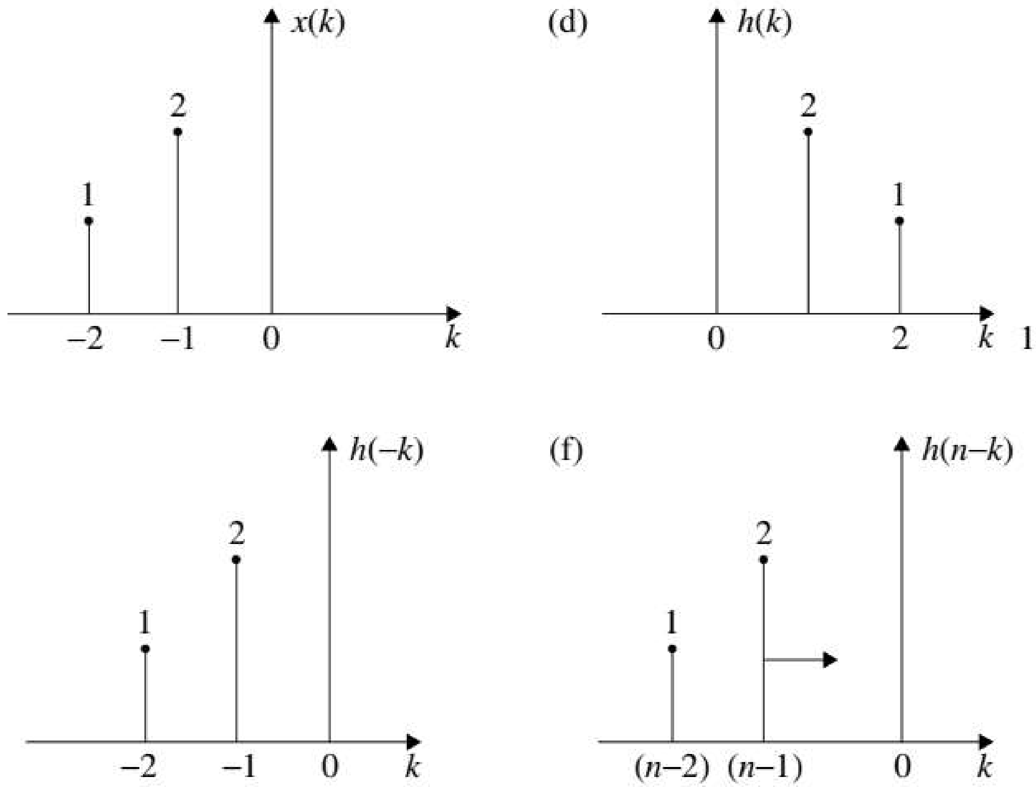
Convolution Sum by Graphical Method

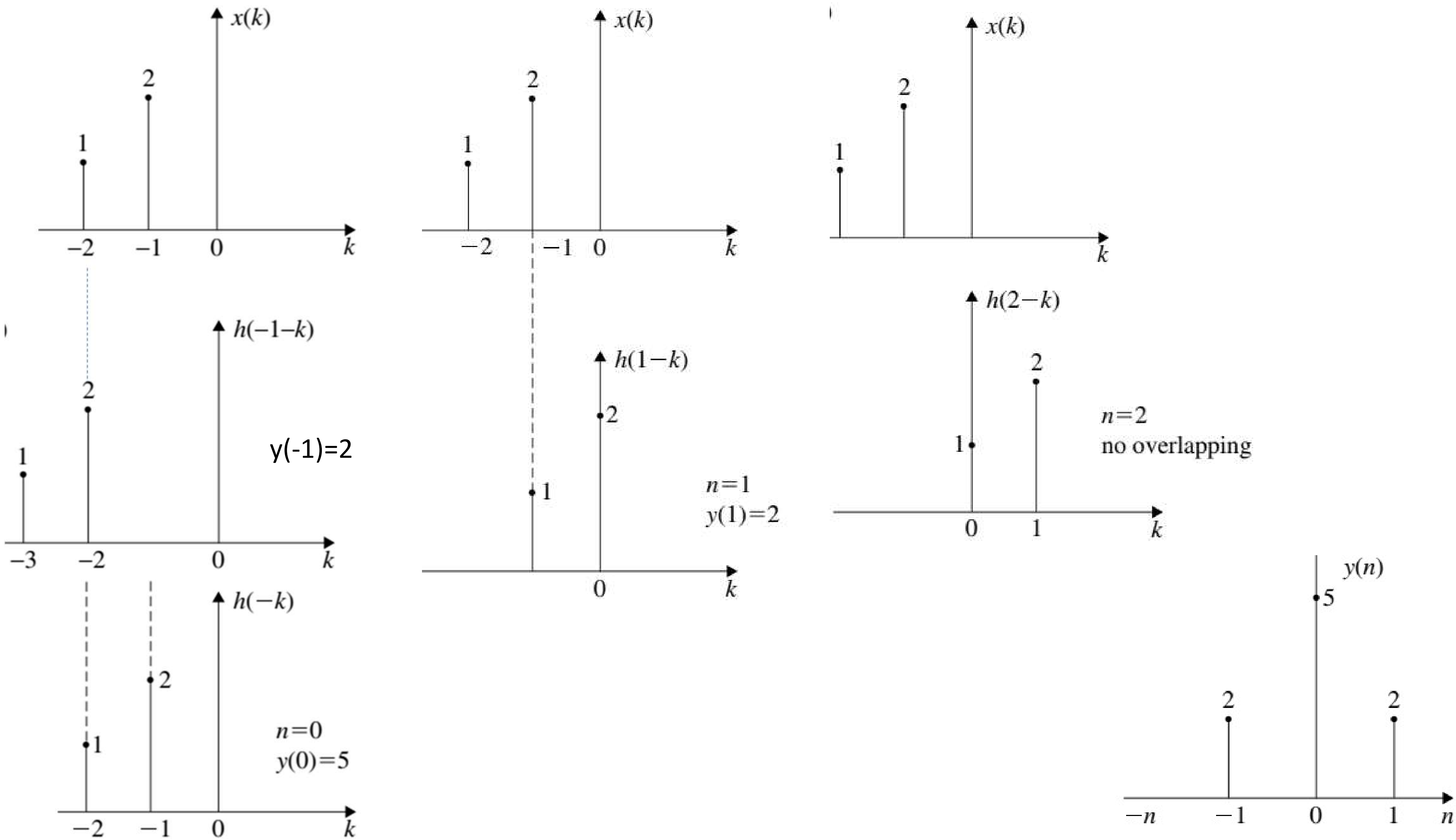
1. Represent $x(n)$ versus n in a graph. Replace n by k and $x(k)$ versus k is represented in a graph. One can straightaway plot $x(k)$ versus k.
2. Similar to Step 1, plot $h(k)$ versus k.
3. By folding $h(k)$, obtain $h(-k)$.
4. By adding "n''" to $h(-k)$, obtain $h(n-k)$. Shift $h(n-k)$ to the extreme left. Start moving $h(n-k)$ toward the right so that $x(k)$ and $h(n-k)$ overlap each other. It is to be noted that $x(k)$ should be kept fixed and $h(n-k)$ alone should be moved by one sample at an instant. Calculate $y(n)$ at that instant.
5. The procedure is repeated at other instants and at each time $y(n)$ is calculated. When there is no overlapping, the movement of $h(n-k)$ is stopped and here $y(n)=0$.

$x(n) = \{1, 2, 3, 4, 5\}$

$h(n) = \{1, 2, 3, 3, 2, 1\}$

Use graphical methods.





x(k)
h(−1−k)
y(-1)=2
h(−k)
n=0
y(0)=5
h(1−k)
n=1
y(1)=2
h(2−k)
n=2
no overlapping
y(n)

HW

$$x(n) = 2u(n+2) - 2u(n) - 3u(n-1) + 3u(n-3)$$
$$h(n) = -u(n+1) + 3u(n) - 5u(n-1) + 3u(n-2)$$

$$x(n) = a^n u(n) \qquad \text{where } 0 < a < 1$$

$$h(n) = \begin{cases} 1 & 0 \le n \le 9 \\ 0 & n \ge 10 \end{cases}$$

Fourier Transform

Fourier Transform

- The Fourier Transform is a mathematical operation that transforms a function or signal from **time** domain, into a representation in the **frequency** domain.
- The Fourier Transform decomposes a signal into its constituent frequencies, providing insight into the signal's frequency content.
- **Fourier Series**: Used for periodic signals. **Fourier Transform**: Used for aperiodic signals.

Mathematical Definition

- The continuous Fourier Transform of a function f(t) is given by

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt$$

- Here, F(ω) is the Fourier Transform of f(t)
- ω is the angular frequency,
- j is the imaginary unit,
- $e^{-j\omega t}$ represents a complex exponential function.

Inverse Fourier Transform

- The original function f(t) can be recovered from its Fourier Transform F(ω) using the inverse Fourier Transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, d\omega$$

Convergence of Fourier Transforms—The Dirichlet Conditions

- For Continuous time signals, the following conditions (**Dirichlet Conditions**) are sufficient for the convergence of $X(j\omega)$.

1. x(t) is absolutely integrable

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

2. *x(t)* should have **finite number of maxima and minima** within any finite interval.
3. x(t) has a **finite number of discontinuities** within any finite interval

Fourier Spectra

$$\boxed{X(j\omega) = |X(j\omega)| \; \underline{\left| X(j\omega) \right.}}$$

- The plot of $|X(j\omega)|$ versus ω is called magnitude spectrum of $X(j\omega)$. The plot of $\angle X(j\omega)$ versus ω is called phase spectrum.
- The amplitude (magnitude) and phase spectra are together called Fourier spectrum

Property	Time signal $x(t)$	Fourier transform $X(j\omega)$
1. Linearity	$x(t) = A\,x_1(t) + B\,x_2(t)$	$X(j\omega) =$ $A\,X_1(j\omega) + B\,X_2(j\omega)$
2. Time shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
3. Conjugation	$x^*(t)$	$X^*(-j\omega)$
4. Differentiation in time	$\dfrac{d^n\,x(t)}{dt^n}$	$(j\omega)^n X(j\omega)$
5. Differentiation in frequency	$tx(t)$	$j\dfrac{d}{d\omega}X(j\omega)$
6. Time integration	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\dfrac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
7. Time scaling	$x(at)$	$\dfrac{1}{\lvert a\rvert}X\left(j\dfrac{\omega}{a}\right)$

Property	Time signal $x(t)$	Fourier transform $X(j\omega)$
8. Time reversal	$x(-t)$	$X(-j\omega)$
9. Frequency shifting	$x(t)e^{j\omega_0 t}$	$X[j(\omega-\omega_0)]$
10. Duality	$X(t)$	$2\pi x(j\omega)$
11. Time convolution	$x(t) * h(t)$	$X(j\omega)H(j\omega)$
12. Parseval's theorem	$E=\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 \, dt$	$E=\frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2 \, d\omega$

- Find Fourier transform of

$$x(t) = \mathrm{sgn}(t)$$

$$\mathrm{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

$$F[\mathrm{sgn}(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$= -\int_{-\infty}^{0} e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-j\omega t}\,dt$$

For the first integral in the right side of the above equations when the lower limit −∞ is applied, it becomes indeterminate

$$F[e^{-a|t|}\mathrm{sgn}(t)]$$
$$a \to 0$$

$$F[e^{-a|t|}\mathrm{sgn}(t)]$$
$$a \to 0$$

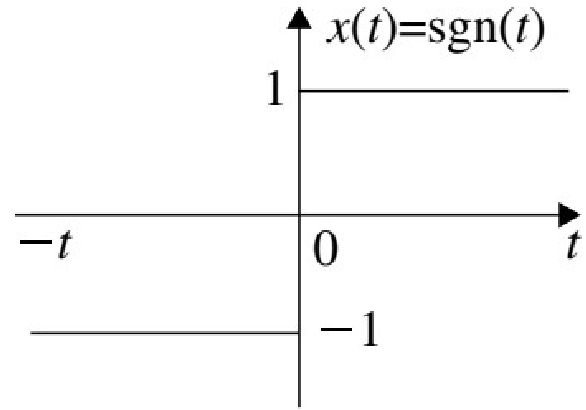
$$= \int_{-\infty}^{0} -e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

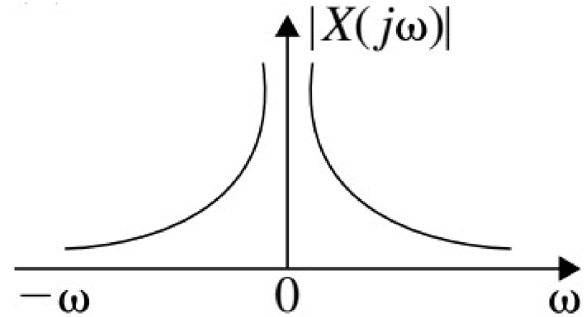
$$= \int_{-\infty}^{0} -e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

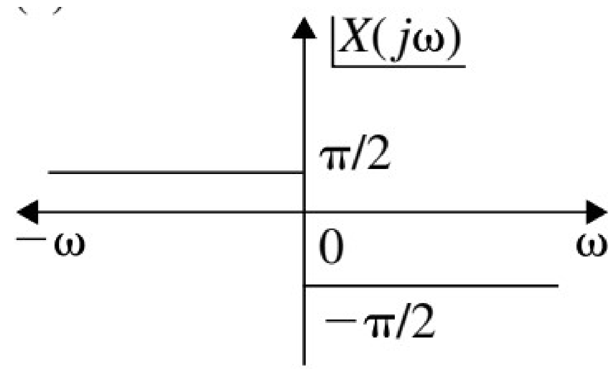
$$= \underset{a\to 0}{Lt}\left[\frac{-1}{a-j\omega}\left\{e^{(a-j\omega)t}\right\}_{-\infty}^{0} - \frac{1}{(a+j\omega)}\left\{e^{-(a+j\omega)t}\right\}_{0}^{\infty}\right]$$

$$= \underset{a\to 0}{Lt}\left[\frac{-1}{(a-j\omega)} + \frac{1}{a+j\omega}\right] = \frac{1}{j\omega} + \frac{1}{j\omega} = \frac{2}{j\omega}$$

$$\boxed{\mathrm{sgn}(t) \xleftrightarrow{\text{FT}} \frac{2}{j\omega}} \quad X(j\omega) = \frac{2}{j\omega} = \begin{cases} \frac{2}{\omega}\angle{-90^\circ} & \omega \geq 0 \\ \frac{2}{\omega}\angle 90^\circ & \omega < 0 \end{cases}$$







Home work

- Find Fourier transform of

$$x(t) = 1 \qquad \text{for all } t$$

$$x(t) = u(t) \text{ and } x(t) = u(-t)$$

$$x(t) = e^{-at}u(t); \quad a > 0$$

$$x(t) = e^{-|a|t}; \quad a > 0$$

$$x(t) = e^{at}u(t); \quad a > 0$$

- For the following signal x(t), find the FT and FT spectra

$$x(t) = \begin{cases} e^{-at} & t > 0 \\ |1| & t = 0 \\ -e^{+at} & t < 0 \end{cases}$$

$$X(j\omega) = \int_{-\infty}^{0} -e^{at} e^{-j\omega t}\, dt + \int_{0^-}^{0^+} 1 e^{-j\omega t}\, dt + \int_{0^+}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= -\int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0^-}^{0^+} e^{-j\omega t}\, dt + \int_{0^+}^{\infty} e^{-(a+j\omega)t}\, dt$$

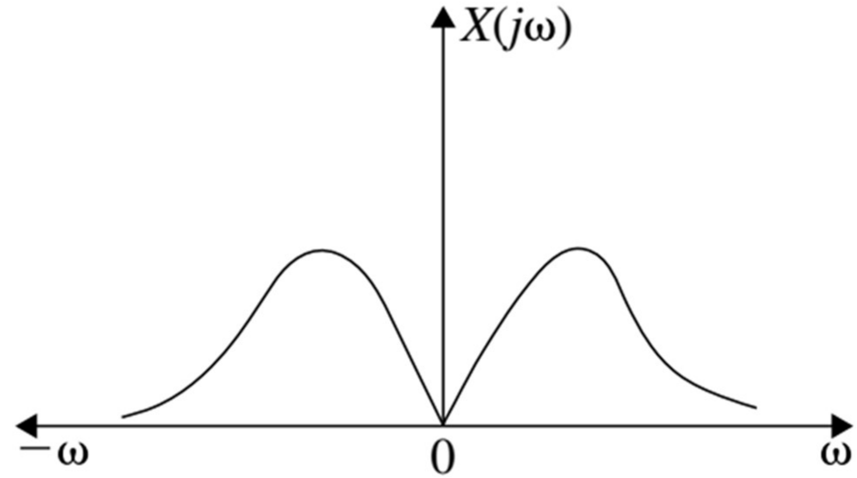
$$X(j\omega) = \frac{-1}{(a-j\omega)} \left[e^{(a-j\omega)t}\right]_{-\infty}^{0} + 0 - \frac{1}{(a+j\omega)} \left[e^{-(a+j\omega)t}\right]_{0^+}^{\infty}$$

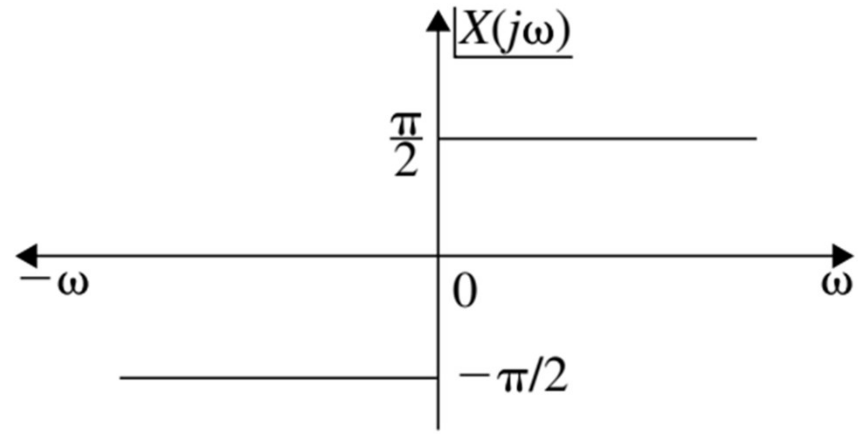
$$= \frac{-1}{(a-j\omega)} + \frac{1}{(a+j\omega)} \quad = \frac{-2j\omega}{(a^2+\omega^2)}$$

$$X(j\omega) = \frac{-2j\omega}{(a^2 + \omega^2)}$$

$$|X(j\omega)| = \frac{2\omega}{(a^2 + \omega^2)}$$

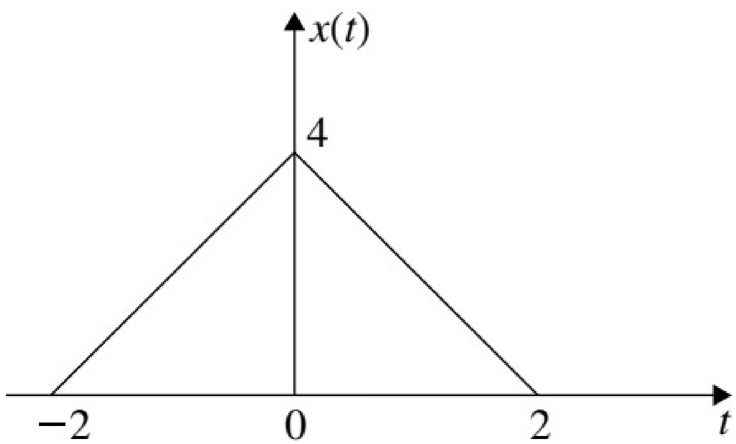
$$\angle X(j\omega) = \begin{cases} -\dfrac{\pi}{2} & \omega > 0 \\ \dfrac{\pi}{2} & \omega < 0 \end{cases}$$





Home work

- Find the Fourier Transform of



Home Work

- Find Fourier transform of

$$x(t) = 1 \qquad \text{for all } t$$

$$x(t) = u(t) \text{ and } x(t) = u(-t)$$

$$x(t) = e^{-at}u(t); \quad a > 0$$

$$x(t) = e^{-|a|t}; \quad a > 0$$

$$x(t) = e^{at}u(t); \quad a > 0$$

Signal	Fourier transform
1. $\delta(t)$	1
2. $u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
3. $\delta(t - t_0)$	$e^{-j\omega t_0}$
4. $te^{-at}u(t)$	$\frac{1}{(a + j\omega)^2}$
5. $u(-t)$	$\pi\delta(\omega) - \frac{1}{j\omega}$
6. $e^{at}u(-t)$	$\frac{1}{(a - j\omega)}$
7. $e^{-a\|t\|}$	$\frac{2a}{a^2 + \omega^2}$

Signal	Fourier transform
8. $\cos \omega_0 t$	$\pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
9. $\sin \omega_0 t$	$-j\pi [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
10. $\dfrac{1}{(a^2 + t^2)}$	$e^{-a\lvert\omega\rvert}$
11. $\mathrm{sgn}(t)$	$\dfrac{2}{j\omega}$
12. 1; for all t	$2\pi\, \delta(\omega)$

Properties of Fourier Transform

Linearity

If $x_1(t) \xleftrightarrow{\text{FT}} X_1(j\omega)$

$x_2(t) \xleftrightarrow{\text{FT}} X_2(j\omega)$

then

$$[A\,x_1(t) + B\,x_2(t)] \xleftrightarrow{\text{FT}} [A\,X_1(j\omega) + B\,X_2(j\omega)]$$

Let $x(t) = A\,x_1(t) + B\,x_2(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} [A\,x_1(t) + B\,x_2(t)]e^{-j\omega t}\,dt$$

$$= A\int_{-\infty}^{\infty} x_1(t)e^{-j\omega t} + B\int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}\,dt$$

$$\boxed{X(j\omega) = A\,X_1(j\omega) + B\,X_2(j\omega)}$$

Time Shifting

If $x(t) \xleftrightarrow{\text{FT}} X(j\omega)$

then $x(t-t_0) \xleftrightarrow{\text{FT}} e^{-j\omega t_0} X(j\omega)$

$$F[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t}\, dt$$

Let $(t-t_0) = p$ and $dt = dp$

$$F[x(t-t_0)] = \int_{-\infty}^{\infty} x(p) e^{-j\omega(p+t_0)}\, dp$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(p) e^{-j\omega p}\, dp$$

$$\boxed{F[x(t-t_0)] = e^{-j\omega t_0}\, X(j\omega)}$$

Differentiation in Time

If $x(t) \xleftrightarrow{\text{FT}} X(j\omega)$

then

$$\frac{dx(t)}{dt} \xleftrightarrow{\text{FT}} j\omega X(j\omega) \qquad F\left[\frac{d^n x(t)}{dt^n}\right] = (j\omega)^n X(j\omega)$$

$$[x(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega$$

Differentiating
Both side

$$\left[\frac{dx(t)}{dt}\right] = \frac{j\omega}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega$$

$$= j\omega X(j\omega)$$

$$\frac{dx(t)}{dt} \xleftrightarrow{\text{FT}} j\omega X(j\omega)$$

Differentiation in Frequency

If $x(t) \overset{\text{FT}}{\longleftrightarrow} X(j\omega)$

then $F[tx(t)] = j\dfrac{d}{d\omega}X(j\omega)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$\frac{d}{d\omega}[X(j\omega)] = \int_{-\infty}^{\infty} -jtx(t)e^{-j\omega t}\,dt$$

$$= -jF[tx(t)]$$

$$\boxed{[tx(t)] \overset{\text{FT}}{\longleftrightarrow} j\frac{dX(j\omega)}{d\omega}}$$

Time Integration

If $x(t) \xleftrightarrow{\text{FT}} X(j\omega)$

then

$$F\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \frac{1}{j\omega}X(j\omega) + \pi X(0)\,\delta(\omega)$$

Let $\quad y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{\infty} x(\tau)\,u(t-\tau)\,d\tau = (x * u)(t).$$

$$Y(\omega) = X(\omega)\,U(\omega),$$

We know $\quad U(\omega) = \pi\,\delta(\omega) + \dfrac{1}{j\omega}.$

$$Y(\omega) = X(\omega)\left(\pi\,\delta(\omega) + \frac{1}{j\omega}\right)$$

$$= \frac{1}{j\omega}X(\omega) + \pi\,X(\omega)\,\delta(\omega).$$

But $X(\omega)\,\delta(\omega) = X(0)\,\delta(\omega)$

$$Y(\omega) = \frac{1}{j\omega}X(\omega) + \pi\,X(0)\,\delta(\omega).$$

Time Scaling

If $x(t) \xleftrightarrow{\text{FT}} X(j\omega)$

then

$$F[x(at)] = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right) \qquad F[x(-t)] = X(-j\omega)$$

$$F[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j\omega t}\, dt$$

Let $at = p$; and $dt = \frac{1}{a} dp$, $a > 0$

$$F[x(p)] = \frac{1}{a} \int_{-\infty}^{\infty} x(p) e^{-\frac{j\omega p}{a}}\, dp$$

$$= \frac{1}{a} X\left(j\frac{\omega}{a}\right)$$

Frequency Shifting

If $x(t) \xleftrightarrow{\text{FT}} X(j\omega)$

then

$F[x(t)e^{j\omega_0 t}] = X[j(\omega - \omega_0)]$

$$
\begin{aligned}
F[x(t)e^{j\omega_0 t}] &= \int_{-\infty}^{\infty} x(t)e^{j\omega_0 t}e^{-j\omega t}\,dt \\
&= \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}\,dt \\
&= X[j(\omega - \omega_0)]
\end{aligned}
$$

Duality

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega) \qquad \Longrightarrow \qquad X(t) \xleftrightarrow{\mathcal{F}} 2\pi\, x(-\omega).$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega.$$

replace t by $-t$:

$$x(-t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega(-t)}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega.$$

Multiply both sides by 2π:

$$2\pi\, x(-t) = \int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega.$$

$$2\pi\, x(-t) = \mathcal{F}\{X(\omega)\}$$

Swap omega and t

$$2\pi\, x(-\omega) = \mathcal{F}\{X(t)\}$$

The Convolution

$$y(t) = x(t) * h(t)$$

$$F[y(t)] = Y(j\omega) = X(j\omega)H(j\omega)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$F[y(t)] = Y(j\omega) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau\right]e^{-j\omega t}\,dt$$

Interchanging the order of integration,

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}\,dt\right]d\tau$$

By time shifting property,

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}H(j\omega)\,d\tau$$

$$= H(j\omega)\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}\,d\tau$$

$$\boxed{Y(j\omega) = H(j\omega)X(j\omega)}$$

Parseval's Theorem

According to Parseval's theorem, the total energy in a signal is obtained by integrating the energy per unit frequency $\frac{|X(j\omega)|^2}{2\pi}$.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \int_{-\infty}^{\infty} x(t)x^*(t) \, dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) e^{-j\omega t} \, d\omega \right] dt$$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) X(j\omega) \, d\omega$$

$$\boxed{E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega}$$

The **Energy Spectral Density (ESD)** tells us **how the energy of a signal is distributed across different frequencies**.

$$\mathrm{ESD}(\omega) = |X(\omega)|^2$$

$X(\omega)$ = Fourier transform of $x(t)$

$|X(\omega)|^2$ = magnitude squared of the spectrum

Parseval's Theorem in terms of **ESD**

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{|X(\omega)|^2}_{\mathrm{ESD}(\omega)} \, d\omega$$

Ques. Based on properties

- Find the Fourier transform of

$$x(t) = e^{j\omega_0 t}$$

Let $y(t) = 1$.

$$Y(j\omega) = 2\pi\,\delta(\omega)$$

By using the frequency shifting property, we get

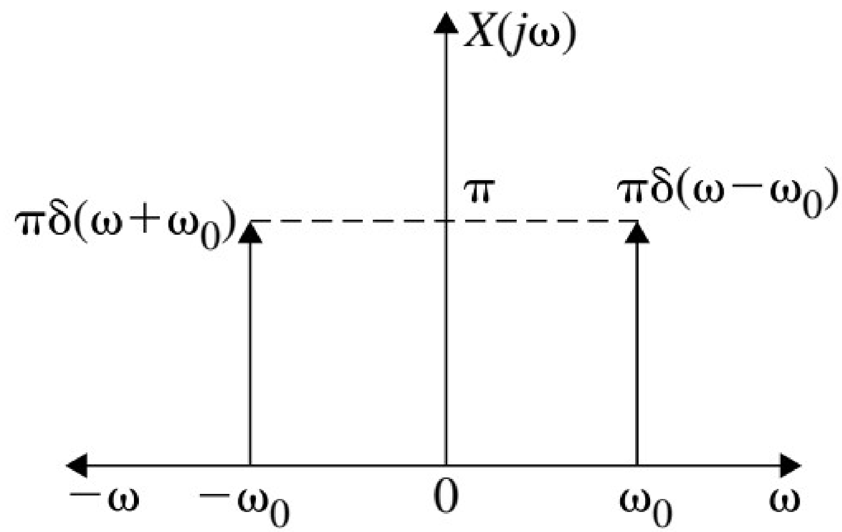
$$\boxed{X(j\omega) = 2\pi\,\delta(\omega - \omega_0)}$$

- Find the Fourier transform of

$$\boldsymbol{x(t) = \cos(\omega_0 t)}$$

$$
\begin{aligned}
x(t) &= \cos(\omega_0 t) \\
&= \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]
\end{aligned}
$$

$$X(j\omega) = \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$



$\boldsymbol{x(t) = \sin \omega_0 t u(t)}$

$$\sin \omega_0 t = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$$

$$\sin \omega_0 t u(t) = \frac{1}{2j}\left[e^{j\omega_0 t}u(t) - e^{-j\omega_0 t}u(t)\right]$$

By using the frequency shifting property

$$F[x(t)] = \frac{1}{2j}\left[\frac{1}{j(\omega - \omega_0)} + \pi\delta(\omega - \omega_0) - \frac{1}{j(\omega + \omega_0)} - \pi\delta(\omega + \omega_0)\right]$$

$$X(j\omega) = \left[\frac{\omega_0}{\omega_0^2 - \omega^2} + \frac{\pi}{2j}\delta(\omega - \omega_0) - \frac{\pi}{2j}\delta(\omega + \omega_0)\right]$$

Fourier Transform Using Differentiation and Integration Properties

$$x(t) \xleftrightarrow{\text{FT}} X(j\omega)$$

$$\frac{dx(t)}{dt} \xleftrightarrow{\text{FT}} j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\text{FT}} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

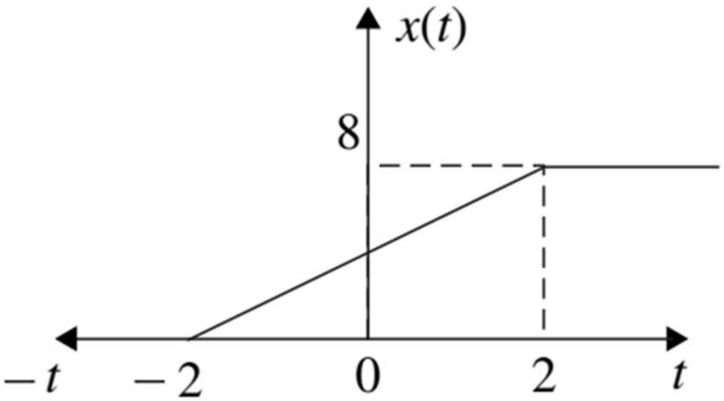
Procedure

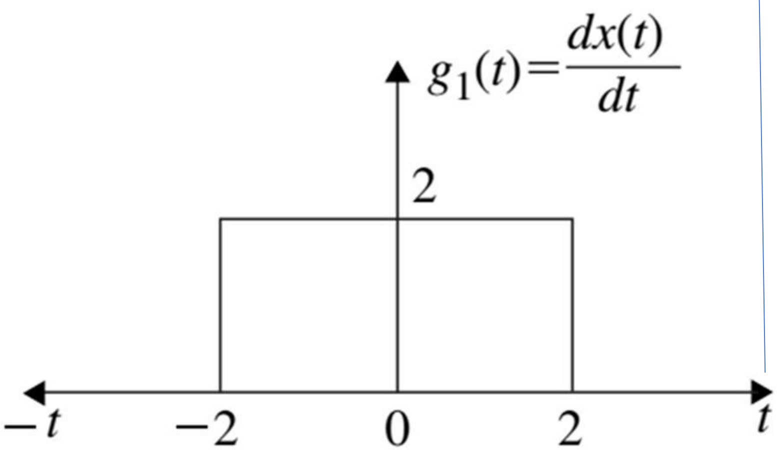
1. The signal *x(t)* is sketched.
2. *x(t)* is differentiated and *dx(t)/dt* is sketched.
3. The differentiation procedure is repeated until the FT could be easily obtained just by observation.
4. Obtain *G(*0*)* which is nothing but *X(*0*)* when *ω* is substituted in the FT of the last derivative of *x(t)*

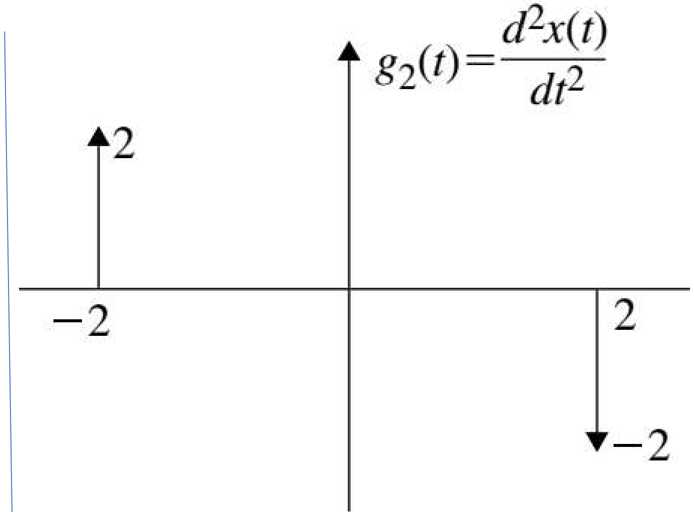
$$X(j\omega) = \frac{G(j\omega)}{(j\omega)^n} + \pi X(0)\delta(\omega)$$

$$G(j\omega) = \text{FT of } n\text{th derivative of } x(t)$$
$$X(0) = G(j\omega)|_{\omega=0}$$

$$
x(t) = \begin{cases} 2t+4 & -2 \le t < 2 \\ 8 & 2 \le t < \infty \\ 0 & \text{otherwise} \end{cases}
$$







$$
\begin{aligned}
G_2(j\omega) &= [2e^{j2\omega} - 2e^{-j2\omega}] \\
&= 2\frac{[e^{j2\omega} - e^{-j2\omega}]2j}{2j} \\
&= j4\sin 2\omega
\end{aligned}
$$

$$
G_2(0) = 0
$$

$$
G_1(j\omega) = \frac{G_2(j\omega)}{j\omega} + \pi\delta(\omega)G_2(0)
$$

$$
\begin{aligned}
G_1(j\omega) &= \frac{j4\sin 2\omega}{j\omega} \\
&= 8\left(\frac{\sin 2\omega}{2\omega}\right)
\end{aligned}
$$

$$
G_1(0) = 8
$$

$$
\left[\left.\frac{\sin 2\omega}{2\omega}\right|_{\omega=0} = 1\right]
$$

$$
X(j\omega) = \frac{G_1(j\omega)}{j\omega} + \pi\delta(\omega)G_1(0)
$$

$$
X(j\omega) = \frac{4\sin 2\omega}{j\omega^2} + 8\pi\delta(\omega)
$$

For the following signals, determine the FT using FT properties.

$$x(t) = 5\sin 10t \qquad y(t) = x(4t-3)$$

$$x(t) \overset{\text{FT}}{\longleftrightarrow} j5\pi\left[\delta(\omega+10) - \delta(\omega-10)\right]$$

$$x(t-3) \overset{\text{FT}}{\longleftrightarrow} j5\pi\left[\delta(\omega+10) - \delta(\omega-10)\right]e^{-j3\omega}$$

$$x(4t-3) \overset{\text{FT}}{\longleftrightarrow} \frac{j5\pi}{4}\left[\delta\left(\frac{\omega}{4}+10\right) - \delta\left(\frac{\omega}{4}-10\right)\right]e^{-j(3/4)\omega}$$

Home Work

For the following signals, determine the FT using FT properties.

$$x(t) = 5 \sin 10t$$

$$y(t) = x(4(t-3))$$

$$y(t) = x(-3t+4)$$

Consider the following CT signal.

$$x(t) = 4\cos 3t \quad y(t) = \tfrac{d^2}{dt^2}x(t-2)$$

$$X(j\omega) = 4\pi\left[\delta(\omega+3) + \delta(\omega-3)\right]$$

$$\begin{aligned} x(t-2) &\overset{\text{FT}}{\longleftrightarrow} 4\pi\left[\delta(\omega+3) + \delta(\omega-3)\right]e^{-j2\omega} \\ &= 4\pi\left[\delta(\omega+3)e^{j6} + \delta(\omega-3)e^{-j6}\right] \end{aligned}$$

$$f(\omega)\,\delta(\omega-\omega_0) = f(\omega_0)\,\delta(\omega-\omega_0).$$

Using differentiation property $\dfrac{d^2x}{dt^2} \overset{\text{FT}}{\longleftrightarrow} (j\omega)^2 X(j\omega)$

$$\begin{aligned} Y(j\omega) &= 4(j\omega)^2\pi\left[\delta(\omega+3)e^{j6} + \delta(\omega-3)e^{-j6}\right] \\ &= 4\pi\left[-\delta(\omega+3)9e^{j6} - \delta(\omega-3)9e^{-j6}\right] \end{aligned}$$

$$Y(j\omega) = -36\pi\left[\delta(\omega+3)\,e^{j6} + \delta(\omega-3)\,e^{-j6}\right]$$

Home Work

Consider the following CT signal.

$$x(t) = 4\cos 3t$$

Determine the FT of the following signals:

(a) $y(t) = x(2 - t) + x(-2 - t)$

(b) $y(t) = x(3t + 5)$

Home Work

A signal has the following FT:

$$X(j\omega) = \frac{\omega^2 + j4\omega + 2}{-\omega^2 + j4\omega + 3}$$

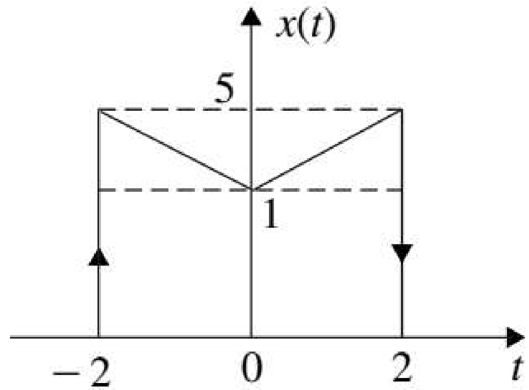
Find the FT of $x(-2t + 1)$.

By using differentiation and integration property of FT, determine the FT of $x(t) = \text{sgn}(t)$.

Consider the signal described by the following signal:

$$x(t) = 1 + 2|t| \qquad |t| \leq 2$$

$$x(t) = \begin{cases} 1+2t & 0 \leq t \leq 2 \\ 1-2t & -2 \leq t \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

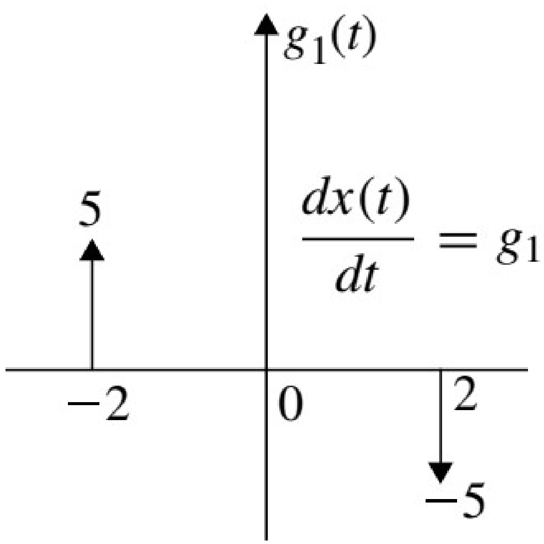


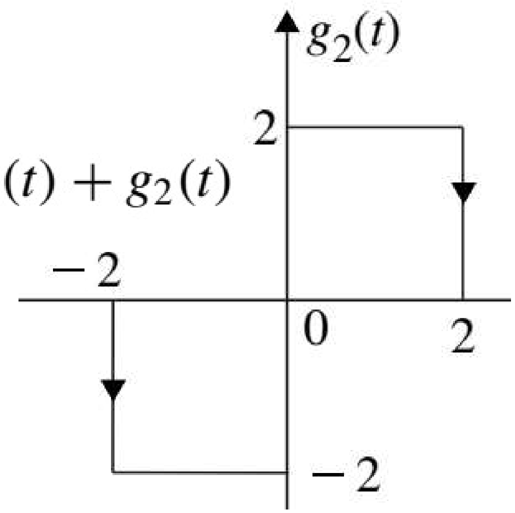
$$\begin{aligned} G_3(j\omega) &= -2[e^{j2\omega} + e^{-j2\omega}] + 4 \\ &= -4\cos 2\omega + 4 \\ &= 4[1 - \cos 2\omega] \\ &= 8\sin^2 \omega \\ G_3(0) &= 0 \end{aligned}$$

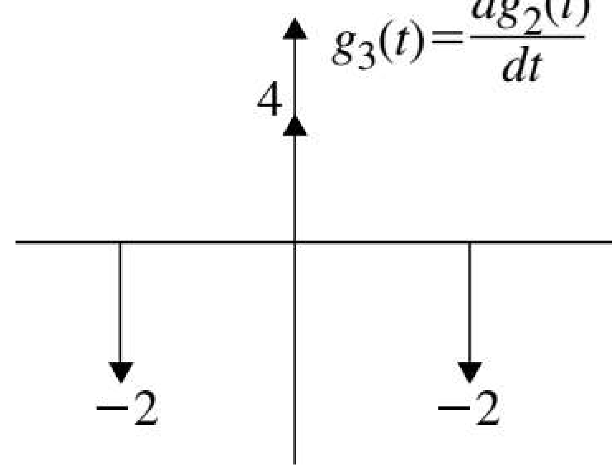
$\cos(2\theta) = 1 - 2\sin^2\theta.$

$$\begin{aligned} G_1(j\omega) &= 5(e^{j2\omega} - e^{-j2\omega}) \\ &= j10 \sin 2\omega \\ G_1(0) &= 0 \end{aligned}$$

$$\frac{dx(t)}{dt} = g_1(t) + g_2(t)$$







Using the integration property,

$$X(j\omega) = \frac{G_1(j\omega)}{(j\omega)} + \frac{G_3(j\omega)}{(j\omega)^2} + \pi[G_1(0)\delta(\omega) + G_3(0)\delta(\omega)]$$

$$X(j\omega) = j10\frac{\sin 2\omega}{j\omega} + 8\frac{\sin^2 \omega}{(j\omega)^2}$$

$$X(j\omega) = 20\text{sinc}2\omega - 8\text{sinc}^2\omega$$

Find the Fourier transform of

$$x(t) = \frac{2a}{a^2 + t^2}$$

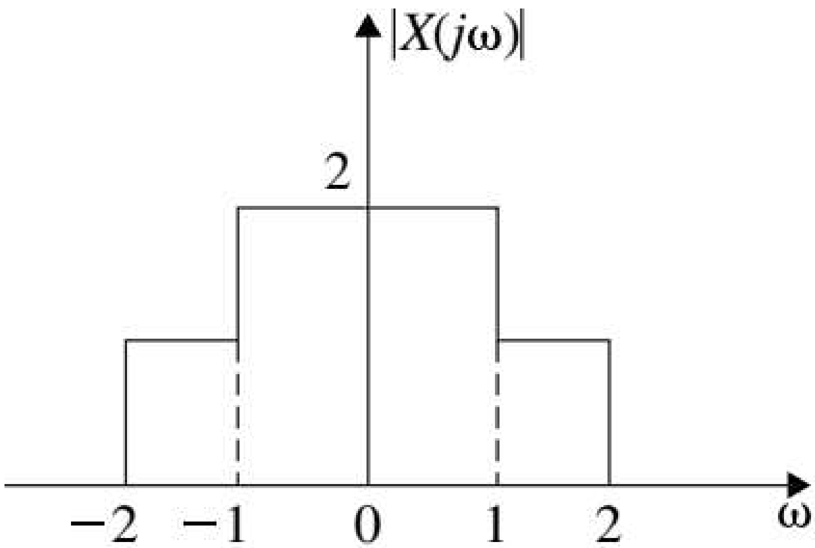
duality property

$$x(t) = e^{-a|t|} \overset{\text{FT}}{\longleftrightarrow} \frac{2a}{a^2 + \omega^2}$$

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \qquad \Longrightarrow \qquad X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, x(-\omega).$$

$$\left[\frac{2a}{(a^2 + t^2)}\right] \overset{\text{FT}}{\longleftrightarrow} 2\pi e^{-a|\omega|}$$

For the Fourier transforms shown in Fig find the energy of the signals using Parseval's theorem



$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

$$E = \frac{1}{2\pi} \left\{ \int_{-2}^{1} 1^2 \, d\omega + \int_{-1}^{1} 2^2 \, d\omega + \int_{1}^{2} 1^2 \, d\omega \right\}$$

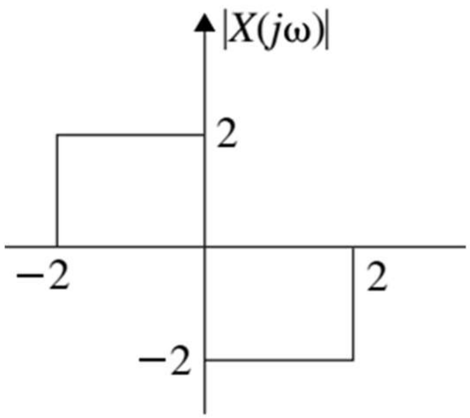
$$= \frac{1}{2\pi} \left\{ \left[\omega\right]_{-2}^{-1} + 4\left[\omega\right]_{-1}^{1} + \left[\omega\right]_{1}^{2} \right\}$$

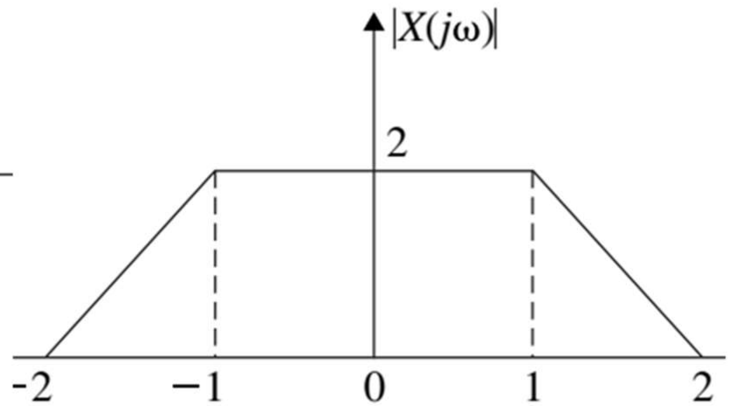
$$= \frac{1}{2\pi} \{-1 + 2 + 4 + 4 + 2 - 1\}$$

$$E = \frac{5}{\pi}$$

Home Work

Find energy using Parseval's theorem





Inverse Fourier Transform

The Inverse Fourier Transform is a mathematical operation that transforms a signal from its frequency domain representation back to its time domain (or spatial domain) representation.
It is the reverse process of the Fourier Transform.

For continuous signals:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

For discrete signals (Inverse Discrete Fourier Transform - IDFT):

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}$$

Find IFT

$$\boldsymbol{X(j\omega) = \delta(\omega - \omega_0)}$$

1; for all t	$2\pi\,\delta(\omega)$

$$\delta(\omega) = \frac{1}{2\pi}$$

$$F^{-1}\left[\delta(\omega - \omega_0)\right] = \frac{1}{2\pi} e^{j\omega_0 t}$$

Find IFT

$$X(j\omega) = \frac{j\omega}{(2 + j\omega)^2}$$

We know

$$F\left[e^{-2t}\right] = \frac{1}{(2 + j\omega)}$$

frequency differentiation

$$F\left[-j\, te^{-2t}\right] = \frac{d}{d\omega}\frac{1}{(2 + j\omega)}$$

$$F\left[-j\, te^{-2t}\right] = \frac{-j.1}{(2 + j\omega)^2}$$

$$F^{-1}\left[\frac{1}{(2 + j\omega)^2}\right] = te^{-2t}$$

applying time differentiation,

$$\frac{dx(t)}{dt} = j\omega X(j\omega)$$

$$F^{-1}\left[\frac{j\omega}{(2 + j\omega^2)}\right] = \frac{d}{dt}\left(te^{-2t}\right)$$

$$\frac{d}{dt}(te^{-2t}) = (1)(e^{-2t}) + (t)(-2e^{-2t})$$

$$= e^{-2t} - 2te^{-2t}$$

$$= (1 - 2t)e^{-2t}.$$

Find IFT

$$X(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-W}^{W} X(j\omega) e^{j\omega t} d\omega \\ &= \frac{1}{2\pi jt} \left[1 e^{j\omega t} \right]_{-W}^{W} \\ &= \frac{1}{2\pi jt} \left[e^{jWt} - e^{-jWt} \right] \\ &= \frac{1}{\pi t} \sin Wt \end{aligned}$$

Home Work

Find IFT

$$X(j\omega) = \frac{6}{(\omega^2+9)}$$

$$X(j\omega) = \frac{(j\omega+2)}{[(j\omega)^2+4j\omega+3]}$$

$$X(j\omega) = \frac{(j\omega+1)}{(j\omega+2)^2(j\omega+3)}$$

Find the Fourier transform of the following signals using convolution theorem.

$$x(t) = e^{-2t}u(t) * e^{-5t}u(t)$$

$$X(j\omega) = F\left[e^{-2t}u(t)\right] F\left[e^{-5t}u(t)\right]$$

$$F\left[e^{-2t}u(t)\right] = \frac{1}{(j\omega + 2)}$$

$$F\left[e^{-5t}u(t)\right] = \frac{1}{(j\omega + 5)}$$

$$X(j\omega) = \frac{1}{(j\omega + 2)(j\omega + 5)}$$

$$X(j\omega) = \frac{1}{3}\left[\frac{1}{j\omega + 2} - \frac{1}{(j\omega + 5)}\right] \qquad x(t) = F^{-1}[X(j\omega)] = \frac{1}{3}[e^{-2t}u(t) - e^{-5t}u(t)]$$

Home Work

Find the Fourier transform of the following signals using convolution theorem.

$$x(t) = \frac{d}{dt}\left[e^{-2t}u(t) * e^{-5t}u(t)\right]$$

$$x(t) = \left[e^{-2t}u(t) * e^{-5t}u(t-5)\right]$$

Fourier Transform in Discrete Time

- DTFT (Discrete-Time Fourier Transform): It operates on **infinite-length sequences** and provides a **continuous frequency representation**.

- DFT (Discrete Fourier Transform): The DFT is specifically designed for discrete, **finite-length sequences**. It transforms a finite set of data points into a **discrete set of frequency components**.

Discrete Fourier Transform (DFT)

- The Fourier transform transforms the sequence $x[n]$ to $X(\Omega)$ which is continuous and periodic.
- The original finite duration signal can be easily recovered from its DFT since there exists one to one correspondence between $x[n]$ and the Fourier transformed discrete signal.
- For the calculation of the DFT of finite duration sequences, a very efficient and fast techniques known as Fast Fourier Transform (FFT) has been developed.

Fourier Transform of x[n]

- Fourier Transform of x[n]

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

- X(Ω) is the continuous function of Ω.
- The range of Ω is from −π to π or 0 to 2π
- When Fourier transform X(Ω) is calculated at only discrete points k, it is called Discrete Fourier Transform (DFT)

Discrete Fourier Transform (DFT)

- The DFT is denoted by X(k)

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

- where k = 0, 1, 2, . . . , (N − 1).

Let us define $W_N = e^{-j2\pi/N}$ W_N is called Twiddle factor.

$$X(k) = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

Discrete Fourier Transform (DFT)

The twiddle factor W_N^{kn} is represented as a matrix with k rows and n column

$$W_N = \begin{array}{r} k=0 \\ \\ \\ \\ k=N-1 \end{array} \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}_{N \times N}$$

Four Point Twiddle Factor

For $N = 4$

$$W_N = \begin{array}{c} \\ k = 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} n = 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \end{array}$$

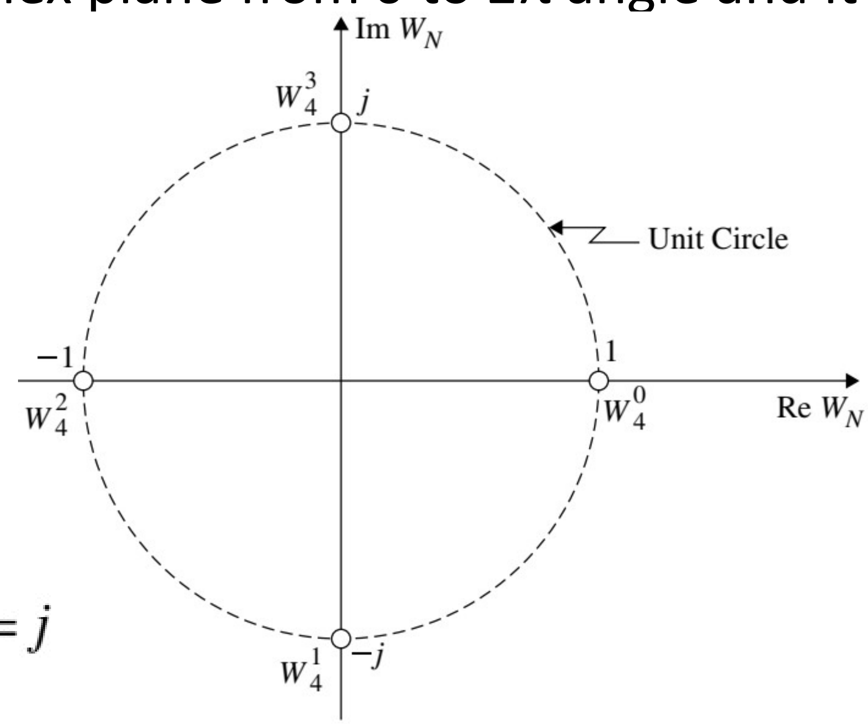
Four Point Twiddle Factor

- the magnitude of the twiddle factor is 1 and the phase angle is − 2Nπ
- It lies on the unit circle in the complex plane from 0 to 2π angle and it gets repeated for every cycle.

$$
\begin{aligned}
W_N &= e^{-j\frac{2\pi}{N}} \\
&= 1\angle -2\pi/N
\end{aligned}
$$

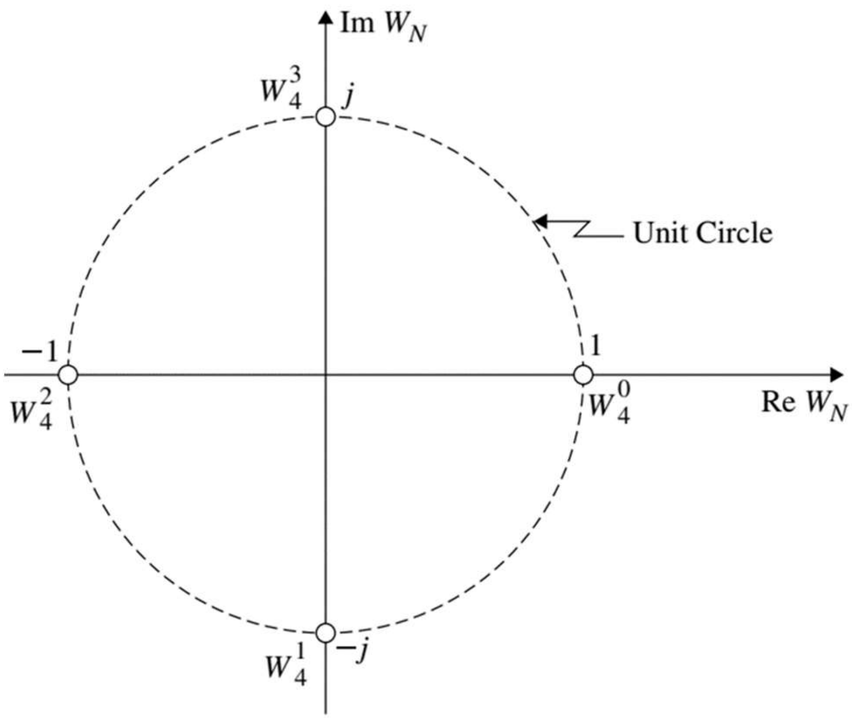
$$W_4^4 = W_4^0;\ W_4^6 = W_4^2 \text{ and } W_4^9 = W_4^1.$$

$$W_4^0 = 1; \quad W_4^1 = -j; \quad W_4^2 = -1; \quad W_4^3 = j$$



Four Point Twiddle Factor

$$W_4^0 = 1; \quad W_4^1 = -j; \quad W_4^2 = -1; \quad W_4^3 = j$$



$$W_N = \begin{array}{c|c} & n = 0 \quad 1 \quad 2 \quad 3 \\ \begin{array}{c} k=0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \end{array}$$

$$W_N = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Compute the DFT of the sequence $x[n] = \{1, j, -1, -j\}$ for $N = 4$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}; \qquad X_4 = \begin{bmatrix} 1 \\ j \\ -1 \\ -j \end{bmatrix}$$

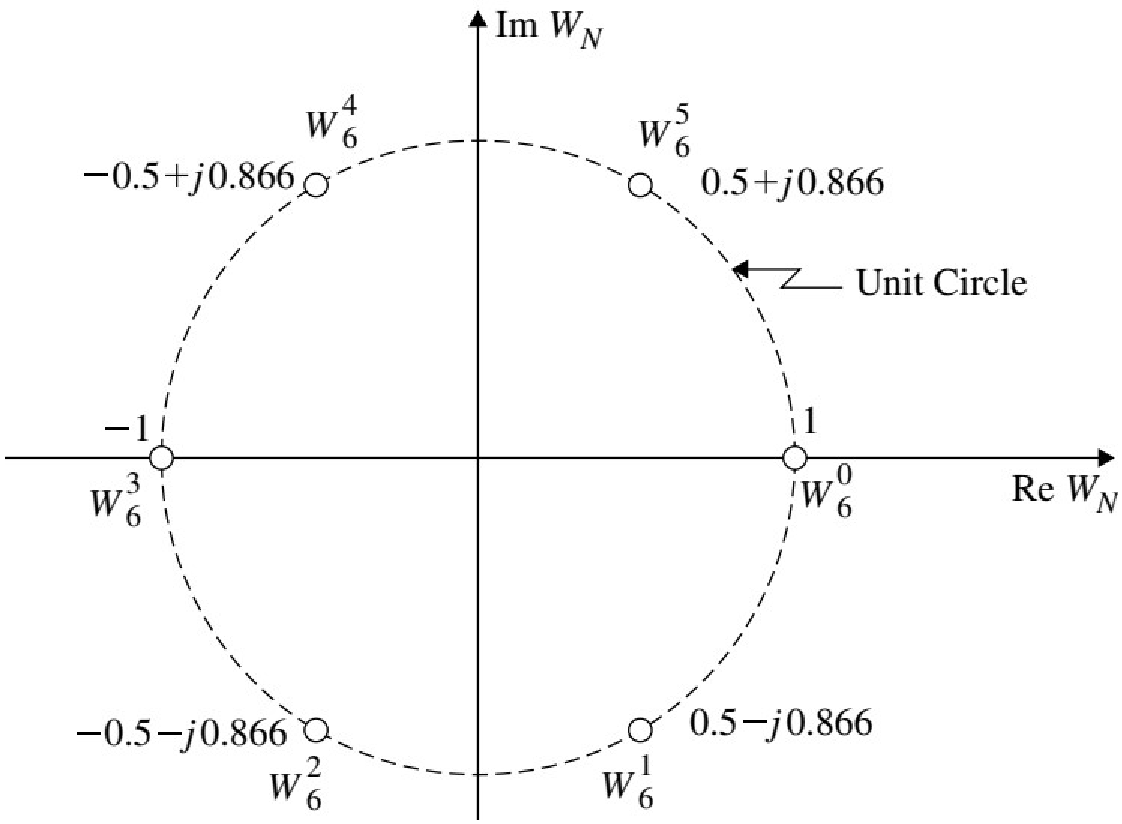
$$X_4 = W_4 x_4$$

$$X_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ j \\ -1 \\ -j \end{bmatrix}$$

$$= \begin{bmatrix} 1 + j - 1 - j \\ 1 + 1 + 1 + 1 \\ 1 - j - 1 + j \\ 1 - 1 + 1 - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{aligned} X(0) &= 0 \\ X(1) &= 4 \\ X(2) &= 0 \\ X(3) &= 0 \end{aligned}}$$

Six Point Twiddle Factor

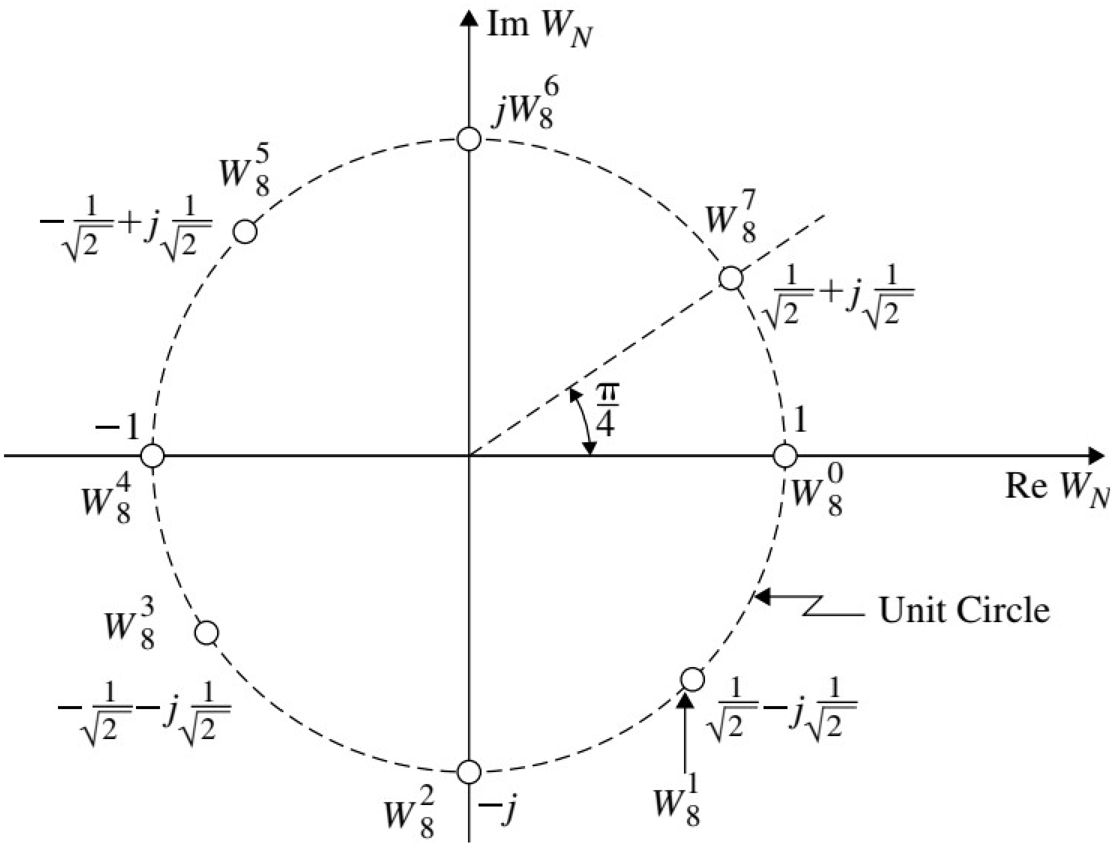


$$W_N = \begin{bmatrix} W_6^0 & W_6^0 & W_6^0 & W_6^0 & W_6^0 & W_6^0 \\ W_6^0 & W_6^1 & W_6^2 & W_6^3 & W_6^4 & W_6^5 \\ W_6^0 & W_6^2 & W_6^4 & W_6^6 & W_6^8 & W_6^{10} \\ W_6^0 & W_6^3 & W_6^6 & W_6^9 & W_6^{12} & W_6^{15} \\ W_6^0 & W_6^4 & W_6^8 & W_6^{12} & W_6^{16} & W_6^{20} \\ W_6^0 & W_6^5 & W_6^{10} & W_6^{15} & W_6^{20} & W_6^{25} \end{bmatrix}$$

Six Point Twiddle Factor

$$W_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0.5 - j0.866 & -0.5 - j0.866 & -1 & -0.5 + j0.866 & 0.5 + j0.866 \\ 1 & -0.5 - j0.866 & -0.5 + j0.866 & 1 & -0.5 - j0.866 & -0.5 + j0.866 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -0.5 + j0.866 & -0.5 - j0.866 & 1 & -0.5 + j0.866 & -0.5 - j0.866 \\ 1 & 0.5 + j0.866 & -0.5 + j0.866 & -1 & -0.5 - j0.866 & 0.5 - j0.866 \end{bmatrix}$$

Eight Point Twiddle Factor



$$W_8 = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ W_8^0 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ W_8^0 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ W_8^0 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\ W_8^0 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ W_8^0 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49} \end{bmatrix}$$

Eight Point Twiddle Factor

$$W_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & j & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & j & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & j & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & j & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \end{bmatrix}$$

Find 8-point DFT of $x[n] = \{1,\ -1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1\}$.

$$X_N = W_N x_N$$

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & j & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \\
1 & -j & -1 & j & 1 & -j & -1 & j \\
1 & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & j & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & j & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \\
1 & j & -1 & -j & 1 & j & -1 & -j \\
1 & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & j & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}
\end{bmatrix}
\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$
\begin{aligned}
X(0) &= 1-1+1-1+1-1+1-1=0 \\
X(1) &= \text{2nd row of} W_8 \text{to multiply} x_8 \\
&= 1-\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}-j+\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}-1+\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}+j=0 \\
X(2) &= \text{3rd row of} W_8 \text{to multiply} x_8 \\
&= 1+j-1-j+1+j-1-j=0 \\
X(3) &= 1+\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}+j-\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}-1-\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}-j+\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}=0 \\
X(4) &= \text{5th row of} W_8 \text{to multiply} x_8 \\
&= 1+1+1+1+1+1+1+1=8 \\
X(5) &= \text{6th row of} W_8 \text{to multiply} x_8 \\
&= 1+\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}-j-\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}-1-\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}+j+\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}=0 \\
X(6) &= 1-j-1+j+1-j-1+j=0 \\
X(7) &= 1-\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}+j+\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}-1+\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}-j-\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}=0
\end{aligned}
$$

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & j & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \\
1 & -j & -1 & j & 1 & -j & -1 & j \\
1 & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & j & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & j & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \\
1 & j & -1 & -j & 1 & j & -1 & -j \\
1 & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & j & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}
\end{bmatrix}
\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}
\qquad
X_8 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 8 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Home Work

Compute the 4-point DFT of the sequence

$$x[n] = 1 \qquad 0 \le n < 2$$

Home Work

Compute the 4-points DFT of the following sequences:

$$(1) \quad x[n] = \{1,\ 1,\ 1,\ 1\}$$

$$(2) \quad x[n] = \{1,\ 1,\ 0,\ 0\}$$

$$(3) \quad x[n] = \cos \pi n$$

$$(4) \quad x[n] = \sin \frac{n\pi}{2}$$

Properties of DFT

- Periodicity
- If x[n] is the input sequence and X(k) is the N-point DFT of x[n], then the periodicity of x[n] and X(k) are defined as

$$x[n+N] = x[n]$$
$$X(k+N) = X(k)$$

Properties of DFT

- Linearity
- Let x1[n] and x2[n] be two N-point sequences whose DFTs are X1(k) and X2(k). Then

$$a_1 x_1[n] + a_2 x_2[n] \underset{N\text{-points}}{\overset{\text{DFT}}{\longleftrightarrow}} a_1 X_1(k) + a_2 X_2(k)$$

Properties of DFT

- Complex Conjugate Symmetry

$$x[n] \xleftrightarrow{\text{DFT}} X(k)$$

$$x^*[n] \xleftrightarrow{\text{DFT}} X(N-k)$$

Properties of DFT

- Circular Time Shifting

$$x[n] \overset{\text{DFT}}{\longleftrightarrow} X(k)$$

$$x[(n-m)]_N \overset{\text{DFT}}{\longleftrightarrow} e^{-j\frac{2\pi km}{N}} X(k)$$

Properties of DFT

- Circular Frequency Shifting

$$x[n] \xleftrightarrow{\text{DFT}} X(k)$$

$$x[n]e^{j\frac{2\pi nl}{N}} \xleftrightarrow{\text{DFT}} X(k-l)_N$$

Properties of DFT

- Multiplication of Two DFTs
- The multiplication of two DFTs is equal to circular convolution of two sequences in time domain.

$$x_1[n] \circledN x_2[n] \xleftrightarrow{\text{DFT}} X_1(k)X_2(k)$$

Properties of DFT

- Parseval's Theorem

$$x[n] \xleftrightarrow{\text{DFT}} X(k)$$

$$y[n] \xleftrightarrow{\text{DFT}} Y(k)$$

$$\sum_{n=0}^{N-1} x[n]y^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$$

Circular Convolution

- The circular convolution of two sequences is symbolically represented as

$$y[n] = x_1[n] \, \textcircled{N} \, x_2[n]$$

Circular Convolution

The following steps are followed to find $y[n]$:

1. Draw two concentric circles of two different diameters. The data points of $x_1[n]$ are placed on the outer circle in the counter-clockwise direction at equidistance.
2. The data points of $x_2[n]$ are placed on the inner circle in the clockwise direction at equidistance.
3. The first data value of both the sequences should be in alignment.
4. Multiply the corresponding values in both the circles and add them. This corresponds to first data value of the circular convolution.
5. Rotate the inner circle in the counter-clockwise direction by one sample and repeat step 4. This corresponds to the second data value of the circular convolution.
6. Repeat step 5 until one revolution is complete. Each time repeat step 4 to get the data value of the circular convolution.

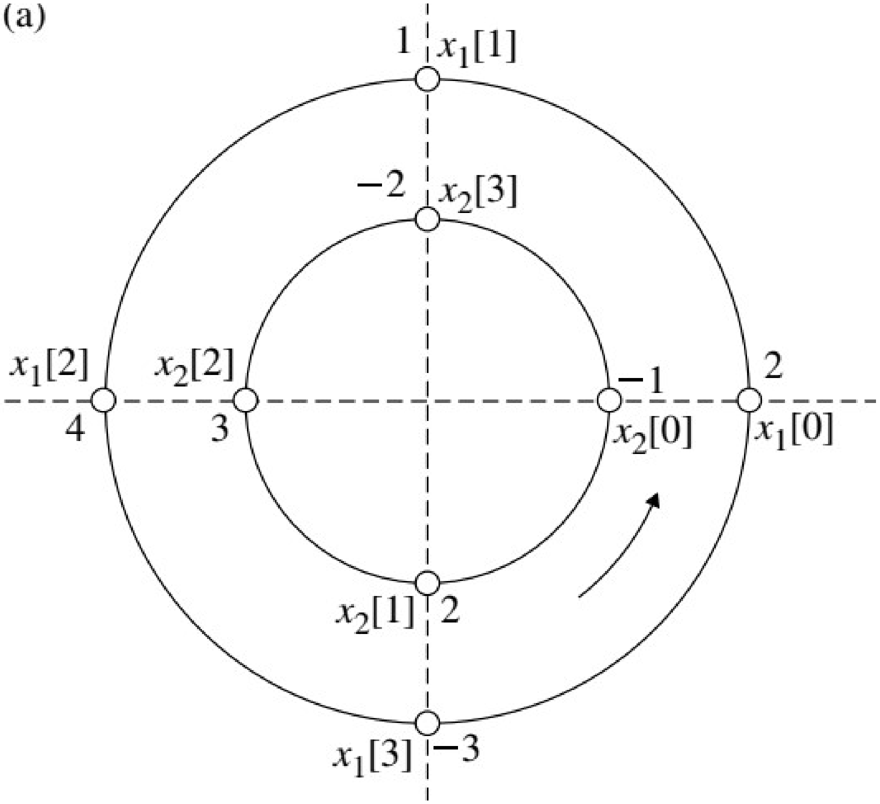
Circular Convolution

Consider the following two sequences:

$$x_1[n] = \{2,\ 1,\ 4,\ -3\}$$
$$x_2[n] = \{-1,\ 2,\ 3,\ -2\}$$

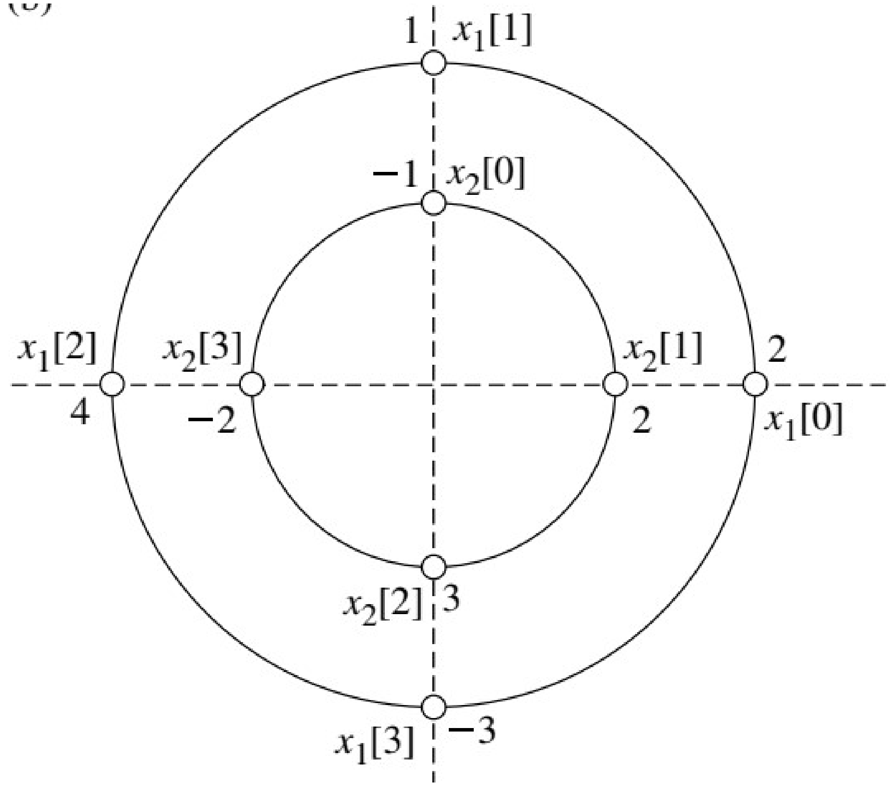
Find the circular convolution

(a)

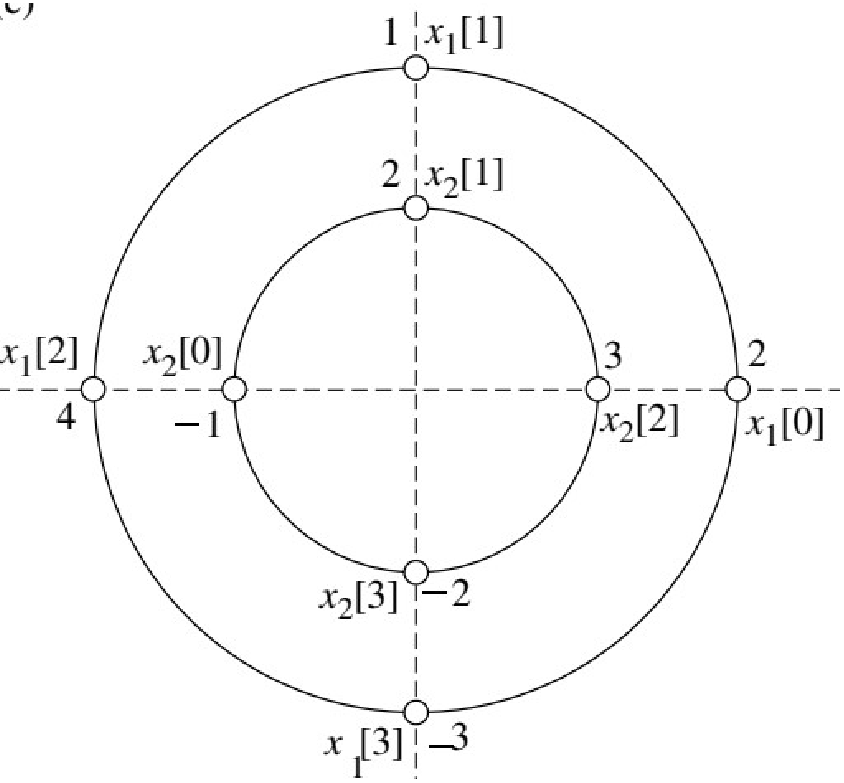


$$y[0] = 2(-1) + 1(-2) + 4(3) + (-3)2 = 2$$

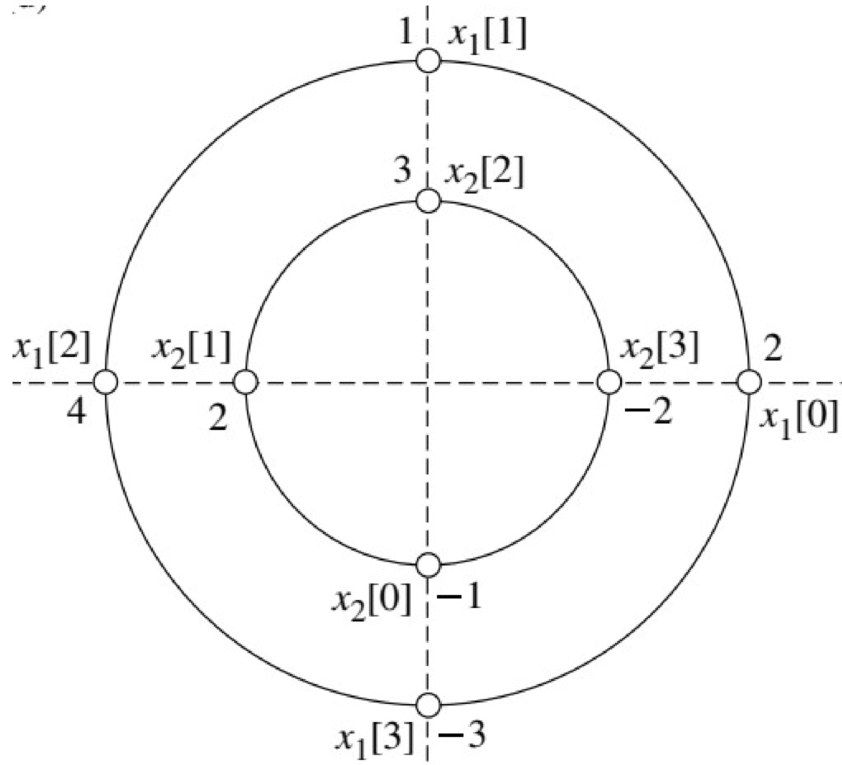
(b)



$$y[1] = 2 \times 2 + 1(-1) + 4(-2) + (-3)3 = -14$$



$$y[2] = 2 \times 3 + 1 \times 2 + 4(-1) + (-3)(-2) = 10$$



$$y[3] = 2(-2) + 1 \times 3 + 4 \times 2 + (-3)(-1) = 10$$

$$\boxed{y[n] = [2,\ -14,\ 10,\ 10]}$$

Circular Convolution-Matrix Multiplication Method

$$\begin{bmatrix} x_2[0] & x_2[N-1] & \cdots & x_2[2] & x_2[1] \\ x_2[1] & x_2[0] & \cdots & x_2[3] & x_2[2] \\ x_2[2] & x_2[1] & \cdots & x_2[4] & x_2[3] \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_2[N-2] & x_2[N-3] & \cdots & x_2[0] & x_2[N-1] \\ x_2[N-1] & x_2[N-2] & \cdots & x_2[1] & x_2[0] \end{bmatrix} \begin{bmatrix} x_1[0] \\ x_1[1] \\ x_1[2] \\ \vdots \\ x_1[N-2] \\ x_1[N-1] \end{bmatrix} = \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \\ y[N-2] \\ y[N-1] \end{bmatrix}$$

Circular Convolution

Consider the following two sequences:

$$x_1[n] = \{2,\ 1,\ 4,\ -3\} \quad \text{and} \quad x_2[n] = \{-1,\ 2,\ 3,\ -2\}$$

Find the circular convolution

$$y[n] = x_1[n] \, Ⓝ \, x_2[n]$$

Use matrix multiplication method.

Circular Convolution

$$\begin{bmatrix} 2 & -3 & 4 & 1 \\ 1 & 2 & -3 & 4 \\ 4 & 1 & 2 & -3 \\ -3 & 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -2-6+12-2=2 \\ -1+4-9-8=-14 \\ -4+2+6+6=10 \\ 3+8+3-4=10 \end{bmatrix} = \begin{bmatrix} 2 \\ -14 \\ 10 \\ 10 \end{bmatrix}$$

$$\boxed{y[n] = \{2,\ -14,\ 10,\ 10\}}$$

Circular Convolution-DFT-IDFT Method

- IDFT
- For the IDFT, the twiddle factor is almost the same but with a positive sign in the exponent

$$W_N^{-1} = e^{j\frac{2\pi}{N}}$$

- the twiddle factor of the IDFT is the complex conjugate of the DFT's twiddle factor

$$W_N^{-1} = \overline{W_N}$$

Circular Convolution-DFT-IDFT Method

Consider the following two sequences:

$$x_1[n] = \{2,\ 1,\ 4,\ -3\} \quad \text{and} \quad x_2[n] = \{-1,\ 2,\ 3,\ -2\}$$

Find the circular convolution

$$y[n] = x_1[n] \text{Ⓝ} x_2[n]$$

Use DFT-IDFT method.

$$X_1(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2-j4 \\ 8 \\ -2+j4 \end{bmatrix}$$

$$X_2(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4-j4 \\ 2 \\ -4+j4 \end{bmatrix}$$

$$\begin{aligned} Y(k) &= X_1(k)X_2(k) \\ &= [4 \quad -2-j4 \quad 8 \quad -2+j4][2 \quad -4-j4 \quad 2 \quad -4+j4] \\ &= \{8 \quad (-8+j24) \quad 16 \quad (-8-j24)\} \end{aligned}$$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\begin{aligned} y[n] &= \frac{1}{4} W_4^* Y_4 \\ &= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 8 \\ -8 + j24 \\ 16 \\ -8 - j24 \end{bmatrix} \end{aligned}$$

$$\boxed{y[n] = \{2,\ -14,\ 10,\ 10\}}$$

Consider the following two sequences:

$$x_1[n] = \{2,\ 3,\ 1,\ 4\}$$
$$x_2[n] = \{5,\ 2,\ 1\}$$

Find the circular convolution

$$X_1(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 1+j \\ -4 \\ 1+j \end{bmatrix}$$

$$X_2 = W_4 x_4$$

$$X_2(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 4-j2 \\ 4 \\ 4+j2 \end{bmatrix}$$

$$\begin{aligned} Y(k) &= X_1(k)X_2(k) \\ &= [10 \quad (1+j) \quad -4 \quad (1-j)][8 \quad (4-j2) \quad 4 \quad (4+j2)] \\ &= \{80 \quad (6+j2) \quad -16 \quad (6-j2)\} \end{aligned}$$

$$y[n] = \frac{1}{4} W_4^* Y_4$$

$$W_N^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$y[n] = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 80 \\ 6+j2 \\ -16 \\ 6-j2 \end{bmatrix} = \begin{bmatrix} 19 \\ 23 \\ 13 \\ 25 \end{bmatrix}$$