The Laplace Transform

Limitation of Fourier Transform: Some useful signals do not have CTFT because these signals do not converge.
Laplace converts a signal from time domain to frequency domain by expressing time signals as linear combinations of complex exponential.

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$s = \sigma + j\omega$$

LT of Causal and Non-causal Systems

$$X(s) = \int_0^{\infty} x(t)e^{-st}dt$$ unilateral LT for causal signals and systems

$$X(s) = \int_{-\infty}^{0} x(t)e^{-st}dt$$

Example

$$x(t) = e^{-at}u(t)$$

$$X(s) = \int_0^\infty e^{-at}e^{-st}dt$$

$$= \int_0^\infty e^{-(s+a)t}dt$$

$$= -\frac{1}{(s+a)}\left[e^{-(s+a)t}\right]_0^\infty$$

$$= -\frac{1}{(s+a)}\left[e^{-(s+a)\infty} - e^{-(s+a)0}\right]$$

$$X(s) = \frac{1}{(s+a)}$$

Home work: Find the Laplace Transform for

$$x(t) = e^{-at}u(-t)$$

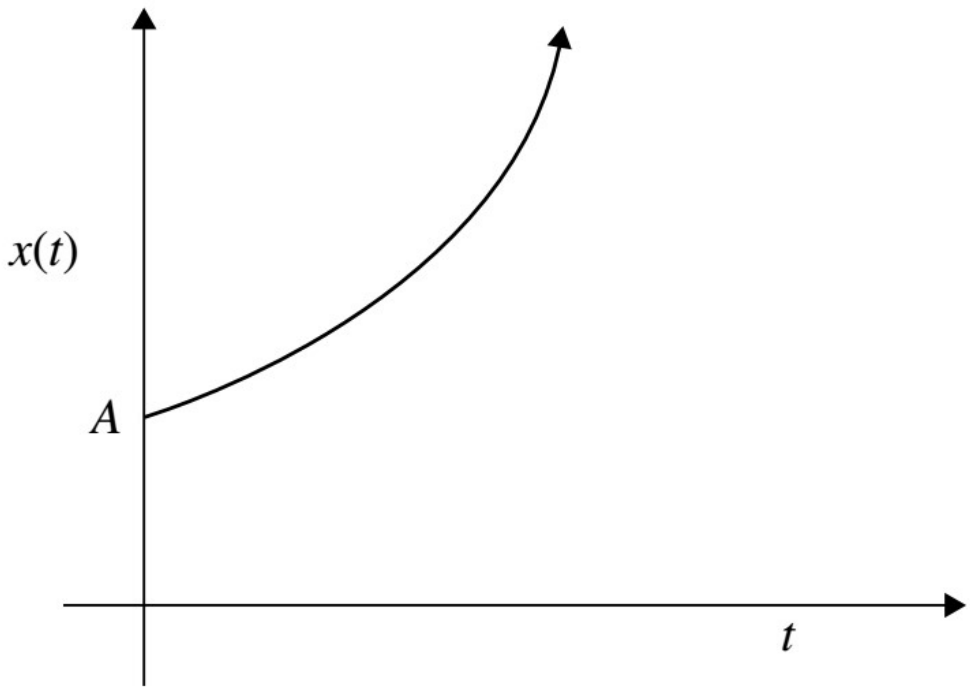
$$X(s) = \frac{-1}{(s+a)}$$

The Region of Convergence

The region of convergence which is denoted as ROC is, defined as the real part of s for which the Laplace integral converges.

The Region of Convergence

$$x(t) = Ae^{at}u(t) \quad a > 0$$

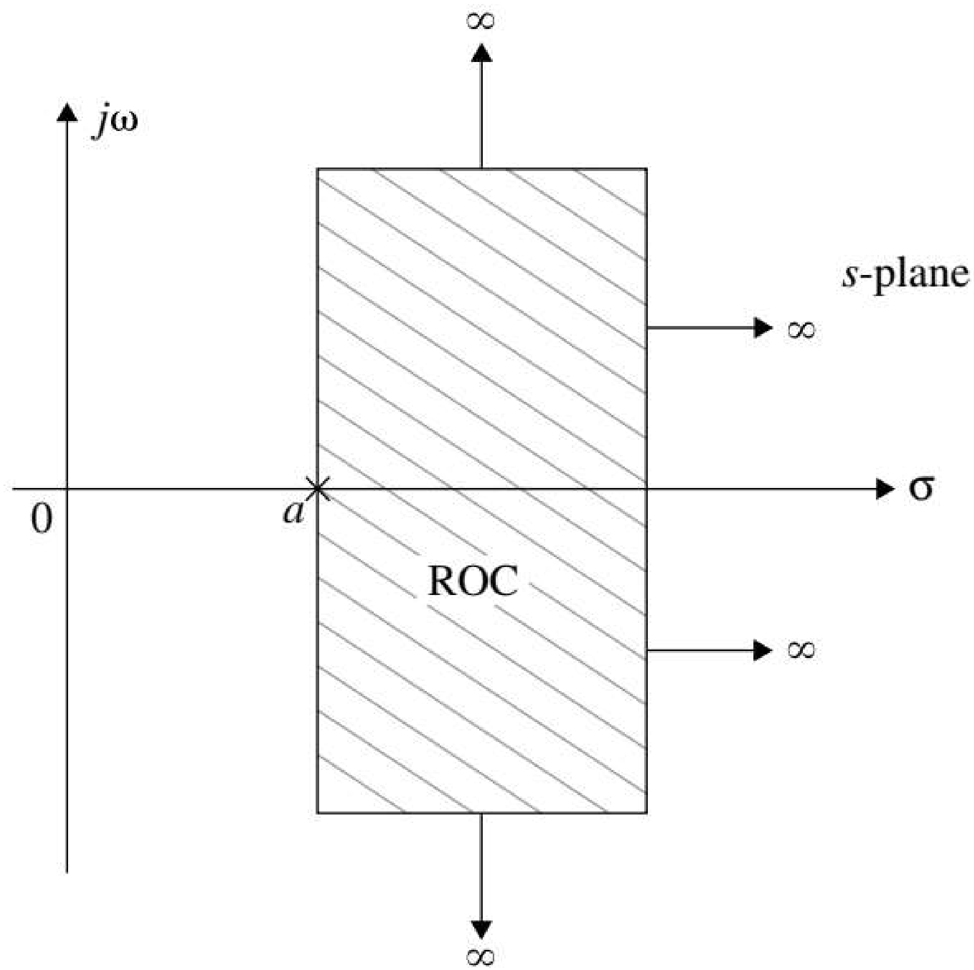


$$X(s) = \int_0^\infty Ae^{at}e^{-st}dt$$
$$= A\int_0^\infty e^{-(s-a)t}dt$$
$$= \frac{-A}{(s-a)}\left[e^{-(s-a)t}\right]_0^\infty \quad \text{.........(1)}$$

$$X(s) = \frac{A}{(s-a)}$$

- Equation 1 converges if (s − a) > 0. In other words, Re s > a.
- when the upper limit of t = ∞ is applied, X(s) = 0 and
- when the lower limit of t = 0 is applied, X(s) is finite.

$$\text{ROC of } X(s) = \frac{A}{(s-a)}$$



Properties of ROCs for LT

1: The ROC of $X(s)$ consists of parallel strips to the imaginary axis.

2: The ROC of LT does not include any pole of $X(s)$.

3: If $x(t)$ is a finite duration signal, and is absolutely integrable then the ROC of $X(s)$ is the entire s-plane.

4: For the right-sided (causal) signal if the $\mathrm{Re}(s) = \sigma_0$ and is in ROC, then for all the values of s for which $\mathrm{Re}(s) > \sigma_0$ is also in ROC.

5: If $x(t)$ is a left-sided (non-causal) signal and if $\mathrm{Re}(s) = \sigma_0$ is in ROC, then for all the values of s for which $\mathrm{Re}(s) < \sigma_0$ is also in ROC.

Determine the LT of the following signal. Mark the poles and ROC in the s-plane.
$x(t) = Ae^{-at}u(t) + Be^{-bt}u(-t)$ where $a > 0$, $b > 0$ and $|a| > |b|$.

The given signal $x(t)$ consists of causal and anti-causal signals and can be written as

$$x(t) = x_1(t) + x_2(t)$$

where

$$x_1(t) = Ae^{-at}u(t)$$

$$x_2(t) = Be^{-bt}u(-t)$$

$$X_1(s) = \int_0^{\infty} Ae^{-at}e^{-st}\,dt$$

$$= A\int_0^{\infty} e^{-(s+a)t}\,dt$$

$$= \frac{-A}{(s+a)}\left[e^{-(s-a)t}\right]_0^{\infty}$$

$$= \frac{A}{(s+a)}$$

The ROC is $\mathrm{Re}(s) > -a$.

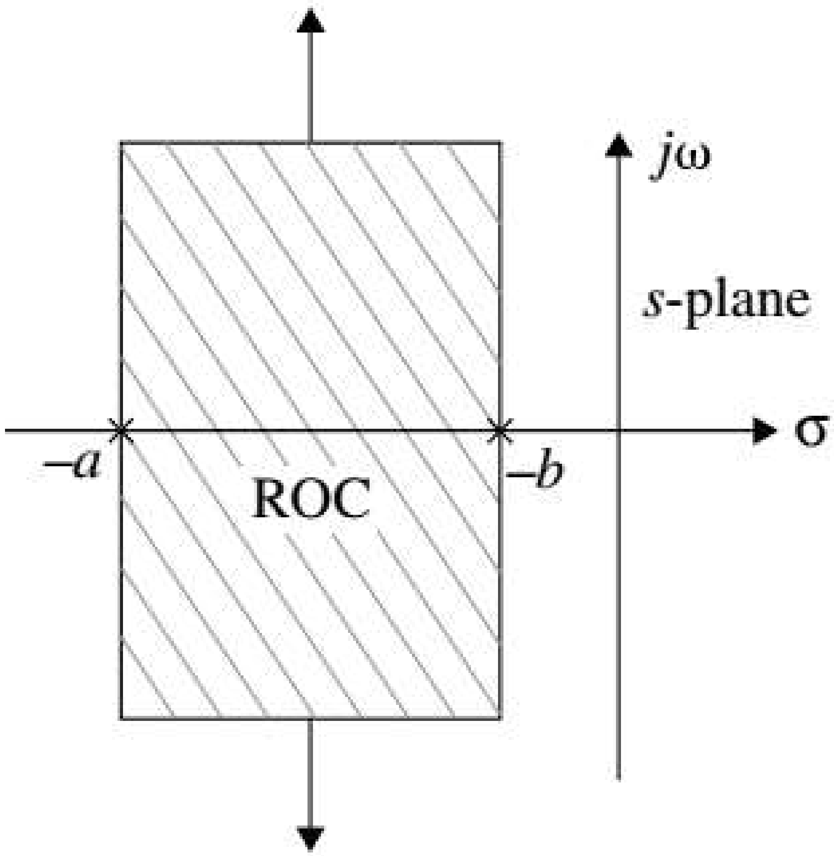
$$x_2(t) = Be^{-bt}u(-t)$$

$$\begin{aligned}
X_2(s) &= \int_{-\infty}^{0} Be^{-bt}e^{-st}\,dt \\
&= B\int_{-\infty}^{0} e^{-(s+b)t}\,dt \\
&= \frac{-B}{(s+b)}\left[e^{-(s+b)t}\right]_{-\infty}^{0} \\
&= -\frac{B}{(s+b)}[1-0] \quad \text{only if } \mathrm{Re}(b+s) < 0 \\
&= \frac{-B}{(s+b)}
\end{aligned}$$

The ROC is $\mathrm{Re}(s) < -b$.

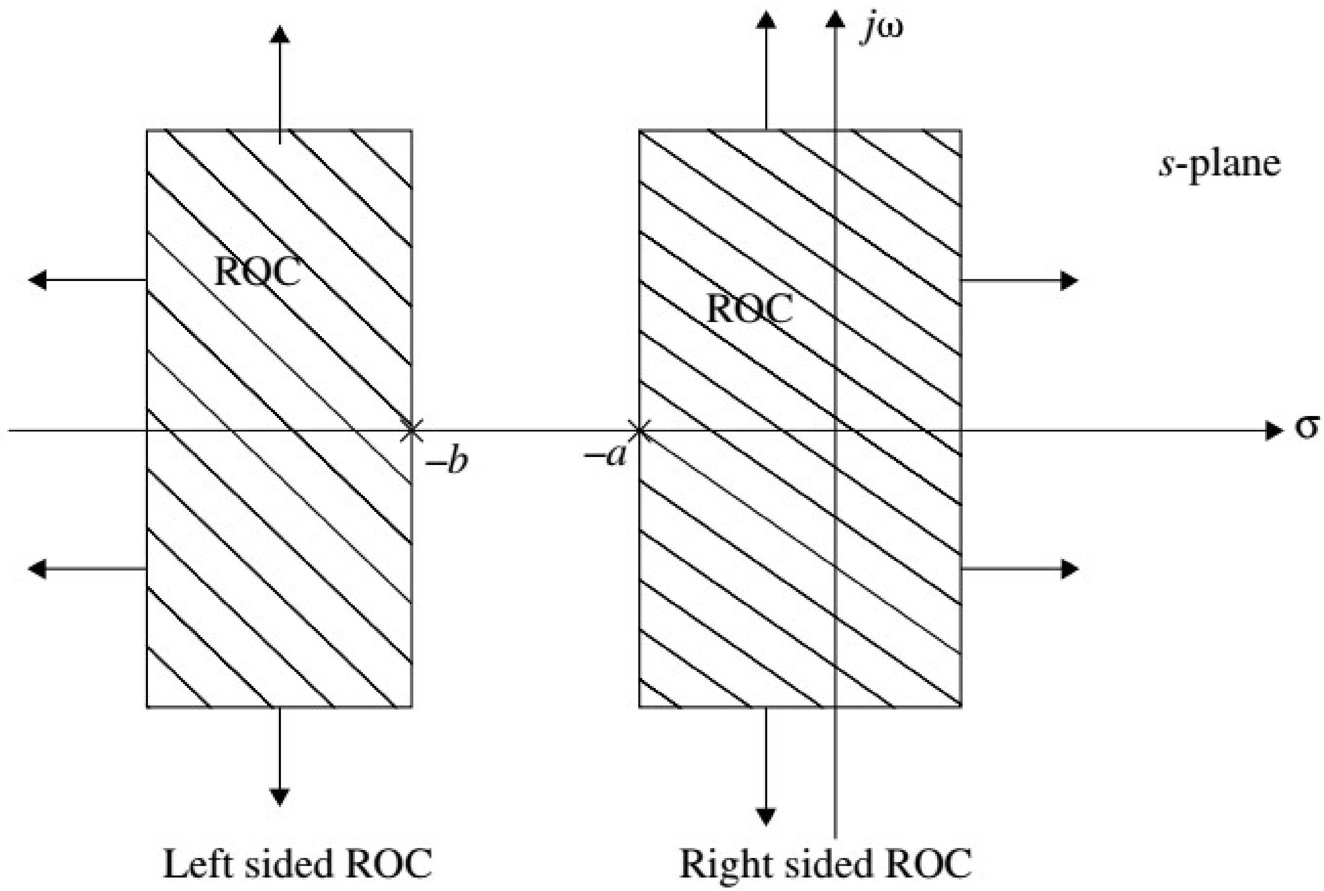
$$\begin{aligned}
X(s) &= X_1(s) + X_2(s) \\
&= \frac{A}{(s+a)} - \frac{B}{(s+b)}
\end{aligned}$$

$|a| > |b|$. The ROC is $\mathrm{Re}(s) > -a$. $\mathrm{Re}(s) < -b$.



(b)

$|b| > |a|$ The ROC is $\text{Re}(s) > -a$. $\text{Re}(s) < -b$.



The ROC s of x1(t) and x2(t) do not overlap and hence x(t) does not have LT

Home work

Determine the LT and sketch the ROC in the s-plane.

$$x(t) = e^{-2t}u(t) + e^{-3t}u(t)$$

$$x(t) = e^{-2t}u(-t) + e^{-3t}u(-t)$$

$$x(t) = e^{-2t}u(-t) + e^{-3t}u(t)$$

$$x(t) = Au(t)$$

The Unilateral Laplace Transform

$$X(s) = \int_0^\infty x(t)e^{-st}dt$$

1. The unilateral LT simplifies the system analysis considerably.
2. The signals are restricted to causal signals.
3. There is one-to-one correspondence between LT and inverse LT.

Properties of Laplace Transform

Linearity

$$x_1(t) \xleftrightarrow{L} X_1(s)$$

$$x_2(t) \xleftrightarrow{L} X_2(s)$$

$$[a_1x_1(t) + a_2x_2(t)] \xleftrightarrow{L} [a_1X_1(s) + a_2X_2(s)]$$

Properties of Laplace Transform

Time Shifting

$$x(t) \xleftrightarrow{L} X(s)$$

$$x(t - t_0) \xleftrightarrow{L} X(s)e^{-st_0}$$

Properties of Laplace Transform

Frequency Shifting

$$x(t) \xleftrightarrow{L} X(s)$$

$$x(t)e^{s_0 t} \xleftrightarrow{L} X(s - s_0)$$

Properties of Laplace Transform

Time Scaling

$$x(t) \xleftrightarrow{L} X(s)$$

$$x(at) \xleftrightarrow{L} \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

Properties of Laplace Transform

Frequency Scaling

$$x(t) \xleftrightarrow{L} X(s)$$

$$\frac{1}{a}x\left(\frac{t}{a}\right) \xleftrightarrow{L} X(as)$$

Properties of Laplace Transform

Time Differentiation

$$x(t) \overset{L}{\longleftrightarrow} X(s)$$

$$\frac{dx(t)}{dt} \overset{L}{\longleftrightarrow} sX(s) - x(0^-)$$

$$\frac{d^2x(t)}{dt^2} \overset{L}{\longleftrightarrow} s^2X(s) - sx(0^-) - \frac{d}{dt}x(0^-)$$

Properties of Laplace Transform

Time Integration

$$x(t) \overset{L}{\longleftrightarrow} X(s)$$

$$\int_0^t x(\tau)\,d\tau \overset{L}{\longleftrightarrow} \frac{X(s)}{s}$$

Properties of Laplace Transform

Time Convolution

$$x_1(t) \stackrel{L}{\longleftrightarrow} X_1(s)$$

$$x_2(t) \stackrel{L}{\longleftrightarrow} X_2(s)$$

$$x_1(t) * x_2(t) \stackrel{L}{\longleftrightarrow} X_1(s)X_2(s)$$

Properties of Laplace Transform

Frequency Differentiation

$$-\,tx(t) \xleftrightarrow{L} \frac{d}{ds}(X(s))$$

Properties of Laplace Transform

Complex Frequency Shifting

$$[e^{s_0 t} x(t)] \xleftrightarrow{L} X(s - s_0)$$

Properties of Laplace Transform

Initial Value Theorem

$$\underset{t \to 0}{Lt}\, x(t) = \underset{s \to \infty}{Lt}\, sX(s)$$

Final Value Theorem

$$\underset{t \to \infty}{Lt}\, x(t) = \underset{s \to 0}{Lt}\, sX(s)$$

Determine the LT of unit impulse function $\delta(t)$

$$L[\delta(t)] = \int_{0^-}^{\infty} \delta(t)e^{-st}dt$$

$$= 1 \qquad \int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0).$$

$$\boxed{\delta(t) \stackrel{L}{\longleftrightarrow} 1} \qquad \text{ROC : all } s$$

Determine the LT of a ramp function of slope R

$$x(t) = Rt\,u(t)$$

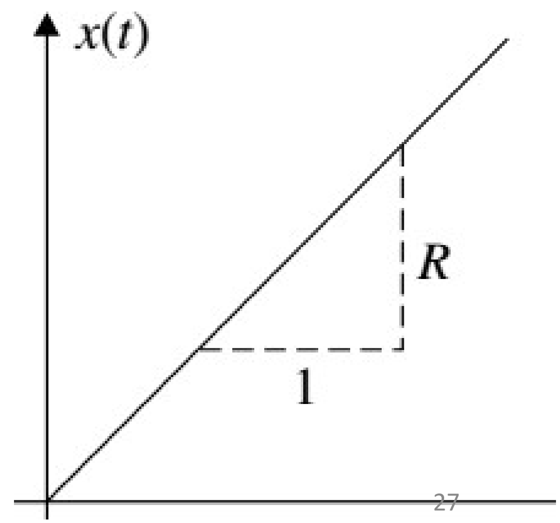
$$L[Rt] = \int_0^{\infty} Rt\,e^{-st}\,dt$$

$$\int ab = a\int b - \int a' \int b$$

$$= R\left[\frac{te^{-st}}{(-s)}\right]_0^{\infty} - R\int_0^{\infty} \frac{e^{-st}}{(-s)}\,dt$$

$$= R[0 - 0] - R\left[\frac{e^{-st}}{-s^2}\right]_0^{\infty}$$

$$= \frac{R}{s^2}$$



Some Important Laplace Transform Pairs

$x(t)$	$X(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$e^{at}u(t)$	$\frac{1}{(s-a)}$
$e^{-at}u(t)$	$\frac{1}{(s+a)}$
$\cos at\, u(t)$	$\frac{s}{(s^2+a^2)}$
$\sin at\, u(t)$	$\frac{a}{(s^2+a^2)}$

Determine the LT of a sine function

$$x(t) = A \sin \omega_0 t \, u(t)$$

We know that

$$\sin \omega_0 t = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$$

So....

$$L[A \sin \omega_0 t] = \frac{A}{2j}[L(e^{j\omega_0 t}) - Le^{-j\omega_0 t}]$$

$$= \frac{A}{2j}\left[\frac{1}{s - j\omega_0} - \frac{1}{s + j\omega_0}\right]$$

$$= \frac{A}{2j}\frac{2j\omega_0}{(s^2 + \omega_0^2)}$$

ROC: Re $s > 0$.

Home Work

Determine the LT of a cosine function

determine the LT of

$x(t) = e^{-at} \sin \omega_0 t.$

We Know

$$L[\sin \omega_0 t] = \frac{\omega_0}{(s^2 + \omega_0^2)}$$

shifting property is

$$L[e^{-at} x(t)] = X(s + a)$$

Applying the above property, we get

$$\boxed{L[e^{-at} \sin \omega_0 t] = \frac{\omega_0}{(s + a)^2 + \omega_0^2}}$$

By applying the complex differentiation property, determine the LT of

$x(t) = t \sin \omega_0 t.$

We Know

$$L[\sin \omega_0 t] = \frac{\omega_0}{(s^2 + \omega_0^2)}$$

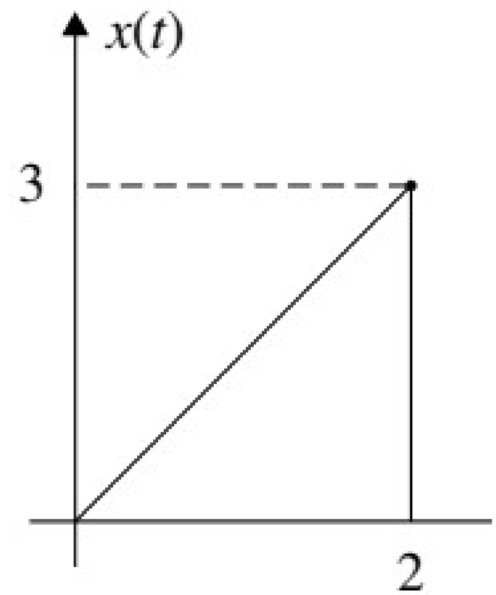
According to the complex differentiation property

$$L[-tx(t)] = \frac{d}{ds} X(s)$$

$$\underset{-t}{L}[\sin \omega_0 t] = \frac{d}{ds} \frac{\omega_0}{(s^2 + \omega_0^2)} = \frac{-2\omega_0 s}{(s^2 + \omega_0^2)^2}$$

$$L[t \sin \omega_0 t] = \frac{2\omega_0 s}{(s^2 + \omega_0^2)^2}$$

Determine the LT of the saw tooth wave form shown



$$x(t) = \frac{3}{2}t \quad 0 \le t \le 2$$

$$X(s) = \int_0^2 \frac{3}{2} t e^{-st} dt$$

using $\int ab = a \int b - \int a' \int b$

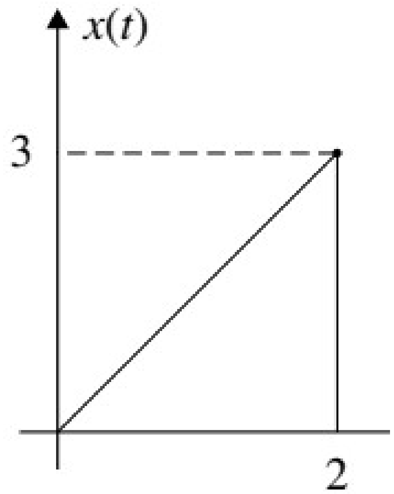
$$X(s) = \left[\frac{3}{2}t\left(-\frac{1}{s}\right)e^{-st}\right]_0^2 + \frac{3}{2}\int_0^2 \frac{1}{s}e^{-st}dt$$

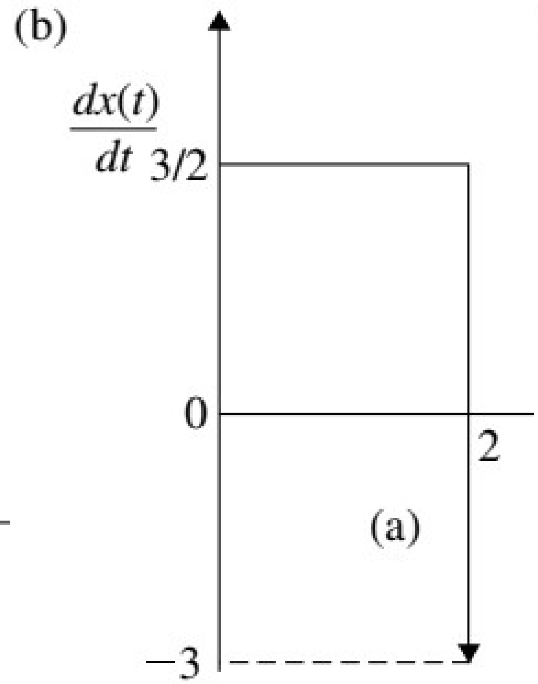
$$= \frac{-3}{s}e^{-2s} + \frac{3}{2s^2}\left[-1e^{-st}\right]_0^2$$

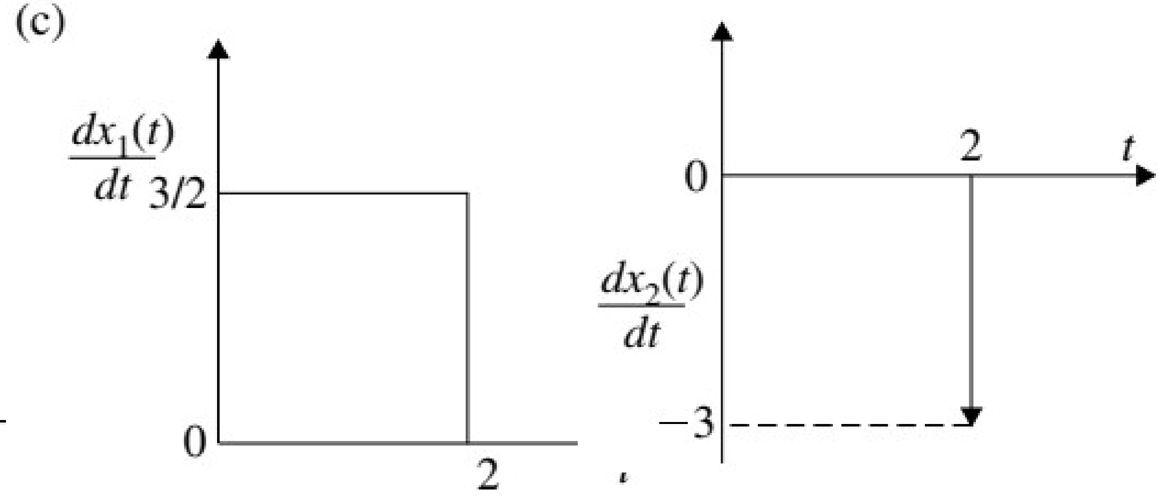
$$= \frac{-3}{s}e^{-2s} - \frac{3}{2s^2}e^{-2s} + \frac{3}{2s^2}$$

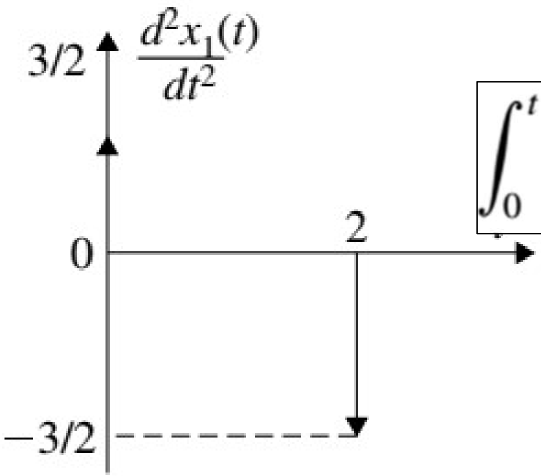
$$X(s) = \frac{3}{2}\frac{1}{s^2} - \left(\frac{3}{s} + \frac{3}{2s^2}\right)e^{-2s}$$

Method 2:









$$\boxed{\int_0^t x(\tau)d\tau \overset{L}{\longleftrightarrow} \frac{X(s)}{s}}$$

$$X_2(s) = \frac{-3}{s}e^{-2s}$$

$$X_1(s) = \frac{1}{s^2}\frac{3}{2}\left[1 - e^{-2s}\right]$$

$$X(s) = X_1(s) + X_2(s)$$

$$\boxed{X(s) = \frac{3}{2s^2}\left[1 - e^{-2s}\right] - \frac{3}{s}e^{-2s}}$$

Home Work

Determine the LT of

$$x(t) = \cos at \sin bt.$$

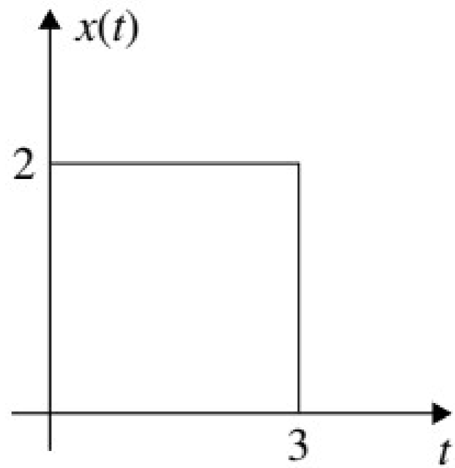
$$x(t) = u(t-3)$$

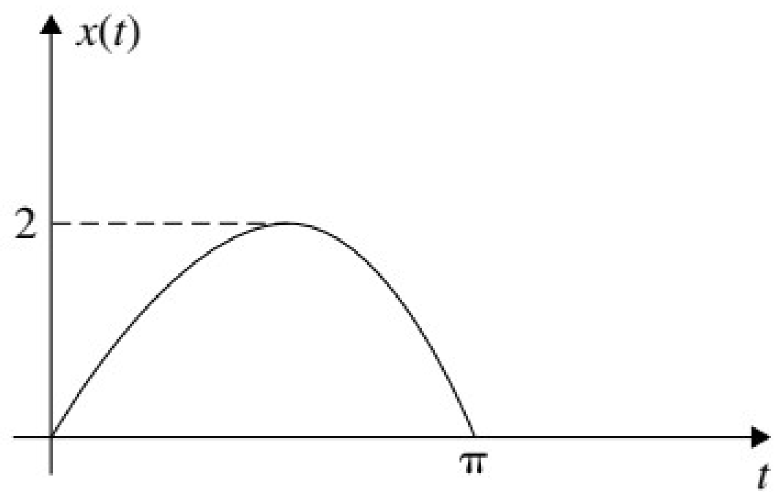
$$x(t) = \sin(at + \theta)$$

$$x(t) = \cos(at + \theta)$$

$$x(t) = \delta(t-2) - \delta(t-5).$$

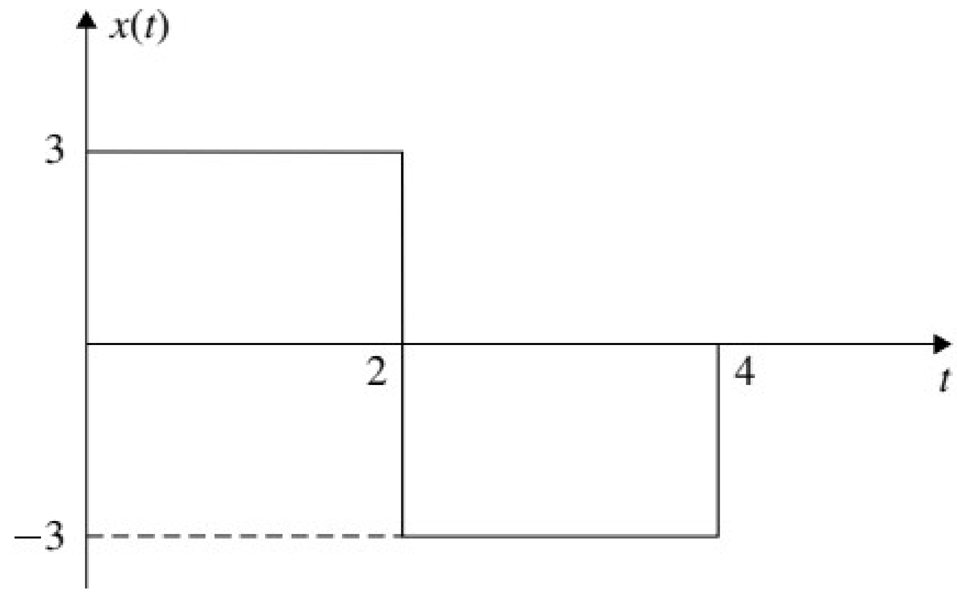
$$x(t) = u(t-2) - u(t-5).$$

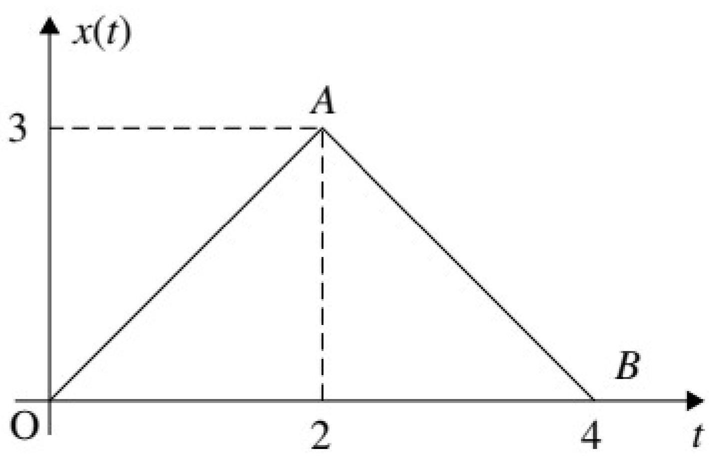


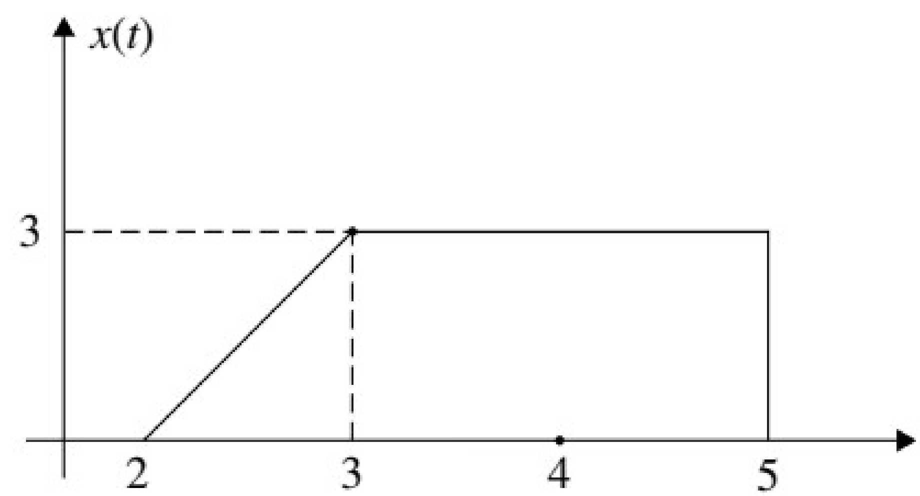


Home Work

Determine the LT of







Consider the following function $X(s) = \dfrac{(5s+4)(s+6)}{s(s+2)(3s+1)}$

Find the initial and final values of $x(t)$.

The initial value is given by

$$\underset{t \to 0}{Lt}\, x(t) = x(0) = \underset{s \to \infty}{Lt}\, sX(s)$$

$$= \underset{s \to \infty}{Lt} \frac{s(5 + \frac{4}{s})(1 + \frac{6}{s})}{s(1 + \frac{2}{s})(3 + \frac{1}{s})}$$

$$= \frac{5 \times 1}{1 \times 3} = \frac{5}{3}$$

$$\boxed{x(0) = \frac{5}{3}}$$

The final value of $x(t)$ is given by

$$\underset{t \to \infty}{x(t)} = x(\infty) = \underset{s \to 0}{Lt}\, sX(s)$$

$$= \underset{s \to 0}{Lt} \frac{s(5s+4)(s+6)}{s(s+2)(3s+1)}$$

$$= \frac{4 \times 6}{2 \times 1} = 12$$

$$\boxed{x(\infty) = 12}$$

Laplace Transform of Periodic Signal

If a signal $x(t)$ is a periodic signal with period T

then the LT of $X(s)$ is given as

$$X(s) = X_1(s)\left[1 + e^{-Ts} + e^{-2Ts} + \ldots\right] = \frac{X_1(s)}{(1 - e^{-Ts})}$$

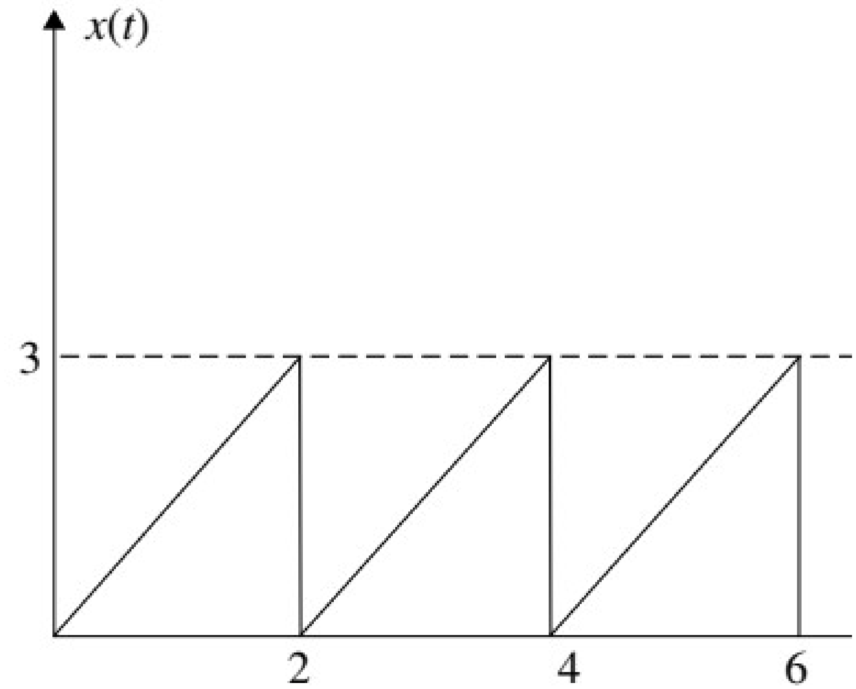
Here, $x_1(t)$ is the signal which is repeated for every T.

Consider the saw tooth wave shown Determine the LT.

$$X_1(s) = \frac{3}{2s^2} - \left(\frac{3}{s} + \frac{3}{2s^2}\right)e^{-2s}$$

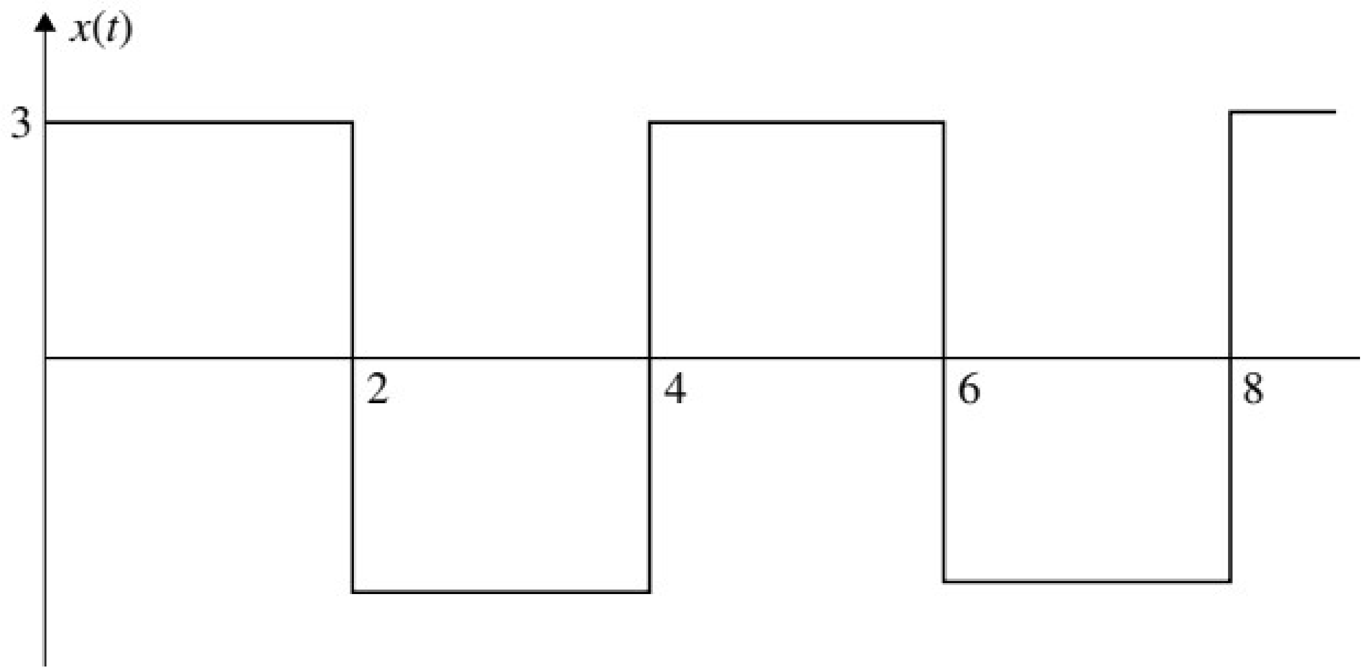
$$X(s) = \frac{X_1(s)}{(1 - e^{-2s})}$$

$$\boxed{X(s) = \frac{3}{2(1 - e^{-2s})}\left[\frac{1}{s^2} - \left(\frac{2}{s} + \frac{1}{s^2}\right)e^{-2s}\right]}$$



Home Work

Determine the LT of



Inverse Laplace Transform

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

Method

express X(s) in polynomial form both in the numerator and the denominator. Both these polynomials are factorized as

$$X(s) = \frac{(s + z_1)(s + z_2) \ldots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)}$$

$$X(s) = \frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

The points in the s-plane at which X(s) = 0 are called **zeros**
Similarly, the points in the s-plane at which X(s) = ∞ are called **poles** of X(s)
For $m < n$ the degree of the numerator polynomial is less than the degree of the denominator polynomial
Under this condition, X(s) is written in the following partial fraction form

$$X(s) = \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} + \frac{A_3}{s+p_3} + \cdots + \frac{A_n}{s+p_n}$$

$$X(s) = \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} + \frac{A_3}{s+p_3} + \cdots + \frac{A_n}{s+p_n}$$

$A_1, A_2, \ldots, A_n$ are called the residues

There are two different methods of finding residues

Method 1. Analytical Method

Method 2. Graphical Method

Find $x(t)$. $\quad X(s) = \dfrac{10(s+2)(s-3)}{s(s+4)(s-5)}$

The given $X(s)$ is expressed in partial fraction form

$$\frac{10(s+2)(s-3)}{s(s+4)(s-5)} = \frac{A_1}{s} + \frac{A_2}{(s+4)} + \frac{A_3}{(s-5)}$$

$$10(s+2)(s-3) = A_1(s+4)(s-5) + A_2 s(s-5) + A_3 s(s+4)$$

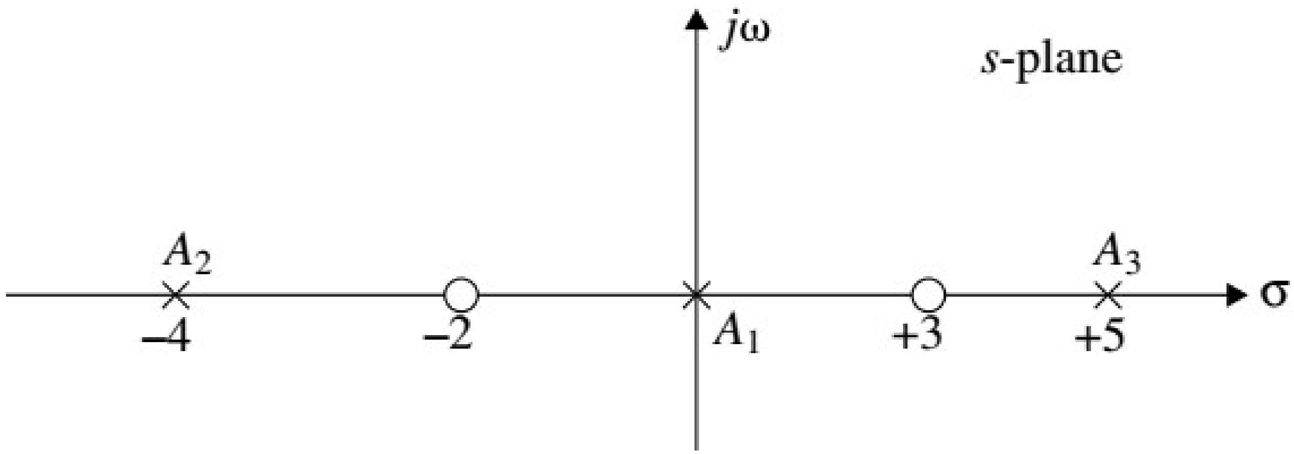
Substitute $s = 0 \Rightarrow 10(2)(-3) = A_1(4)(-5) + 0 + 0 \Rightarrow A_1 = \dfrac{60}{20} = 3$

Substitute $s = -4 \Rightarrow \quad 10(-4+2)(-4-3) = 0 - A_2 4(-4-5) + 0$

$$A_2 = \frac{35}{9} \qquad A_3 = \frac{28}{9}$$

Method 2. Graphical Method

$$A = \frac{\text{Constant term} \times \text{Directed Vector distances drawn from all zeros to the concerned point}}{\text{Directed vector distances drawn from all poles to the concerned point}}$$



$$X(s) = \frac{10(s+2)(s-3)}{s(s+4)(s-5)}$$

$$A_1 = \frac{10(2)(-3)}{4(-5)} = 3 \qquad A_2 = \frac{10(-2)(-7)}{(-4)(-9)} = \frac{35}{9} \qquad A_3 = \frac{10(7)(2)}{(5)(9)} = \frac{28}{9}$$

$$X(s) = \frac{3}{s} + \frac{35}{9}\frac{1}{(s+4)} + \frac{28}{9}\frac{1}{(s-5)}$$

$$\boxed{x(t) = \left(3 + \frac{35}{9}e^{-4t} + \frac{28}{9}e^{5t}\right)u(t)}$$

Home Work

Find the inverse LT of

$$X(s) = \frac{10e^{-3s}}{(s-2)(s+2)}$$

$$X(s) = \frac{(s+1)+3e^{-4s}}{(s+2)(s+3)}.$$

$$X(s) = \frac{(s+1)(s+3)}{(s+2)(s+4)} = 1 - \frac{(2s+5)}{(s+2)(s+4)}$$

Solving Differential Equation

Solving Differential Equation without Initial Conditions

$$L[y(t)] = Y(s)$$

$$L\left[\frac{dy}{dt}\right] = sY(s)$$

$$L\left[\frac{d^2y}{dt}\right] = s^2Y(s)$$

Consider an LTIC system with the following differential equation with zero initial conditions for the input and output.

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

Find the impulse response of the system.

Taking LT on both sides

$$(s^2 + 4s + 3)Y(s) = (s + 2)X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{(s+2)}{(s^2 + 4s + 3)}$$

$$= \frac{(s+2)}{(s+1)(s+3)}$$

$$= \frac{A_1}{(s+1)} + \frac{A_2}{(s+3)}$$

$$A_1 = 0.5 \quad A_2 = 0.5$$

$$Y(s) = \frac{0.5}{(s+1)} + \frac{0.5}{(s+3)}$$

$$\boxed{y(t) = 0.5\left[e^{-t} + e^{-3t}\right]u(t)}$$

Home Work

Using LT, find the impulse response of an LTI system

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$$

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t)$$

$$\frac{d^2y(t)}{dt} + 2\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + x(t)$$

Solving Differential Equation with the Initial Conditions

$$L\left[\frac{dy}{dt}\right] = sY(s) - y(0^-)$$

$$L\left[\frac{d^2y}{dt^2}\right] = s^2Y(s) - sy(0^-) - \dot{y}(0^-)$$

$$L\left[\frac{d^3y}{dt^3}\right] = s^3Y(s) - s^2y(0^-) - s\dot{y}(0^-) - \ddot{y}(0^-)$$

A certain system is described by the following differential equation

$$\frac{d^2y(t)}{dt^2} + 7\frac{dy(t)}{dt} + 12y(t) = x(t)$$

Use LT to determine the response of the system to unit step input applied at $t = 0$
Assume the initial conditions are $y(0^-) = -2$ and $\frac{dy(0^-)}{dt} = 0$.

Taking LT on both sides

$$s^2Y(s) - sy(0^-) - \dot{y}(0^-) + 7Y(s) - 7y(0^-) + 12Y(s) = X(s)$$

$$(s^2 + 7s + 12)Y(s) + 2s + 14 = \frac{1}{s}$$

$$Y(s) = \frac{(-2s^2 - 14s + 1)}{s(s^2 + 7s + 12)}$$

$$= \frac{(-2s^2 - 14s + 1)}{s(s + 3)(s + 4)}$$

$$= \frac{A_1}{s} + \frac{A_2}{(s+3)} + \frac{A_3}{(s+4)}$$

$$A_1 = \frac{1}{12} \quad A_2 = \frac{-25}{3} \quad A_3 = \frac{25}{4}$$

$$Y(s) = \frac{1}{12}\frac{1}{s} - \frac{25}{3}\frac{1}{(s+3)} + \frac{25}{4}\frac{1}{(s+4)}$$

$$\boxed{y(t) = \left[\frac{1}{12} - \frac{25}{3}e^{-3t} + \frac{25}{4}e^{-4t}\right]u(t)}$$

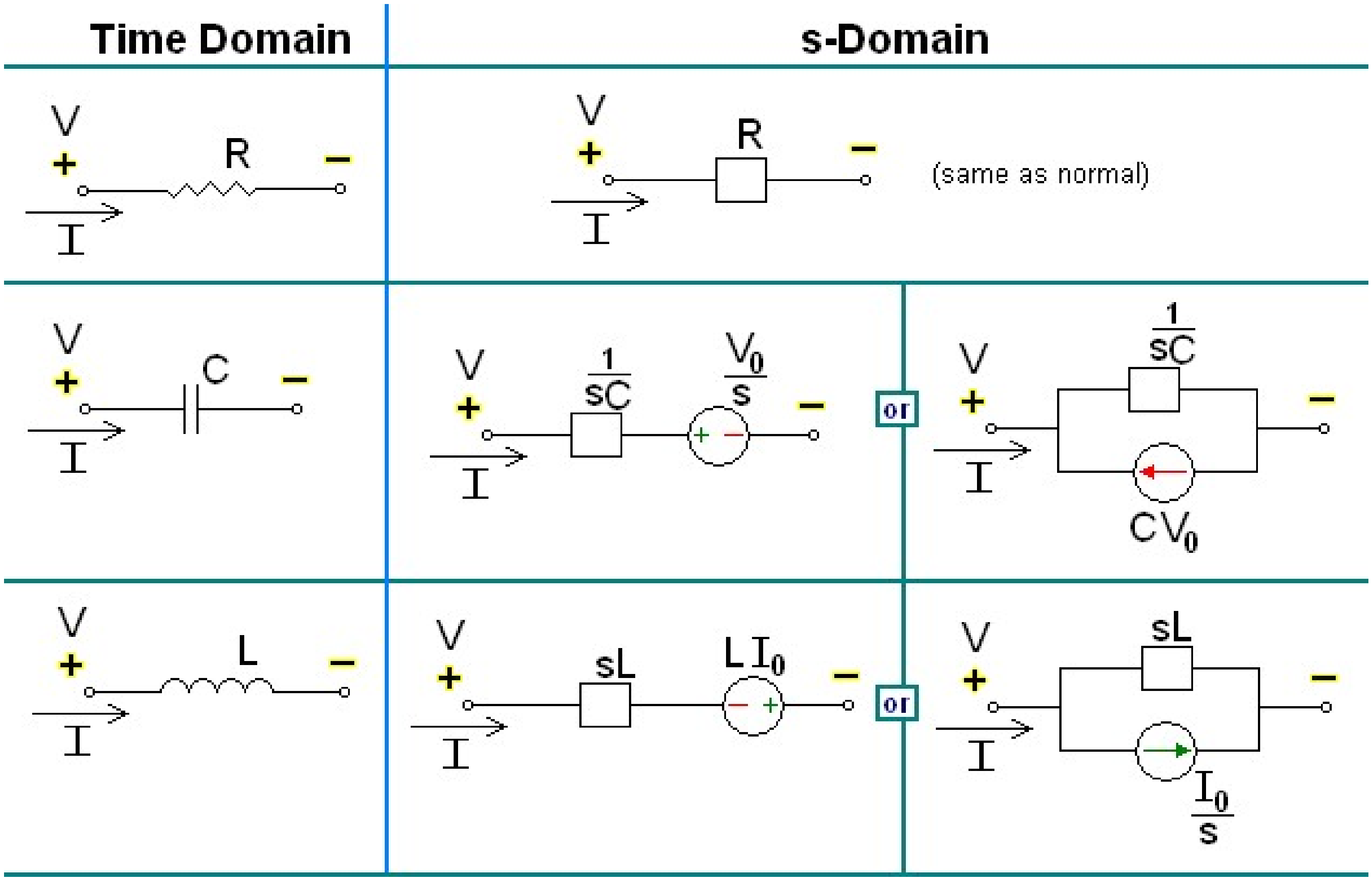
Home Work

Solve

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt} + x(t)$$

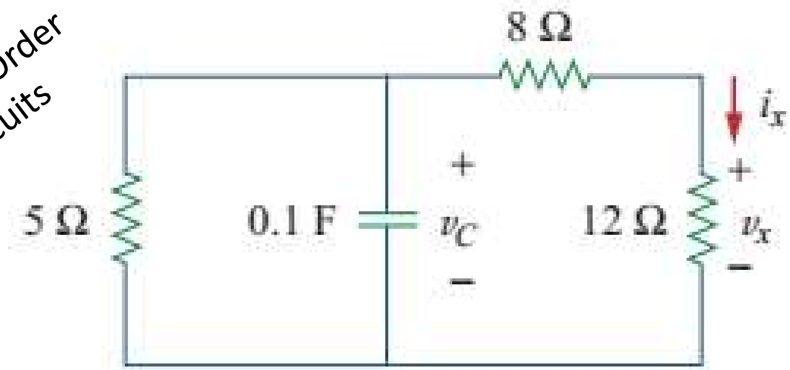
if the initial conditions are $y(0^+) = \frac{9}{4}$; $\dot{y}(0^+) = 5$, if the input is $e^{-3t}u(t)$.

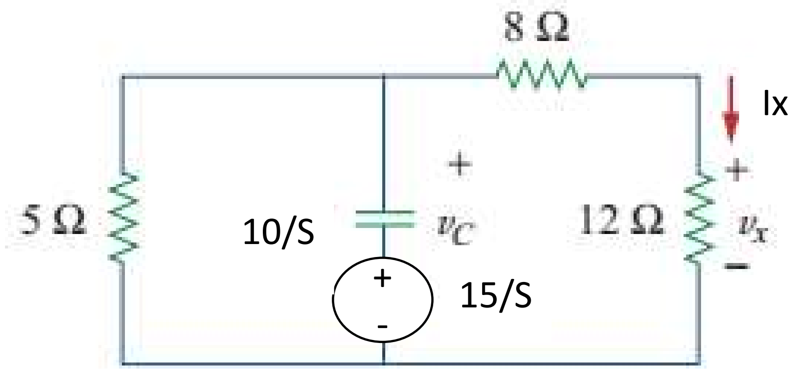
Circuit Analysis



First Order Circuits

let $v_C(0) = 15$ V. Find v_C, v_x, and i_x for $t > 0$.





$$5.I1 + \left(\frac{10}{s}(I1 - I2)\right) = -\frac{15}{S} \quad \text{....(1)}$$

$$\frac{10}{S}(I2 - I1) + 20.I2 = \frac{15}{S} \quad \text{....(2)}$$

$$\left(5 + \frac{10}{S}\right)I1 + \left(-\frac{10}{S}\right)I2 = -\frac{15}{S} \quad \text{....(3)}$$

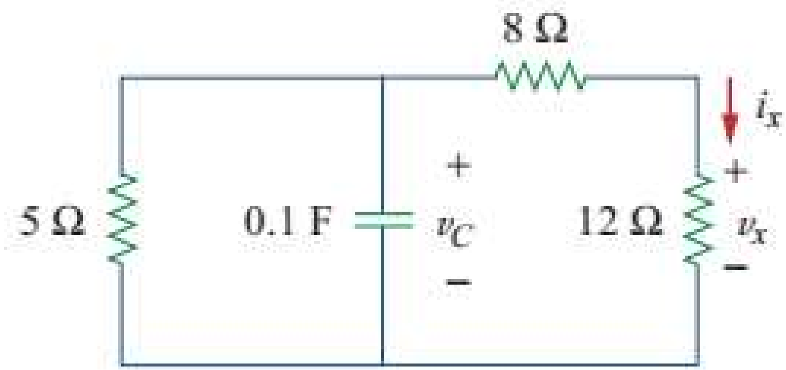
$$\left(-\frac{10}{S}\right)I1 + \left(\frac{10}{S} + 20\right)I2 = \frac{15}{S} \quad \text{....(4)}$$

$$\left(5 + \frac{10}{S}\right)I1 + \left(-\frac{10}{S}\right)I2 = -\frac{15}{S} \quad \text{....(3)}$$

$$\left(-\frac{10}{S}\right)I1 + \left(\frac{10}{S} + 20\right)I2 = \frac{15}{S} \quad \text{....(4)}$$

$$I2 = \frac{\left(5 + \frac{10}{S}\right).\left(\frac{15}{S}\right) - \left(-\frac{10}{S}\right).\left(-\frac{15}{S}\right)}{\left(5 + \frac{10}{S}\right).\left(\frac{10}{S} + 20\right) - \left(-\frac{10}{S}\right).(-\frac{10}{S})}$$

$$I2 = \frac{\frac{3}{4}}{S + \frac{5}{2}} \qquad\qquad i2 = \frac{3}{4}e^{-\frac{5}{2}t}$$



$$i2 = \frac{3}{4}e^{-\frac{5}{2}t}$$

The RC **Time Constant**, also called tau, the time constant (in seconds) of an RC circuit, is equal to the product of the circuit resistance (in ohms) and the circuit capacitance (in farads), i.e.

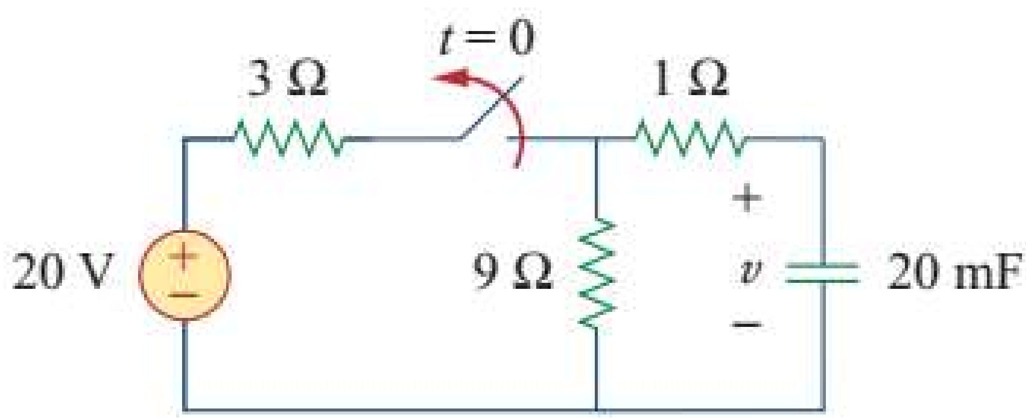
$$\tau = RC \quad V_0: \quad V(t) = V_0(e^{-t/\tau})$$

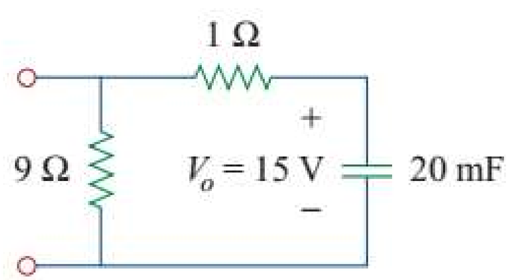
It is the time required to charge the capacitor, through the resistor, from an initial charge voltage of zero to approximately 63.2% of the value of an applied DC voltage,

or to discharge the capacitor through the same resistor to approximately 36.8% of its initial charge voltage.

Homework

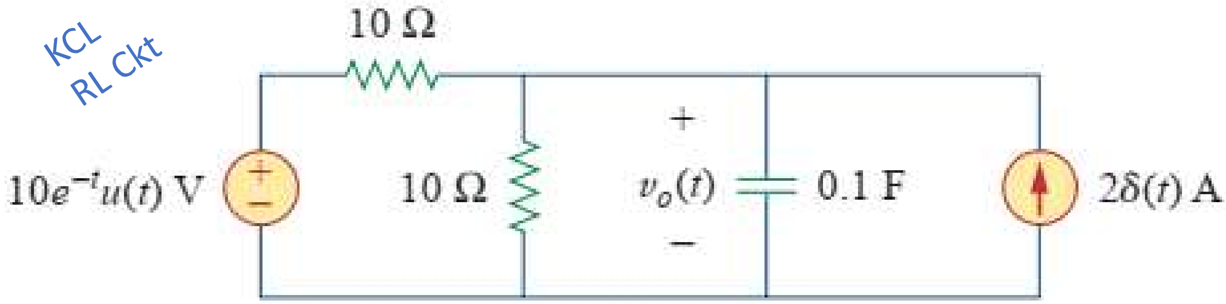
Find $v(t)$ for $t \geq 0$.

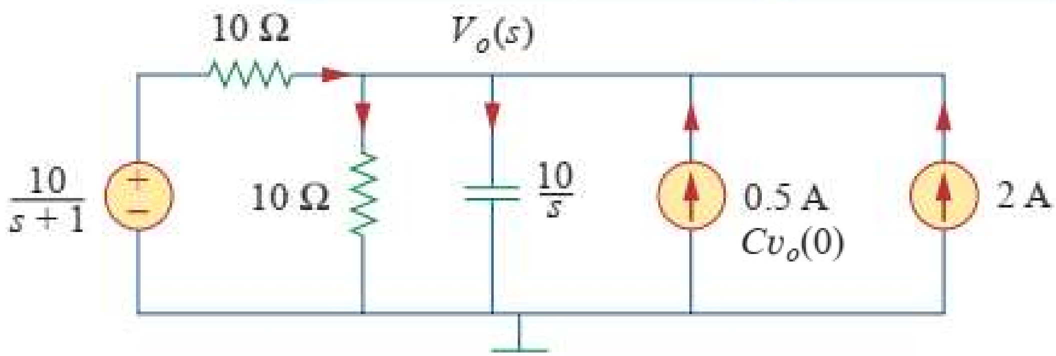




Find $v_o(t)$ in the circuit of Fig. Assume $v_o(0) = 5$ V.

KCL
RL Ckt





$$\frac{10/(s+1) - V_o}{10} + 2 + 0.5 = \frac{V_o}{10} + \frac{V_o}{10/s}$$

$$\frac{1}{s+1} + 2.5 = \frac{2V_o}{10} + \frac{sV_o}{10}$$

$$\frac{1}{s+1} + 2.5 = \frac{1}{10}V_o(s+2)$$

$$\frac{10}{s+1} + 25 = V_o(s+2)$$

$$V_o = \frac{25s + 35}{(s+1)(s+2)}$$

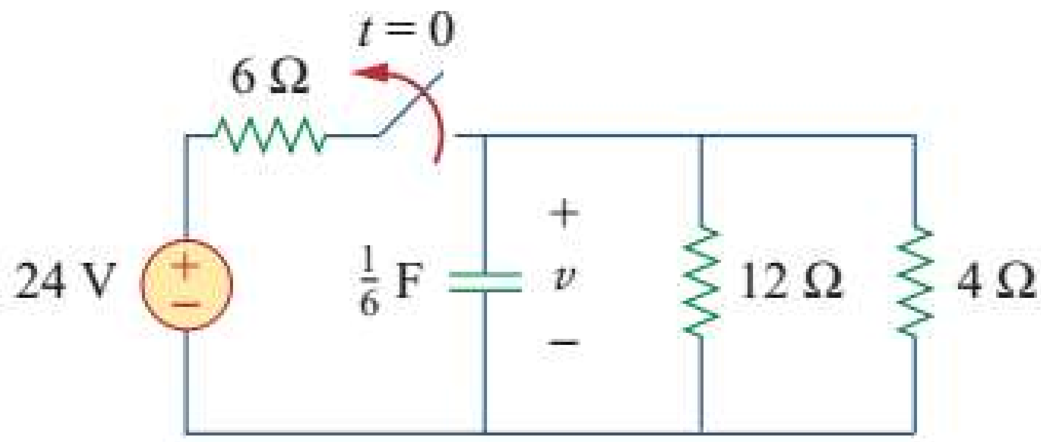
$$= \frac{A}{s+1} + \frac{B}{s+2}$$

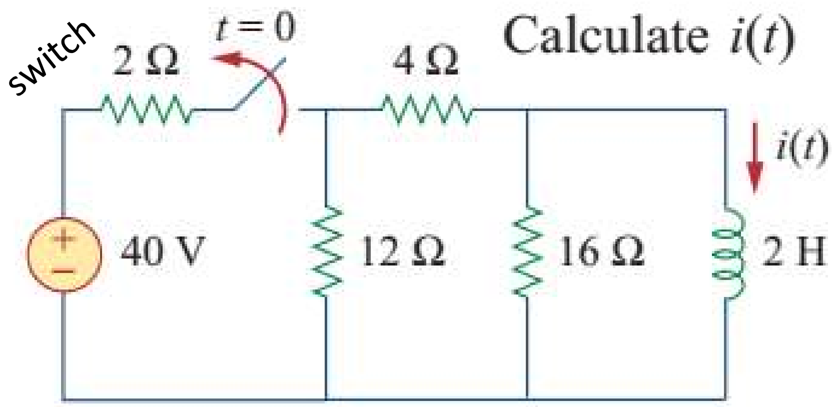
$$V_o(s) = \frac{10}{s+1} + \frac{15}{s+2}$$

$$v_o(t) = (10e^{-t} + 15e^{-2t})u(t) \text{ V}$$

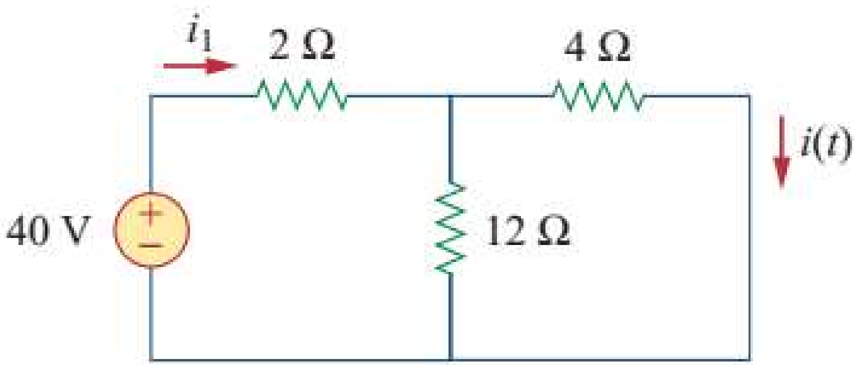
Homework

find $v(t)$





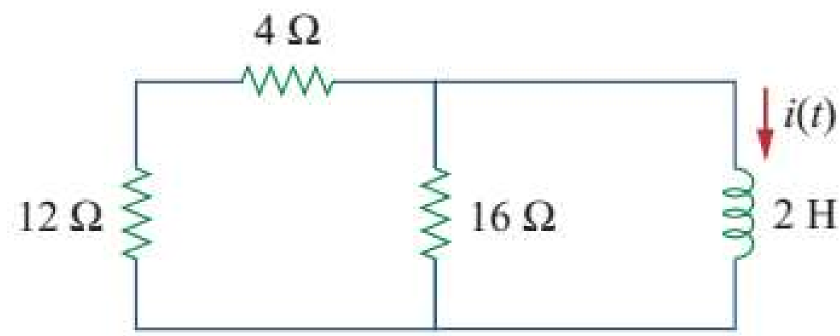
For t<0

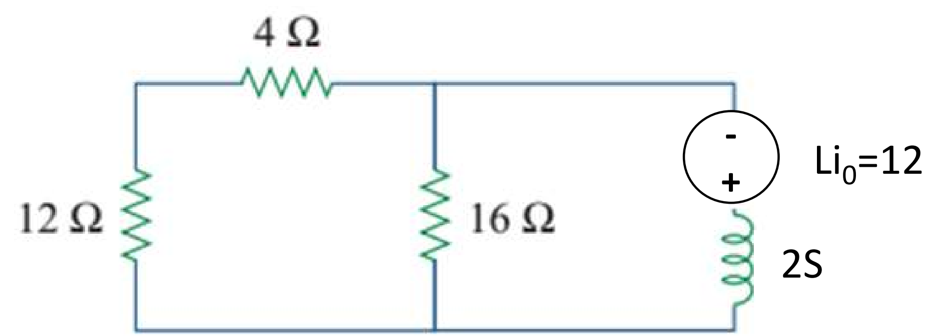


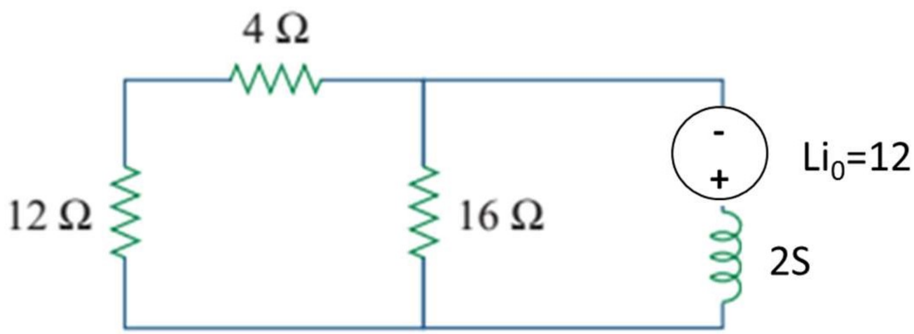
$$\frac{4 \times 12}{4 + 12} = 3\ \Omega \qquad i_1 = \frac{40}{2 + 3} = 8$$

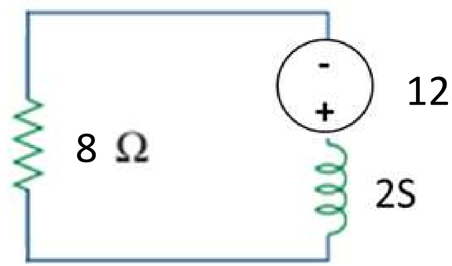
$$i(t) = \frac{12}{12 + 4} i_1 = 6$$

for $t > 0$.









Req=8

$$I = \frac{12}{2S+8}$$

$$I = \frac{6}{(S+4)}$$

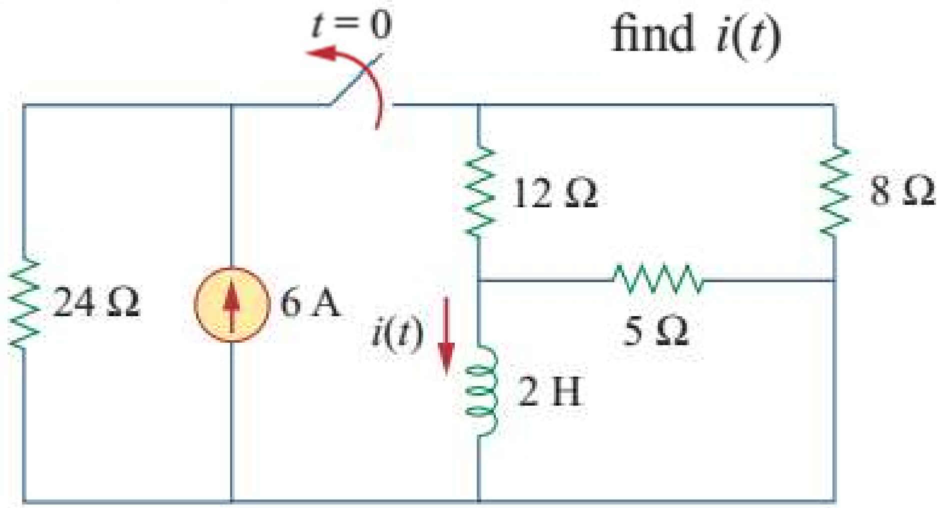
$$i = 6e^{-4t}$$

The time constant for an RL circuit is defined by τ=L/R .

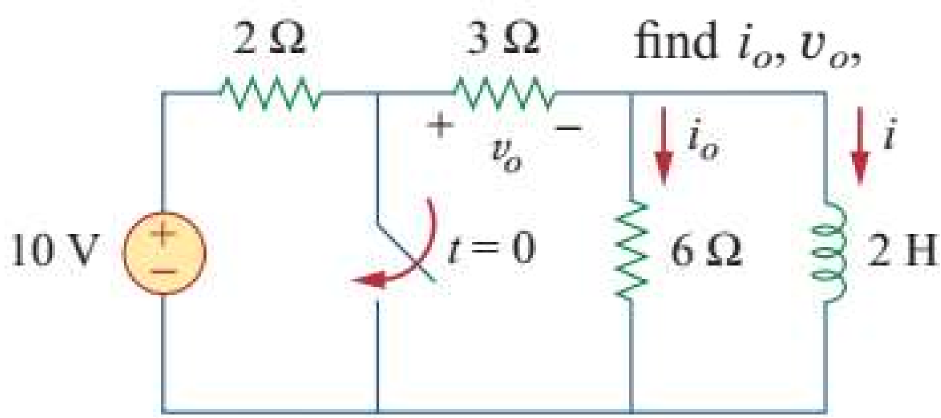
$I = I_0 e^{-t/\tau}$

τ=2/8

Homework

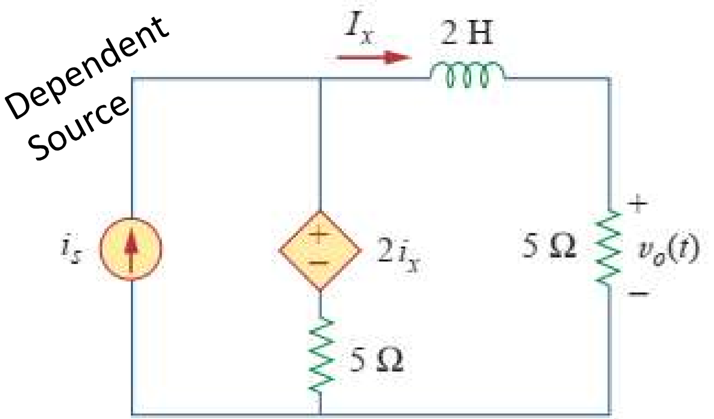


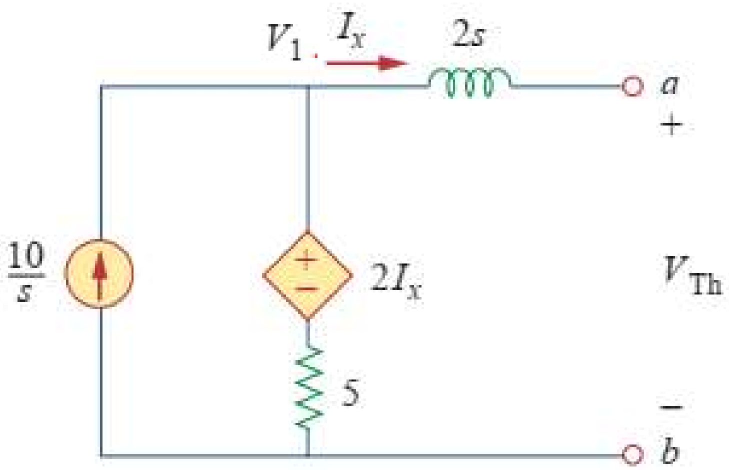
$t = 0$
find $i(t)$
12 Ω
8 Ω
24 Ω
6 A
$i(t)$
5 Ω
2 H



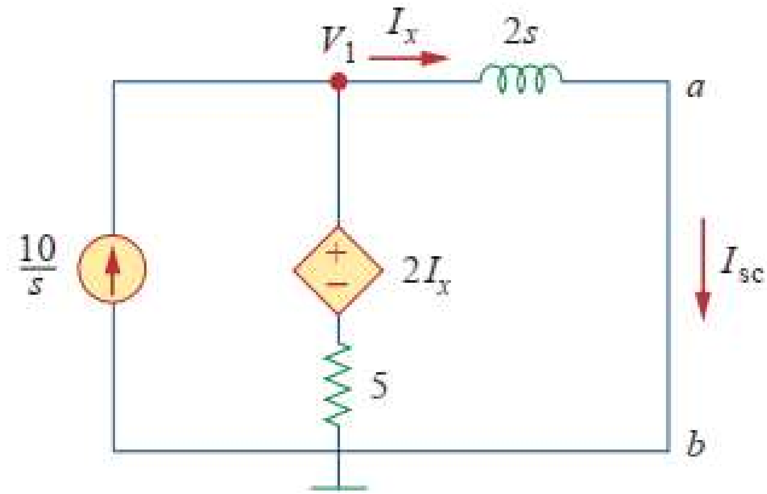
2 Ω
3 Ω
find i_o, v_o,
+
v_o
−
i_o
i
10 V
$t = 0$
6 Ω
2 H

$i_s = 10\,u(t)$ A. Determine $v_o(t)$.





$$V_{oc} = V_{Th} = 5\left(\frac{10}{s}\right) = \frac{50}{s}$$



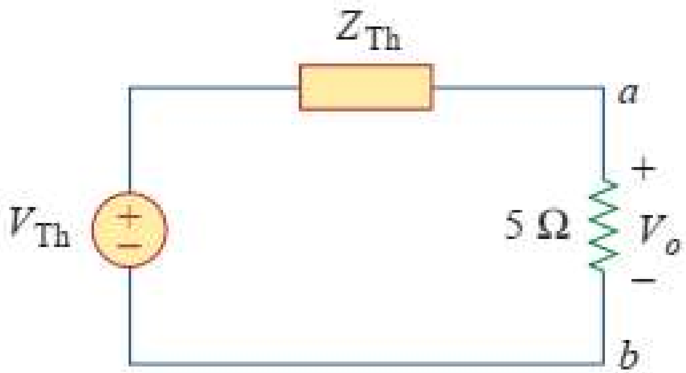
$$I_{sc} = I_x = V_1/2s$$

$$-\frac{10}{s} + \frac{(V_1 - 2I_x) - 0}{5} + \frac{V_1 - 0}{2s} = 0$$

$$V_1 = \frac{100}{2s + 3}$$

$$I_{sc} = \frac{V_1}{2s} = \frac{100/(2s + 3)}{2s} = \frac{50}{s(2s + 3)}$$

$$Z_{\mathrm{Th}} = \frac{V_{\mathrm{oc}}}{I_{\mathrm{sc}}} = \frac{50/s}{50/[s(2s+3)]} = 2s + 3$$



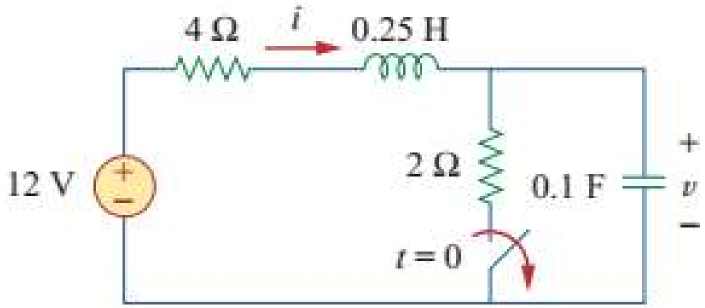
$$V_o = \frac{5}{5 + Z_{\mathrm{Th}}} V_{\mathrm{Th}} = \frac{5}{5 + 2s + 3}\left(\frac{50}{s}\right) = \frac{250}{s(2s+8)} = \frac{125}{s(s+4)}$$

$$V_o = \frac{125}{s(s+4)} = \frac{A}{s} + \frac{B}{s+4}$$

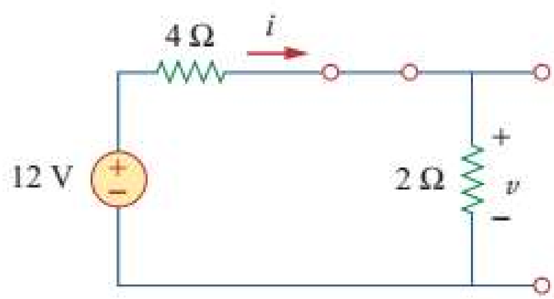
$$V_o = \frac{31.25}{s} - \frac{31.25}{s+4}$$

$$v_o(t) = 31.25(1 - e^{-4t})u(t)\ \mathrm{V}$$

Second-Order Circuits



For t<0



$i(0^-) = 2\text{ A}, \quad v(0^-) = 4\text{ V}$

$$4I + \frac{S}{4}I + \frac{10}{S}I = \frac{12}{S} + \frac{1}{2} - \frac{4}{S}$$

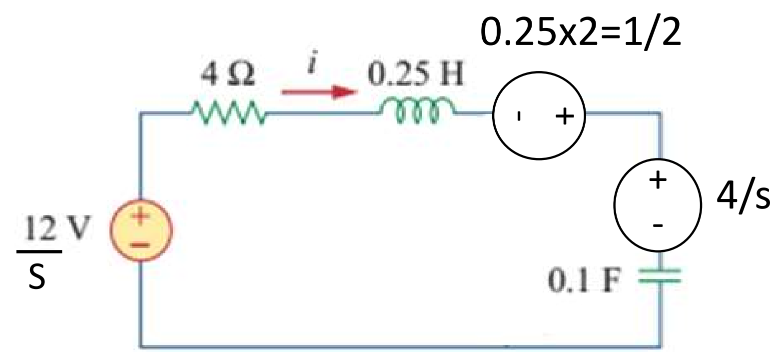
$$\left(4 + \frac{S}{4} + \frac{10}{S}\right)I = \frac{8}{S} + \frac{1}{2}$$

$$\left(\frac{16S + S^2 + 40}{4S}\right)I = \frac{16 + S}{2S}$$

$$I(S^2 + 16S + 40) = 32 + 2S$$

$$I = \frac{32 + 2S}{S^2 + 16S + 40}$$

for $t > 0$.



$$I = \frac{32 + 2S}{(S + 3.101)(S + 12.89)}$$

$$I = \frac{A1}{S + 3.101} + \frac{A2}{S + 12.89}$$

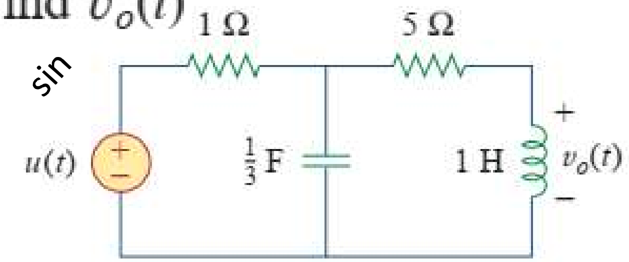
$$A1 = \frac{25.798}{9.789} = 2.635$$

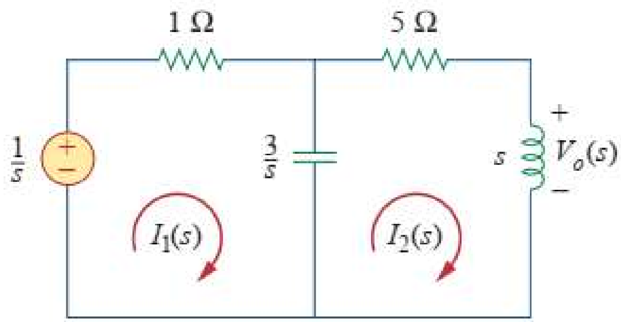
$$A2 = \frac{6.22}{-9.789} = -0.635$$

$$I = \frac{2.635}{S + 3.101} - \frac{0.635}{S + 12.89}$$

$$i(t) = 2.635e^{-3.101t} - 0.635e^{-12.89t}$$

Find $v_o(t)$





$$\frac{1}{s} = \left(1 + \frac{3}{s}\right)I_1 - \frac{3}{s}I_2 \quad \text{For mesh 1}$$

$$0 = -\frac{3}{s}I_1 + \left(s + 5 + \frac{3}{s}\right)I_2 \quad \text{For mesh 2,}$$

$$I_1 = \frac{1}{3}(s^2 + 5s + 3)I_2$$

$$\frac{1}{s} = \left(1 + \frac{3}{s}\right)\frac{1}{3}(s^2 + 5s + 3)I_2 - \frac{3}{s}I_2$$

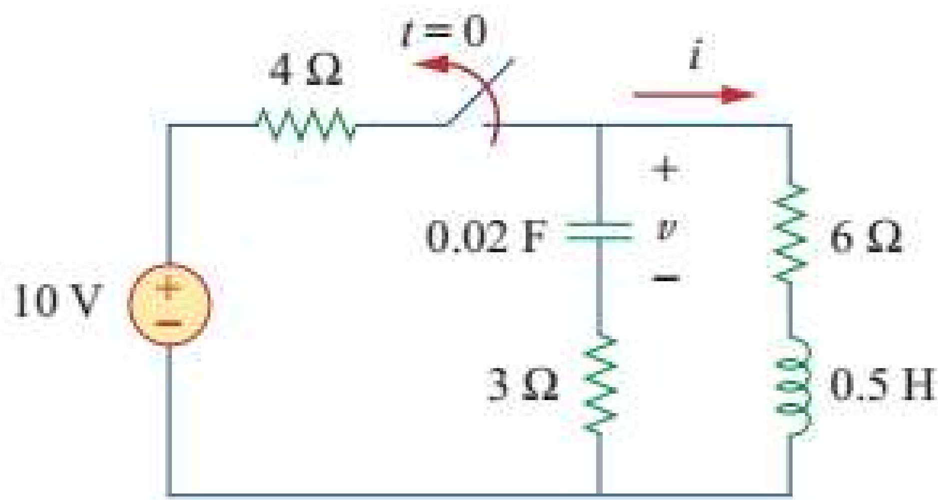
$$I_2 = \frac{3}{s^3 + 8s^2 + 18s}$$

$$V_o(s) = sI_2 = \frac{3}{s^2 + 8s + 18} = \frac{3}{\sqrt{2}}\frac{\sqrt{2}}{(s+4)^2 + (\sqrt{2})^2}$$

$$v_o(t) = \frac{3}{\sqrt{2}}e^{-4t}\sin\sqrt{2}t \text{ V,}$$

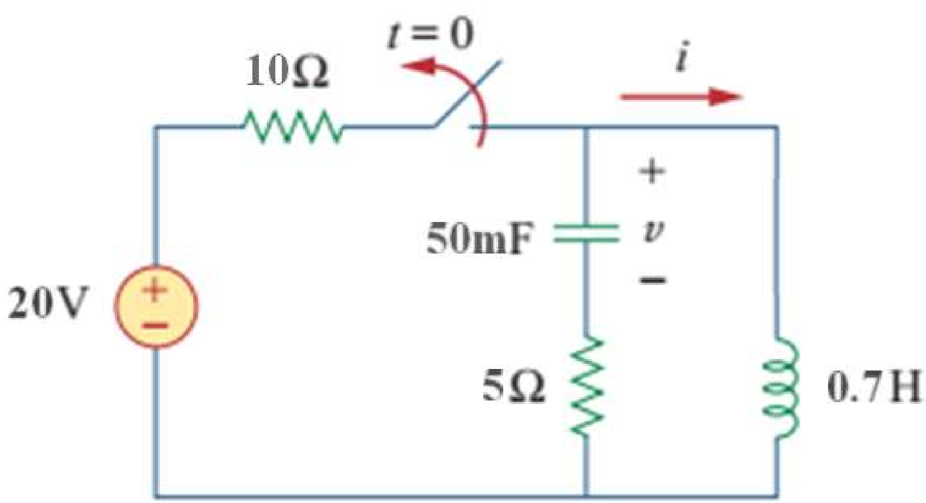
Homework

Find v



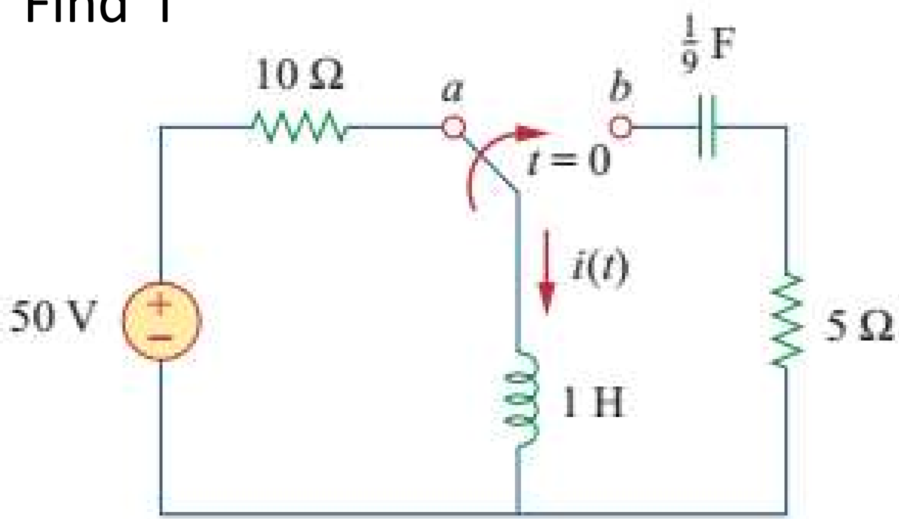
Homework

Find i



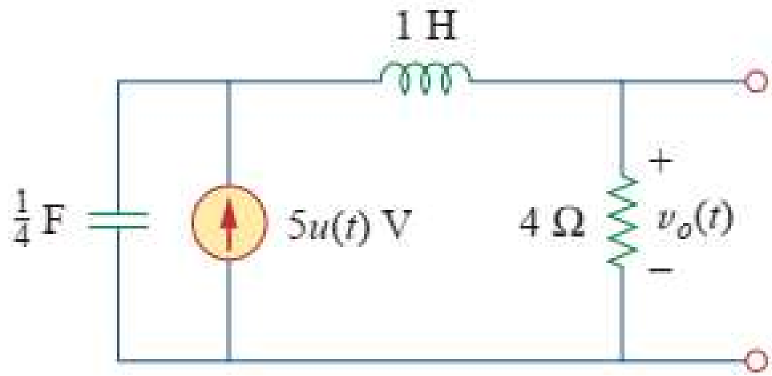
Homework

Find i



Homework

Determine $v_o(t)$



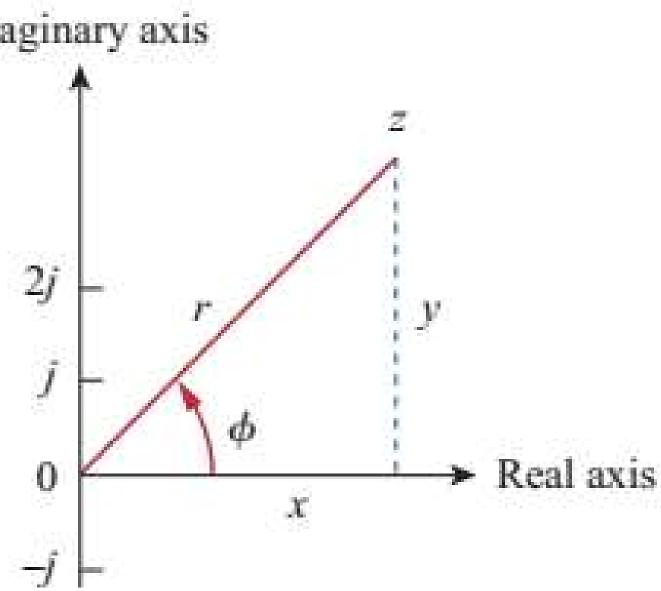
AC Circuits

Phasors

A phasor is a complex number that represents the amplitude and phase of a sinusoid.

$$z = x + jy = r\angle\phi = r(\cos\phi + j\sin\phi)$$

Time domain representation	Phasor domain representation
$V_m \cos(\omega t + \phi)$	$V_m \angle\phi$
$V_m \sin(\omega t + \phi)$	$V_m \angle\phi - 90°$



Phasor representation of circuit elements

Element	Impedance
R	$\mathbf{Z} = R$
L	$\mathbf{Z} = j\omega L$
C	$\mathbf{Z} = \dfrac{1}{j\omega C}$

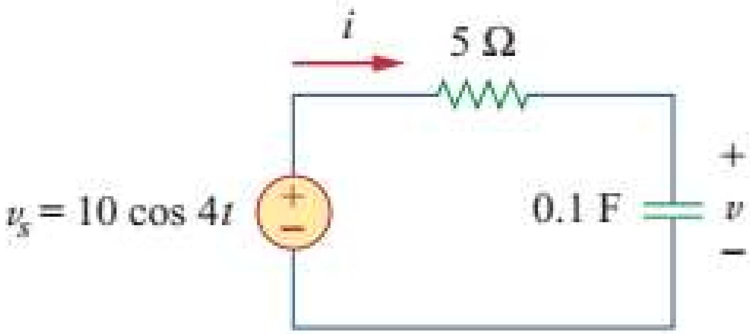
How phasors simplify AC circuit analysis (in short):

- Phasors convert **sinusoidal time-varying signals** into **complex numbers**.
- This removes the need to handle time functions like sin(ωt+φ) during calculations.
- So instead of solving differential equations in time, AC circuit problems become **simple algebraic equations** in the frequency domain.

Role of phasors in steady-state sinusoidal analysis:

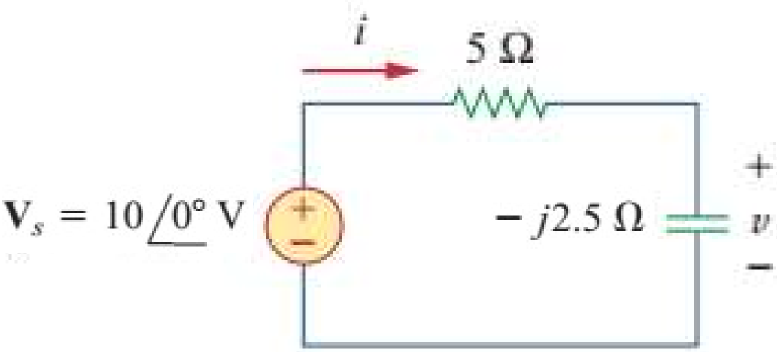
- Phasors represent the **amplitude and phase** of a sinusoid at a fixed frequency.
- They allow resistors, inductors, and capacitors to be replaced by their **complex impedances**.
- Using impedances, we can apply **Ohm's law, KVL, and KCL** just like DC circuits.
- After solving, the final phasor is converted back to the **time-domain sinusoid**.

Find $v(t)$ and $i(t)$



$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{10\angle 0^\circ}{5 - j2.5}$$

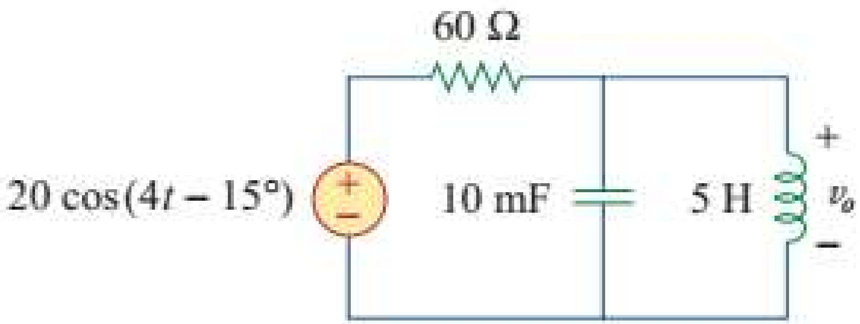
$$= 1.6 + j0.8$$

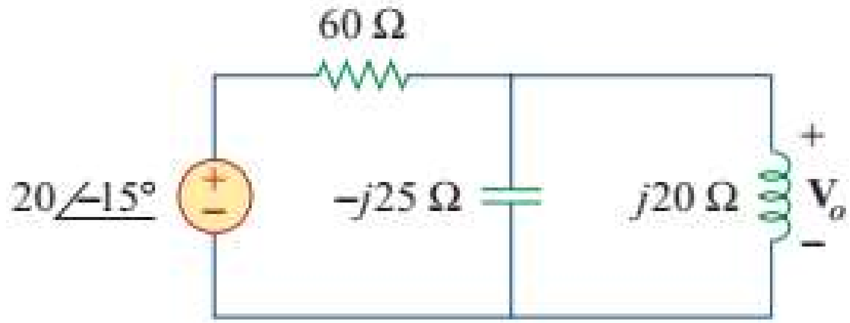


$$\mathbf{V} = \mathbf{I}\mathbf{Z}_C = \frac{\mathbf{I}}{j\omega C} = \frac{1.789\angle 26.57^\circ}{j4 \times 0.1}$$

$$= 4.47\angle -63.43^\circ \text{ V}$$

Determine $v_o(t)$



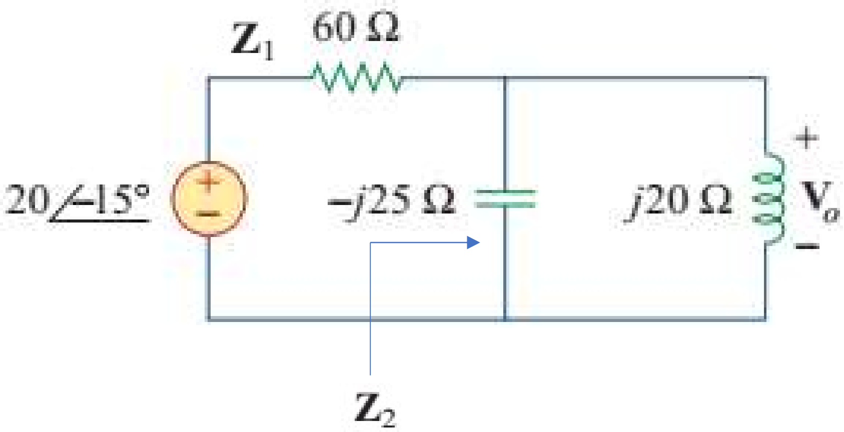


$$v_s = 20\cos(4t - 15^\circ) \quad \Rightarrow \quad \mathbf{V}_s = 20\underline{/-15^\circ}\ \text{V}, \qquad \omega = 4$$

$$10\ \text{mF} \quad \Rightarrow \quad \frac{1}{j\omega C} = \frac{1}{j4 \times 10 \times 10^{-3}}$$

$$= -j25\ \Omega$$

$$5\ \text{H} \quad \Rightarrow \quad j\omega L = j4 \times 5 = j20\ \Omega$$



$$\mathbf{Z}_2 = -j25 \parallel j20 = \frac{-j25 \times j20}{-j25 + j20} = j100\ \Omega$$

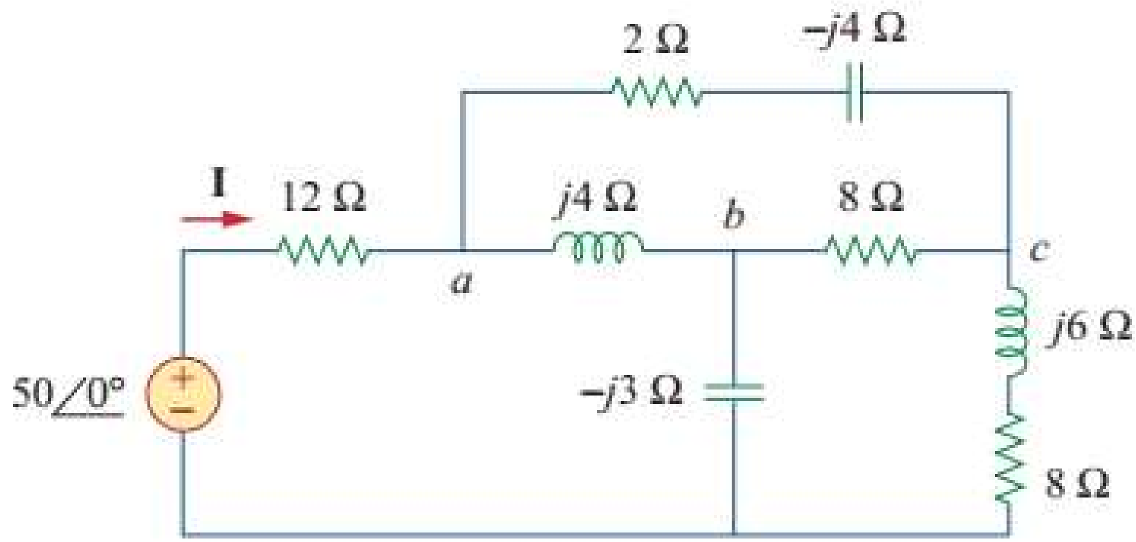
$$\mathbf{V}_o = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}_s = \frac{j100}{60 + j100}(20\underline{/-15^\circ})$$

$$\mathbf{V}_o = 17.15\underline{/15.96^\circ}\ \text{V}$$

$$v_o(t) = 17.15\cos(4t + 15.96^\circ)\ \text{V}$$

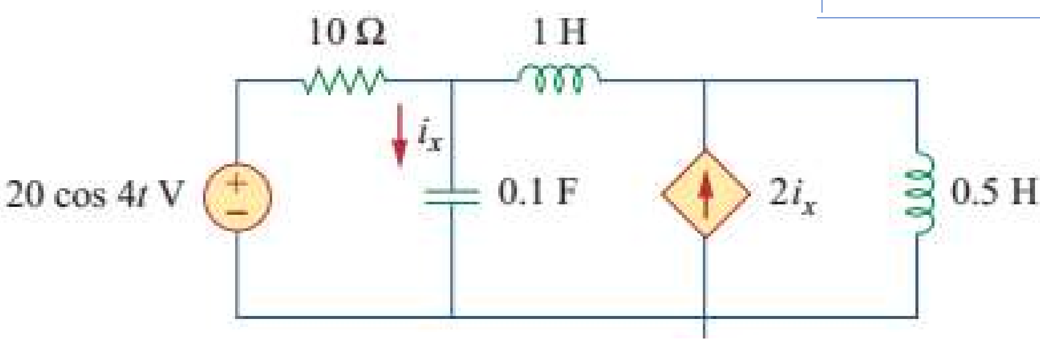
Homework

Find current **I**



Nodal Analysis

Find i_x in the circuit



$$20\cos 4t \Rightarrow 20\angle 0^\circ, \qquad \omega = 4 \text{ rad/s}$$

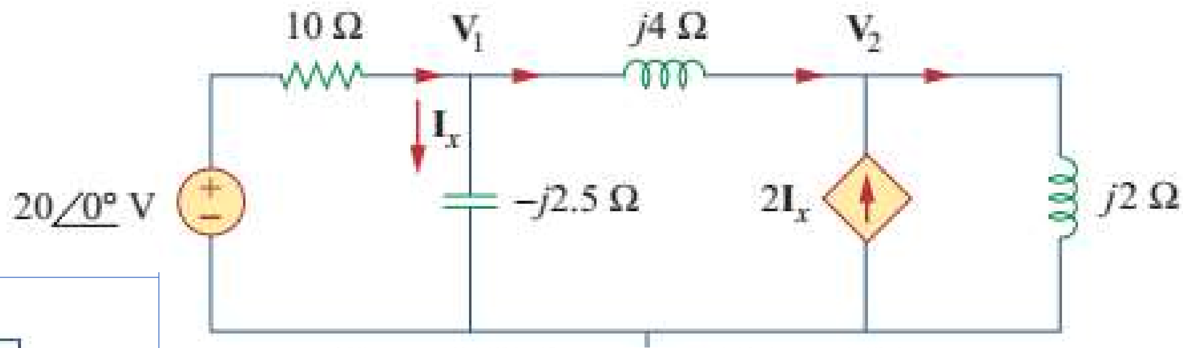
$$1\text{ H} \Rightarrow j\omega L = j4$$

$$0.5\text{ H} \Rightarrow j\omega L = j2$$

$$0.1\text{ F} \Rightarrow \frac{1}{j\omega C} = -j2.5$$

at node 1,

$$\frac{20 - \mathbf{V}_1}{10} = \frac{\mathbf{V}_1}{-j2.5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4}$$



$$(1 + j1.5)\mathbf{V}_1 + j2.5\mathbf{V}_2 = 20$$

At node 2,

$$2\mathbf{I}_x + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4} = \frac{\mathbf{V}_2}{j2} \qquad \mathbf{I}_x = \mathbf{V}_1/-j2.5.$$

$$\frac{2\mathbf{V}_1}{-j2.5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4} = \frac{\mathbf{V}_2}{j2}$$

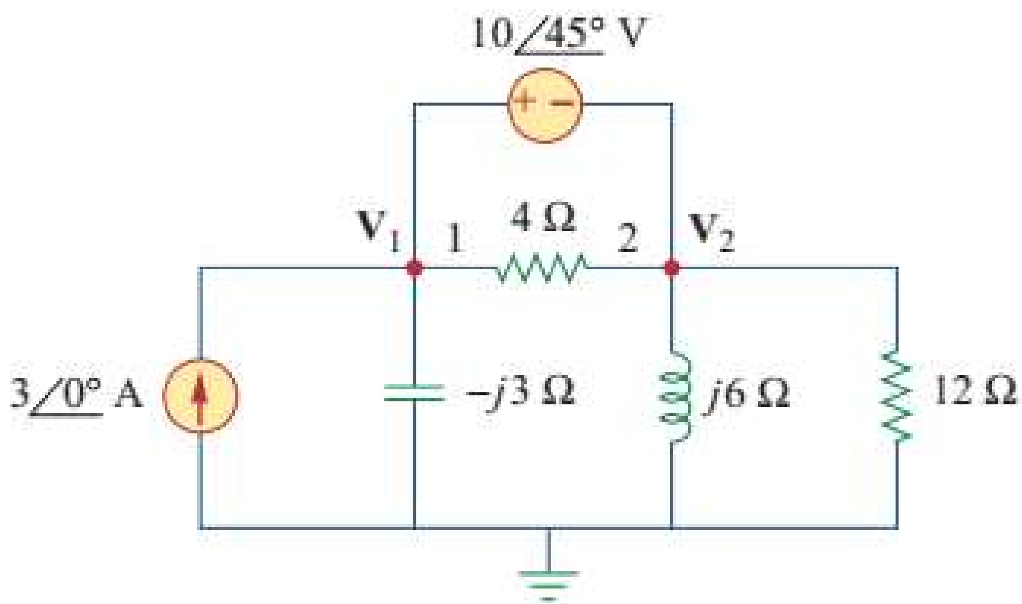
$$11\mathbf{V}_1 + 15\mathbf{V}_2 = 0$$

$$\mathbf{V}_1 = 18.97\angle 18.43^\circ \text{ V} \qquad \mathbf{V}_2 = 13.91\angle 198.3^\circ \text{ V}$$

$$\mathbf{I}_x = \frac{\mathbf{V}_1}{-j2.5} = \frac{18.97\angle 18.43^\circ}{2.5\angle -90^\circ} = 7.59\angle 108.4^\circ \text{ A}$$

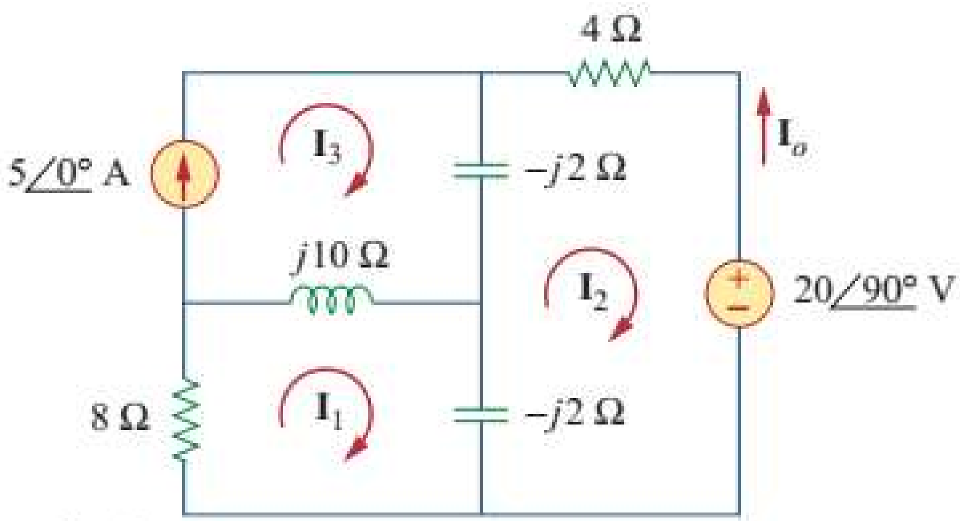
Homework

Compute $\mathbf{V}_1$ and $\mathbf{V}_2$ in the circuit



Mesh Analysis

Determine current $\mathbf{I}_o$ in the circuit



mesh 1.

$$(8 + j10 - j2)\mathbf{I}_1 - (-j2)\mathbf{I}_2 - j10\mathbf{I}_3$$

mesh 2,

$$(4 - j2 - j2)\mathbf{I}_2 - (-j2)\mathbf{I}_1 - (-j2)\mathbf{I}_3 + 20\underline{/90°} = 0$$

$$\mathbf{I}_3 = 5.$$

$$(8 + j8)\mathbf{I}_1 + j2\mathbf{I}_2 = j50$$

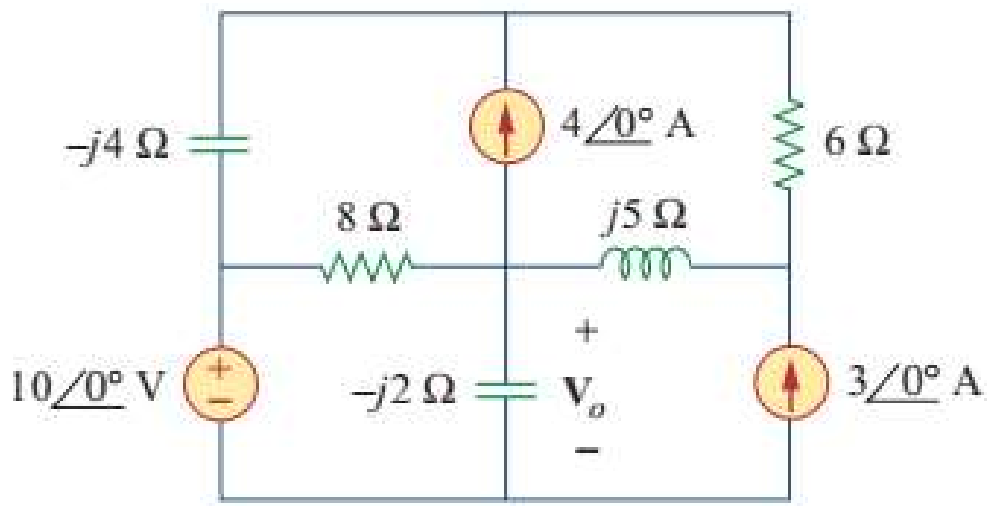
$$j2\mathbf{I}_1 + (4 - j4)\mathbf{I}_2 = -j20 - j10$$

$$\mathbf{I}_2 = 6.12\underline{/-35.22°}\text{ A}$$

$$\mathbf{I}_o = -\mathbf{I}_2 = 6.12\underline{/144.78°}\text{ A}$$

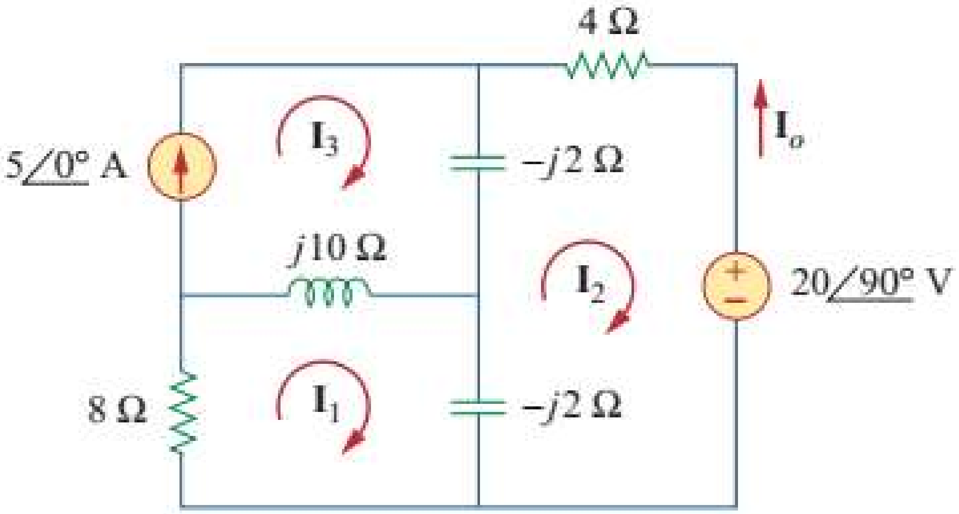
Homework

Solve for $\mathbf{V}_o$

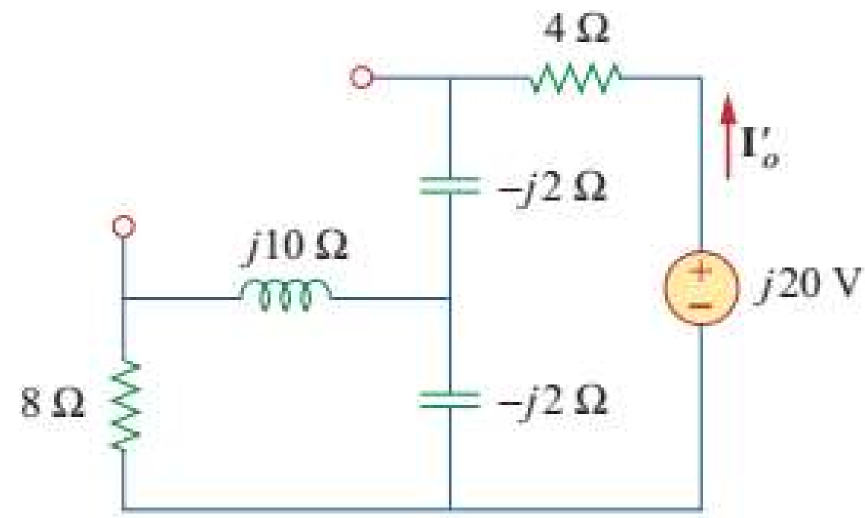


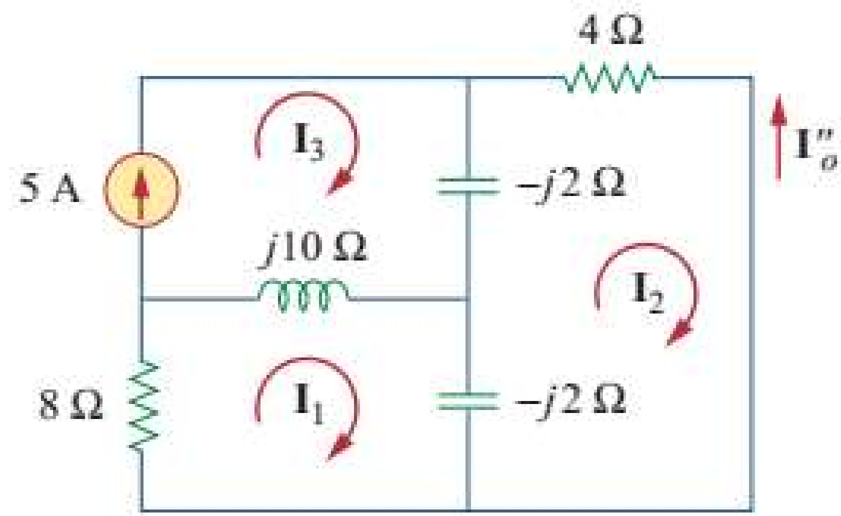
Superposition Theorem

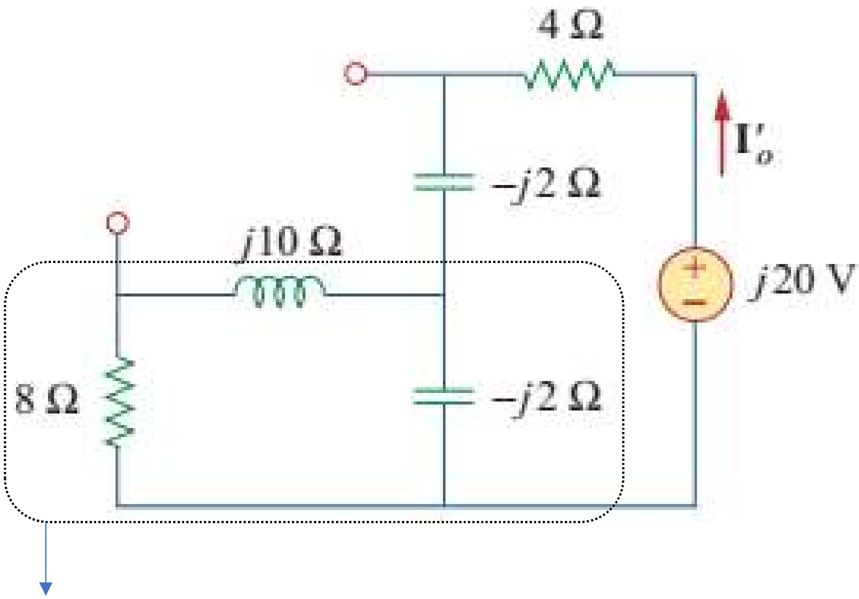
Use the superposition theorem to find $\mathbf{I}_o$



$$\mathbf{I}_o = \mathbf{I}'_o + \mathbf{I}''_o$$



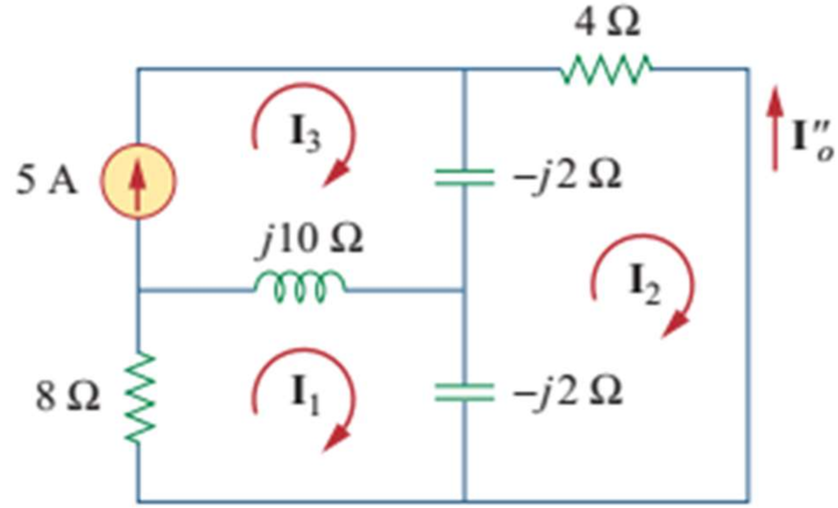




$$\mathbf{Z} = \frac{-j2(8 + j10)}{-2j + 8 + j10} = 0.25 - j2.25$$

$$\mathbf{I}'_o = \frac{j20}{4 - j2 + \mathbf{Z}} = \frac{j20}{4.25 - j4.25}$$

$$\mathbf{I}'_o = -2.353 + j2.353$$



$$(8 + j8)\mathbf{I}_1 - j10\mathbf{I}_3 + j2\mathbf{I}_2 = 0$$

$$(4 - j4)\mathbf{I}_2 + j2\mathbf{I}_1 + j2\mathbf{I}_3 = 0$$

$$\mathbf{I}_3 = 5$$

$$(8 + j8)\mathbf{I}_1 + j2\mathbf{I}_2 = j50$$

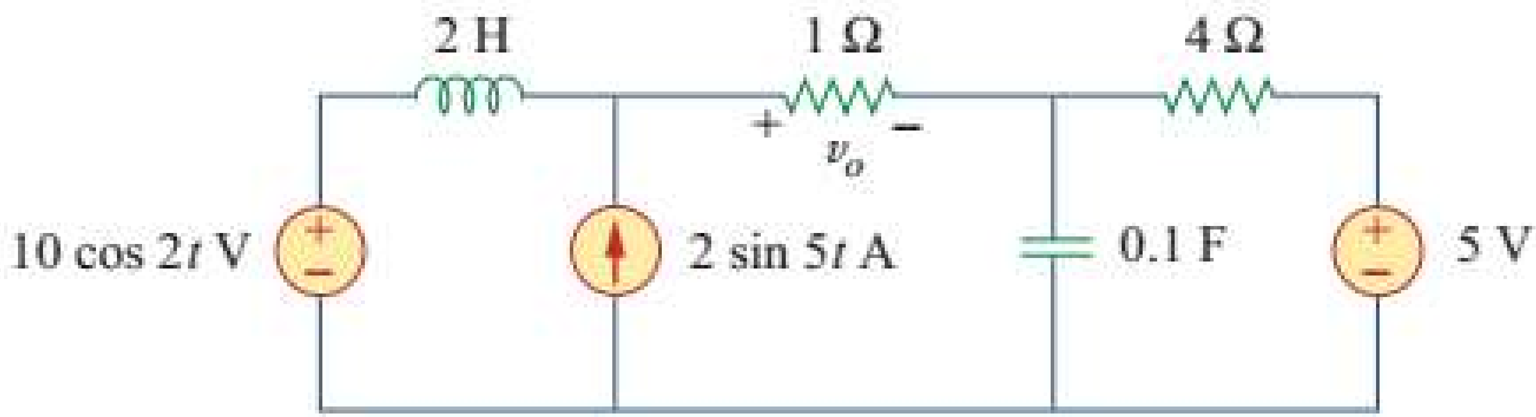
$$(4 - j4)\mathbf{I}_2 + j2\mathbf{I}_1 = -j10$$

$$\mathbf{I}''_o = -\mathbf{I}_2 = -2.647 + j1.176$$

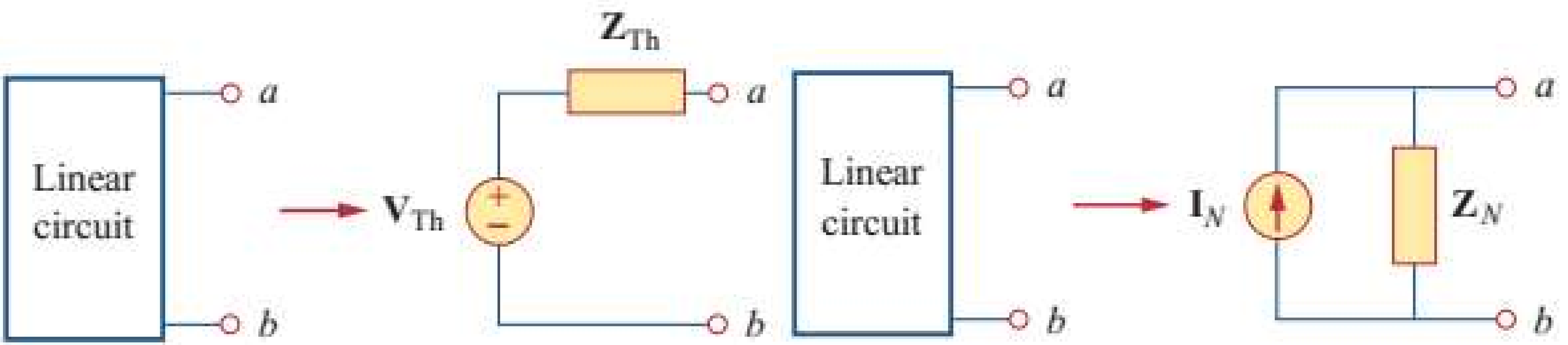
$$\mathbf{I}_o = \mathbf{I}'_o + \mathbf{I}''_o = -5 + j3.529 = 6.12\underline{/144.78^\circ}\text{ A}$$

Homework

Find v_o of the circuit using the superposition theorem.

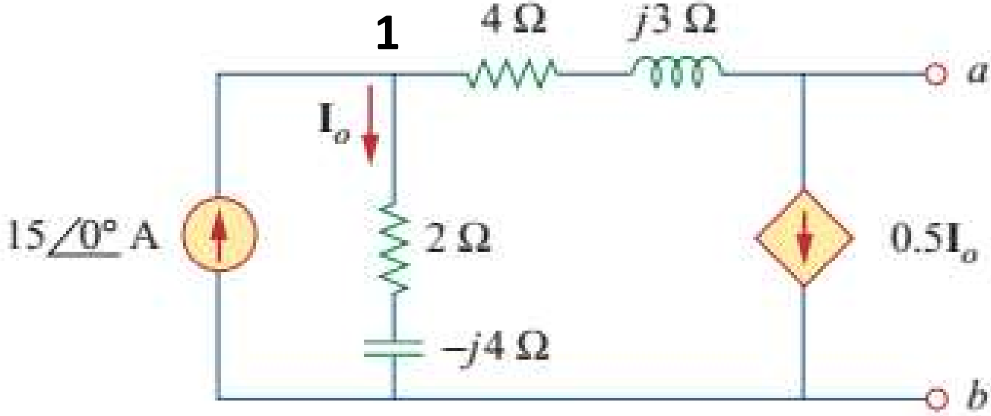


Thevenin and Norton Equivalent Circuits



$$\mathbf{V}_{\text{Th}} = \mathbf{Z}_N \mathbf{I}_N, \qquad \mathbf{Z}_{\text{Th}} = \mathbf{Z}_N$$

Find the Thevenin equivalent



KCL at node 1

$$15 = \mathbf{I}_o + 0.5\mathbf{I}_o \quad \mathbf{I}_o = 10\text{ A}$$

KVL to the loop

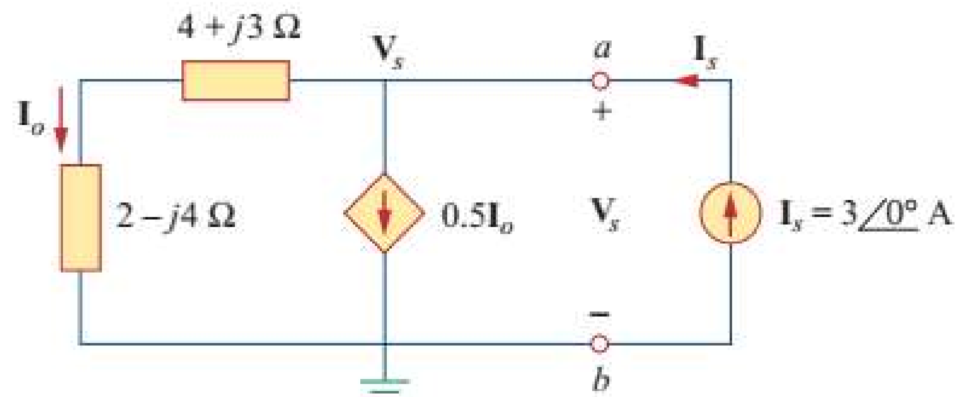
$$-\mathbf{I}_o(2 - j4) + 0.5\mathbf{I}_o(4 + j3) + \mathbf{V}_{\text{Th}} = 0$$

$$\mathbf{V}_{\text{Th}} = 10(2 - j4) - 5(4 + j3) = -j55$$

To obtain $\mathbf{Z}_{\text{Th}}$,

remove the independent source.

connect a 3-A current source



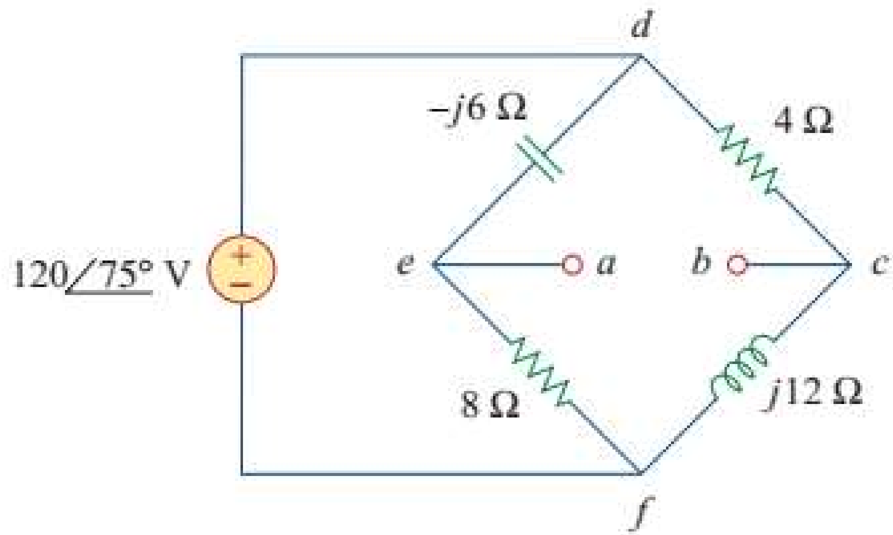
$$3 = \mathbf{I}_o + 0.5\mathbf{I}_o \quad \mathbf{I}_o = 2\text{ A}$$

$$\mathbf{V}_s = \mathbf{I}_o(4 + j3 + 2 - j4) = 2(6 - j)$$

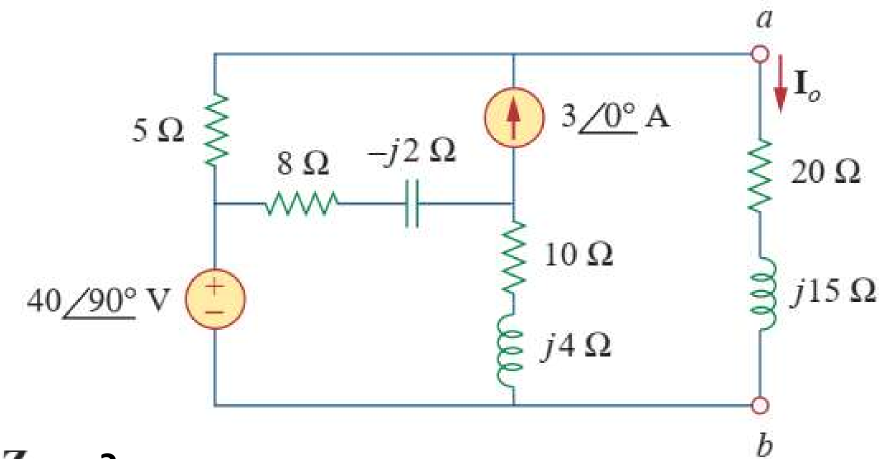
$$\mathbf{Z}_{\text{Th}} = \frac{\mathbf{V}_s}{\mathbf{I}_s} = \frac{2(6 - j)}{3} = 4 - j0.6667\ \Omega$$

Homework

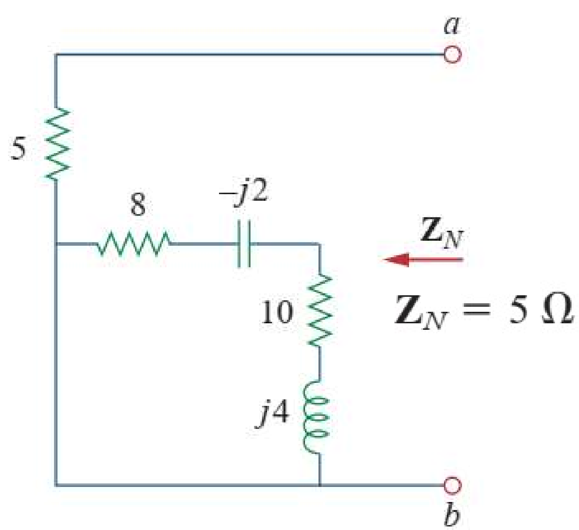
Obtain the Thevenin equivalent Norton equivalent



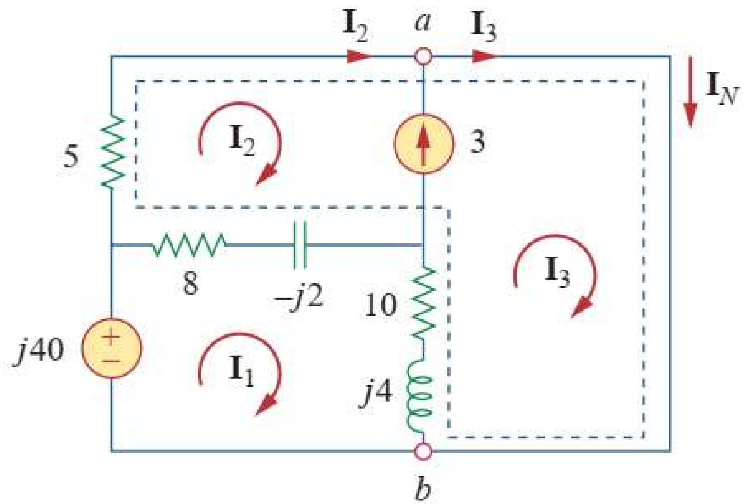
Obtain current in Figure using Norton's theorem



$\mathbf{Z}_N$ =?



$\mathbf{I}_N$ =?



In loop I1

$$-j40 + (18 + j2)\mathbf{I}_1 - (8 - j2)\mathbf{I}_2 - (10 + j4)\mathbf{I}_3 = 0 \quad ..(1)$$

For the supermesh,

$$(13 - j2)\mathbf{I}_2 + (10 + j4)\mathbf{I}_3 - (18 + j2)\mathbf{I}_1 = 0 \quad ..(2)$$

$$I3 - I2 = 3 \quad ..(3)$$

Adding (1), (2)

$$-j40 + 5\mathbf{I}_2 = 0$$

$$\mathbf{I}_2 = j8$$

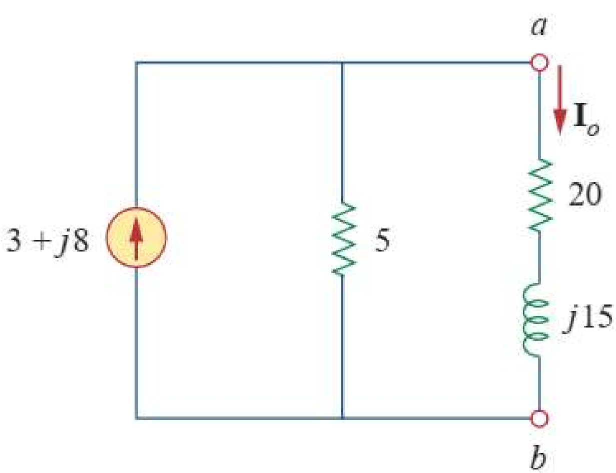
From equation 3

$$\mathbf{I}_3 = \mathbf{I}_2 + 3 = 3 + j8$$

The Norton current is

$$\mathbf{I}_N = \mathbf{I}_3 = (3 + j8)\text{ A}$$

Norton equivalent circuit



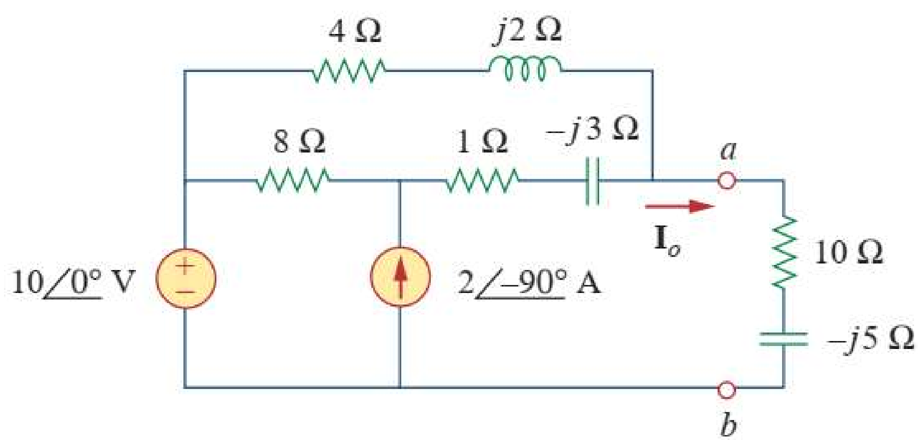
By current division,

$$\mathbf{I}_o = \frac{5}{5 + 20 + j15}\mathbf{I}_N$$

$$= \frac{3 + j8}{5 + j3} = 1.465\underline{/38.48^\circ}\text{ A}$$

Home Work

Determine the Norton equivalent of the circuit in Fig. as seen from terminals a-b. Use the equivalent to find Io



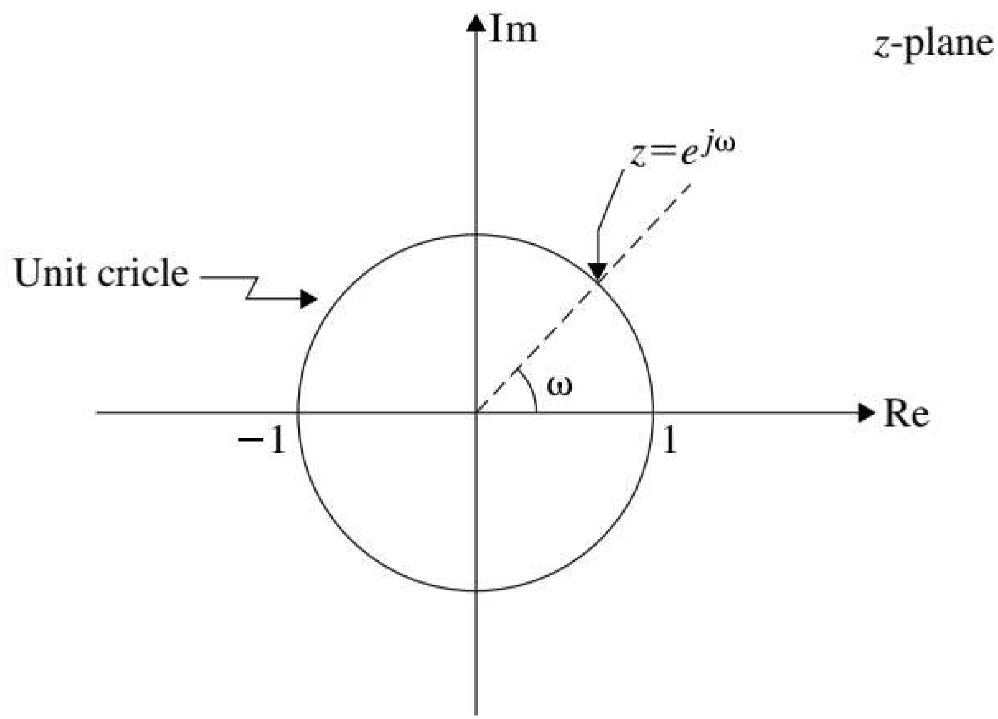
The z-Transform Analysis of Discrete Time Signals and Systems

- The z-transform is the **discrete counterpart of the Laplace transform**
- The Laplace transform converts **differential equations** into algebraic equations.
- In the same way, the z-transform converts **difference equations** of discrete time systems to algebraic equations.
- z-transform can be applied even to **unstable systems** which means that z-transform can be used to a larger class of systems and signals.

$$\boxed{\begin{aligned} X[z] &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ x[n] &= \frac{1}{2\pi j}\oint X(z)z^{n-1}dz \end{aligned}}$$ bilateral z-transform.

$$\boxed{X[z] = \sum_{n=0}^{\infty} x[n]z^{-n}}$$ unilateral or right-sided z-transform.

In the context of Z-transforms, the Z-plane is a complex plane that represents the frequency-domain (z-domain) of a discrete-time signal.



The Region of Convergence (ROC)

The values of z in the complex z-plane for which the sum in the z-transform equation converges is called the region of convergence

1. The ROC is a concentric ring in the z-plane.
2. The ROC does not contain any pole.
3. If $x[n]$ is a finite sequence in a finite interval $N_1 \leq n \leq N_2$, then the ROC is the entire z-plane except $z = 0$ and $z = \infty$.
4. If $x[n]$ is a right-sided sequence (causal), then the ROC is the exterior of the circle $|z| = r_{\text{max}}$ where r_{max} is the radius of the outermost pole of $X(z)$.
5. If $x[n]$ is a left-sided sequence (non-causal), then the ROC is the interior of the circle $|z| = r_{\text{min}}$ where r_{min} is the radius of the innermost pole of $X(z)$.
6. If $x[n]$ is a two-sided sequence, then the ROC is given by $r_1 < |z| < r_2$ where r_1 and r_2 are the magnitudes of the two poles of $X(z)$. Here, ROC is an annular ring between the circle $|z| = r_1$ and $|z| = r_2$ which does not include any poles.

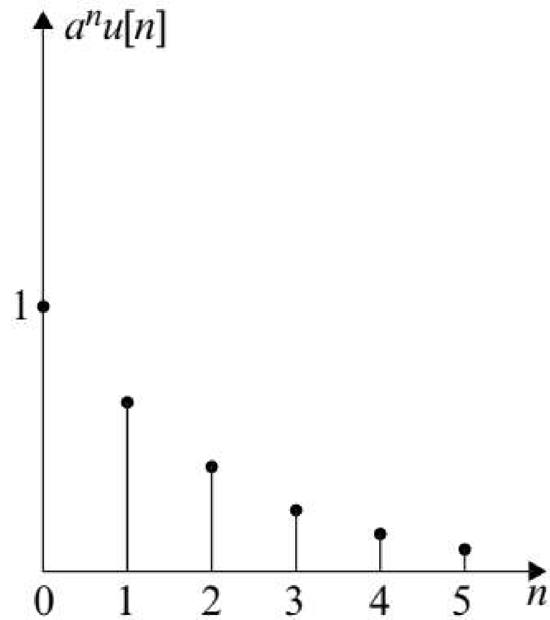
If unit circle lies inside ROC → system is stable
If unit circle lies outside ROC → system is unstable

Physical meaning:
Stability means that if you give a bounded input, output remains bounded.
ROC directly tells whether that summation is finite or infinite.

A discrete-time LTI system is **causal** if and only if the ROC of its Z-transform lies outside the outermost pole (i.e., ROC = $|z|>rmax$; rmax is the magnitude of the pole with largest radius).

A system is **non-causal** if its ROC is **not** outside the outermost pole.

$$x[n] = a^n u[n]$$



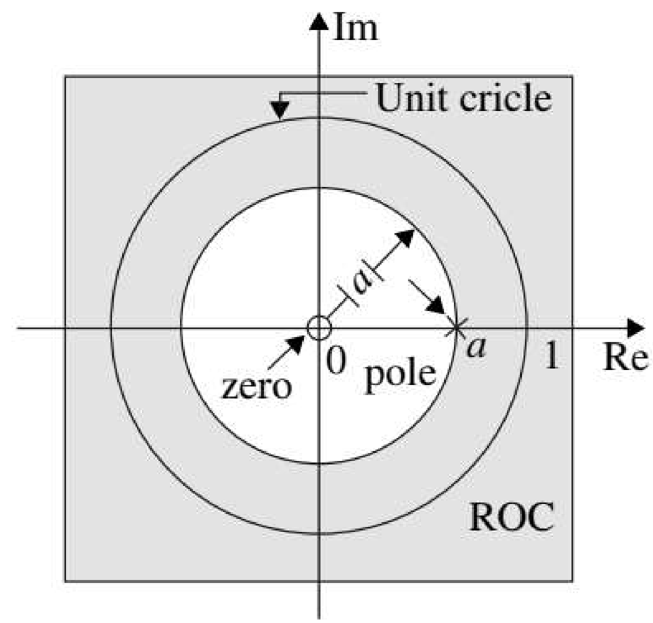
$$X(z) = \sum_{n=0}^{\infty} a^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} \qquad [\because\ u[n] = 1 \text{ all } n \geq 0]$$

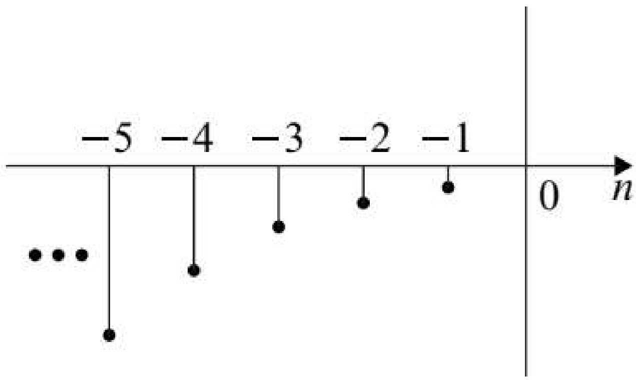
$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$X(z) = \frac{1}{\left[1 - \frac{a}{z}\right]}$$

$$\boxed{X(z) = \frac{z}{(z-a)} \qquad X(z) = \frac{1}{1 - az^{-1}}}$$



$$\boldsymbol{x[n] = -a^n[u[-n-1]]}$$



$$Z[-a^n u[-n-1]] =$$

$$= \sum_{n=-\infty}^{-1} -a^n z^{-n} \qquad \because [u(-n-1)] = 1 \text{ for all } -n$$

$$= \sum_{n=-\infty}^{-1} -\left[\frac{a}{z}\right]^n = \sum_{n=1}^{\infty} -\left[\frac{z}{a}\right]^n$$

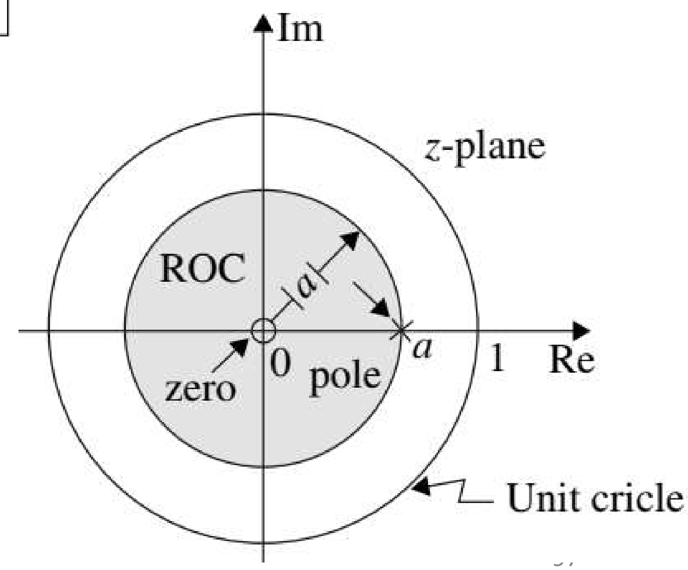
$$= -\left[\frac{z}{a} + \frac{z^2}{a^2} + \frac{z^3}{a^3} + \cdots\right]$$

$$= 1 - \left[1 + \frac{z}{a} + \left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right)^3 + \cdots\right]$$

$$= 1 - \frac{1}{1 - \frac{z}{a}} \qquad \text{if } \left|\frac{z}{a}\right| < 1$$

$$\boxed{Z[-a^n u[-n-1]] = \frac{z}{(z-a)}}$$

ROC $|z| < a$

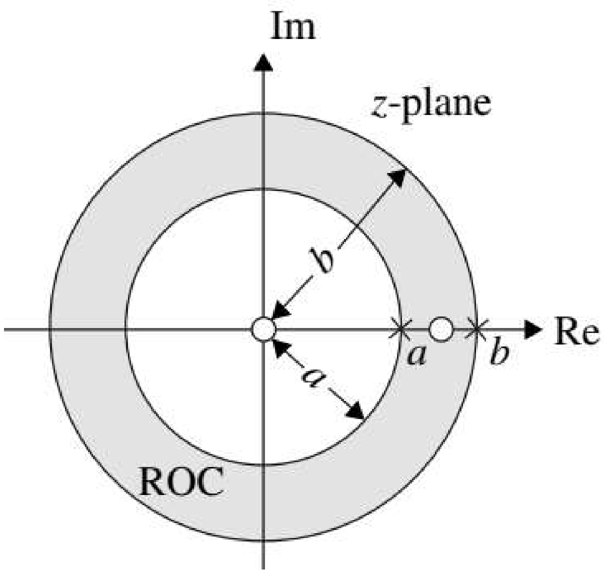


$$\boldsymbol{x[n] = a^n u[n] - b^n u[-n-1]}$$

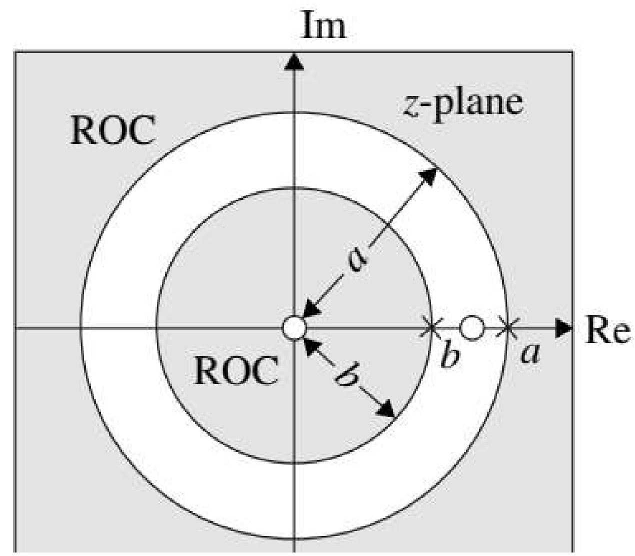
$$X(z) = \frac{z}{(z-a)} + \frac{z}{(z-b)}$$

ROC $\quad |z| > a \qquad |z| < b$

$|a| < |b|$



$|a| > |b|$



Find the z-transform and the ROC

$$\mathbf{x[n] = \{2, -1, 0, 3, 4\}}$$

$$X[z] = \sum_{n=0}^{4} x[n]z^{-n}$$

$$X[z] = 2 - z^{-1} + 0 + 3z^{-3} + 4z^{-4}$$

ROC is $|z| > 0$.

$$\mathbf{x[n] = \{1, -2, 3, -1, \underset{\uparrow}{2}\}}$$

$$X[z] = \sum_{n=-4}^{0} x[n]z^{-n}$$

$$X[z] = z^4 - 2z^3 + 3z^2 - z + 2$$

ROC is $|z| < \infty$.

Home Work

$$x[n] = \{5,\ 3,\ -2,\ \underset{\uparrow}{0},\ 4,\ -3\}$$

$$\boldsymbol{x[n] = \delta[n]}$$

$$X[z] = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n}$$

$$X[z] = 1$$ For n=0

ROC is entire z-plane

$$\boldsymbol{x[n] = u[n]}$$

$$X[z] = \sum_{n=0}^{\infty} z^{-n}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots$$

$$= \frac{1}{1 - \frac{1}{z}}$$

$$X[z] = \frac{z}{(z-1)}$$

$$X[z] = \frac{1}{(1 - z^{-1})}$$

ROC: $|z| > 1$

Home Work

$$x[n] = u[-n]$$

$$x[n] = a^{-n}u[-n]$$

$$x[n] = (-a)^n u[-n]$$

$$x[n] = a^{|n|};\ a < 1$$

$$x[n] = e^{j\omega_0 n}u[n]$$

$$x[n] = \sin \omega_0 n u[n]$$

$$\boldsymbol{x[n] = a^{-n}u[-n-1]}$$

$$X[z] = \sum_{n=-\infty}^{-1} a^{-n}z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (az)^{-n}$$

$$= \sum_{n=1}^{\infty}(az)^{n}$$

$$= az + (az)^2 + (az)^3 + \cdots$$

$$X[z] = az[1 + az + (az)^2 + \cdots]$$

$$= \frac{az}{1-az}$$

$$X[z] = \frac{-z}{\left(z - \frac{1}{a}\right)} \quad \text{ROC: } |z| < \frac{1}{a}$$

$$\boldsymbol{x[n] = \cos\omega_0 n u[n]}$$

$$x[n] = \frac{1}{2}[e^{j\omega_0 n} + e^{-j\omega_0 n}]$$

We know $$Z[e^{j\omega_0 n}] = \frac{z}{(z - e^{j\omega_0})}$$

and $$Z[e^{-j\omega_0 n}] = \frac{z}{(z - e^{-j\omega_0})}$$

$$X[z] = \frac{1}{2}\left[\frac{z}{(z - e^{j\omega_0})} + \frac{z}{(z - e^{-j\omega_0})}\right]$$

$$= \frac{z}{2}\frac{\left[z - e^{-j\omega_0} + z - e^{j\omega_0}\right]}{\left[z^2 - z(e^{-j\omega_0} + e^{j\omega_0}) + 1\right]}$$

$$X[z] = \frac{z}{2}\frac{[2z - 2\cos\omega_0]}{[z^2 - 2z\cos\omega_0 + 1]}$$

$$X[z] = \frac{\cancel{z^2}\,(1 - z^{-1}\cos\omega_0)}{\cancel{z^2}(1 - z^{-1}2\cos\omega_0 + z^{-2})} \qquad \text{ROC: } |z| > 1$$

$$\boldsymbol{x[n] = u[n] - u[n-6]}$$

$$x[n] = \{1,\ 1,\ 1,\ 1,\ 1,\ 1\}$$
$$X[z] = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

ROC: all z except $z \neq 0$

Properties of z-Transform

Linearity

If $x_1[n] \overset{Z}{\longleftrightarrow} X_1[z]$ and $x_2[n] \overset{Z}{\longleftrightarrow} X_2[z]$

then

$$\{a_1 x_1[n] + a_2 x_2[n]\} \overset{Z}{\longleftrightarrow} [a_1 X_1[z] + a_2 X_2[z]]$$

Time Shifting

If $\quad x[n] \stackrel{Z}{\longleftrightarrow} X[z]$

then

$$x[n-k] \stackrel{Z}{\longleftrightarrow} z^{-k} X[z]$$

Time Reversal

If $x[n] \xleftrightarrow{Z} X[z] \quad \text{ROC: } r_1 < |z| < r_2$

then

$$x[-n] \xleftrightarrow{Z} X[z^{-1}] \quad \text{ROC: } \frac{1}{r_1} < |z| < \frac{1}{r_2}$$

Multiplication by n

If $Z[x[n]] = X[z]$

then

$$Z[nx[n]] = -z\frac{d}{dz}X[z]$$

Multiplication by an Exponential

If
$$Z[x[n]] = X[z]$$

then
$$Z[a^n x[n]] = X[a^{-1} z]$$

Time Expansion

If $\quad Z[x[n]] = X[z]$

then $\quad Z[x_k[n]] = X[z^k]$

$$x_k[n] = x\left[\frac{n}{k}\right]$$

Convolution Theorem

If $\quad y[n] = x[n] * h[n]$

then $\quad Y[z] = X[z]H[z]$

Final Value Theorem

$$\text{If } Z[x[n]] = X[z]$$

$$x[\infty] = \underset{z \to 1}{Lt}\,(z-1)X[z]$$

Initial Value Theorem

$$x(0) = \lim_{n \to 0} x(n) = \lim_{z \to \infty} X(z)$$

$$\boldsymbol{x[n] = \delta[n - n_0]}$$

$$\delta[n] \xleftrightarrow{Z} 1$$

time shifting property,

$$Z[\delta[n - n_0]] = z^{-n_0}$$

ROC: all z excluding $|z| = 0$.

$$\boldsymbol{x[n] = u[n - n_0]}$$

$$u[n] \xleftrightarrow{Z} \frac{z}{(z-1)}$$

$$Z[u[n - n_0]] = \frac{z^{-n_0} z}{(z-1)} = \frac{z^{-(n_0-1)}}{(z-1)}$$

ROC: $1 < |z| < \infty$

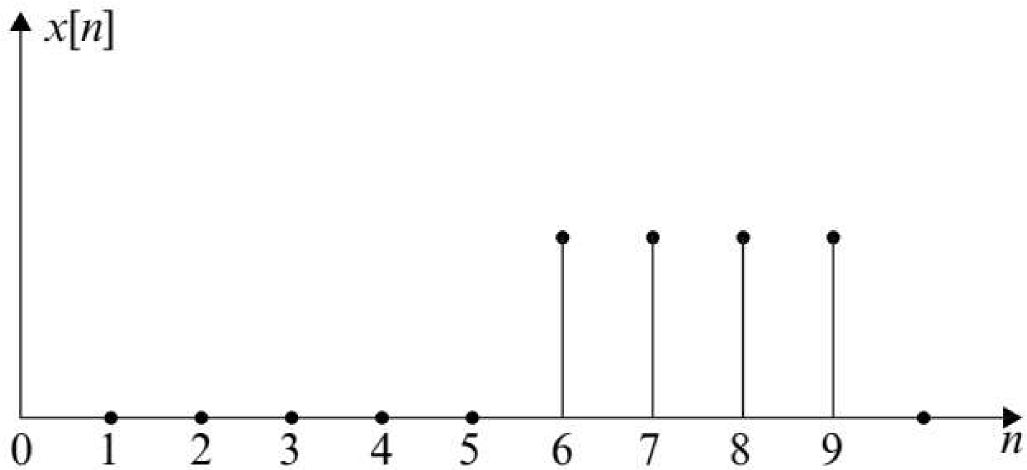
Home Work

$$x[n] = a^{n+1}u[n+1]$$

$$x[n] = a^{n-1}u[n-1]$$

$$x[n] = \left(\frac{1}{2}\right)^n u[-n]$$

$$x[n] = u[n-6] - u[n-10]$$



$$X[z] = z^{-6} + z^{-7} + z^{-8} + z^{-9}$$

ROC: all z except $z \neq 0$.

$$x[n] = nu[n]$$

We know

$$Z[u[n]] = \frac{z}{(z-1)}$$

differentiation property in z,

$$Z[nu[n]] = -z\frac{dX[z]}{dz}$$

$$Z[nu[n]] = -z\frac{d}{dz}\left[\frac{z}{(z-1)}\right]$$

$$\frac{d}{dz}\left(\frac{z}{z-1}\right) = \frac{(z-1)\cdot 1 - z\cdot 1}{(z-1)^2} = \frac{z-1-z}{(z-1)^2} = -\frac{1}{(z-1)^2}$$

If f(z) = u(z) / v(z)
f'(z) = [v(z) * u'(z) - u(z) * v'(z)] / [v(z)]^2

$$X[z] = \frac{z}{(z-1)^2}$$

$$\boldsymbol{x[n] = n[u[n] - u[n-8]]}$$

$$u[n] \overset{Z}{\longleftrightarrow} \frac{z}{(z-1)}$$

using the shift theorem,

$$Z[u[n] - u[n-8]] = \frac{z}{(z-1)}[1 - z^{-8}]$$

$$= \frac{(z - z^{-7})}{(z-1)}$$

differentiation property in z,

$$Z[nx[n]] = -z\frac{d}{dz}X[z]$$

If f(z) = u(z) / v(z)
f'(z) = [v(z) * u'(z) - u(z) * v'(z)] / [v(z)]^2

$$Z[n[u[n] - u[n-8]]] = -z\frac{d}{dz}\frac{[z - z^{-7}]}{z-1}$$

$$X[z] = -z\frac{[(z-1)(1+7z^{-8}) - (z - z^{-7})]}{(z-1)^2}$$

$$X[z] = \frac{(-8z^{-6} + 7z^{-7} + z)}{(z-1)^2}$$

$$X[z] = \frac{[z^8 - 8z + 7]}{z^7(z-1)^2}$$

Show that $u[n] * u[n-1] = nu[n]$

$$Z[u[n]] = \frac{z}{(z-1)}$$

$$Z[u[n-1]] = \frac{1}{(z-1)}$$

$$\begin{aligned} Z[u[n] * u[n-1]] &= Z[u[n]]Z[u[n-1]] \\ &= \frac{z}{(z-1)} \frac{1}{(z-1)} \\ &= \frac{z}{(z-1)^2} \end{aligned}$$

$$u[n] * u[n-1] = Z^{-1}\left[\frac{z}{(z-1)^2}\right]$$

We know

$$u[n] \overset{Z}{\longleftrightarrow} \frac{z}{(z-1)}$$

$$Z[nu[n]] = -z\frac{dX[z]}{dz}$$

$$Z[nu[n]] = -z\frac{d}{dz}\left[\frac{z}{(z-1)}\right]$$

$$X[z] = \frac{z}{(z-1)^2}$$

$$u[n] * u[n-1] = n[u[n]]$$

If f(z) = u(z) / v(z)

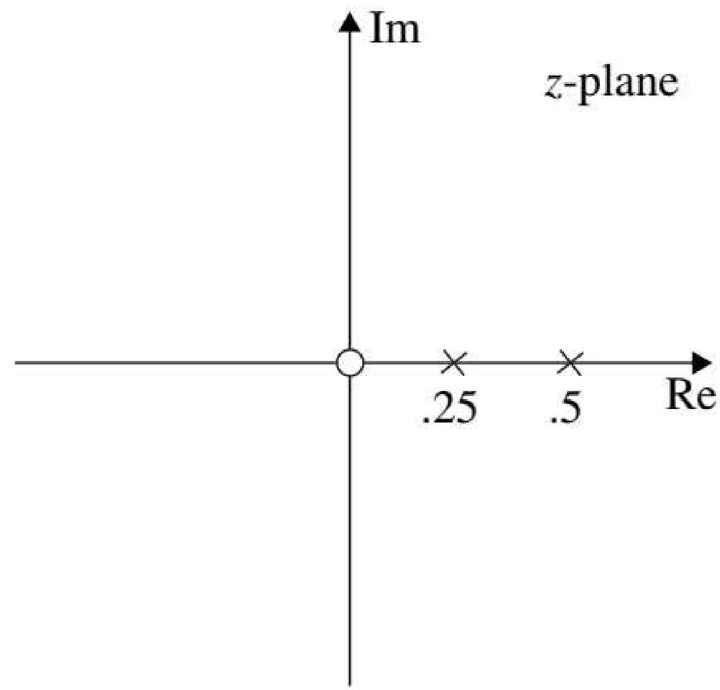
f'(z) = [v(z) * u'(z) - u(z) * v'(z)] / [v(z)]^2

$$\boldsymbol{x[n] = \left[\left(\tfrac{1}{2}\right)^n - \left(\tfrac{1}{4}\right)^n\right] u[n]}$$

Find $X[z]$ and plot the poles and zeros.

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{z}{\left(z - \frac{1}{2}\right)}$$

$$x_2[n] = \left(\frac{1}{4}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{z}{\left(z - \frac{1}{4}\right)}$$



$$\begin{aligned} X[z] &= X_1[z] - X_2[z] \\ &= \frac{z}{(z - 0.5)} - \frac{z}{(z - 0.25)} \end{aligned}$$

$$X[z] = \frac{z0.25}{(z - 0.5)(z - 0.25)}$$

Home Work

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] + 3\left(\frac{1}{6}\right)^n u[n]$$

$$x[n] = \left[\left(-\frac{1}{3}\right)^n u[-n] + 3\left(\frac{1}{6}\right)^n\right] u[n]$$

$$x[n] = \left[\left(-\frac{1}{3}\right)^n + 3\left(\frac{1}{6}\right)^n\right] u[-n]$$

$$x[n] = \begin{cases} (4)^n & n < 0 \\ \left(\frac{1}{4}\right)^n & n = 0, 2, 4, \ldots \\ \left(\frac{1}{5}\right)^n & n = 1, 3, 5, \ldots \end{cases}$$

$$\begin{aligned} x[n] &= 1 \qquad && n \geq 0 \\ &= 3^n && n < 0 \end{aligned}$$

Find the initial and final values

$$X[z] = \frac{z}{(4z^2 - 5z - 1)} \qquad \text{ROC: } |z| > 1$$

$$x[0] = \underset{z\to\infty}{Lt}\, X[z]$$

$$= \underset{z\to\infty}{Lt}\, \frac{z}{z^2\left(4 - \frac{5}{z} - \frac{1}{z^2}\right)}$$

$$= \underset{z\to\infty}{Lt}\, \frac{1}{z\left(4 - \frac{5}{z} - \frac{1}{z^2}\right)}$$

$$x[0] = 0$$

$$x[\infty] = \underset{z\to 1}{Lt}\,(z-1)X[z]$$

$$x[\infty] = \underset{z\to 1}{Lt}\, \frac{\cancel{(z-1)}}{} \frac{z}{4\cancel{(z-1)}\left(z - \frac{1}{4}\right)}$$

$$x[\infty] = \frac{1}{3}$$

Home Work

Find the initial and final values

$$x[z] = \frac{10z(z - 0.4)}{(z - 0.5)(z - 0.3)} \qquad \text{ROC: } |z| > 0.5$$

Inverse z-Transform

There are 3 different methods to perform inverse Z transform

1. Partial fraction method;
2. Power series expansion;
3. Residue method.

Partial Fraction Method

$$X[z] = \frac{1-\frac{1}{3}z^{-1}}{(1-z^{-1})(1+2z^{-1})} \qquad \text{ROC: } |z| > 2$$

$$X[z] = \frac{1-\frac{1}{3}z^{-1}}{(1-z^{-1})(1+2z^{-1})}$$

$$= \frac{z\left(z-\frac{1}{3}\right)}{(z-1)(z+2)}$$

$$\frac{X[z]}{z} = \frac{A_1}{(z-1)} + \frac{A_2}{(z+2)}$$

$$\frac{\cancel{z}\left(z-\frac{1}{3}\right)}{\cancel{z}(z-1)(z+2)} = \frac{A_1(z+2)+A_2(z-1)}{(z-1)(z+2)}$$

$$\left(z-\frac{1}{3}\right) = A_1(z+2) + A_2(z-1)$$

Substitute $z = 1 \quad A_1 = \dfrac{2}{9}$

Substitute $z = -2 \quad A_2 = \dfrac{7}{9}$

$$X[z] = \frac{1}{9}\left[\frac{2z}{z-1} + \frac{7z}{z+2}\right]$$

$$x[n] = \frac{1}{9}\left[2(1)^n + 7(-2)^n\right]u[n]$$

Home Work

$$X[z] = \frac{1}{1024}\left[\frac{1024 - z^{-10}}{1 - \frac{1}{2}z^{-1}}\right] \qquad \text{ROC: } |z| > 0$$

$$X[z] = \frac{z^2}{(1 - az)(z - a)}$$

$$X[z] = \frac{(7z - 23)}{(z - 3)(z - 4)} \qquad \text{ROC: } |z| > 4$$

$$X[z] = \frac{z(z^2 + z - 30)}{(z - 2)(z - 4)^3} \qquad \text{ROC: } |z| > 4$$

$$\boldsymbol{X[z] = \frac{4}{(z-5)}}$$

$$\mathbf{X[z] = z(1 - z^{-1})(1 + 2z^{-1})}$$

$$X[z] = z(1 - z^{-1})(1 + 2z^{-1})$$

$$X[z] = z[1 + 2z^{-1} - z^{-1} - 2z^{-2})$$

$$= [z + 1 - 2z^{-1}]$$

$$x[n] = \{1, \underset{\uparrow}{1}, \ -2\}$$

Inverse z-Transform Using Power Series Expansion

$$X[z] = \frac{4z}{(z^2-3z+2)}; \text{ROC: } |z| > 2$$

ROC: $|z| > 2$, $x[n]$ is a right-sided sequence

$X[z]$ is expressed in the power of z^{-1}

$$\begin{array}{r l}
 & 4z^{-1} + 12z^{-2} + 28z^{-3} + \cdots \\
z^2 - 3z + 2 \,\big)\!\!\!\! & \overline{4z \qquad\qquad\qquad\qquad} \\
 & \underline{4z - 12 + 8z^{-1}} \\
 & \quad 12 - 8z^{-1} \\
 & \quad \underline{12 - 36z^{-1} + 24z^{-2}} \\
 & \qquad 28z^{-1} - 24z^{-2} \\
 & \qquad \underline{28z^{-1} - 84z^{-2} + 56z^{-3}}
\end{array}$$

$$X[z] = 4z^{-1} + 12z^{-2} + 28z^{-3} + \cdots \qquad x[n] = \{\underset{\uparrow}{0},\ 4,\ 12,\ 28, \ldots\}$$

$$\mathbf{X[z] = \frac{4z}{(z^2-3z+2)};\ ROC: |z| < 1}$$

$X[z]$ is expressed in the power of z

$$\begin{array}{r l}
 & \quad 2z + 3z^2 + \frac{7}{2}z^3 \\ \cline{2-2}
2 - 3z + z^2 \big) & 4z \\
 & \underline{4z - 6z^2 + 2z^3} \\
 & \qquad 6z^2 - 2z^3 \\
 & \qquad \underline{6z^2 - 9z^3 + 3z^4} \\
 & \qquad\qquad 7z^3 - 3z^4 \\
 & \qquad\qquad \underline{7z^3 - \frac{21}{2}z^4 + \frac{7}{2}z^5}
\end{array}$$

$$X[z] = 2z + 3z^2 + \frac{7}{2}z^3 + \cdots \qquad x[n] = \left\{\cdots \frac{7}{2},\ 3,\ 2, \underset{\uparrow}{0}\right\}$$

Home Work

$$\boldsymbol{X[z] = \frac{1}{(1-az^{-1})}};\ \textbf{ROC: } |z| > |a|$$

$X[z] = \log(1 - 2z), \qquad |z| < \dfrac{1}{2}$ by using the power series

$$\log(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \qquad |x| < 1$$

Inverse z-Transform Using Contour Integration or the Method of Residue

$$x[n] = \sum \quad \text{(Residues of } X[z]z^{-n} \text{ at the poles inside } (c))$$

$$\boldsymbol{X[z] = \frac{(1+z^{-1})}{(1+8z^{-1}+15z^{-2})}; \ |z| > 5}$$

$$X[z] = \frac{z(z+1)}{(z+3)(z+5)}$$

$$x[n] = \sum \text{Residue of } \frac{z(z+1)}{(z+3)(z+5)} z^{n-1}$$

$$= \text{Residue of } (z+3)\frac{z(z+1)}{(z+3)(z+5)} z^{n-1}\Big|_{z=-3}$$

$$+ \text{Residue of } (z+5)\frac{z(z+1)z^{n-1}}{(z+3)(z+5)}\Big|_{z=-5}$$

$$x[n] = -(-3)^n + 2(-5)^n$$

$$\boldsymbol{X[z] = \frac{z^{-1}}{(1-10z^{-1}+24z^{-2})}; \; 4 < |z| < 6}$$

$$= \frac{z}{(z-4)(z-6)}$$

For $n \geq 0$, Pole is z=4

$$x[n] = \text{ Residue of } X[z]z^{n-1}\Big|_{z=4}$$

$$= (z-4)\frac{z(z^{n-1})}{(z-4)(z-6)}\Big|_{z=4}$$

$$= -\frac{1}{2}(4)^n u[n]$$

For $n < 0$, Pole is z=6

$$x[n] = -\left[(z-6)\frac{zz^{n-1}}{(z-4)(z-6)}\right]_{z=6}$$

$$= -\frac{1}{2}(6)^n u(-n-1)$$

$$x[n] = -\frac{1}{2}[(4)^n u[n] + (6)^n u(-n-1)]$$

The System Function of DT Systems

$x[n]$ = Input of the system;
$y[n]$ = Output of the system;
$h[n]$ = Impulse response of the system.

$$y[n] = x[n] * h[n]$$

$$Y[z] = X[z]H[z]$$

$$H[z] = \frac{Y[z]}{X[z]}$$ system function or the transfer function

System function is defined as the ratio of the z-transforms of the output y[n] and the input x[n]

System Described by Difference Equation

- A discrete-time system is often described using a difference equation, which relates the current output of the system to past outputs, current input, and past inputs.

- This is analogous to a differential equation used in continuous-time systems.

- General Form of a Difference Equation

$$y[n] + a_1 y[n-1] + a_2 y[n-2] + \cdots + a_M y[n-M] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + \cdots + b_N x[n-N],$$

$$\sum_{k=0}^{M} a_k y[n-k] = \sum_{k=0}^{N} b_k x[n-k].$$

- $y[n]$: Output signal at time n.
- $x[n]$: Input signal at time n.
- $a_1, a_2, \ldots, a_M$: Coefficients associated with past outputs ($y[n-k]$).
- $b_0, b_1, \ldots, b_N$: Coefficients associated with the input signal ($x[n-k]$).
- M: Order of the system with respect to outputs.
- N: Order of the system with respect to inputs.

z-Transform Solution of Linear Difference Equations

Right Shift (Delay)

$$x[n]u[n] \overset{Z}{\longleftrightarrow} X[z]$$

$$x[n-1]u[n] \overset{Z}{\longleftrightarrow} \frac{1}{z}X[z] + x[-1]$$

$$x[n-2]u[n] \overset{Z}{\longleftrightarrow} \frac{1}{z^2}X[z] + \frac{1}{z}x[-1] + x[-2]$$

Left Shift (Advance)

$$x[n]u[n] \overset{Z}{\longleftrightarrow} X[z]$$

$$x[n+1]u[n] \overset{Z}{\longleftrightarrow} zX[z] - zx[0]$$

$$x[n+2]u[n] \overset{Z}{\longleftrightarrow} z^2X[z] - z^2x[0] - zx[1]$$

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n-1] \text{ when } x[n] = \delta[n] \text{ and } y[n] = 0,\ n < 0.$$

$y[n] = 0,\ n < 0.$ implies the initial conditions are zero.

$$\left[1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right] Y[z] = 2z^{-1}X[z]$$

For $\delta[n]$, $X[z] = 1$,

$$Y[z] = \frac{2z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

$$= \frac{2z}{z^2 - \frac{3}{4}z + \frac{1}{8}}$$

$$\frac{Y[z]}{z} = \frac{2}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)}$$

$$= \frac{A_1}{\left(z - \frac{1}{2}\right)} + \frac{A_2}{\left(z - \frac{1}{4}\right)}$$

$$A_1 = 8 \qquad A_2 = -8$$

$$Y[z] = 8\left[\frac{z}{\left(z - \frac{1}{2}\right)} - \frac{z}{\left(z - \frac{1}{4}\right)}\right]$$

$$y[n] = 8\left[\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right]u[n] \qquad \text{ROC: } |z| > \frac{1}{2}$$

Home Work

$$y[n+2] + 1.1y[n+1] + 0.3y[n] = x[n+1] + x[n]$$

where $x[n] = (-4)^{-n}u[n]$. Find $y[n]$ if the initial conditions are zero.

A causal LTI system is described by the difference equation

$$y[n] = y[n-1] + y[n-2] + x[n-1]$$

Find (a) System function for this system and (b) Unit impulse response of the system.

Home Work

Find the output of the system whose input-output is related by the difference equation

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] - \frac{1}{2}x[n-1]$$

for the step input. Assume initial conditions to be zero.

Find the output response of the discrete time system described by the following difference equation:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n]$$

The initial conditions are $y[-1] = 0$ and $y = [-2] = 1$. The input $x[n] = \left(\frac{1}{5}\right)^n u[n]$.

Taking z-transform on both sides

$$Y[z] - \frac{3}{4}[z^{-1}Y[z] + y[-1]] + \frac{1}{8}[z^{-2}Y[z]$$

$$+z^{-1}y[-1] + y[-2]] = X[z]$$

$$\left[1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right]Y[z] = -\frac{1}{8} + \frac{z}{\left(z - \frac{1}{5}\right)}$$

$$\frac{\left[z^2 - \frac{3}{4}z + \frac{1}{8}\right]}{z^2}Y[z] = -\frac{1}{8} + \frac{z}{\left(z - \frac{1}{5}\right)}$$

$$\frac{Y[z]}{z} = -\frac{z}{8\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)} + \frac{z^2}{\left(z - \frac{1}{5}\right)\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)}$$

$$= Y_1[z] + Y_2[z]$$

$$Y_1[z] = -\frac{z}{8\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)}$$

$$Y_1[z] = \frac{1}{8\left(z - \frac{1}{4}\right)} - \frac{1}{4\left(z - \frac{1}{2}\right)}$$

$$Y_2[z] = \frac{z^2}{\left(z - \frac{1}{5}\right)\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)}$$

$$Y_2[z] = \frac{8}{3}\frac{1}{\left(z - \frac{1}{5}\right)} - \frac{5}{\left(z - \frac{1}{4}\right)} + \frac{10}{3}\frac{1}{\left(z - \frac{1}{2}\right)}$$

$$Y[z] = \frac{z}{8\left(z - \frac{1}{4}\right)} - \frac{z}{4\left(z - \frac{1}{2}\right)} + \frac{8}{3}\frac{z}{\left(z - \frac{1}{5}\right)} - \frac{5z}{\left(z - \frac{1}{4}\right)} + \frac{10}{3}\frac{z}{\left(z - \frac{1}{2}\right)}$$

$$= -\frac{39}{8}\frac{z}{\left(z - \frac{1}{4}\right)} + \frac{37}{12}\frac{z}{\left(z - \frac{1}{2}\right)} + \frac{8}{3}\frac{z}{\left(z - \frac{1}{5}\right)}$$

$$y[n] = \left[-\frac{39}{8}\left(\frac{1}{4}\right)^n + \frac{37}{12}\left(\frac{1}{2}\right)^n + \frac{8}{3}\left(\frac{1}{5}\right)^n\right]u[n]$$

Home Work

Consider the following difference equation:

$$y[n] + 2y[n-1] + 2y[n-2] = x[n]$$

The initial conditions are $y[-1] = 0$ and $y = [-2] = 2$. Find the step response of the system.

Digital Filters

- Digital filters are signal processing tools designed to manipulate or modify digital signals by enhancing desired components (e.g., specific frequencies) and attenuating unwanted ones.
- They operate on discrete-time signals using mathematical algorithms.
- **Two Main Types**:
- **FIR Filters**: Finite Impulse Response, non-recursive, inherently stable.
- **IIR Filters**: Infinite Impulse Response, recursive, compact but may be unstable.

Introduction to FIR Filters

- Finite Impulse Response (FIR) filters are digital filters with a finite number of terms in their impulse response.
- Mathematical Representation:

$$y[n] = \sum_{k=0}^{M-1} h[k]x[n-k]$$

- Where:
  y[n] Output signal
  h[k] Filter coefficients (impulse response)
  x[n−k] Input samples
- Convolution between input signal $x[n]$ and filter impulse response $h[k]$.

Key Characteristics

- Key Characteristics:
  - Always stable
  - Linear phase (if coefficients are symmetric)
  - Non-recursive (does not use feedback)

Characteristics

- **Always stable**
- FIR stands for **Finite Impulse Response**, meaning the filter's response to any finite input signal will also be finite.
- FIR filters do not have a **feedback** loop (i.e., output values are not fed back into the filter).

$$y[n] = \sum_{k=0}^{M-1} h[k]x[n-k]$$

Properties of FIR Filters

- **Linear Phase**: Symmetric coefficients ensure no phase distortion.
- **Stability**: Guaranteed due to the absence of feedback.
- **Flexibility in Design**: Easier to design with exact specifications compared to IIR.

Advantages and Disadvantages of FIR Filters

- Advantages:
  - **Stable**
  - **Linear phase** achievable: filter's phase response is a straight line
  - **Simple implementation** for low orders

- Disadvantages:
  - Requires more **computational power** (high-order filters)
  - Can be inefficient for **sharp frequency cut**offs

FIR Filter Applications

- Fields of Use:
  - Audio signal processing (e.g., equalizers)
  - Communication systems (e.g., channel equalization)
  - Medical imaging (e.g., noise reduction in ECG signals)
  - Radar systems (e.g., signal smoothing)

Introduction to IIR Filters

- Infinite Impulse Response (IIR) filters are digital filters that have an **impulse response extending indefinitely** due to the use of **feedback**.
- Mathematical Representation:

$$y[n] = \sum_{k=0}^{N-1} b[k] \cdot x[n-k] - \sum_{k=1}^{M-1} a[k] \cdot y[n-k]$$

  - $y[n]$: Output signal
  - $x[n-k]$: Input samples
  - $b[k]$: Feedforward coefficients
  - $a[k]$: Feedback coefficients

Characteristics

- Can achieve the **same frequency response with lower filter order** compared to FIR filters.
- May **not** have **linear phase**.
- Uses **feedback**, making them recursive.

Applications

- Audio processing (e.g., equalizers).
- Biomedical signal filtering (e.g., ECG, EEG).
- Communication systems (e.g., noise reduction).

Properties of IIR Filters

- **Efficiency**: Lower order than FIR for the same specifications, saving computation.
- **Frequency Response**: Provides **sharp roll-offs** with fewer coefficients.
- **Stability**: Requires careful design; poles must lie within the unit circle in the Z-plane.
- **Phase Response**: Typically **nonlinear**, which may distort signals.

Advantages and Disadvantages of IIR Filters

- **Advantages**:
  - Requires fewer coefficients for a given performance.
  - Efficient for real-time applications.
  - Mimics analog filters well.
- **Disadvantages**:
  - May be unstable if not designed properly.
  - Nonlinear phase can distort signals.
  - Harder to design with exact specifications.

Applications of IIR Filters

- Fields of Use:
- **Audio Processing**: Graphic equalizers, reverb effects.
- **Communication Systems**: Noise cancellation, channel equalization.
- **Biomedical Engineering**: ECG filtering, brainwave signal analysis.
- **Radar and Sonar Systems**: Signal detection and smoothing.

Difference Between FIR and IIR Filters

- FIR Filters:
  - Finite impulse response.
  - Always stable.
  - Linear phase achievable.
- IIR Filters:
  - Infinite impulse response.
  - Can become unstable if not properly designed.
  - May not have linear phase.