Semi-abelian categories, cocommutative Hopf algebras and Hopf braces

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Outline

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- Semi-abelian categories
- Cocommutative Hopf algebras
- Hopf braces
- A torsion theory in HBR_{coc}
- Commutators in HBR_{coc}

Abelian categories

The notion of abelian category (Buchsbaum, 1955) captures some crucial properties Ab and R-Mod have in common.

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Definition

A category $\mathbb{C}$ is abelian if

- $\mathbb{C}$ has a zero-object 0
- $\mathbb{C}$ has binary products $A \times B$
- any arrow f in $\mathbb{C}$ has a factorization $f = i \circ p$

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, two heads, "p"'] & & Y \\
& I \arrow[ur, tail, "i"'] &
\end{tikzcd}$$

where p is a *normal epimorphism*, i is a *normal monomorphism*.

Normal epimorphism

When $\mathbb{C}$ is pointed, an arrow $A \overset{p}{\twoheadrightarrow} P$ is a normal epimorphism if it is the cokernel of some arrow in $\mathbb{C}$: there is an i such that

$$\begin{tikzcd}
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is a pushout.

In the categories Ab of abelian groups and R-Mod of R-modules

- normal epimorphism = surjective homomorphism.

Normal monomorphism

An arrow $K \rightarrowtail^{k} A$ is called a normal monomorphism if it is the kernel of some arrow in $\mathbb{C}$. This means that there is an $f\colon A \to B$ such that

$$\begin{tikzcd}
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Groups

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Abelian groups

In the category Ab : any monomorphism $k\colon K \to A$ is normal !

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- Ab has a 0
- the product $A \times B$ exists for any A, B
- any homomorphism f in Ab has a factorization $f = i \circ p$

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, two heads, "p"'] & & Y \\
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Examples

Ab, R-Mod, Ab(Comp) are all abelian categories.

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What about the category Grp of groups ?

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- Problem : an arrow f in Grp does not have a factorization $f = i \circ p$

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with p a *normal epimorphism* and i is a *normal* monomorphism.

The category Rng of rings is not abelian :

- an arrow f in Rng does not have a factorization $f = i \circ p$

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\
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\end{tikzcd}$$

  with p a normal epimorphism and i a normal monomorphism.

This is due to the fact that normal monomorphisms in Rng correspond to inclusions of ideals.

Question : is there a list of simple axioms to conceptually understand some typical properties the categories Grp, Rng, Lie_K have in common ?

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Aim : find an axiomatic context for

- isomorphism theorems
- non-abelian homological algebra
- radical and torsion theories
- commutator theory

Historical remarks

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- A. Grothendieck, Sur quelques points d'algèbre homologique, Tohoku Math. J. (1957)
- Several proposals of "non-abelian contexts" :
  S. A. Amitsur (1954), A.G. Kurosh (1959),
  P. Higgins (1956), A. Frölich (1961), S.A. Huq (1968), M. Gerstenhaber (1970),
  O. Wyler (1971), G. Orzech (1972), etc.

A solution of Mac Lane’s problem :

Definition (G. Janelidze, L. Márki, W.Tholen, JPAA, 2002)

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- $\mathbb{C}$ is (Barr)-exact : a) any morphism factors as a regular epimorphism followed by a monomorphism ; b) regular epimorphisms are pullback stable, c) any equivalence relation is a kernel pair.

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- $\mathbb{C}$ is (Barr)-exact : a) any morphism factors as a regular epimorphism followed by a monomorphism ; b) regular epimorphisms are pullback stable, c) any equivalence relation is a kernel pair.
- $\mathbb{C}$ is (Bourn)-protomodular : the Split Short Five Lemma holds in $\mathbb{C}$

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, shift right, "f"'] \arrow[d, "v"] & B \arrow[l, shift right, "s"'] \arrow[d, "w"] \\
0 \arrow[r] & K' \arrow[r, "k'"'] & A' \arrow[r, shift right, "f'"'] & B' \arrow[l, shift right, "s'"']
\end{tikzcd}$$

u, w isomorphisms $\Rightarrow$ v isomorphism.

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Any abelian category ! In particular : Ab, R-Mod, Ab(Comp), etc.

[$\mathbb{C}$ is abelian] $\Leftrightarrow$ [$\mathbb{C}$ and $\mathbb{C}^{op}$ are semi-abelian] !

In an abelian category the canonical morphism Σ from $A + B$ to $A \times B$ is an isomorphism :

$$\begin{tikzcd}
 & A \times B \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
A \arrow[dr, "i_1"'] & & B \arrow[dl, "i_2"] \\
 & A + B \arrow[uu, dotted, "\Sigma", "\cong"'] &
\end{tikzcd}$$

In an abelian category the canonical morphism Σ from $A + B$ to $A \times B$ is an isomorphism :

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\end{tikzcd}$$

One then has the notion of biproduct

$$\begin{tikzcd}
A \arrow[r, shift left, "i_1"] & A \oplus B \arrow[l, shift left, "p_1"] \arrow[r, shift left, "p_2"] & B, \arrow[l, shift left, "i_2"]
\end{tikzcd}$$

and any $\mathrm{Hom}_{\mathbb{C}}(C, D)$ has an abelian group structure.

In a semi-abelian category the canonical morphism Σ from $A+B$ to $A\times B$ is only a normal epimorphism (not an isomorphism)

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A semi-abelian category is not additive !

The “general idea” :

Whereas

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the “non-additive” version of this “equation” is

semi-abelian = exact + 0 + *protomodular*.

One replaces the “additivity” with the validity of the “Split Short Five Lemma”.

Outline

Semi-abelian categories

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Commutators in HBR_{coc}

Let K be a field.

Bialgebras

A K-bialgebra $(A, m, u, \Delta, \epsilon)$ is both a K-algebra (A, m, u) and a K-coalgebra (A, Δ, ϵ), where m, u, Δ, ϵ are linear maps such that the following diagrams commute, and m and u are K-coalgebra morphisms.

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "1_A \otimes m"] \arrow[d, swap, "m \otimes 1_A"] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, swap, "m"] & A
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes K \arrow[r, "1_A \otimes u"] \arrow[dr, swap, "r_A"] & A \otimes A \arrow[d, "m"] & K \otimes A \arrow[l, swap, "u \otimes 1_A"] \arrow[dl, "l_A"] \\
 & A &
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, swap, "\Delta"] & A \otimes A \arrow[d, "\Delta \otimes 1_A"] \\
A \otimes A \arrow[r, swap, "1_A \otimes \Delta"] & A \otimes A \otimes A
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes K & A \otimes A \arrow[l, swap, "1_A \otimes \epsilon"] \arrow[r, "\epsilon \otimes 1_A"] & K \otimes A \\
 & A \arrow[ul, "r_A^{-1}"] \arrow[u, swap, "\Delta"] \arrow[ur, swap, "l_A^{-1}"] &
\end{tikzcd}$$

Cocommutative Hopf algebras

A Hopf K-algebra $(A, m, u, \Delta, \epsilon, S)$ is a K-bialgebra with a linear map $S\colon A \to A$, the antipode, making the following diagram commute :

$$\begin{tikzcd}
 & A \otimes A \arrow[rr, shift left, "1_A \otimes S"] \arrow[rr, shift right, "S \otimes 1_A"'] & & A \otimes A \arrow[dr, "m"] & \\
A \arrow[ur, "\Delta"] \arrow[drr, "\epsilon_A"'] & & & & A \\
 & & K \arrow[urr, "u_A"'] & &
\end{tikzcd}$$

A Hopf algebra $(A, m, u, \Delta, \epsilon, S)$ is cocommutative if the following triangle commutes :

$$\begin{tikzcd}
 & A \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
A \otimes A \arrow[rr, "\cong", "tw"'] & & A \otimes A,
\end{tikzcd}$$

where $tw \colon A \otimes A \to A \otimes A$ is the “flip map”.

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A morphism $f\colon (A, m, u, \Delta, \epsilon, S) \to (B, m, u, \Delta, \epsilon, S)$ of Hopf algebras is a linear map $f\colon A \to B$ that is both an algebra and a coalgebra map.

Example

Any group G gives the group-algebra

$$K[G] = \{\sum_g \alpha_g g \mid g \in G, \alpha_g \in K\},$$

which becomes a cocommutative Hopf algebra with

$$\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1}.$$

The category $\mathrm{GrpHopf}_K$ of group Hopf algebras (= generated by “grouplike elements”) is full in the category $\mathrm{Hopf}_{K,coc}$ of cocommutative Hopf algebras :

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$\mathrm{Hopf}_{K,coc}$ is finitely complete, with the product of two Hopf algebras A and B being given by the tensor product

$$\begin{tikzcd} A & A \otimes B \arrow[l, "p_1"'] \arrow[r, "p_2"] & B, \end{tikzcd}$$

where $p_1(a \otimes b) = a\epsilon(b)$ and $p_2 = \epsilon(a)b$, and the zero object is K.

Theorem (M.G., F. Sterck and J. Vercruysse, JPAA, 2019)

For any field K, the category $\mathrm{Hopf}_{K,coc}$ is semi-abelian.

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Main steps of the proof :

To prove that $\mathsf{Hopf}_{K,coc}$ is a *regular category* one first shows the existence of the normal epi/mono factorization.

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A key fact to prove this is the *"Newman correspondence"* :

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A key fact to prove this is the *"Newman correspondence"* :

Theorem (K. Newman, J. Algebra, 1975)

For $A \in \mathrm{Hopf}_{K,coc}$ there is a bijective correspondence between :

$$\{\, Hopf\ subalgebras\ of\ A\} \cong \{left\ ideal\ two-sided\ coideals\ of\ A\}$$

Given a Hopf subalgebra D of A, the bijection

$$\phi_A\colon \{\ \textit{Hopf subalgebras of } A\} \to \{\textit{left ideal two}-\textit{sided coideals of } A\}$$

sends D to the left ideal two-sided coideal

$$\phi_A(D) = AD^+,$$

where $D^+ = \{x \in D \mid \epsilon(x) = 0\}$.

The kernel of a morphism $f\colon A \to B$ in $\mathsf{Hopf}_{K,coc}$ is given by the inclusion $\mathsf{HKer}(f) \to A$ of the following Hopf subalgebra of A :

$$\mathsf{HKer}(f) = \{a \in A \mid f(a_1) \otimes a_2 = 1_B \otimes a\} \subset A.$$

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The vector space kernel $\ker(f)$ of the linear map f turns out to be a Hopf ideal :

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This implies that

$$f(A) = \{f(a) \mid a \in A\} \cong \frac{A}{\ker(f)} = \frac{A}{A(\mathrm{HKer}(f)^+)A} \in \mathrm{Hopf}_{K,coc}.$$

The canonical factorization in Vect_K of a morphism $f\colon A \to B$ in $\mathrm{Hopf}_{K,coc}$

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\pi"'] & & B \\
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is then also a factorization in the category $\mathsf{Hopf}_{K,coc}$.

Consequently, any morphism f has a canonical factorization $f = i\pi$ in $\mathsf{Hopf}_{K,coc}$ as a normal epi π followed by a monomorphism i.

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The protomodularity of $\mathsf{Hopf}_{K,coc}$ directly follows from the fact that

$$\mathsf{Hopf}_{K,coc} = \mathsf{Grp}(CoAlg_{k,coc}).$$

Finally, to check the (Barr)-exactness of $\mathsf{Hopf}_{K,coc}$, one observes that in any commutative diagram

$$\begin{tikzcd}
N \arrow[r, tail, "n"] \arrow[d, two heads, "\overline{p}"'] & A \arrow[d, two heads, "p"] \\
p(N) \arrow[r, tail, "m"'] & B
\end{tikzcd}$$

where n is a normal mono, p and $\overline{p}$ are regular epimorphisms and m is a mono, then m is a normal mono. □

In particular, the previous result implies

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Proof :

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Proof :

$$\mathrm{Hopf}_{K,coc}^{comm} = \mathrm{Ab}(\mathrm{Hopf}_{K,coc}).$$

$A \in \mathrm{Hopf}_{K,coc}$ is abelian $\Leftrightarrow \Delta \colon A \to A \otimes A$ is a normal mono

$\Leftrightarrow$ the product in A is commutative : $ab = ba$

$\Leftrightarrow$ $A \in \mathrm{Hopf}_{K,coc}^{comm}$ $\square$

Strong protomodularity

The category $\mathrm{Hopf}_{K,coc}$ is even strongly protomodular (in the sense of D. Bourn) : in any diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, shift right, "f"'] \arrow[d, "v"] & B \arrow[l, shift right, "s"'] \arrow[d, equal, "1_B"] \\
0 \arrow[r] & K' \arrow[r, "k'"'] & A' \arrow[r, shift right, "f'"'] & B \arrow[l, shift right, "s'"']
\end{tikzcd}$$

where f and f' are split epimorphisms and u is a normal mono, then $k'u$ is also a normal mono.

Commutators

Strong protomodularity is useful to develop commutator theory (the “Smith=Huq” condition holds).

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For normal Hopf subalgebras M, N of $A \in \mathsf{Hopf}_{K,coc}$

$$\begin{tikzcd} M \arrow[r, tail] & A & N \arrow[l, tail] \end{tikzcd}$$

one can “compute” their categorical commutator :

$$[M, N]_{Huq} = \langle \{ m_1 n_1 S(m_2) S(n_2) \mid m \in M, n \in N \} \rangle_A$$

(where $\Delta(m) = m_1 \otimes m_2$ and $\Delta(n) = n_1 \otimes n_2$).

In $\mathrm{Hopf}_{K,coc}$ the condition $[M,N]_{Huq}=0$ is equivalent to the existence of a (unique) morphism $p\colon M\otimes N\to A$ making the diagram

$$\begin{tikzcd}
 & M\otimes N \arrow[dd, dotted, "p"] & \\
M \arrow[ur, "{(1_M,0)}"] \arrow[dr] & & N \arrow[ul, "{(0,1_N)}"'] \arrow[dl] \\
 & A &
\end{tikzcd}$$

commute, where $p(m\otimes n)=mn$, for any $m\otimes n\in M\otimes N$.

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$$a \circ (b + c) = a \circ b - a + a \circ c, \qquad\qquad \forall a, b, c \in A.$$

Skew braces produce the solutions of the Yang-Baxter equation :

$$r(a, b) = (-a + a \circ b, (-a + a \circ b)' \circ a \circ b).$$

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SKB is a semi-abelian variety of algebras, containing the variety Grp of groups as a subvariety : any group $(G, +)$ can be seen as a skew brace $(G, +, +)$. Up to this identification of $(G, +)$ with $(G, +, +)$, one has

$$\begin{tikzcd}
\text{Grp} \arrow[r, shift right=1.5ex, "U"'] \arrow[r, phantom, "\perp"] & \text{SKB} \arrow[l, shift right=1.5ex, "F"']
\end{tikzcd}$$

SKB also contains the variety of RadRng of radical rings as a subvariety :

$$\begin{tikzcd}\mathrm{RadRng} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathrm{SKB}, \arrow[l, bend right, "F"']\end{tikzcd}$$

which is determined by the identities

$$(a+b)\circ c = a\circ c - c + b\circ c$$

and

$$a+b=b+a.$$

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In other words, there is an analogy :

$$\mathsf{Grp} : \mathsf{Hopf}_{K,coc} = \mathsf{SKB} : \mathsf{HBR}_{coc}.$$

Definition (Angiono, Galindo, Vendramin, Proc. Amer. Math. Soc., 2017)

A Hopf brace is a datum $(H, \cdot, \bullet, 1, \Delta, \varepsilon, S, T)$ where $(H, \cdot, 1, \Delta, \varepsilon, S)$ and $(H, \bullet, 1, \Delta, \varepsilon, T)$ are Hopf algebras satisfying the compatibility condition

$$a \bullet (b \cdot c) = (a_1 \bullet b) \cdot S(a_2) \cdot (a_3 \bullet c), \quad \text{for all } a, b, c \in H. \tag{1}$$

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A morphism of Hopf braces $f : (H, \cdot, \bullet) \to (K, \cdot, \bullet)$ is a map that is a Hopf algebra morphism with respect to both the Hopf algebra structures :

$$f(a \cdot b) = f(a) \cdot f(b), \quad f(a \bullet b) = f(a) \bullet f(b), \quad f(1_H) = 1_K$$

and

$$\varepsilon(f(a)) = \varepsilon(a), \quad \Delta(f(a)) = f(a_1) \otimes f(a_2), \qquad \forall a, b \in H.$$

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The category HBR$_{coc}$ of cocommutative Hopf braces is monoidal :

if $(H, \cdot, \bullet, 1, \Delta, \varepsilon, S, T)$, $(K, \cdot', \bullet', 1', \Delta', \varepsilon', S', T')$ are Hopf braces, we have that $H^{\cdot} \otimes K^{\cdot'}$ and $H^{\bullet} \otimes K^{\bullet'}$ are Hopf algebras on the vector space $H \otimes K$.

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The multiplications are given by

$$m_{\cdot_{\otimes}} := (m_{\cdot} \otimes m_{\cdot'})(\mathrm{Id}_H \otimes tw \otimes \mathrm{Id}_K), \qquad m_{\bullet_{\otimes}} := (m_{\bullet} \otimes m_{\bullet'})(\mathrm{Id}_H \otimes tw \otimes \mathrm{Id}_K),$$

the coalgebra structure by

$$\Delta_{\otimes} := (\mathrm{Id}_H \otimes tw \otimes \mathrm{Id}_K)(\Delta \otimes \Delta'), \quad \varepsilon_{\otimes} := \varepsilon \otimes \varepsilon',$$

and the antipodes by $S_{\otimes} := S \otimes S'$, $\quad T_{\otimes} := T \otimes T'$.

Propositon (A. Agore, Internat. J. Math., 2019)

The category HBR_{coc} is finitely complete, with products coinciding with tensor products, and the monoidal unit coinciding with the terminal object K.

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a) $(B, \cdot, \bullet)$ is a normal sub-Hopf brace (= the kernel of some morphism)

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a) $(B, \cdot, \bullet)$ is a normal sub-Hopf brace (= the kernel of some morphism)

b) $a_1 \cdot b \cdot S(a_2) \in B$, $a_1 \bullet b \bullet S(a_2) \in B$, $S(a_1) \cdot (a_2 \bullet b) \in B$, for any $a \in A$, $b \in B$

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In other words, B is a normal Hopf subalgebra of both $(A, \cdot)$ and $(A, \bullet)$ and, moreover, B is stable for the action $\rightharpoonup$ of A on B, i.e.

$$a \rightharpoonup b \in B \qquad \text{for all } a \in A, b \in B,$$

where

$$a \rightharpoonup b = S(a_1) \cdot (a_2 \bullet b).$$

Proposition (M.G. and A. Sciandra, 2024)

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  $$U^{\cdot}(A, \cdot, \bullet) = (A, \cdot),$$

  and $U^{\bullet} : \mathrm{HBR}_{coc} \to \mathrm{Hopf}_{K,coc}$ by

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- the strong protomodularity of $\mathrm{Hopf}_{K,coc}$. □

Hopf brace ideals

With A. Sciandra we define the notion of “ideal” in the context of Hopf braces to extend the results we know for cocommutative Hopf algebras.

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Let $(H, \cdot, \bullet)$ be a Hopf brace.
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A Hopf brace ideal is a vector subspace $I \subseteq H$ such that :

- $H \cdot I \subseteq I, I \cdot H \subseteq I, \quad H \bullet I \subseteq I, I \bullet H \subseteq I,$
- $\Delta(I) \subseteq I \otimes H + H \otimes I, \quad \varepsilon(I) = 0,$
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Lemma (M. G. and A. Sciandra, 2024)

A sub-Hopf brace B of a Hopf brace $(A, \cdot, \bullet)$ is normal if and only if

$$A \cdot B^+ = A \bullet B^+,$$

and $A \cdot B^+$ is a Hopf brace ideal.

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The notion of Hopf brace ideal allows one to establish a new “Newman-type” correspondence for Hopf braces :

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Given any $f\colon (A, \cdot, \bullet) \to (B, \cdot, \bullet)$ in HBR_{coc}, from the previous Lemma it follows that the quotient $\frac{A}{\mathsf{ker}(f)}$ of A by the kernel $\mathsf{ker}(f)$ of f (in Vect_K) is a Hopf brace :

$$\frac{A}{\mathsf{ker}(f)} = \frac{A}{A \cdot (\mathsf{HKer}(f)^+)} = \frac{A}{A \bullet (\mathsf{HKer}(f)^+)} \in \mathsf{HBR}_{coc}$$

This implies that any morphism $f\colon (A,\cdot,\bullet) \to (B,\cdot,\bullet)$ admits a normal epi-mono factorization in HBR_{coc} :

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, two heads, "\pi"'] & & B \\
& f(A) \cong \frac{A}{\ker(f)} \arrow[ur, hook, "i"'] &
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Theorem (M.G. and A. Sciandra, 2024)

The category HBR_{coc} is semi-abelian.

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Theorem (M.G. and A. Sciandra, 2024)

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Idea of the proof :

Once the factorization above is established, the next step is to prove that both the functors $U^{\cdot}\colon \mathsf{HBR} \to \mathsf{Hopf}_{K,coc}$ and $U^{\bullet}\colon \mathsf{HBR}_{coc} \to \mathsf{Hopf}_{K,coc}$ are *regular* and *conservative* (= isomorphism reflecting). The existence of binary coproducts in HBR_{coc} was proved by Agore and Chirvasitu (preprint, in preparation).

The Noether Isomorphism Theorems, the classical homological lemmas (Snake Lemma, 3×3-Lemma, etc.), the Zassenhaus Lemma, and other classical results from Group Theory hold in HBR_{coc}... simply because this category is semi-abelian !

Outline

Semi-abelian categories

Cocommutative Hopf algebras

Hopf braces

A torsion theory in HBR_{coc}

Commutators in HBR_{coc}

In this section we assume that $\mathbb{K}$ is an *algebraically closed field of characteristic* 0.

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The category SKB turns out to be equivalent to the full (replete) subcategory of HBR_{coc}, whose objects are the cocommutative Hopf braces whose “underlying” Hopf algebras are “group Hopf algebras”.

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The category SKB turns out to be equivalent to the full (replete) subcategory of HBR_{coc}, whose objects are the cocommutative Hopf braces whose “underlying” Hopf algebras are “group Hopf algebras”.

The “compatibility condition” in the case of a group Hopf brace $(\mathbb{K}[G], \cdot, \bullet)$ becomes

$$g \bullet (h \cdot k) = (g \bullet h) \cdot g^{-\cdot} \cdot (g \bullet k),$$

and we’ll write SKB also for the category of “group Hopf braces”.

Another interesting subcategory of HBR_{coc} is the category PHBR_{coc} of primitive Hopf braces.

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The objects of PHBR_{coc} are the cocommutative Hopf braces whose “underlying” Hopf algebras are universal enveloping algebras.

These Hopf braces are generated by “primitive” elements, i.e. by the elements x such that $\Delta(x) = 1 \otimes x + x \otimes 1$.

Proposition (M.G. and A. Sciandra, 2024)

The pair $(\mathsf{PHBR}, \mathsf{SKB})$ is a hereditary torsion theory in HBR_{coc}.

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Idea of the proof :

The first axiom of torsion theory follows from the fact that, for any Hopf brace $(H, \cdot, \bullet)$, there is a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & (\mathrm{HKer}(\pi), \cdot, \bullet) \arrow[r] & (H, \cdot, \bullet) \arrow[r, "\pi"] & (\mathbb{K}[G], \cdot, \bullet) \arrow[r] & 0 \end{tikzcd}$$

where $(\mathbb{K}[G], \cdot, \bullet) \in$ SKB is a skew brace (G is the set of group-like elements of H), and $(\mathrm{HKer}(\pi), \cdot, \bullet) \in \mathrm{PHBR}_{coc}$ turns out to be a primitive Hopf brace.

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where $(\mathbb{K}[G], \cdot, \bullet) \in$ SKB is a skew brace (G is the set of group-like elements of H), and $(\text{HKer}(\pi), \cdot, \bullet) \in \text{PHBR}_{coc}$ turns out to be a primitive Hopf brace.

The second axiom of torsion theory is that any morphism $f\colon (P, \cdot, \bullet) \to (G, \cdot, \bullet)$ from a “primitive” Hopf brace $(P, \cdot, \bullet)$ to a “skew brace” $(G, \cdot, \bullet)$ is the zero arrow. This follows from the fact that any morphism preserves primitive elements. $\square$

Proposition (M.G. and A. Sciandra, 2024)

The category SKB is a localization of HBR_{coc} : in the adjunction

$$\begin{tikzcd}
\mathsf{SKB} \arrow[r, phantom, "\perp"] \arrow[r, leftarrow, shift left=2, "F"] & \mathsf{HBR}_{coc} \arrow[l, leftarrow, shift left=2, "U"]
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the left adjoint $F\colon \mathsf{HBR}_{coc} \to \mathsf{SKB}$ preserves finite limits.

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- F preserves finite products because SKB is a Birkhoff subcategory of HBR_{coc},
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- F preserves finite products because SKB is a Birkhoff subcategory of HBR_{coc},
- F preserves monomorphisms because the torsion theory $(\mathsf{PHBR}, \mathsf{SKB})$ is hereditary.
- One then checks that F also preserves equalizers. ☐

Remark

This result is analogous to the one in the article (M.G., G. Kadjo and J. Vercruysse, Bull. Belgian Math. Soc., 2016), where it is proved that the category Grp of groups is a localization of $\mathsf{Hopf}_{K,coc}$:

$$\begin{tikzcd}
\mathsf{Grp} \arrow[r, shift right=1.5ex, "U"'] \arrow[r, phantom, "\perp"] & \mathsf{Hopf}_{K,coc} \arrow[l, shift right=1.5ex, "F"']
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Strong protomodularity is again useful to develop commutator theory in HBR_{coc}.

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For normal sub Hopf braces X, Y of $(A, \cdot, \bullet) \in \mathrm{HBR}_{coc}$

$$\begin{tikzcd} X \arrow[r, tail] & A & Y \arrow[l, tail] \end{tikzcd}$$

one can actually “compute” the categorical commutator.

Lemma (M.G. and A. Sciandra, 2024)

Let $(A, \cdot, \bullet)$ be a cocommutative Hopf brace and $(X, \cdot, \bullet)$, $(Y, \cdot, \bullet)$ two sub-Hopf braces of $(A, \cdot, \bullet)$. The following conditions are equivalent :

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- there is a unique morphism $p : (X \otimes Y, \cdot_{\otimes}, \bullet_{\otimes}) \to (A, \cdot, \bullet)$ in HBR_{coc} such that

$$\begin{tikzcd}
& X \otimes Y \arrow[dd, dotted, "p"] & \\
X \arrow[ur, "{(1_X,0)}"] \arrow[dr] & & Y \arrow[ul, "{(0,1_Y)}"'] \arrow[dl] \\
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- $x_1 \cdot y_1 \cdot S(x_2) \cdot S(y_2) = S(x_1) \cdot (x_2 \bullet y_1) \cdot S(y_2) = x_1 \bullet y_1 \bullet T(x_2) \bullet T(y_2) = \varepsilon(x)\varepsilon(y)1.$

Corollary

The Huq commutator $[X, Y]_{Huq} = [X, Y]$ of two normal sub-Hopf braces of $(A, \cdot, \bullet)$ is the normal sub-Hopf brace $([X, Y], \cdot, \bullet)$, where

$$[X, Y] = \langle\{x_1 \cdot y_1 \cdot S(x_2) \cdot S(y_2), S(x_1) \cdot (x_2 \bullet y_1) \cdot S(y_2), x_1 \bullet y_1 \bullet T(x_2) \bullet T(y_2) \mid x \in X, y \in Y\}\rangle_N.$$

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Abelian objects

The abelian objects in the category HBR_{coc} can be characterized as the Hopf braces $(A, \cdot, \bullet)$ such that $\cdot = \bullet$, and the product $\cdot$ is commutative : $x \cdot y = y \cdot x$.

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In other words : $\mathsf{Ab}(\mathsf{HBR}_{\mathsf{coc}}) \cong \mathsf{Ab}(\mathsf{Hopf}_{K,coc})$. This is an abelian category.

The abelianisation functor of HBR_{coc} can then be seen as the composite of the left adjoints in

$$\begin{tikzcd}\mathsf{Ab}(\mathsf{Hopf}_{K,coc}) \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathsf{Hopf}_{K,coc} \arrow[l, shift right] \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathsf{HBR}_{coc}, \arrow[l, shift right]\end{tikzcd}$$

which is a “linearization” of

$$\begin{tikzcd}\mathsf{Ab} \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathsf{Grp} \arrow[l, shift right] \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathsf{SKB}. \arrow[l, shift right]\end{tikzcd}$$

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$$\begin{tikzcd} \mathrm{Ab}(\mathrm{Hopf}_{K,coc}) \arrow[r, shift right=1.5ex] \arrow[r, phantom, "\perp"] & \mathrm{Hopf}_{K,coc} \arrow[l, shift right=1.5ex] \arrow[r, shift right=1.5ex] \arrow[r, phantom, "\perp"] & \mathrm{HBR}_{coc}, \arrow[l, shift right=1.5ex] \end{tikzcd}$$

which is a “linearization” of

$$\begin{tikzcd} \mathrm{Ab} \arrow[r, shift right=1.5ex] \arrow[r, phantom, "\perp"] & \mathrm{Grp} \arrow[l, shift right=1.5ex] \arrow[r, shift right=1.5ex] \arrow[r, phantom, "\perp"] & \mathrm{SKB}. \arrow[l, shift right=1.5ex] \end{tikzcd}$$

The central extensions of Hopf braces can be described as the surjective morphisms $f\colon (A, \cdot, \bullet) \to (B, \cdot, \bullet)$ having the property that

$$a \cdot k = k \cdot a = a \bullet k = k \bullet a, \qquad \forall a \in A, \forall k \in K.$$

Final remarks

- The category HBR_{coc} has many strong “exactness properties” (it is semi-abelian and strongly protomodular), and the descriptions of some basic categorical constructions (commutator, semi-direct product) are available.

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- The Hopf formulas for homology of skew braces have been studied in a joint work with L. Vendramin and T. Letourmy (preprint, 2024). These results are based on Categorical Galois Theory.
- It would be interesting to extend these results to cocommutative Hopf braces, describe "crossed skew braces", etc.

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