# $L$-algebras: the Yang–Baxter equation and algebraic logic

Leandro Vendramin

Vrije Universiteit Brussel

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# The Yang–Baxter equation

**Problem (Drinfeld)**

Study set-theoretic solutions (to the YBE).

A set-theoretic solution (to the YBE) is a pair $(X, r)$, where $X$ is a set and $r\colon X \times X \to X \times X$ is a bijective map such that

$$(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r).$$

**First works:** Gateva–Ivanova and Van den Bergh and Etingof, Schedler and Soloviev.

**Examples:**

- The flip: $r(x, y) = (y, x)$.
- Let $X$ be a set and $\sigma, \tau \colon X \to X$ be bijections such th at $\sigma\tau = \tau\sigma$. Then

$$r(x, y) = (\sigma(y), \tau(x))$$

  is a solution.
- Let $X = \mathbb{Z}/n$. Then

$$r(x, y) = (2x - y, x) \quad \text{and} \quad r(x, y) = (y - 1, x + 1)$$

  are solutions.

**More examples:**
If $X$ is a group, then

$$r(x, y) = (xyx^{-1}, x) \quad \text{and} \quad r(x, y) = (xy^{-1}x^{-1}, xy^2)$$

are solutions.

## Problem

Construct (finite) set-theoretical solutions.

We deal with non-degenerate solutions, i.e. solutions

$$r(x, y) = (\sigma_x(y), \tau_y(x)),$$

where all maps $\sigma_x \colon X \to X$ and $\tau_x \colon X \to X$ are bijective. We consider involutive solutions, i.e. $r^2 = \mathrm{id}$.

**Convention:**
A solution will be a non-degenerate involutive solution.

How many involutive solutions are there?

The number of solutions (up to isomorphism).

| size | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|
| | 23 | 88 | 595 | 3456 | 34530 | 321931 | 4895272 |

These solutions were constructed with Akgün and Mereb using constraint programming techniques.

Constraint programming is a paradigm for solving combinatorial problems. The idea is to search for variables that satisfy a certain number of constraints.

Involutive solutions are easier to construct than arbitrary solutions. Let us write

$$r(x,y) = (\sigma_x(y), \tau_y(x)).$$

Assume that $r^2 = \mathrm{id}$. Then

$$\sigma_y(x) = \tau^{-1}_{\tau_x(y)}(x)$$

for all $x, y$.

This means that to construct involutive solutions over a set $X$, one needs, only the set $\{\tau_x : x \in X\}$.

Which conditions on the set $\{\tau_x : x \in X\}$ are needed to construct involutive solutions?

This is how you find cycle sets!

# Cycle sets

A cycle set is a pair $(X, \cdot)$, where $X$ is a set and $X \times X \to X$, $(x, y) \mapsto x \cdot y$, is a binary operation such that

1. The cycloid equation

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$$

   holds for all $x, y, z \in X$, and
2. the maps $\varphi_x \colon X \to X$, $y \mapsto x \cdot y$, are bijective for all $x \in X$.

**Theorem (Rump)**

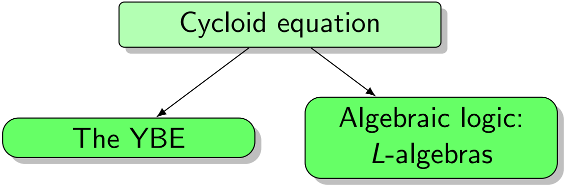
There exists a bijective correspondence between finite cycle sets and finite non-degenerate involutive solutions to the YBE.

The correspondence is given as follows: If $(X, \cdot)$ is a cycle set, then

$$r(x, y) = ((y * x) \cdot y, y * x),$$

where $y * x = z$ if and only if $y \cdot z = x$, is a solution. Conversely, if $(X, r)$ is a solution, then $X$ with $x \cdot y = \tau_x^{-1}(y)$ is a cycle set.

The cycloid equation is relevant in extensions of classical logic, like the Birkhoff and Von Neumann approach[1] to quantum logic.




[1] Ann. Math. 37(4) (1936), 823–843.

# $L$-algebras

A set $X$ with a binary operation $X \times X \to X$, $(x, y) \mapsto x \cdot y$, is an $L$-algebra if there exists an element $e \in X$ such that

$$e \cdot x = x \quad \text{and} \quad x \cdot e = x \cdot x = e \quad \text{for all } x \in X, \tag{1}$$

$$x \cdot y = y \cdot x = e \implies x = y, \tag{2}$$

and the cycloid equation

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z) \tag{3}$$

holds for all $x, y, z \in X$.

The element $e \in X$ is the logical unit.

Let $X$ be an $L$-algebra. Then

$$x \leq y \Longleftrightarrow x \cdot y = e$$

defines a partial order on $X$ with greatest element $e$.

If you like algebraic logic, maybe you should write the binary operation $\cdot$ with an arrow (e.g. $\to$) for "implication". The logical unit is the "truth".

Moreover, $x \leq y$ means that $x$ entails $y$. (This means strong implication: $x$ is true, so $y$ is also true.)

# Example

For a cycle set $X$ and a formal symbol $e$, let $L_X = X \cup \{e\}$. The binary operation

$$L_X \times L_X \to L_X, \qquad (x,y) \mapsto \begin{cases} e & \text{if } x = y \text{ or } y = e, \\ y & \text{if } x = e, \\ x \cdot y & \text{if } x \neq y, \end{cases}$$

turns $L_X$ into a discrete $L$-algebra (i.e. $x < y \implies y = e$).

An $L$-algebra $X$ is self-similar if for each $x, y \in X$ there exists an element $z = z(x, y) \in X$ such that $z \leq y$ and $y \cdot z = x$.

**Notation:** $z = xy$.

**Facts:**

1. $xy$ is uniquely determined by $xy \leq y$ and $y \cdot (xy) = x$.
2. The operation $X \times X \to X$, $(x, y) \mapsto xy$, is well-defined, associative and

$$xe = ex = x, \quad (xy) \cdot z = x \cdot (y \cdot z)$$

hold for all $x, y, z \in X$.

**Theorem (Rump)**

Let $X$ be an $L$-algebra $X$. Then there exists a unique (up to isomorphism) self-similar $L$-algebra $S(X)$ generated (as a monoid) by $X$ and there is an embedding $X \hookrightarrow S(X)$ of $L$-algebras.

So $X$ embedds into a "nicer" $L$-algebra $S(X)$.

Since $S(X)$ is left Ore, it admits a left quotient group $G(X)$, known as the structure group of $X$. There there exists a canonical map

$$X \hookrightarrow S(X) \to G(X).$$

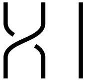
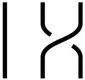
**Theorem (Rump)**

*Let $X$ be an $L$-algebra. Then $G(X)$ is torsion-free.*

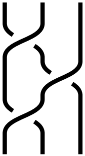
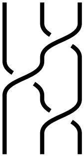
# Example

Recall that the braid group $\mathbb{B}_3$ in three strands is the group with generators $r$ and $s$ and the relation relation $rsr = srs$.

Generators:

$r =$  $s =$ 

The defining relation $rsr = srs$ is the Yang–Baxter equation:

 $=$ 

# Example

Let $X = \{e, x, y, xy, yx\}$ with the $L$-algebra structure given by

$$x \cdot y = xy, \quad y \cdot x = yx.$$

Then $G(X) \simeq \mathbb{B}_3$, the braid group in three strands. In particular, $\mathbb{B}_3$ is torsion-free.

**Fact:**
The braid group $\mathbb{B}_n$ is the structure group of an $L$-algebra.

One can use the connection between the YBE and $L$-algebras to construct finite $L$-algebras of small size.

Let $X = \{1, \dots, n\}$. The element $n$ will be the logical unit. An $L$-algebra structure on $X$ is a matrix $(M_{ij})_{1 \leq i,j \leq n} \in \mathbb{Z}^{n \times n}$ satisfying the following conditions:

1. $M_{n,j} = j$ for all $j \in \{1, \dots, n\}$.
2. $M_{i,n} = n$ for all $i \in \{1, \dots, n\}$.
3. $M_{k,k} = n$ for all $k \in \{1, \dots, n\}$.
4. $M_{Mi,j,Mi,k} = M_{M_{j,i},M_{j,k}}$ for all $i, j, k \in \{1, \dots, n\}$.
5. $M_{i,j} = n = M_{j,i} \implies i = j$.

There is a correspondence between finite $L$-algebras and matrices satisfying (1)–(5):

$$X \rightsquigarrow M_X,$$

where $(M_X)_{ij} = i \cdot j$.

Over the set of $n \times n$ matrices satisfying conditions (1)–(5) we consider the following equivalence relation:

$$M \sim N \Longleftrightarrow \exists g \in \mathrm{Sym}_{n-1} : N_{i,j} = g^{-1}(M_{g(i),g(j)}) \,\forall i, j.$$

Then

$$X \simeq Y \Longleftrightarrow M_X \sim M_Y.$$

## Example

Let $X = \{x, y, e\}$ with

$$e \cdot y = y, \quad x \cdot y = y \cdot x = e \cdot x = x.$$

Then $X$ is an $L$-algebra.

Let us compute $M_X$. For this, we need to change the labelling of the elements of $X$:

$$f\colon \{1, 2, 3\} \to \{x, y, e\}, \quad f(1) = x, \quad f(2) = y, \quad f(3) = e.$$

Then

$$M_X = \begin{pmatrix} 3 & 1 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

The number of $L$-algebras (up to isomorphism).

| size | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| | 5 | 44 | 632 | 15582 | 907806 | 377322225 |

The $L$-algebras were constructed with Dietzel and Menchón. The calculations use constraint programming techniques. The enumeration for size eight requires other ideas, like the underlying poset structure of the $L$-algebras.

What can you do with the database?

An $L$-algebra is then said to be linear if the partial order

$$x \leq y \iff x \cdot y = e$$

is a total order.

**Theorem (with Dietzel and Menchón)**

There are $B(n-1)$ isomorphism classes of linear $L$-algebras of size $n$, where $B(n)$ denotes the $n$-th Bell number.

The first Bell numbers are 1,1,2,5,15,52,203,877,4140... This is the sequence A000110 in the OEIS.

Bell numbers count the number of partitions of sets. For example, the set $\{a, b, c\}$ admits five partitions:

$$\begin{aligned} &\{\{a, b, c\}\}, \\ &\{\{a, b\}, \{c\}\}, \\ &\{\{b, c\}, \{a\}\}, \\ &\{\{a, c\}, \{b\}\}, \\ &\{\{a\}, \{b\}, \{c\}\}. \end{aligned}$$

Thus $B(3) = 5$.

## Problem

Let $n$ be a positive integer. Find an explicit bijection between the $L$-algebras on the ordered set

$$\{1 < 2 < \cdots < n\},$$

where $n$ is the logical unit, and partitions of the set $\{1, \ldots, n-1\}$.

An $L$-algebra $X$ is of type (F) if it satisfies

$$x \cdot y = x \cdot (x \cdot y) \quad \text{and} \quad x \cdot y = y \Longleftrightarrow y \cdot x = x$$

for all $x, y \in X$; this class of (symmetric) $L$-algebras appears in the literature.

**Conjecture**

The number of $L$-algebras of type (F) and size $n$ is $F_n$, the $n$-th Fibonacci number.

# Hilbert algebras

An important family of $L$-algebras is that of Hilbert algebras. This is an $L$-algebra $X$ such that

$$x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$$

for all $x, y, z \in X$.

The number of Hilbert algebras (up to isomorphism).

| size | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| | 2 | 6 | 21 | 95 | 550 | 4036 | 37602 | 1043328 |

A geometric theory of $L$-algebras?

An ideal is an $L$-algebra $X$ is a subset $I$ of $X$ such that the following conditions hold:

1. $e \in X$.
2. $x \in I$ and $x \cdot y \in I \implies y \in I$.
3. $x \in I \implies (x \cdot y) \cdot y \in I$.
4. $x \in I \implies y \cdot x \in I$ and $y \cdot (x \cdot y) \in I$.

**Examples:**
$\{e\}$ and $X$ are ideals. The intersection of ideals is an ideal.

**Theorem (Rump)**

Let $X$ be an $L$-algebra. There exists a bijective correspondence between ideals of $X$ and congruences $\sim$ on $X$ for which the quotient $X/\sim$ is an $L$-algebra.

The correspondence is given as follows $x \sim y \iff x \cdot y \in I$ and $y \cdot x \in I$. Conversely, if $\sim$ is a congruence, then $I = \{x \in X : x \sim e\}$ is an ideal of $X$.

As usual, a congruence $\sim$ on $X$ is an equivalence relation on $X$ compatible with the binary operation, i.e.

$$x \sim x_1 \text{ and } y \sim y_1 \implies x \cdot y \sim x_1 \cdot y_1.$$

An $L$-algebra $X$ is said to be distributive if

$$I \cap (J \vee K) = (I \cap J) \vee (I \cap K)$$

for all ideals $I$, $J$ and $K$, where $A \vee B$ denotes the ideal of $X$ generated by $A \cup B$.

**Example:** Hilbert algebras are distributive.

### Theorem (with Rump)

Finite $L$-algebras are distributive.

What now?

The ideals of an $L$-algebra $X$ can be identified with the open sets of a topological space $\mathrm{Spec}X$, the spectrum of $X$.

**General problem**

Study the spectrum of $L$-algebras.

Some questions:

1. Determine the spectrum in particular classes (e.g. linear).
2. What about simple $L$-algebras?