

Semi-abelian categories, cocommutative Hopf algebras and Hopf braces

Marino Gran



20 January 2025

*“The Interplay Between Skew Braces and Hopf-Galois Theory”
Vrije Universiteit Brussel*

Outline

Semi-abelian categories

Cocommutative Hopf algebras

Hopf braces

A torsion theory in \mathbf{HBR}_{coc}

Commutators in \mathbf{HBR}_{coc}

Outline

Semi-abelian categories

Cocommutative Hopf algebras

Hopf braces

A torsion theory in \mathbf{HBR}_{coc}

Commutators in \mathbf{HBR}_{coc}

Abelian categories

The notion of **abelian category** (Buchsbaum, 1955) captures some crucial properties **Ab** and **R-Mod** have in common.

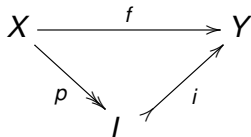
Abelian categories

The notion of **abelian category** (Buchsbaum, 1955) captures some crucial properties **Ab** and **R-Mod** have in common.

Definition

A category \mathbb{C} is **abelian** if

- ▶ \mathbb{C} has a zero-object 0
- ▶ \mathbb{C} has binary products $A \times B$
- ▶ any arrow f in \mathbb{C} has a factorization $f = i \circ p$



where p is a **normal epimorphism**, i is a **normal monomorphism**.

Normal epimorphism

When \mathbb{C} is pointed, an arrow $A \xrightarrow{p} P$ is a **normal epimorphism** if it is the **cokernel** of some arrow in \mathbb{C} : there is an i such that

$$\begin{array}{ccc} I & \xrightarrow{i} & A \\ \downarrow & & \downarrow p \\ 0 & \longrightarrow & P \end{array}$$

is a **pushout**.

Normal epimorphism

When \mathbb{C} is pointed, an arrow $A \xrightarrow{p} P$ is a **normal epimorphism** if it is the **cokernel** of some arrow in \mathbb{C} : there is an i such that

$$\begin{array}{ccc} I & \xrightarrow{i} & A \\ \downarrow & & \downarrow p \\ 0 & \longrightarrow & P \end{array}$$

is a **pushout**.

In the categories **Ab** of abelian groups and **R-Mod** of R-modules

- **normal epimorphism** = **surjective homomorphism**.

Normal monomorphism

An arrow $K \xrightarrow{k} A$ is called a **normal monomorphism** if it is the **kernel** of some arrow in \mathbb{C} . This means that there is an $f: A \rightarrow B$ such that

$$\begin{array}{ccc} K & \xrightarrow{k} & A \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & B \end{array}$$

is a **pullback**.

Normal monomorphism

An arrow $K \xrightarrow{k} A$ is called a **normal monomorphism** if it is the **kernel** of some arrow in \mathbb{C} . This means that there is an $f: A \rightarrow B$ such that

$$\begin{array}{ccc} K & \xrightarrow{k} & A \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & B \end{array}$$

is a **pullback**.

Groups

In the category **Grp** : **normal monomorphism** = **normal subgroup inclusion**

Normal monomorphism

An arrow $K \xrightarrow{k} A$ is called a **normal monomorphism** if it is the **kernel** of some arrow in \mathbb{C} . This means that there is an $f: A \rightarrow B$ such that

$$\begin{array}{ccc} K & \xrightarrow{k} & A \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & B \end{array}$$

is a **pullback**.

Groups

In the category \mathbf{Grp} : **normal monomorphism** = **normal subgroup inclusion**

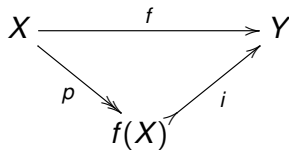
Abelian groups

In the category \mathbf{Ab} : **any monomorphism** $k: K \rightarrow A$ is **normal** !

The category **Ab** of abelian groups is **abelian** :

The category **Ab** of abelian groups is **abelian** :

- ▶ **Ab** has a **0**
- ▶ the product **$A \times B$** exists for any A, B
- ▶ any homomorphism f in **Ab** has a factorization $f = i \circ p$



where p is a **normal monomorphism** and i a **normal monomorphism**.

Examples

\mathbf{Ab} , $\mathbf{R}\text{-Mod}$, $\mathbf{Ab}(\mathbf{Comp})$ are all abelian categories.

Examples

\mathbf{Ab} , $\mathbf{R}\text{-Mod}$, $\mathbf{Ab(Comp)}$ are all abelian categories.

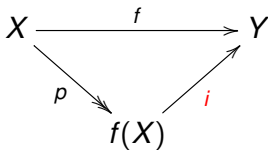
What about the category \mathbf{Grp} of groups?

Grp is not abelian :

- ▶ Grp has a 0-object : the trivial group $\{1\}$
- ▶ the direct product $A \times B$ exists

Grp is **not** abelian :

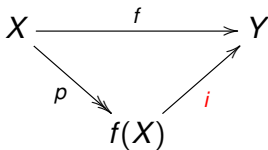
- ▶ **Grp** has a 0-object : the trivial group $\{1\}$
- ▶ the direct product $A \times B$ exists
- ▶ **Problem** : an arrow f in **Grp** does **not** have a factorization $f = i \circ p$



with p a *normal epimorphism* and i is a normal monomorphism.

The category **Rng** of rings is **not** abelian :

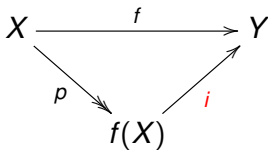
- ▶ an arrow f in **Rng** does **not** have a factorization $f = i \circ p$



with p a normal epimorphism and i a normal monomorphism.

The category **Rng** of rings is **not** abelian :

- ▶ an arrow f in **Rng** does **not** have a factorization $f = i \circ p$



with p a normal epimorphism and i a normal monomorphism.

This is due to the fact that **normal monomorphisms** in **Rng** correspond to inclusions of **ideals**.

Question : is there a list of simple axioms to conceptually understand some typical properties the categories \mathbf{Grp} , \mathbf{Rng} , \mathbf{Lie}_K have in common ?

Question : is there a list of simple axioms to conceptually understand some typical properties the categories **Grp**, **Rng**, **Lie_K** have in common ?

S. Mac Lane, Duality for groups, Bull. Amer. Math. Soc. (1950)



Roughly speaking, the problem is to find the “fourth proportional” in

$$\text{Ab} : \text{abelian category} = \text{Grp} : ?$$

Roughly speaking, the problem is to find the “fourth proportional” in

$$\text{Ab} : \text{abelian category} \quad = \quad \text{Grp} : ?$$

Aim : find an axiomatic context for

- ▶ isomorphism theorems
- ▶ non-abelian homological algebra
- ▶ radical and torsion theories
- ▶ commutator theory

Historical remarks

- ▶ S. Mac Lane, Duality for groups, Bull. Amer. Math. Soc. (1950)

Historical remarks

- ▶ S. Mac Lane, Duality for groups, Bull. Amer. Math. Soc. (1950)
- ▶ D. Buchsbaum, Exact categories and duality, Trans. A.M.S. (1955)
- ▶ A. Grothendieck, Sur quelques points d'algèbre homologique, Tohoku Math. J. (1957)

Historical remarks

- ▶ S. Mac Lane, Duality for groups, Bull. Amer. Math. Soc. (1950)
- ▶ D. Buchsbaum, Exact categories and duality, Trans. A.M.S. (1955)
- ▶ A. Grothendieck, Sur quelques points d'algèbre homologique, Tohoku Math. J. (1957)
- ▶ Several proposals of “non-abelian contexts” :
S. A. Amitsur (1954), A.G. Kurosh (1959),
P. Higgins (1956), A. Frölich (1961), S.A. Huq (1968), M. Gerstenhaber (1970),
O. Wyler (1971), G. Orzech (1972), etc.

A solution of Mac Lane's problem :

Definition (G. Janelidze, L. Márki, W.Tholen, JPAA, 2002)

A finitely complete category \mathbb{C} is **semi-abelian** if

A solution of Mac Lane's problem :

Definition (G. Janelidze, L. Márki, W.Tholen, JPAA, 2002)

A finitely complete category \mathbb{C} is **semi-abelian** if

- ▶ \mathbb{C} has a 0 object

A solution of Mac Lane's problem :

Definition (G. Janelidze, L. Márki, W.Tholen, JPAA, 2002)

A finitely complete category \mathbb{C} is **semi-abelian** if

- ▶ \mathbb{C} has a 0 object
- ▶ \mathbb{C} has binary coproducts $A + B$

A solution of Mac Lane's problem :

Definition (G. Janelidze, L. Márki, W.Tholen, JPAA, 2002)

A finitely complete category \mathbb{C} is **semi-abelian** if

- ▶ \mathbb{C} has a 0 object
- ▶ \mathbb{C} has binary coproducts $A + B$
- ▶ \mathbb{C} is (Barr)-**exact** : a) any morphism factors as a regular epimorphism followed by a monomorphism ; b) regular epimorphisms are pullback stable, c) any equivalence relation is a kernel pair.

A solution of Mac Lane's problem :

Definition (G. Janelidze, L. Márki, W.Tholen, JPAA, 2002)

A finitely complete category \mathbb{C} is **semi-abelian** if

- ▶ \mathbb{C} has a 0 object
- ▶ \mathbb{C} has binary coproducts $A + B$
- ▶ \mathbb{C} is (Barr)-**exact** : a) any morphism factors as a regular epimorphism followed by a monomorphism ; b) regular epimorphisms are pullback stable, c) any equivalence relation is a kernel pair.
- ▶ \mathbb{C} is (Bourn)-**protomodular** : the Split Short Five Lemma holds in \mathbb{C}

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \xrightarrow{k} & A & \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{f} \end{array} & B \\ & & \downarrow u & & \downarrow v & & \downarrow w \\ 0 & \longrightarrow & K' & \xrightarrow{k'} & A' & \begin{array}{c} \xleftarrow{s'} \\ \xrightarrow{f'} \end{array} & B' \end{array}$$

u, w isomorphisms $\Rightarrow v$ isomorphism.

The axioms of semi-abelian category are not self-dual.

The axioms of semi-abelian category are not self-dual.

Examples

Grp, Rng, Lie_K (more generally, any variety of Ω -groups)

The axioms of semi-abelian category are not self-dual.

Examples

Grp, Rng, Lie_K (more generally, any variety of Ω -groups)

Grp(Comp), C*-Alg, etc.

The axioms of semi-abelian category are not self-dual.

Examples

Grp , Rng , Lie_K (more generally, any variety of Ω -groups)

$\text{Grp}(\text{Comp})$, $\text{C}^*\text{-Alg}$, etc.

Any **abelian** category ! In particular : Ab , R-Mod , $\text{Ab}(\text{Comp})$, etc.

The axioms of semi-abelian category are not self-dual.

Examples

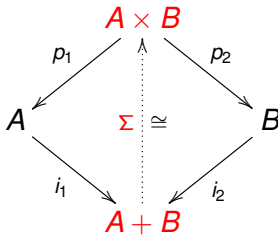
Grp , Rng , Lie_K (more generally, any variety of Ω -groups)

$\text{Grp}(\text{Comp})$, $C^*\text{-Alg}$, etc.

Any **abelian** category ! In particular : Ab , R-Mod , $\text{Ab}(\text{Comp})$, etc.

$[\mathbb{C} \text{ is abelian}] \Leftrightarrow [\mathbb{C} \text{ and } \mathbb{C}^{op} \text{ are semi-abelian}]!$

In an **abelian** category the canonical morphism Σ from $A + B$ to $A \times B$ is an **isomorphism** :



In an **abelian** category the canonical morphism Σ from $A + B$ to $A \times B$ is an **isomorphism** :

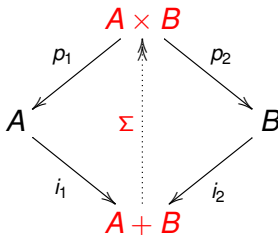
$$\begin{array}{ccccc}
 & & A \times B & & \\
 & p_1 \swarrow & \uparrow & \searrow p_2 & \\
 A & & \Sigma \cong & & B \\
 & i_1 \searrow & \downarrow & \swarrow i_2 & \\
 & & A + B & &
 \end{array}$$

One then has the notion of **biproduct**

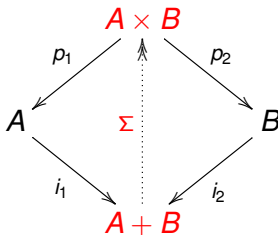
$$\begin{array}{ccccc}
 A & \xrightarrow{i_1} & A \oplus B & \xrightarrow{p_2} & B, \\
 & \xleftarrow{p_1} & & \xleftarrow{i_2} &
 \end{array}$$

and any $\text{Hom}_{\mathbb{C}}(C, D)$ has an abelian group structure.

In a **semi-abelian** category the canonical morphism Σ from $A + B$ to $A \times B$ is only a **normal epimorphism** (not an isomorphism)



In a **semi-abelian** category the canonical morphism Σ from $A + B$ to $A \times B$ is only a **normal epimorphism** (not an isomorphism)



A **semi-abelian** category is **not additive** !

The “general idea” :

Whereas

abelian = exact + additive,

The “general idea” :

Whereas

$$\text{abelian} = \text{exact} + \textit{additive},$$

the “non-additive” version of this “equation” is

$$\text{semi-abelian} = \text{exact} + 0 + \textit{protomodular}.$$

One replaces the “additivity” with the validity of the “Split Short Five Lemma”.

Outline

Semi-abelian categories

Cocommutative Hopf algebras

Hopf braces

A torsion theory in \mathbf{HBR}_{coc}

Commutators in \mathbf{HBR}_{coc}

Let K be a field.

Bialgebras

A **K -bialgebra** $(A, m, u, \Delta, \epsilon)$ is both a K -algebra (A, m, u) and a K -coalgebra (A, Δ, ϵ) , where m, u, Δ, ϵ are linear maps such that the following diagrams commute, and m and u are K -coalgebra morphisms.

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{1_A \otimes m} & A \otimes A \\ m \otimes 1_A \downarrow & & \downarrow m \\ A \otimes A & \xrightarrow{m} & A \end{array}$$

$$\begin{array}{ccccc} A \otimes K & \xrightarrow{1_A \otimes u} & A \otimes A & \xleftarrow{u \otimes 1_A} & K \otimes A \\ & \searrow r_A & \downarrow m & \swarrow l_A & \\ & & A & & \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \Delta \downarrow & & \downarrow \Delta \otimes 1_A \\ A \otimes A & \xrightarrow{1_A \otimes \Delta} & A \otimes A \otimes A \end{array}$$

$$\begin{array}{ccccc} A \otimes K & \xleftarrow{1_A \otimes \epsilon} & A \otimes A & \xrightarrow{\epsilon \otimes 1_A} & K \otimes A \\ & \swarrow r_A^{-1} & \uparrow \Delta & \searrow l_A^{-1} & \\ & & A & & \end{array}$$

Cocommutative Hopf algebras

A **Hopf K -algebra** $(A, m, u, \Delta, \epsilon, S)$ is a K -bialgebra with a linear map $S: A \rightarrow A$, the **antipode**, making the following diagram commute :

$$\begin{array}{ccccc} & & A \otimes A & \xrightleftharpoons[S \otimes 1_A]{1_A \otimes S} & A \otimes A \\ & \nearrow \Delta & & & \searrow m \\ A & & & & A \\ & \searrow \epsilon_A & & & \nearrow u_A \\ & & K & & \end{array}$$

A Hopf algebra $(A, m, u, \Delta, \epsilon, S)$ is **cocommutative** if the following triangle commutes :

$$\begin{array}{ccc} & A & \\ \Delta \swarrow & & \searrow \Delta \\ A \otimes A & \xrightarrow[\text{tw}]{\cong} & A \otimes A \end{array}$$

where $tw: A \otimes A \rightarrow A \otimes A$ is the “flip map”.

A Hopf algebra $(A, m, u, \Delta, \epsilon, S)$ is **cocommutative** if the following triangle commutes :

$$\begin{array}{ccc} & A & \\ \Delta \swarrow & & \searrow \Delta \\ A \otimes A & \xrightarrow[\text{tw}]{\cong} & A \otimes A \end{array}$$

where $tw: A \otimes A \rightarrow A \otimes A$ is the “flip map”.

A **morphism** $f: (A, m, u, \Delta, \epsilon, S) \rightarrow (B, m, u, \Delta, \epsilon, S)$ of Hopf algebras is a linear map $f: A \rightarrow B$ that is both an algebra and a coalgebra map.

Example

Any group G gives the **group-algebra**

$$K[G] = \left\{ \sum_g \alpha_g g \mid g \in G, \alpha_g \in K \right\},$$

which becomes a **cocommutative Hopf algebra** with

$$\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1}.$$

The category GrpHopf_K of group Hopf algebras (= generated by “grouplike elements”) is full in the category $\text{Hopf}_{K,\text{coc}}$ of cocommutative Hopf algebras :

$$\text{GrpHopf}_K \subset \text{Hopf}_{K,\text{coc}}.$$

The category GrpHopf_K of group Hopf algebras (= generated by “grouplike elements”) is full in the category $\text{Hopf}_{K,\text{coc}}$ of cocommutative Hopf algebras :

$$\text{GrpHopf}_K \subset \text{Hopf}_{K,\text{coc}}.$$

$\text{Hopf}_{K,\text{coc}}$ is finitely complete, with the product of two Hopf algebras A and B being given by the tensor product

$$A \xleftarrow{p_1} A \otimes B \xrightarrow{p_2} B,$$

where $p_1(a \otimes b) = a\epsilon(b)$ and $p_2 = \epsilon(a)b$, and the zero object is K .

Theorem (M.G., F. Sterck and J. Vercruysse, JPAA, 2019)

For any field K , the category $\text{Hopf}_{K,\text{coc}}$ is semi-abelian.

Theorem (M.G., F. Sterck and J. Vercruysse, JPAA, 2019)

For any field K , the category $\text{Hopf}_{K,\text{coc}}$ is semi-abelian.

Main steps of the proof :

To prove that $\text{Hopf}_{K,\text{coc}}$ is a *regular category* one first shows the existence of the normal epi/mono factorization.

Theorem (M.G., F. Sterck and J. Vercruysse, JPAA, 2019)

For any field K , the category $\mathbf{Hopf}_{K, \text{coc}}$ is semi-abelian.

Main steps of the proof :

To prove that $\mathbf{Hopf}_{K, \text{coc}}$ is a *regular category* one first shows the existence of the normal epi/mono factorization.

A key fact to prove this is the “*Newman correspondence*”:

Theorem (M.G., F. Sterck and J. Vercruysse, JPAA, 2019)

For any field K , the category $\mathbf{Hopf}_{K,\text{coc}}$ is **semi-abelian**.

Main steps of the proof :

To prove that $\mathbf{Hopf}_{K,\text{coc}}$ is a *regular category* one first shows the existence of the **normal epi/mono factorization**.

A key fact to prove this is the “*Newman correspondence*”:

Theorem (K. Newman, J. Algebra, 1975)

For $A \in \mathbf{Hopf}_{K,\text{coc}}$ there is a bijective correspondence between :

$$\{ \text{Hopf subalgebras of } A \} \cong \{ \text{left ideal two-sided coideals of } A \}$$

Given a Hopf subalgebra D of A , the **bijection**

$$\phi_A: \{ \text{Hopf subalgebras of } A \} \rightarrow \{ \text{left ideal two-sided coideals of } A \}$$

sends D to the left ideal two-sided coideal

$$\phi_A(D) = AD^+,$$

where $D^+ = \{x \in D \mid \epsilon(x) = 0\}$.

The **kernel** of a morphism $f: A \rightarrow B$ in $\mathbf{Hopf}_{K, coc}$ is given by the inclusion $\mathbf{HKer}(f) \rightarrow A$ of the following Hopf subalgebra of A :

$$\mathbf{HKer}(f) = \{a \in A \mid f(a_1) \otimes a_2 = 1_B \otimes a\} \subset A.$$

The **kernel** of a morphism $f: A \rightarrow B$ in $\mathbf{Hopf}_{K, \text{coc}}$ is given by the inclusion $\text{HKer}(f) \rightarrow A$ of the following Hopf subalgebra of A :

$$\text{HKer}(f) = \{a \in A \mid f(a_1) \otimes a_2 = 1_B \otimes a\} \subset A.$$

The vector space kernel $\ker(f)$ of the linear map f turns out to be a **Hopf ideal** :

$$\ker(f) = \phi_A(\text{HKer}(f)) =: A(\text{HKer}(f)^+) = A(\text{HKer}(f)^+)A.$$

The **kernel** of a morphism $f: A \rightarrow B$ in $\mathbf{Hopf}_{K, \text{coc}}$ is given by the inclusion $\text{HKer}(f) \rightarrow A$ of the following Hopf subalgebra of A :

$$\text{HKer}(f) = \{a \in A \mid f(a_1) \otimes a_2 = 1_B \otimes a\} \subset A.$$

The vector space kernel $\ker(f)$ of the linear map f turns out to be a **Hopf ideal**:

$$\ker(f) = \phi_A(\text{HKer}(f)) =: A(\text{HKer}(f)^+) = A(\text{HKer}(f)^+)A.$$

This implies that

$$f(A) = \{f(a) \mid a \in A\} \cong \frac{A}{\ker(f)} = \frac{A}{A(\text{HKer}(f)^+)A} \in \mathbf{Hopf}_{K, \text{coc}}.$$

The canonical factorization in \mathbf{Vect}_K of a morphism $f: A \rightarrow B$ in $\mathbf{Hopf}_{K, \text{coc}}$

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \searrow \pi & & \nearrow i \\
 f(A) \cong \frac{A}{\ker(f)} & &
 \end{array}$$

is then also a factorization in the category $\mathbf{Hopf}_{K, \text{coc}}$.

The canonical factorization in \mathbf{Vect}_K of a morphism $f: A \rightarrow B$ in $\mathbf{Hopf}_{K, \text{coc}}$

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \searrow \pi & & \nearrow i \\
 f(A) \cong \frac{A}{\ker(f)} & &
 \end{array}$$

is then also a factorization in the category $\mathbf{Hopf}_{K, \text{coc}}$.

Consequently, any morphism f has a canonical factorization $f = i\pi$ in $\mathbf{Hopf}_{K, \text{coc}}$ as a **normal epi** π followed by a **monomorphism** i .

One then verifies that normal epimorphisms (=surjective homomorphisms) are pullback stable along monomorphisms.

One then verifies that normal epimorphisms (=surjective homomorphisms) are pullback stable along monomorphisms.

This, together with the fact that normal epimorphisms are stable under binary products (=tensor products), implies the **regularity** of $\mathbf{Hopf}_{K, coc}$.

One then verifies that normal epimorphisms (=surjective homomorphisms) are pullback stable along monomorphisms.

This, together with the fact that normal epimorphisms are stable under binary products (=tensor products), implies the **regularity** of $\mathbf{Hopf}_{K, coc}$.

The **protomodularity** of $\mathbf{Hopf}_{K, coc}$ directly follows from the fact that

$$\mathbf{Hopf}_{K, coc} = \mathbf{Grp}(\mathbf{CoAlg}_{k, coc}).$$

Finally, to check the (Barr)-**exactness** of $\mathbf{Hopf}_{K, \text{coc}}$, one observes that in any commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{n} & A \\ \bar{p} \downarrow & & \downarrow p \\ p(N) & \xrightarrow{m} & B \end{array}$$

where n is a **normal mono**, p and \bar{p} are regular epimorphisms and m is a **mono**, then m is a **normal mono**. □

In particular, the previous result implies

Theorem (M. Takeuchi, Manuscr. Math., 1972)

The category $\text{Hopf}_{K, \text{coc}}^{\text{comm}}$ of commutative and cocommutative Hopf algebras is abelian.

In particular, the previous result implies

Theorem (M. Takeuchi, Manuscr. Math., 1972)

The category $\text{Hopf}_{K,\text{coc}}^{\text{comm}}$ of commutative and cocommutative Hopf algebras is abelian.

Proof :

$$\text{Hopf}_{K,\text{coc}}^{\text{comm}} = \text{Ab}(\text{Hopf}_{K,\text{coc}}).$$

In particular, the previous result implies

Theorem (M. Takeuchi, Manuscr. Math., 1972)

The category $\mathbf{Hopf}_{K,\text{coc}}^{\text{comm}}$ of commutative and cocommutative Hopf algebras is **abelian**.

Proof :

$$\mathbf{Hopf}_{K,\text{coc}}^{\text{comm}} = \mathbf{Ab}(\mathbf{Hopf}_{K,\text{coc}}).$$

$A \in \mathbf{Hopf}_{K,\text{coc}}$ is **abelian** $\Leftrightarrow \Delta: A \rightarrow A \otimes A$ is a **normal mono**

\Leftrightarrow the product in A is **commutative** : $ab = ba$

$\Leftrightarrow A \in \mathbf{Hopf}_{K,\text{coc}}^{\text{comm}}$



Strong protomodularity

The category $\mathbf{Hopf}_{K, \text{coc}}$ is even **strongly protomodular** (in the sense of D. Bourn) :
in any diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \xrightarrow{k} & A & \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{f} \end{array} & B \\ & & \downarrow u & & \downarrow v & & \parallel 1_B \\ 0 & \longrightarrow & K' & \xrightarrow{k'} & A' & \begin{array}{c} \xleftarrow{s'} \\ \xrightarrow{f'} \end{array} & B \end{array}$$

where f and f' are split epimorphisms and u is a **normal mono**,
then $k'u$ is also a **normal mono**.

Commutators

Strong protomodularity is useful to develop commutator theory (the “Smith=Huq” condition holds).

Commutators

Strong protomodularity is useful to develop commutator theory (the “Smith=Huq” condition holds).

For normal Hopf subalgebras M, N of $A \in \mathbf{Hopf}_{K, coc}$

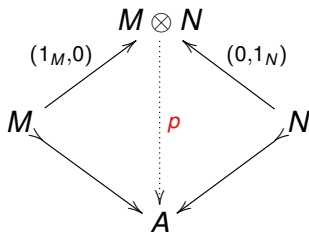
$$M \rightrightarrows A \leftrightsquigarrow N$$

one can “compute” their categorical commutator :

$$[M, N]_{Huq} = \langle \{m_1 n_1 S(m_2) S(n_2) \mid m \in M, n \in N\} \rangle_A$$

(where $\Delta(m) = m_1 \otimes m_2$ and $\Delta(n) = n_1 \otimes n_2$).

In $\text{Hopf}_{K, \text{coc}}$ the condition $[M, N]_{\text{Huq}} = 0$ is equivalent to the existence of a (unique) morphism $p: M \otimes N \rightarrow A$ making the diagram



commute, where $p(m \otimes n) = mn$, for any $m \otimes n \in M \otimes N$.

Outline

Semi-abelian categories

Cocommutative Hopf algebras

Hopf braces

A torsion theory in \mathbf{HBR}_{coc}

Commutators in \mathbf{HBR}_{coc}

Skew braces

In 2017 L. Guarnieri and L. Vendramin introduced the structure of **skew (left) brace**, that produce *bijective* solutions of the Yang-Baxter equation, generalizing the structure of **brace** due to W. Rump.

Skew braces

In 2017 L. Guarnieri and L. Vendramin introduced the structure of **skew (left) brace**, that produce *bijective* solutions of the Yang-Baxter equation, generalizing the structure of **brace** due to W. Rump.

A **skew brace** is a set A with two group structures, $(A, +)$ and (A, \circ) such that

$$a \circ (b + c) = a \circ b - a + a \circ c, \quad \forall a, b, c \in A.$$

Skew braces

In 2017 L. Guarnieri and L. Vendramin introduced the structure of **skew (left) brace**, that produce *bijective* solutions of the Yang-Baxter equation, generalizing the structure of **brace** due to W. Rump.

A **skew brace** is a set A with two group structures, $(A, +)$ and (A, \circ) such that

$$a \circ (b + c) = a \circ b - a + a \circ c, \quad \forall a, b, c \in A.$$

Skew braces produce the solutions of the Yang-Baxter equation :

$$r(a, b) = (-a + a \circ b, (-a + a \circ b)' \circ a \circ b).$$

Morphisms in the category **SKB** preserve the operations $+$ and \circ , and also the unique constant 0, which is the neutral element for both $+$ and \circ .

Morphisms in the category **SKB** preserve the operations $+$ and \circ , and also the unique constant 0, which is the neutral element for both $+$ and \circ .

SKB is a **semi-abelian variety** of algebras, containing the variety **Grp** of groups as a subvariety : any group $(G, +)$ can be seen as a skew brace $(G, +, +)$. Up to this identification of $(G, +)$ with $(G, +, +)$, one has

$$\text{Grp} \begin{array}{c} \xleftarrow{F} \\ \perp \\ \xrightarrow{U} \end{array} \text{SKB}$$

SKB also contains the variety of RadRng of radical rings as a subvariety :

$$\text{RadRng} \begin{array}{c} \xleftarrow{F} \\ \perp \\ \xrightarrow{U} \end{array} \text{SKB},$$

which is determined by the identities

$$(a + b) \circ c = a \circ c - c + b \circ c$$

and

$$a + b = b + a.$$

There is a natural “Hopf-theoretic generalization” of **SKB**, namely the category $\mathbf{HBR}_{\text{coc}}$ of **cocommutative Hopf braces**.

There is a natural “Hopf-theoretic generalization” of **SKB**, namely the category **HBR_{coc}** of **cocommutative Hopf braces**.

In other words, there is an analogy :

$$\text{Grp} : \text{Hopf}_{K, \text{coc}} = \text{SKB} : \text{HBR}_{\text{coc}}.$$

Definition (Angiono, Galindo, Vendramin, Proc. Amer. Math. Soc., 2017)

A **Hopf brace** is a datum $(H, \cdot, \bullet, 1, \Delta, \varepsilon, S, T)$ where $(H, \cdot, 1, \Delta, \varepsilon, S)$ and $(H, \bullet, 1, \Delta, \varepsilon, T)$ are Hopf algebras satisfying the compatibility condition

$$a \bullet (b \cdot c) = (a_1 \bullet b) \cdot S(a_2) \cdot (a_3 \bullet c), \quad \text{for all } a, b, c \in H. \quad (1)$$

We'll write (H, \cdot, \bullet) , for short.

Definition (Angiono, Galindo, Vendramin, Proc. Amer. Math. Soc., 2017)

A **Hopf brace** is a datum $(H, \cdot, \bullet, 1, \Delta, \varepsilon, S, T)$ where $(H, \cdot, 1, \Delta, \varepsilon, S)$ and $(H, \bullet, 1, \Delta, \varepsilon, T)$ are Hopf algebras satisfying the compatibility condition

$$a \bullet (b \cdot c) = (a_1 \bullet b) \cdot S(a_2) \cdot (a_3 \bullet c), \quad \text{for all } a, b, c \in H. \quad (1)$$

We'll write (H, \cdot, \bullet) , for short.

A **morphism of Hopf braces** $f : (H, \cdot, \bullet) \rightarrow (K, \cdot, \bullet)$ is a map that is a Hopf algebra morphism with respect to both the Hopf algebra structures :

$$f(a \cdot b) = f(a) \cdot f(b), \quad f(a \bullet b) = f(a) \bullet f(b), \quad f(1_H) = 1_K$$

and

$$\varepsilon(f(a)) = \varepsilon(a), \quad \Delta(f(a)) = f(a_1) \otimes f(a_2), \quad \forall a, b \in H.$$

The category $\mathbf{HBR}_{\text{coc}}$ of cocommutative Hopf braces is monoidal :

The category \mathbf{HBR}_{coc} of cocommutative Hopf braces is monoidal :

if $(H, \cdot, \bullet, 1, \Delta, \varepsilon, S, T)$, $(K, \cdot', \bullet', 1', \Delta', \varepsilon', S', T')$ are Hopf braces, we have that $H^\cdot \otimes K^{\cdot'}$ and $H^\bullet \otimes K^{\bullet'}$ are Hopf algebras on the vector space $H \otimes K$.

The category \mathbf{HBR}_{coc} of cocommutative Hopf braces is monoidal :

if $(H, \cdot, \bullet, 1, \Delta, \varepsilon, S, T)$, $(K, \cdot', \bullet', 1', \Delta', \varepsilon', S', T')$ are Hopf braces, we have that $H \cdot \otimes K \cdot'$ and $H \bullet \otimes K \bullet'$ are Hopf algebras on the vector space $H \otimes K$.

The multiplications are given by

$$m_{\cdot \otimes} := (m_{\cdot} \otimes m_{\cdot'})(\mathrm{Id}_H \otimes tw \otimes \mathrm{Id}_K), \quad m_{\bullet \otimes} := (m_{\bullet} \otimes m_{\bullet'})(\mathrm{Id}_H \otimes tw \otimes \mathrm{Id}_K),$$

the coalgebra structure by

$$\Delta_{\otimes} := (\mathrm{Id}_H \otimes tw \otimes \mathrm{Id}_K)(\Delta \otimes \Delta'), \quad \varepsilon_{\otimes} := \varepsilon \otimes \varepsilon',$$

and the antipodes by $S_{\otimes} := S \otimes S'$, $T_{\otimes} := T \otimes T'$.

Propositon (A. Agore, Internat. J. Math., 2019)

The category \mathbf{HBR}_{coc} is **finitely complete**, with products coinciding with tensor products, and the monoidal unit coinciding with the terminal object K .

Propositon (A. Agore, Internat. J. Math., 2019)

The category \mathbf{HBR}_{coc} is **finitely complete**, with products coinciding with tensor products, and the monoidal unit coinciding with the terminal object K .

Lemma

Given a sub-Hopf brace (B, \cdot, \bullet) of a Hopf brace (A, \cdot, \bullet) , the following are equivalent :

Propositon (A. Agore, Internat. J. Math., 2019)

The category \mathbf{HBR}_{coc} is **finitely complete**, with products coinciding with tensor products, and the monoidal unit coinciding with the terminal object K .

Lemma

Given a sub-Hopf brace (B, \cdot, \bullet) of a Hopf brace (A, \cdot, \bullet) , the following are equivalent :

- a) (B, \cdot, \bullet) is a **normal sub-Hopf brace** (= the kernel of some morphism)

Propositon (A. Agore, Internat. J. Math., 2019)

The category \mathbf{HBR}_{coc} is **finitely complete**, with products coinciding with tensor products, and the monoidal unit coinciding with the terminal object K .

Lemma

Given a sub-Hopf brace (B, \cdot, \bullet) of a Hopf brace (A, \cdot, \bullet) , the following are equivalent :

- a) (B, \cdot, \bullet) is a **normal sub-Hopf brace** (= the kernel of some morphism)
- b) $a_1 \cdot b \cdot S(a_2) \in B$, $a_1 \bullet b \bullet S(a_2) \in B$, $S(a_1) \cdot (a_2 \bullet b) \in B$, for any $a \in A$, $b \in B$

Propositon (A. Agore, Internat. J. Math., 2019)

The category \mathbf{HBR}_{coc} is **finitely complete**, with products coinciding with tensor products, and the monoidal unit coinciding with the terminal object K .

Lemma

Given a sub-Hopf brace (B, \cdot, \bullet) of a Hopf brace (A, \cdot, \bullet) , the following are equivalent :

- a) (B, \cdot, \bullet) is a **normal sub-Hopf brace** (= the kernel of some morphism)
- b) $a_1 \cdot b \cdot S(a_2) \in B$, $a_1 \bullet b \bullet S(a_2) \in B$, $S(a_1) \cdot (a_2 \bullet b) \in B$, for any $a \in A$, $b \in B$

In other words, B is a **normal Hopf subalgebra** of both (A, \cdot) and (A, \bullet) and, moreover, **B is stable for the action \rightharpoonup of A on B** , i.e.

$$a \rightharpoonup b \in B \quad \text{for all } a \in A, b \in B,$$

where

$$a \rightharpoonup b = S(a_1) \cdot (a_2 \bullet b).$$

Proposition (M.G. and A. Sciandra, 2024)

The category \mathbf{HBR}_{coc} is strongly protomodular.

Proposition (M.G. and A. Sciandra, 2024)

The category \mathbf{HBR}_{coc} is strongly protomodular.

The proof mainly uses :

- the description of normal subobjects in \mathbf{HBR}_{coc} ,

Proposition (M.G. and A. Sciandra, 2024)

The category \mathbf{HBR}_{coc} is strongly protomodular.

The proof mainly uses :

- the description of normal subobjects in \mathbf{HBR}_{coc} ,
- the fact that the two forgetful functors $U : \mathbf{HBR}_{coc} \rightarrow \mathbf{Hopf}_{K,coc}$ defined by

$$U(A, \cdot, \bullet) = (A, \cdot),$$

and $U^\bullet : \mathbf{HBR}_{coc} \rightarrow \mathbf{Hopf}_{K,coc}$ by

$$U^\bullet(A, \cdot, \bullet) = (A, \bullet)$$

preserve limits,

Proposition (M.G. and A. Sciandra, 2024)

The category \mathbf{HBR}_{coc} is strongly protomodular.

The proof mainly uses :

- the description of normal subobjects in \mathbf{HBR}_{coc} ,
- the fact that the two forgetful functors $U : \mathbf{HBR}_{coc} \rightarrow \mathbf{Hopf}_{K,coc}$ defined by

$$U(A, \cdot, \bullet) = (A, \cdot),$$

and $U^\bullet : \mathbf{HBR}_{coc} \rightarrow \mathbf{Hopf}_{K,coc}$ by

$$U^\bullet(A, \cdot, \bullet) = (A, \bullet)$$

preserve limits,

- the strong protomodularity of $\mathbf{Hopf}_{K,coc}$.



Hopf brace ideals

With A. Sciandra we define the notion of “ideal” in the context of Hopf braces to extend the results we know for cocommutative Hopf algebras.

Hopf brace ideals

With A. Sciandra we define the notion of “ideal” in the context of Hopf braces to extend the results we know for cocommutative Hopf algebras.

Definition

Let (H, \cdot, \bullet) be a Hopf brace.

A **Hopf brace ideal** is a vector subspace $I \subseteq H$ such that :

Hopf brace ideals

With A. Sciandra we define the notion of “ideal” in the context of Hopf braces to extend the results we know for cocommutative Hopf algebras.

Definition

Let (H, \cdot, \bullet) be a Hopf brace.

A **Hopf brace ideal** is a vector subspace $I \subseteq H$ such that :

- $H \cdot I \subseteq I, I \cdot H \subseteq I, \quad H \bullet I \subseteq I, I \bullet H \subseteq I,$
- $\Delta(I) \subseteq I \otimes H + H \otimes I, \quad \varepsilon(I) = 0,$
- $S(I) \subseteq I, T(I) \subseteq I.$

Hopf brace ideals

With A. Sciandra we define the notion of “ideal” in the context of Hopf braces to extend the results we know for cocommutative Hopf algebras.

Definition

Let (H, \cdot, \bullet) be a Hopf brace.

A **Hopf brace ideal** is a vector subspace $I \subseteq H$ such that :

- $H \cdot I \subseteq I, I \cdot H \subseteq I, \quad H \bullet I \subseteq I, I \bullet H \subseteq I,$
- $\Delta(I) \subseteq I \otimes H + H \otimes I, \quad \varepsilon(I) = 0,$
- $S(I) \subseteq I, T(I) \subseteq I.$

Lemma (M. G. and A. Sciandra, 2024)

A sub-Hopf brace B of a Hopf brace (A, \cdot, \bullet) is **normal** if and only if

$$A \cdot B^+ = A \bullet B^+,$$

and $A \cdot B^+$ is a **Hopf brace ideal**.

Semi-abelianness of HBR_{coc}

The notion of **Hopf brace ideal** allows one to establish a new “Newman-type” correspondence for Hopf braces :

Semi-abelianness of \mathbf{HBR}_{coc}

The notion of **Hopf brace ideal** allows one to establish a new “Newman-type” correspondence for Hopf braces :

$$\phi_A : \{ \text{normal sub – Hopf braces of } A \} \cong \{ \text{Hopf brace ideals of } A \}$$

Semi-abelianness of \mathbf{HBR}_{coc}

The notion of **Hopf brace ideal** allows one to establish a new “Newman-type” correspondence for Hopf braces :

$$\phi_A : \{ \text{normal sub – Hopf braces of } A \} \cong \{ \text{Hopf brace ideals of } A \}$$

Given any $f: (A, \cdot, \bullet) \rightarrow (B, \cdot, \bullet)$ in \mathbf{HBR}_{coc} , from the previous Lemma it follows that the quotient $\frac{A}{\ker(f)}$ of A by the kernel $\ker(f)$ of f (in \mathbf{Vect}_K) is a Hopf brace :

$$\frac{A}{\ker(f)} = \frac{A}{A \cdot (\mathbf{HKer}(f)^+)} = \frac{A}{A \bullet (\mathbf{HKer}(f)^+)} \in \mathbf{HBR}_{coc}$$

This implies that any morphism $f: (A, \cdot, \bullet) \rightarrow (B, \cdot, \bullet)$ admits a normal epi-mono factorization in \mathbf{HBR}_{coc} :

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow \pi & \nearrow i \\ & f(A) \cong \frac{A}{\ker(f)} & \end{array}$$

This implies that any morphism $f: (A, \cdot, \bullet) \rightarrow (B, \cdot, \bullet)$ admits a normal epi-mono factorization in \mathbf{HBR}_{coc} :

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \searrow \pi & & \nearrow i \\
 f(A) \cong \frac{A}{\ker(f)} & &
 \end{array}$$

Theorem (M.G. and A. Sciandra, 2024)

The category \mathbf{HBR}_{coc} is **semi-abelian**.

This implies that any morphism $f: (A, \cdot, \bullet) \rightarrow (B, \cdot, \bullet)$ admits a normal epi-mono factorization in \mathbf{HBR}_{coc} :

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \searrow \pi & & \nearrow i \\
 & f(A) \cong \frac{A}{\ker(f)} &
 \end{array}$$

Theorem (M.G. and A. Sciandra, 2024)

The category \mathbf{HBR}_{coc} is **semi-abelian**.

Idea of the proof :

Once the factorization above is established, the next step is to prove that both the functors $U: \mathbf{HBR} \rightarrow \mathbf{Hopf}_{K,coc}$ and $U^\bullet: \mathbf{HBR}_{coc} \rightarrow \mathbf{Hopf}_{K,coc}$ are *regular* and *conservative* (= isomorphism reflecting). The existence of binary coproducts in \mathbf{HBR}_{coc} was proved by Agore and Chirvasitu (preprint, in preparation).



The Noether Isomorphism Theorems, the classical homological lemmas (Snake Lemma, 3×3 -Lemma, etc.), the Zassenhaus Lemma, and other classical results from Group Theory hold in \mathbf{HBR}_{coc} ... simply because this category is **semi-abelian**!

Outline

Semi-abelian categories

Cocommutative Hopf algebras

Hopf braces

A torsion theory in \mathbf{HBR}_{coc}

Commutators in \mathbf{HBR}_{coc}

In this section we assume that \mathbb{K} is an *algebraically closed field of characteristic 0*.

In this section we assume that \mathbb{K} is an algebraically closed field of characteristic 0.

The category **SKB** turns out to be equivalent to the full (replete) subcategory of **HBR_{coc}**, whose objects are the cocommutative Hopf braces whose “underlying” Hopf algebras are “group Hopf algebras”.

In this section we assume that \mathbb{K} is an algebraically closed field of characteristic 0.

The category **SKB** turns out to be equivalent to the full (replete) subcategory of **HBR_{coc}**, whose objects are the cocommutative Hopf braces whose “underlying” Hopf algebras are “group Hopf algebras”.

The “compatibility condition” in the case of a group Hopf brace $(\mathbb{K}[G], \cdot, \bullet)$ becomes

$$g \bullet (h \cdot k) = (g \bullet h) \cdot g^{-\cdot} \cdot (g \bullet k),$$

and we'll write **SKB** also for the category of “group Hopf braces”.

Another interesting subcategory of \mathbf{HBR}_{coc} is the category \mathbf{PHBR}_{coc} of **primitive Hopf braces**.

Another interesting subcategory of \mathbf{HBR}_{coc} is the category \mathbf{PHBR}_{coc} of **primitive Hopf braces**.

The objects of \mathbf{PHBR}_{coc} are the cocommutative Hopf braces whose “underlying” Hopf algebras are **universal enveloping algebras**.

Another interesting subcategory of \mathbf{HBR}_{coc} is the category \mathbf{PHBR}_{coc} of **primitive Hopf braces**.

The objects of \mathbf{PHBR}_{coc} are the cocommutative Hopf braces whose “underlying” Hopf algebras are **universal enveloping algebras**.

These Hopf braces are generated by “primitive” elements, i.e. by the elements x such that $\Delta(x) = 1 \otimes x + x \otimes 1$.

Proposition (M.G. and A. Sciandra, 2024)

The pair $(\text{PHBR}, \text{SKB})$ is a hereditary torsion theory in HBR_{coc} .

Proposition (M.G. and A. Sciandra, 2024)

The pair $(\text{PHBR}, \text{SKB})$ is a hereditary torsion theory in HBR_{coc} .

Idea of the proof :

The first axiom of torsion theory follows from the fact that, for any Hopf brace (H, \cdot, \bullet) , there is a short exact sequence

$$0 \longrightarrow (\text{HKer}(\pi), \cdot, \bullet) \longrightarrow (H, \cdot, \bullet) \xrightarrow{\pi} (\mathbb{K}[G], \cdot, \bullet) \longrightarrow 0$$

where $(\mathbb{K}[G], \cdot, \bullet) \in \text{SKB}$ is a skew brace (G is the set of group-like elements of H), and $(\text{HKer}(\pi), \cdot, \bullet) \in \text{PHBR}_{\text{coc}}$ turns out to be a primitive Hopf brace.

Proposition (M.G. and A. Sciandra, 2024)

The pair $(\text{PHBR}, \text{SKB})$ is a hereditary torsion theory in HBR_{coc} .

Idea of the proof :

The first axiom of torsion theory follows from the fact that, for any Hopf brace (H, \cdot, \bullet) , there is a short exact sequence

$$0 \longrightarrow (\text{HKer}(\pi), \cdot, \bullet) \longrightarrow (H, \cdot, \bullet) \xrightarrow{\pi} (\mathbb{K}[G], \cdot, \bullet) \longrightarrow 0$$

where $(\mathbb{K}[G], \cdot, \bullet) \in \text{SKB}$ is a skew brace (G is the set of group-like elements of H), and $(\text{HKer}(\pi), \cdot, \bullet) \in \text{PHBR}_{\text{coc}}$ turns out to be a primitive Hopf brace.

The second axiom of torsion theory is that any morphism $f: (P, \cdot, \bullet) \rightarrow (G, \cdot, \bullet)$ from a “primitive” Hopf brace (P, \cdot, \bullet) to a “skew brace” (G, \cdot, \bullet) is the zero arrow. This follows from the fact that any morphism preserves primitive elements. \square

Proposition (M.G. and A. Sciandra, 2024)

The category **SKB** is a **localization** of **HBR_{coc}** : in the adjunction

$$\mathbf{SKB} \begin{array}{c} \xleftarrow{F} \\ \perp \\ \xrightarrow{U} \end{array} \mathbf{HBR}_{coc},$$

the left adjoint $F: \mathbf{HBR}_{coc} \rightarrow \mathbf{SKB}$ preserves **finite limits**.

Proposition (M.G. and A. Sciandra, 2024)

The category **SKB** is a **localization** of **HBR_{coc}** : in the adjunction

$$\mathbf{SKB} \begin{array}{c} \xleftarrow{F} \\ \perp \\ \xrightarrow{U} \end{array} \mathbf{HBR}_{coc},$$

the left adjoint $F: \mathbf{HBR}_{coc} \rightarrow \mathbf{SKB}$ preserves **finite limits**.

Idea of the proof :

- F preserves finite products because **SKB** is a Birkhoff subcategory of **HBR_{coc}**,

Proposition (M.G. and A. Sciandra, 2024)

The category **SKB** is a **localization** of **HBR_{coc}** : in the adjunction

$$\mathbf{SKB} \begin{array}{c} \xleftarrow{F} \\ \perp \\ \xrightarrow{U} \end{array} \mathbf{HBR}_{coc},$$

the left adjoint $F: \mathbf{HBR}_{coc} \rightarrow \mathbf{SKB}$ preserves **finite limits**.

Idea of the proof :

- F preserves finite products because **SKB** is a Birkhoff subcategory of **HBR_{coc}**,
- F preserves monomorphisms because the torsion theory (**PHBR**, **SKB**) is hereditary.

Proposition (M.G. and A. Sciandra, 2024)

The category **SKB** is a **localization** of **HBR_{coc}** : in the adjunction

$$\mathbf{SKB} \begin{array}{c} \xleftarrow{F} \\ \perp \\ \xrightarrow{U} \end{array} \mathbf{HBR}_{coc},$$

the left adjoint $F: \mathbf{HBR}_{coc} \rightarrow \mathbf{SKB}$ preserves **finite limits**.

Idea of the proof :

- F preserves finite products because **SKB** is a Birkhoff subcategory of **HBR_{coc}**,
- F preserves monomorphisms because the torsion theory (**PHBR**, **SKB**) is hereditary.
- One then checks that F also preserves equalizers. □

Remark

This result is analogous to the one in the article (M.G., G. Kadjo and J. Vercruysse, Bull. Belgian Math. Soc., 2016), where it is proved that the category **Grp** of groups is a localization of **Hopf**_{K,coc} :

$$\text{Grp} \begin{array}{c} \xleftarrow{F} \\ \perp \\ \xrightarrow{U} \end{array} \text{Hopf}_{K,\text{coc}}$$

Outline

Semi-abelian categories

Cocommutative Hopf algebras

Hopf braces

A torsion theory in \mathbf{HBR}_{coc}

Commutators in \mathbf{HBR}_{coc}

Strong protomodularity is again useful to develop commutator theory in \mathbf{HBR}_{coc} .

Strong protomodularity is again useful to develop commutator theory in \mathbf{HBR}_{coc} .

For normal sub Hopf braces X, Y of $(A, \cdot, \bullet) \in \mathbf{HBR}_{coc}$

$$X \rightrightarrows A \leftleftarrows Y$$

one can actually “compute” the categorical commutator.

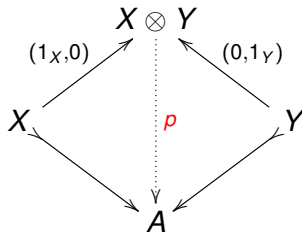
Lemma (M.G. and A. Sciandra, 2024)

Let (A, \cdot, \bullet) be a cocommutative Hopf brace and (X, \cdot, \bullet) , (Y, \cdot, \bullet) two sub-Hopf braces of (A, \cdot, \bullet) . The following conditions are equivalent :

Lemma (M.G. and A. Sciandra, 2024)

Let (A, \cdot, \bullet) be a cocommutative Hopf brace and $(X, \cdot, \bullet), (Y, \cdot, \bullet)$ two sub-Hopf braces of (A, \cdot, \bullet) . The following conditions are equivalent :

- there is a unique morphism $p : (X \otimes Y, \cdot_{\otimes}, \bullet_{\otimes}) \rightarrow (A, \cdot, \bullet)$ in \mathbf{HBR}_{coc} such that

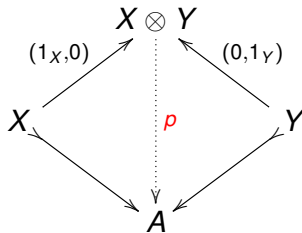


commutes ;

Lemma (M.G. and A. Sciandra, 2024)

Let (A, \cdot, \bullet) be a cocommutative Hopf brace and (X, \cdot, \bullet) , (Y, \cdot, \bullet) two sub-Hopf braces of (A, \cdot, \bullet) . The following conditions are equivalent :

- there is a unique morphism $p : (X \otimes Y, \cdot_{\otimes}, \bullet_{\otimes}) \rightarrow (A, \cdot, \bullet)$ in \mathbf{HBR}_{coc} such that



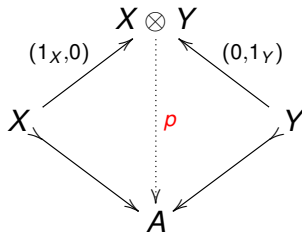
commutes ;

- $y \cdot x = x \cdot y = x \bullet y = y \bullet x, \quad \forall x \in X, y \in Y;$

Lemma (M.G. and A. Sciandra, 2024)

Let (A, \cdot, \bullet) be a cocommutative Hopf brace and $(X, \cdot, \bullet), (Y, \cdot, \bullet)$ two sub-Hopf braces of (A, \cdot, \bullet) . The following conditions are equivalent :

- there is a unique morphism $p : (X \otimes Y, \cdot_{\otimes}, \bullet_{\otimes}) \rightarrow (A, \cdot, \bullet)$ in \mathbf{HBR}_{coc} such that



commutes ;

- $y \cdot x = x \cdot y = x \bullet y = y \bullet x, \quad \forall x \in X, y \in Y;$
- $x_1 \cdot y_1 \cdot S(x_2) \cdot S(y_2) = S(x_1) \cdot (x_2 \bullet y_1) \cdot S(y_2) = x_1 \bullet y_1 \bullet T(x_2) \bullet T(y_2) = \varepsilon(x)\varepsilon(y)1.$

Corollary

The **Huq commutator** $[X, Y]_{Huq} = [X, Y]$ of two normal sub-Hopf braces of (A, \cdot, \bullet) is the normal sub-Hopf brace $([X, Y], \cdot, \bullet)$, where

$$[X, Y] = \langle \{x_1 \cdot y_1 \cdot S(x_2) \cdot S(y_2), S(x_1) \cdot (x_2 \bullet y_1) \cdot S(y_2), x_1 \bullet y_1 \bullet T(x_2) \bullet T(y_2) \mid x \in X, y \in Y\} \rangle_N.$$

Corollary

The **Huq commutator** $[X, Y]_{Huq} = [X, Y]$ of two normal sub-Hopf braces of (A, \cdot, \bullet) is the normal sub-Hopf brace $([X, Y], \cdot, \bullet)$, where

$$[X, Y] = \langle \{x_1 \cdot y_1 \cdot S(x_2) \cdot S(y_2), S(x_1) \cdot (x_2 \bullet y_1) \cdot S(y_2), x_1 \bullet y_1 \bullet T(x_2) \bullet T(y_2) \mid x \in X, y \in Y\} \rangle_N.$$

Abelian objects

The abelian objects in the category \mathbf{HBR}_{coc} can be characterized as the Hopf braces (A, \cdot, \bullet) such that $\cdot = \bullet$, and the product \cdot is commutative : $x \cdot y = y \cdot x$.

Corollary

The **Huq commutator** $[X, Y]_{Huq} = [X, Y]$ of two normal sub-Hopf braces of (A, \cdot, \bullet) is the normal sub-Hopf brace $([X, Y], \cdot, \bullet)$, where

$$[X, Y] = \langle \{x_1 \cdot y_1 \cdot S(x_2) \cdot S(y_2), S(x_1) \cdot (x_2 \bullet y_1) \cdot S(y_2), x_1 \bullet y_1 \bullet T(x_2) \bullet T(y_2) \mid x \in X, y \in Y\} \rangle_N.$$

Abelian objects

The abelian objects in the category \mathbf{HBR}_{coc} can be characterized as the Hopf braces (A, \cdot, \bullet) such that $\cdot = \bullet$, and the product \cdot is commutative : $x \cdot y = y \cdot x$.

In other words : $\mathbf{Ab}(\mathbf{HBR}_{coc}) \cong \mathbf{Ab}(\mathbf{Hopf}_{K, coc})$. This is an **abelian** category.

The **abelianisation** functor of \mathbf{HBR}_{coc} can then be seen as the composite of the left adjoints in

$$\mathbf{Ab}(\mathbf{Hopf}_{K,coc}) \begin{array}{c} \xleftarrow{\quad} \\ \perp \\ \xrightarrow{\quad} \end{array} \mathbf{Hopf}_{K,coc} \begin{array}{c} \xleftarrow{\quad} \\ \perp \\ \xrightarrow{\quad} \end{array} \mathbf{HBR}_{coc},$$

which is a “linearization” of

$$\mathbf{Ab} \begin{array}{c} \xleftarrow{\quad} \\ \perp \\ \xrightarrow{\quad} \end{array} \mathbf{Grp} \begin{array}{c} \xleftarrow{\quad} \\ \perp \\ \xrightarrow{\quad} \end{array} \mathbf{SKB}.$$

The **abelianisation** functor of \mathbf{HBR}_{coc} can then be seen as the composite of the left adjoints in

$$\mathbf{Ab}(\mathbf{Hopf}_{K,coc}) \begin{array}{c} \xleftarrow{\quad} \\ \perp \\ \xrightarrow{\quad} \end{array} \mathbf{Hopf}_{K,coc} \begin{array}{c} \xleftarrow{\quad} \\ \perp \\ \xrightarrow{\quad} \end{array} \mathbf{HBR}_{coc},$$

which is a “linearization” of

$$\mathbf{Ab} \begin{array}{c} \xleftarrow{\quad} \\ \perp \\ \xrightarrow{\quad} \end{array} \mathbf{Grp} \begin{array}{c} \xleftarrow{\quad} \\ \perp \\ \xrightarrow{\quad} \end{array} \mathbf{SKB}.$$

The **central extensions** of Hopf braces can be described as the surjective morphisms $f: (A, \cdot, \bullet) \rightarrow (B, \cdot, \bullet)$ having the property that

$$a \cdot k = k \cdot a = a \bullet k = k \bullet a, \quad \forall a \in A, \forall k \in K.$$

Final remarks

- ▶ The category \mathbf{HBR}_{coc} has many strong “exactness properties” (it is semi-abelian and strongly protomodular), and the descriptions of some basic categorical constructions (commutator, semi-direct product) are available.

Final remarks

- ▶ The category \mathbf{HBR}_{coc} has many strong “exactness properties” (it is semi-abelian and strongly protomodular), and the descriptions of some basic categorical constructions (commutator, semi-direct product) are available.
- ▶ The Hopf formulas for **homology of skew braces** have been studied in a joint work with L. Vendramin and T. Letourmy (preprint, 2024). These results are based on Categorical Galois Theory.

Final remarks

- ▶ The category \mathbf{HBR}_{coc} has many strong “exactness properties” (it is semi-abelian and strongly protomodular), and the descriptions of some basic categorical constructions (commutator, semi-direct product) are available.
- ▶ The Hopf formulas for **homology of skew braces** have been studied in a joint work with L. Vendramin and T. Letourmy (preprint, 2024). These results are based on Categorical Galois Theory.
- ▶ It would be interesting to extend these results to **cocommutative Hopf braces**, describe “crossed skew braces”, etc.

References

- A. Agore, *Constructing Hopf braces*, Internat. J. Math. (2019)
- A. Agore and A. Chirvasitu, *On the category of Hopf braces*, in preparation.
- I. Angiono, C. Galindo, L. Vendramin, *Hopf braces and Yang-Baxter operators*, Proc. Amer. Math. Soc. (2017)
- M. Gran, G. Kadjo and J. Vercruysse, *Split extension classifiers in the category of cocommutative Hopf algebras*, Bull. Belgian Math. Soc. (2016)
- M. Gran, F. Sterck and J. Vercruysse, *A semi-abelian extension of a theorem by Takeuchi*, J. Pure Appl. Algebra (2019)
- M. Gran, T. Letourmy and L. Vendramin, *Hopf formulae for homology of skew braces*, preprint (2024)
- M. Gran and A. Sciandra, *Hopf braces and semi-abelian categories*, preprint (2024)
- L. Guarnieri, L. Vendramin, *Skew braces and the Yang-Baxter equation*, Math. Comp. (2017)
- G. Janelidze, L. Márki and W. Tholen, *Semi-abelian categories*, J. Pure Appl. Algebra (2002)
- S. Mac Lane, *Duality for groups*, Bull. Amer. Math. Soc. (1950)
- M. Takeuchi, *A correspondence between Hopf ideals and sub-Hopf algebras*, Manuscr. Math. (1972)