Lecture 8 - Applying the advection equation

Goals

- Introduce the advection-diffusion equation
- Find solution for steady-state with a constant surface heat flux
- Find solution for steady-state with a constant basal temperature

Advection-diffusion equation in 1D

If we add advection into our diffusion equation from before for heat conduction, we get the advection-diffusion equation. This is a versatile and widely applied equation in many different fields. For us, we can now simulate 1D heat advection by processes such as sedimentation and erosion. Mathematically, we'll start with our two equations: (1) The diffusion equation without heat production and (2) the advection equation, then combine them.

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2} \qquad \text{Diffusion} \tag{20}$$

$$\frac{\partial T}{\partial t} = v_z \frac{\partial T}{\partial z} \qquad \text{Advection} \tag{21}$$

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2} + v_z \frac{\partial T}{\partial z} \qquad \text{Diffusion + Advection} \tag{22}$$

In steady state, we can ignore the transient term $\partial T/\partial t$, so

$$\cancelto{0}{\frac{\partial T}{\partial t}} = \kappa \frac{\partial^2 T}{\partial z^2} + v_z \frac{\partial T}{\partial z} \qquad \text{Steady-state advection-diffusion equation} \tag{23}$$

$$\frac{\partial^2 T}{\partial z^2} = -\frac{v_z}{\kappa}\frac{\partial T}{\partial z} \qquad \text{Rearranged} \tag{24}$$

$$\tag{25}$$

Another way to write the previous equation is

$$\frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right) = -\frac{v_z}{\kappa}\frac{\partial T}{\partial z} \tag{26}$$

In this case, we can make some substitutions and find something quite useful. Assume $f = \partial T/\partial z$ and $c = v_z/\kappa$. With this, we can say $f'(z) = -cf(z)$. This is a common form of differential equation with a solution $f(z) = f(0)\mathrm{e}^{cz}$. Thus, in terms of our equation we can say

$$\frac{\partial T}{\partial z} = -\left.\frac{\partial T}{\partial z}\right|_{(z=0)} \mathrm{e}^{-(v_z z/\kappa)} \tag{27}$$

Solutions to the steady-state advection-diffusion equation

Constant gradient g at surface

The simplest solution to the previous equation is to assume a constant temperature gradient g at $z = 0$.

$$\frac{\partial T}{\partial z} = -\frac{\partial T}{\partial z}\bigg|_{(z=0)} \mathrm{e}^{-(v_z z/\kappa)} = g\mathrm{e}^{-(v_z z/\kappa)} \tag{28}$$

$$\int \frac{\partial T}{\partial z} = g \int \mathrm{e}^{-(v_z z/\kappa)} \qquad \text{Integrate} \tag{29}$$

$$T(z) = -\frac{g\kappa}{v_z}\mathrm{e}^{-(v_z z/\kappa)} + c_1 \tag{30}$$

Assume $T(0) = 0$.

$$T(z) = -\frac{g\kappa}{v_z}\mathrm{e}^{-(v_z z/\kappa)} + c_1 \tag{31}$$

$$0 = -\frac{g\kappa}{v_z}\cancelto{1}{\mathrm{e}^{-(v_z 0/\kappa)}} + c_1 \tag{32}$$

$$c_1 = \frac{g\kappa}{v_z} \tag{33}$$

Thus, we find

$$T(z) = -\frac{g\kappa}{v_z}\mathrm{e}^{-(v_z z/\kappa)} + \frac{g\kappa}{v_z} \tag{34}$$

$$T(z) = \frac{g\kappa}{v_z}\left(1 - \mathrm{e}^{-(v_z z/\kappa)}\right) \qquad \text{Rearranged} \tag{35}$$

Check your understanding:

What should our temperature profile look like?
At constant z, what happens to T if v_z gets large?

Constant temperature T_L at $z = L$

A more useful second boundary condition is to assume $T(L) = T_L$. In this case

$$T(z) = T_L\left(\frac{1 - \mathrm{e}^{-(v_z z/\kappa)}}{1 - \mathrm{e}^{-(v_z L/\kappa)}}\right) \tag{36}$$

The Peclet number

The Peclet number is a useful value for estimating the relative influence of advective versus diffusive heat transfer processes.

$$\mathrm{Pe} = \frac{v_z L}{\kappa} \tag{37}$$

Where κ...

If a typical rock thermal diffusivity is $\kappa = 10^{-6}$ and typical continental crust is 35 km thick, how fast does it need to erode for advection exceed the effects of diffusion?
How would this be different for erosion of the entire lithosphere?

Advection-diffusion equation take-home messages

- Math gets a bit more complex, even for the 'simplest' cases; Often need numerical methods for more complex geometries
- Behavior of the equation is strongly controlled by the boundary conditions
- Even these simple equations can be quite useful. Advection can be a significant influence on the thermal field and these simple calculations allow you to estimate when it is a factor and its magnitude of influence.

Caveats

- Steady-state
- 1-D
- Constants assumed to be constant :)
- No heat production