Introduction to quantitative geology

Lecture 6

Solving the diffusion equation

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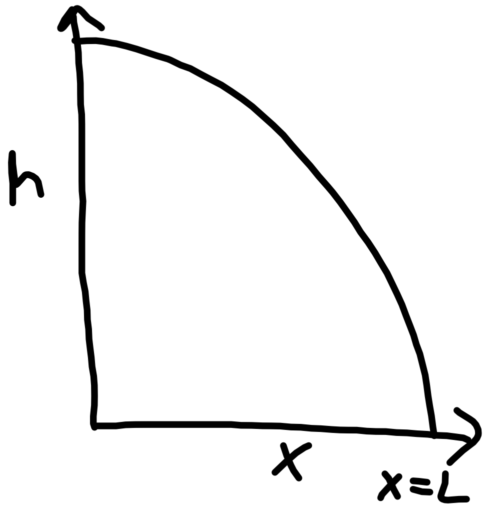
Goals for this lecture:

- Introduce the diffusion equation
- Find solution for steady-state hillslope diffusion



Diffusion components

1. $q = -D \frac{dc}{dx}$ flux $\propto$ gradient

$q = -\rho K \frac{\partial h}{\partial x}$ (hillslope)

2. $\frac{\partial c}{\partial t} = - \frac{\partial q}{\partial x}$ mass conservation

$$\frac{\partial h}{\partial t} = -\frac{1}{\rho} \frac{\partial q}{\partial x}$$

hillslope

Symbols

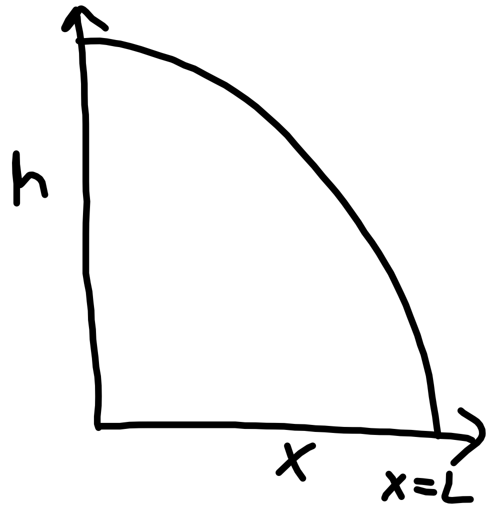
q = sediment flux per unit length [M/L/T]

ρ = sed. density

K = sed. diffusivity

h = elevation

x = dist. from divide



Diffusion components

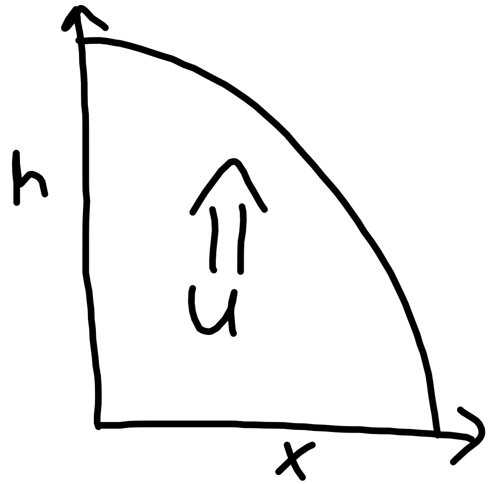
① $q = -\rho k \frac{\partial h}{\partial x}$ flux $\propto$ gradient

② $\frac{\partial h}{\partial t} = -\frac{1}{\rho} \frac{\partial q}{\partial x}$ mass conservation

Put ① in ②

$$\frac{\partial h}{\partial t} = -\frac{1}{\rho} \frac{\partial}{\partial x}\left(-\rho k \frac{\partial h}{\partial x}\right) = k \frac{\partial^2 h}{\partial x^2}$$

$$\boxed{\frac{\partial h}{\partial t} = k \frac{\partial^2 h}{\partial x^2}}$$ diff. eqn

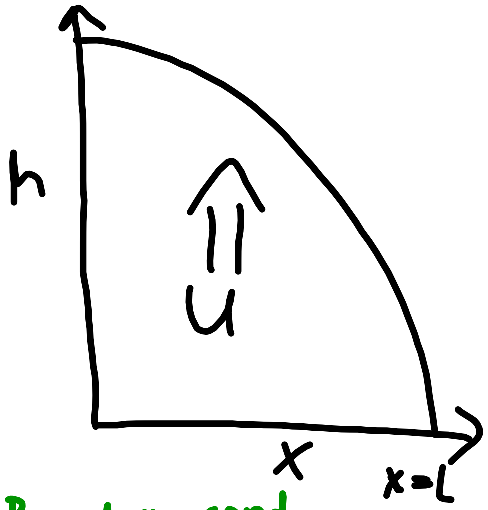


Diffusion equation

$$\frac{\partial h}{\partial t} = K \frac{\partial^2 h}{\partial x^2} + U$$

$$0 = K \frac{\partial^2 h}{\partial x^2} + U \quad \text{(steady state)}$$

$$\frac{d^2 h}{dx^2} = -U/K$$



Boundary cond.

① $h = 0$ at $x = L$

② $\frac{dh}{dx} = 0$ at $x = 0$

$$\frac{d^2h}{dx^2} = -U/K \rightarrow \text{now solve}$$

$$\int \frac{d^2h}{dx^2}\,dx = -\frac{U}{K}\int dx \quad \text{(integrate)}$$

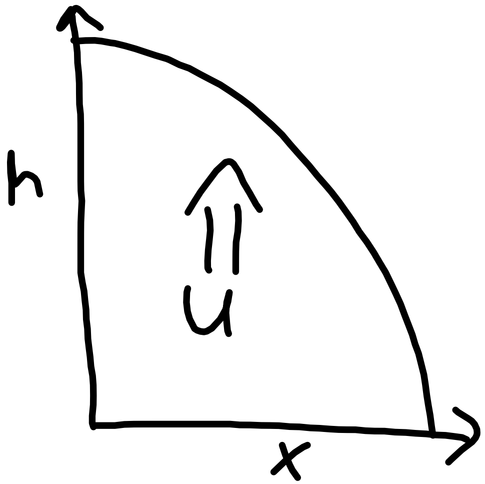
$$\frac{dh}{dx} = -\frac{U}{K}x + C_1 \Rightarrow C_1 = 0$$

$$\int \frac{dh}{dx}\,dx = -\frac{U}{K}\int x\,dx + C_1\int dx$$

$$\boxed{h(x) = -\frac{U}{K}\frac{x^2}{2} + C_1 x + C_2}$$

$$0 = -\frac{UL^2}{2K} + C_2$$

$$C_2 = \frac{UL^2}{2K}$$



General solution

$$h(x) = -\frac{U}{K}\frac{x^2}{2} + C_1 x + C_2$$

$C_1 = 0$; $C_2 = \frac{UL^2}{2K}$

$$h(x) = -\frac{U}{2K}x^2 + \frac{U}{2K}L^2$$

$$\boxed{h(x) = \frac{U}{2K}(L^2 - x^2)}$$