Introduction to Quantitative Geology

Lecture 8 - Solving the advection-diffusion equation

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$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2} \quad \text{Diff.}$$

$$\frac{\partial T}{\partial t} = v_z \frac{\partial T}{\partial z} \quad \text{Adv.}$$

$$\cancel{\frac{\partial T}{\partial t}} = 0 = k \frac{\partial^2 T}{\partial z^2} + v_z \frac{\partial T}{\partial z}$$

BC 1: steady state: $\frac{\partial T}{\partial t} = 0$.

$$\cancel{\frac{\partial T}{\partial t}}^{=0} = \kappa \frac{\partial^2 T}{\partial z^2} + v_z \frac{\partial T}{\partial z}$$

$$\frac{\partial^2 T}{\partial z^2} = -\frac{v_z}{\kappa} \frac{\partial T}{\partial z}$$

$$f := \frac{\partial T}{\partial z}$$

$$f' = \frac{\partial^2 T}{\partial z^2}$$

$$f' = -\frac{v}{\kappa} f$$

$$c = \frac{v}{\kappa}$$

$$f'(z) = -c f(z)$$

$$f(z) = f(0) e^{-cz}$$

$$f(z) = \exp^{cz} = c f(z)$$

$$f'(z) = -c\,f(z)$$
$$f(z) = f(0)\,e^{-cz}$$

$$f := \frac{\partial T}{\partial z}$$
$$f' = \frac{\partial^2 T}{\partial z^2}$$

$$\frac{\partial T}{\partial z} = -\frac{\partial T}{\partial z}\Big|_{z=0} e^{-cz}$$
$$\frac{\partial T}{\partial z}\Big|_{z=0} =: g$$

$$\frac{\partial T}{\partial z} = -g e^{-cz}$$

$$T(z) = \dots$$

$$\frac{\partial T}{\partial z} = -g e^{-cz} \quad \Big| \int_z$$

$$c = \frac{v}{\kappa}$$

$$\int_z \frac{\partial T}{\partial z} = -g \int_z e^{-cz}$$

$$T(z) = -g \frac{1}{c} e^{-cz} + C_1$$

$$T(z) = -g\frac{1}{c}e^{-\frac{v}{\kappa}z} + C_1$$

$$BC: \quad T(z=0) \overset{!}{=} 0$$

$$0 = -g\frac{\kappa}{v}\underbrace{e^{-\frac{v}{\kappa}\cdot 0}}_{=1} + C_1$$

$$\Rightarrow C_1 = g\frac{\kappa}{v}$$

$$T(z) = -g\frac{\kappa}{v}e^{-\frac{v}{\kappa}z} + \frac{g\kappa}{v}$$

$$= g\frac{\kappa}{v}\left(1 - e^{-\frac{v}{\kappa}z}\right)$$

$$T(z=0)=0$$

$$T(z) = -g\frac{K}{V} e^{-\frac{V}{K}z} + \frac{gK}{V}$$

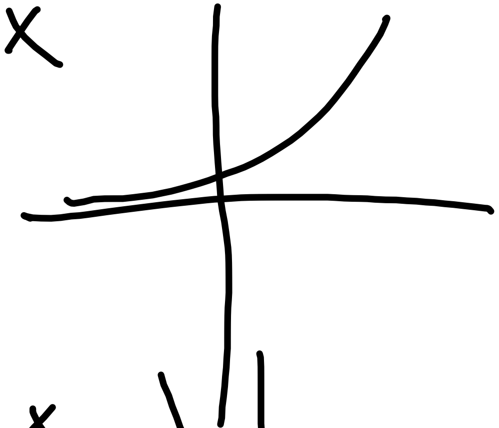
$$= g\frac{K}{V}\left(1 - e^{-\frac{V}{K}z}\right)$$

$$V \to \infty \qquad g\frac{K}{V} \to 0$$

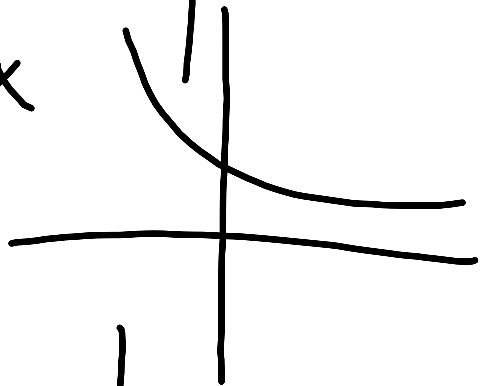
$$T(z=L) = T_L$$

$$T(z) = T_L\left(\frac{1-e^{-\frac{V}{K}z}}{1-e^{-\frac{VL}{K}}}\right) \qquad \frac{VL}{K} = Pe$$

e^{x}



e^{-x}



$-e^{-x}+1$