Introduction to Quantitative Geology (Course 54070)
Spring 2016

Lecture 6 - Applying the diffusion equation

Goals

- Introduce diffusion equation
- Find solution for steady-state hillslope diffusion

General requirements - diffusion

The diffusion equation has two general requirements: Transport/transfer proportional to gradient and conservation of mass/energy. We expect some form of flux $\propto$ gradient.

Hillslope transport

Consider a cross-section through a hillslope where the drainage divide (ridge crest) is at $x = 0$ and the river that defines the minimum elevation is at $x = L$. Assume the elevation of the river is equal to zero with respect to the *y* axis.

Transfer proportional to gradient

In lecture we saw that

$$q = -D \frac{\partial C_{\mathrm{A}}}{\partial x} \tag{1}$$

In our case, we can say

$$q = -\rho \kappa \frac{\partial h}{\partial x} \tag{2}$$

- q: Sediment flux per unit length; mass flux $[M/L/T]$
- ρ: bulk sediment density
- κ: Sediment diffusivity $[L^2/T]$
- h: Elevation
- x: Distance from divide

Mass conservation

In our example in lecture we saw that

$$\frac{\partial C_{\mathrm{A}}}{\partial t} = -\frac{\partial q}{\partial x} \tag{3}$$

Here, we assume any change in flux results in a change in elevation. Consider a simple example of more material entering than leaving. Mathematically, change in elevation is equal to the change in flux per unit length divided by the bulk density.

$$\frac{\partial h}{\partial t} = -\frac{1}{\rho}\frac{\partial q}{\partial x} \tag{4}$$

Alternative is to move ρ to other side.

Diffusion equation for hillslope transport

Substitute one into the other. If we assume ρ is constant, they cancel and we are left with the "classical" diffusion equation.

$$\frac{\partial h}{\partial t} = \kappa\frac{\partial^2 h}{\partial x^2} \tag{5}$$

Solving the diffusion equation in steady state

General scenario

Assume landscape is being uplifted at a rate U and river is incising at the same rate, but opposite direction at $x = L$.

$$\frac{\partial h}{\partial t} = \kappa\frac{\partial^2 h}{\partial x^2} + U \tag{6}$$

U is the uplift rate. Steady-state, so $\partial h/\partial t = 0$.

$$0 = \kappa\frac{\partial^2 h}{\partial x^2} + U \tag{7}$$

At this point, since we only have derivatives with respect to x we can say

$$0 = \kappa\frac{d^2 h}{dx^2} + U \tag{8}$$

Now, we can put the constants on one side:

$$\frac{d^2 h}{dx^2} = -\frac{U}{\kappa} \tag{9}$$

At this point, we are ready to solve the equation for $h(x)$ by integrating twice. First integration:

$$\int \frac{d^2 h}{dx^2}dx = -\frac{U}{\kappa}\int dx \tag{10}$$

$$\frac{dh}{dx} = -\frac{U}{\kappa}x + c_1 \tag{11}$$

Now we simply integrate a second time:

$$\int \frac{dh}{dx}dx = -\frac{U}{\kappa}\int x dx + c_1\int dx \tag{12}$$

$$h(x) = -\frac{U}{2\kappa}x^2 + c_1 x + c_2 \tag{13}$$

Applying the boundary conditions
At this point, we have a solution, but in order to use it we will need to apply boundary conditions. The boundary conditions will allow us to solve for c_1 and c_2, the two constants of integration. Typically, this means that we have certain places in the solution domain where we know either the value of $h(x)$ or the value of its first derivative $h'(x) = \frac{dh}{dx}$, the slope.
For the hillslope cross-section, are there any places where we might claim to know $h(x)$ or $h'(x)$?

Finding integration constant 1
To get this, need value of dh/dx at one end of the slope. The divide is a good choice, slope there must be 0.

$$\left|\frac{dh}{dx}\right|_{x=0} = 0 \tag{14}$$

Plug that value in to Equation 11 and we find $c_1 = 0$.

Finding integration constant 2
We know that $h(L) = 0$, so we plug that in.

$$h(x) = -\frac{U}{2\kappa}x^2 + c_2 \tag{15}$$
$$0 = -\frac{U}{2\kappa}L^2 + c_2 \tag{16}$$
$$c_2 = \frac{U}{2\kappa}L^2 \tag{17}$$

Plug that in and we find.

$$h(x) = -\frac{U}{2\kappa}x^2 + \frac{U}{2\kappa}L^2 \tag{18}$$
$$h(x) = \frac{U}{2\kappa}\left(L^2 - x^2\right) \tag{19}$$

Check your understanding:

Looking at our equation for $h(x)$, what should our hillslope look like?
How with the geometry of the hillslope change with the different variables?

Features of our predictive model

Relief: $R = \frac{UL^2}{2\kappa}$. Relief is the difference in the elevation at the drainage divide ($h(0)$) and the river ($h(l)$).

Max slope: $|\frac{\partial h}{\partial x}|_{\max} = \frac{UL}{\kappa}$ (slope at $x = L$)

Time constant*: $\tau = L^2/\kappa$ (time required for response to change)