Lecture 10 - Example of viscous flow

Goals

- Linear viscous flow down an inclined plane

Linear viscous flow

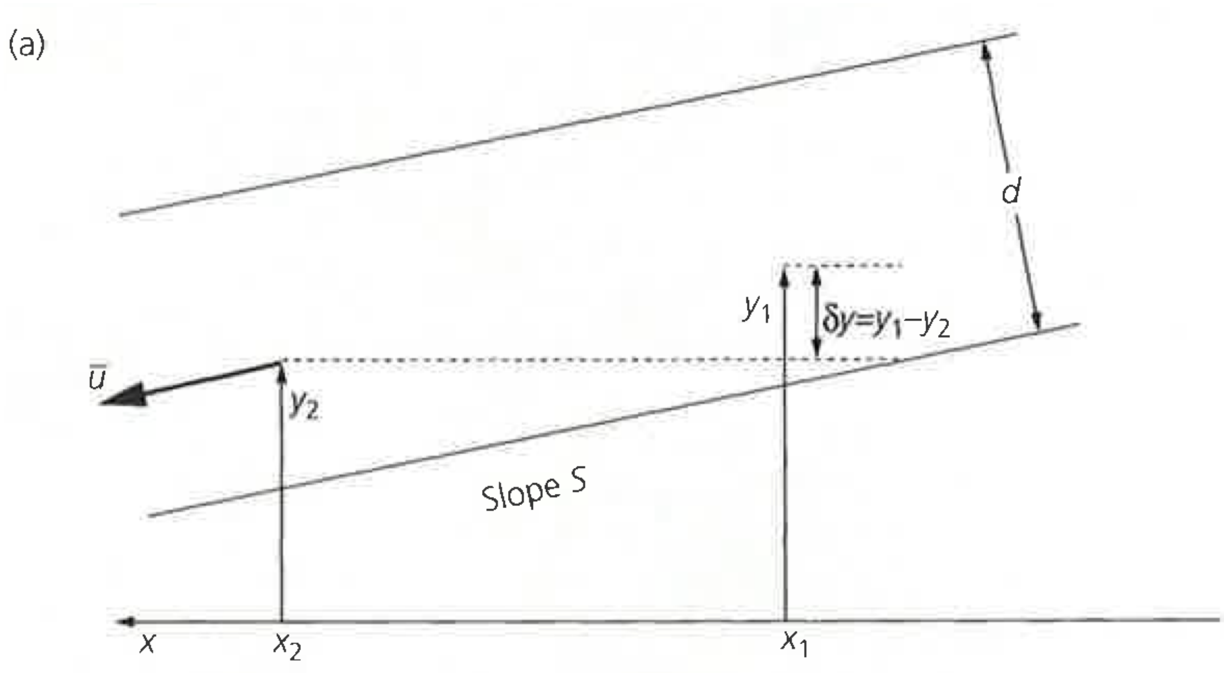

Figure 1: Viscous flow down an incline

Shear stress in the fluid

For a fluid flowing down an inclined slope, the change in potential energy per unit area of the contact surface along some length of the slope δx is

$$\Delta E_{\mathrm{P}} = \rho g h \delta z \tag{38}$$

$$= \rho g h (\bar{u} S) \tag{39}$$

where ΔE_{P} is the change in potential energy,
ρ is the fluid density,
g is the acceleration due to gravity,
h is the thickness of the flow,
δz is the elevation change over the distance x,
$\bar{u}$ is the average flow velocity, and
S is the slope of the incline.

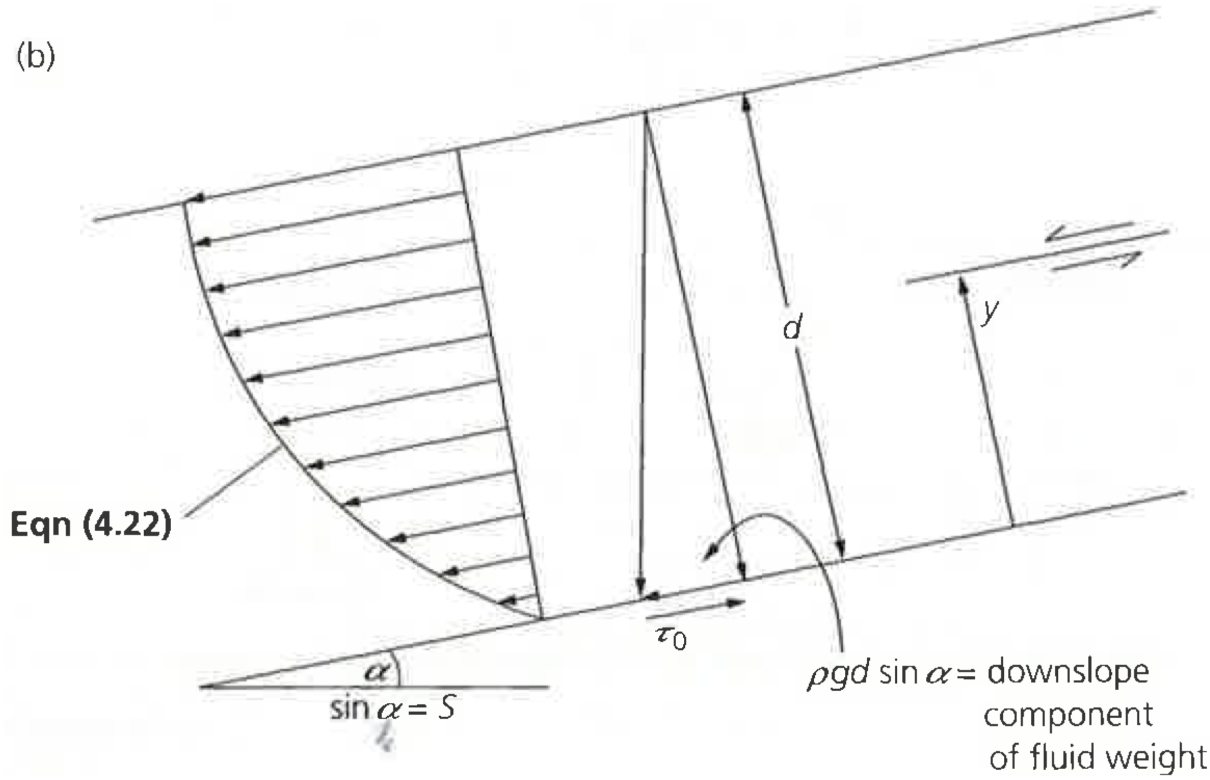

Figure 2: Viscous flow down an incline

The downslope component of the gravity force on the flow is thus $\rho g h S$, which must be opposed by the drag force at the base of the flow τ_0. Thus, we can say

$$\tau_0 = \rho g h S \tag{40}$$

For any position z above the base of the flow, we can also calculate the shear stress at that position, which is a function of the thickness of the overlying fluid $(h - z)$

$$\tau_z = \rho g S(h - z) \qquad \text{Since we know that } \tau_0 = \rho g h S \tag{41}$$

$$\tau_z = \tau_0 \left(1 - \frac{z}{h}\right) \tag{42}$$

What does this suggest about the shear strength as a function of depth in the fluid?

Linking to viscous flow

For a laminar flow, we know $\tau = \eta du/dz$, so we can rewrite the resistance equation as

$$\tau_z = \eta \frac{du}{dz} \tag{43}$$

$$\tau_z = \rho g S(h - z) \tag{44}$$

$$\eta \frac{du}{dz} = \rho g S(h - z) \tag{45}$$

$$\frac{du}{dz} = \frac{\rho g S(h - z)}{\eta} \tag{46}$$

$$\frac{du}{dz} = \frac{\rho g S}{\eta}(h - z) \tag{47}$$

If we integrate this equation with respect to z, we find

$$\int \frac{du}{dz} = \frac{\rho g S}{\eta} \int (h - z) \tag{48}$$

$$u = \frac{\rho g S}{\eta} \left(zh - \frac{z^2}{2} \right) + c_1 \tag{49}$$

If we assume the flow velocity $u = 0$ at $z = 0$ (the base of the flow), the constant $c_1 = 0$, so we are left with

$$u = \frac{\rho g S}{\eta} \left(zh - \frac{z^2}{2} \right) \tag{50}$$

Check your understanding:

What would the velocity profile look like in this flow?
Where is the maximum velocity?
What happens if the viscosity decreases? Slope increases? Thickness increases?

Viscous flow take-home messages

- Flow is a balance between the gravitational force on the fluid and resistance (drag) at the base
- Flow velocity increases following a parabolic geometry from $u = 0$ at the base to $\frac{\rho g S}{\eta} \frac{z^2}{2}$

Caveats

- Steady-state
- 1-D
- Laminar flow!
- Constants are constant
- No temperature dependence